ESE 531: Digital Signal Processing

Lec 14: March 1, 2018

Review, Generalized Linear Phase Systems

Lecture Outline

- Review: All Pass Systems
- Review: Minimum Phase Systems
- General Linear Phase Systems

All-Pass Systems

All-Pass Filters

- A system is an all-pass system if

$$\left|H(e^{j\omega})\right| = 1, \quad \text{all } \omega$$

- Its phase response $\theta(\omega)$ may be non-trivial

$$H_{\text{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

General All-Pass Filter

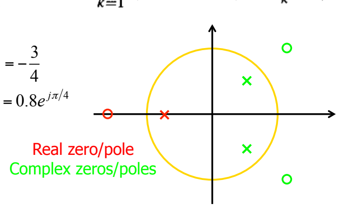
- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\text{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$



All-Pass Properties

- Claim: For a stable, causal ($r < 1$) all-pass system:
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
    - Unwrapped phase always non-positive and decreasing
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$
    - Group delay always positive

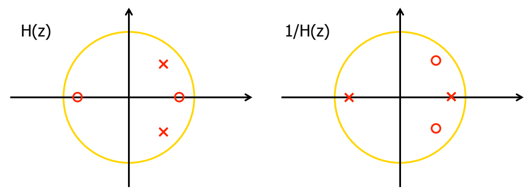
Pg 309 in text book

  - Intuition
    - delay is positive → system is causal
    - Phase negative → phase lag

Minimum-Phase Systems

Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle

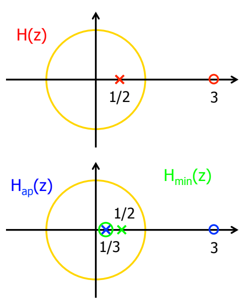


Min-Phase Decomposition Example

$$H(z)=\frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

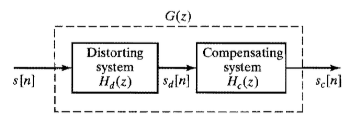
- Set $H_{ap}(z)=\dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$

$$H_{\min}(z)=-3\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$



Min-Phase Decomposition Purpose

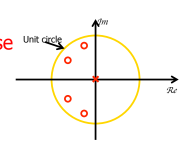
- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\, H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z)$ → $H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

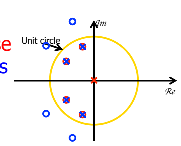
Minimum Energy-Delay Property

Min phase



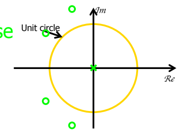
Minimum Energy-Delay Property

Min phase
x All Pass



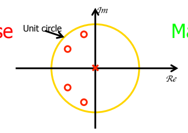
Minimum Energy-Delay Property

Max phase

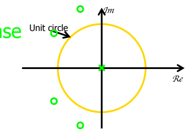


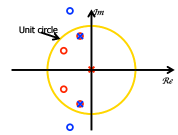
Minimum Energy-Delay Property

Min phase



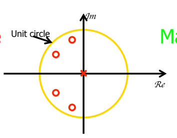
Max phase



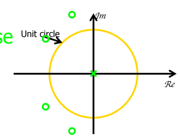


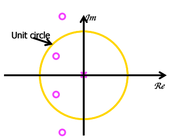
Minimum Energy-Delay Property

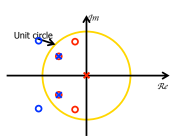
Min phase



Max phase

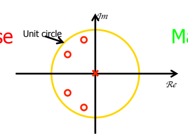




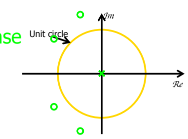


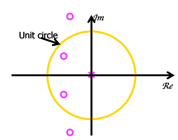
Minimum Energy-Delay Property

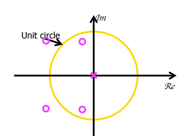
Min phase



Max phase







Energy Delay Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

- $\arg[H_{max}(e^{j\omega})] = \arg[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

  $= \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$

  $= \quad \leq 0 \quad + \quad \leq 0$

Energy Delay Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

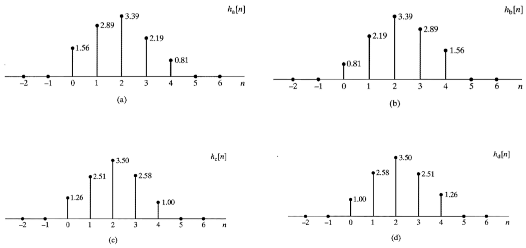
- $\text{grd}[H_{max}(e^{j\omega})] = \text{grd}[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

  $= \text{grd}[H_{min}(e^{j\omega})] + \text{grd}[H_{ap}(e^{j\omega})]$

  $= \quad \geq 0 \quad + \quad \geq 0$

Minimum Energy-Delay Property

Min phase | Max phase



Generalized Linear Phase Systems

Generalized Linear Phase

- An LTI system has generalized linear phase if frequency response $H(e^{j\omega})$ can be expressed as:

$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha + j\beta}, \quad |\omega| < \pi$$

- Where $A(\omega)$ is a real function.

Generalized Linear Phase

- An LTI system has generalized linear phase if frequency response $H(e^{j\omega})$ can be expressed as:

$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha + j\beta}, \quad |\omega| < \pi$$

- Where $A(\omega)$ is a real function.
- What is the group delay?

Causal FIR Systems

$$y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M], \quad \text{all } n$$

$$H(z) = b_0 + b_1 z^{-1} + \ldots + b_M z^{-M} = b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})$$

$$h[n] = \begin{cases} b_n, & n = 0,1,\ldots,M \\ 0, & otherwise \end{cases}$$

Causal FIR Systems

- Causal FIR systems have generalized linear phase if they have impulse response length (M+1)
- It can be shown if

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$

- then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \ldots + x[n]}{M+1}$$

Impulse response

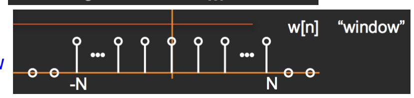
Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

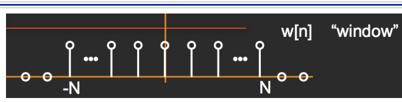
$$y[n] = \frac{x[n-M] + \ldots + x[n]}{M+1}$$

Impulse response

Scaled &Time Shifted window



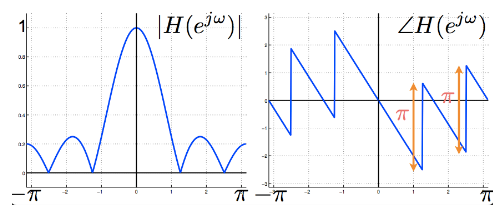
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N+1/2)\omega)}{\sin(\omega/2)}$$

$$\frac{1}{M+1} w[n-M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2+1/2)\omega)}{\sin(\omega/2)}$$

Example: Moving Average



Causal FIR Systems

- Causal FIR systems have generalized linear phase if they have impulse response length (M+1)
- It can be shown if

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$

- Then

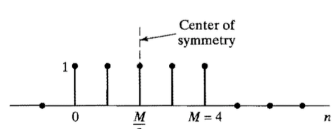
$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

- Sufficient conditions to guarantee GLP, not necessary
  - Causal IIR can also have GLP

FIR GLP: Type I

Type I Even Symmetry, M even

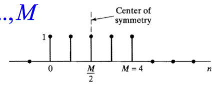
$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Even}} \; e^{-j\omega M/2}$ ← integer delay

FIR GLP: Type I – Example, M=4

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$

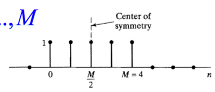
Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Even}} \; e^{-j\omega M/2}$ ← integer delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$
$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega}\right]$$
$$= \underbrace{\left[2h[0]\cos(2\omega) + 2h[1]\cos(\omega) + h[2]\right]}_{A(\omega)\ (even)} e^{-j2\omega}$$

FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Even}} \; e^{-j\omega M/2}$ ← integer delay

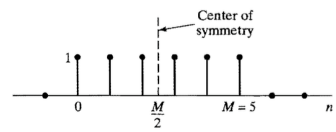
$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right)$$

$$a[0] = h[M/2],$$
$$a[k] = 2h[(M/2)-k], \qquad k = 1,2,\ldots,M/2.$$

FIR GLP: Type II

Type II Even Symmetry, M odd

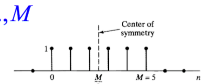
$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



FIR GLP: Type II

Type II Even Symmetry, M odd

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Even}} \; e^{-j\omega M/2}$ ← integer delay

FIR GLP: Type II – Example, M=3

Type II Even Symmetry, M odd

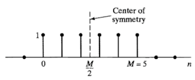
Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Even}} \; e^{-j\omega M/2}$ ← integer delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$$

$$H(e^{j\omega}) = \underbrace{\left[2h[0]\cos(3\omega/2) + 2h[1]\cos(\omega/2)\right]}_{A(\omega)} e^{-j3\omega/2}$$

FIR GLP: Type II

Type II Even Symmetry, M odd

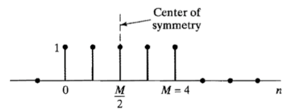


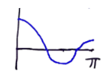
Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

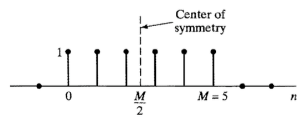
$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right\}$$

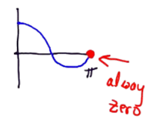
$$b[k] = 2h[(M+1)/2 - k], \qquad k = 1, 2, \ldots, (M+1)/2.$$

FIR GLP: Type I and II



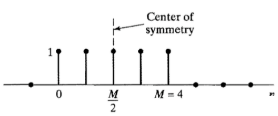


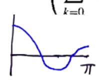




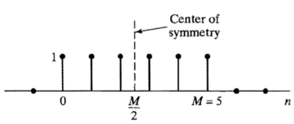
FIR GLP: Type I and II

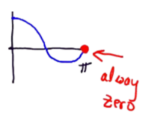
$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right)$$





$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right\}$$

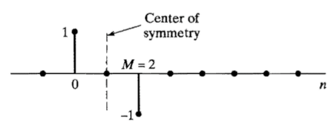




FIR GLP: Type III

Type III Odd Symmetry, M even

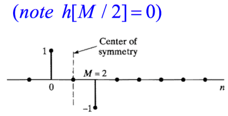
$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note \;\; h[M/2]=0)$



FIR GLP: Type III

Type III Odd Symmetry, M even

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note \;\; h[M/2]=0)$



$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Odd}} \; e^{-j\omega M/2 + j\pi/2}$$

FIR GLP: Type III – Example, M=4

Type III Odd Symmetry, M even

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note \;\; h[M/2]=0)$

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Odd}} \; e^{-j\omega M/2 + j\pi/2}$$

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$

$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} - h[1]e^{-j\omega} - h[0]e^{-j2\omega}\right]$$

$$= \underbrace{\left[2h[0]\sin(2\omega) + 2h[1]\sin(\omega)\right]}_{A(\omega)\;(odd)} \; j\,e^{-j2\omega}$$

FIR GLP: Type III

Type III Odd Symmetry, M even

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note\ h[M/2] = 0)$

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

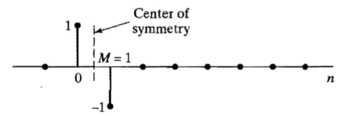
$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin\omega k\right]$$

$$c[k] = 2h[(M/2)-k], \qquad k = 1, 2, \ldots, M/2.$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

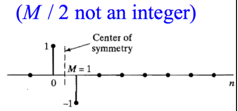
$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$



FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$



$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

fractional delay

FIR GLP: Type IV – Example, M=3

Type IV Odd Symmetry, M Odd

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

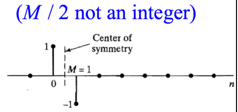
fractional delay

$$H(e^{j\omega}) = \underbrace{\left[2h[0]\sin(3\omega/2) + 2h[1]\sin(\omega/2)\right]}_{A(\omega)}\ je^{-j3\omega/2}$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$



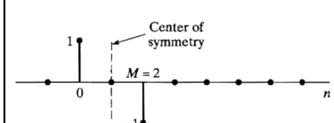
$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

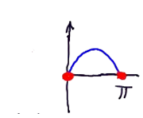
fractional delay

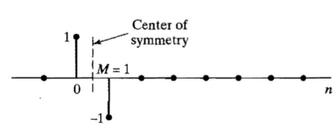
$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k - \tfrac{1}{2}\right)\right]\right]$$

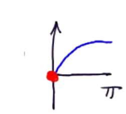
$$d[k] = 2h[(M+1)/2 - k], \qquad k = 1, 2, \ldots, (M+1)/2.$$

FIR GLP: Type III and IV



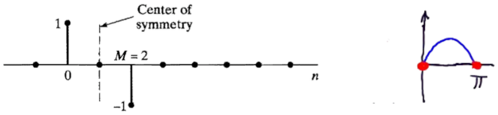


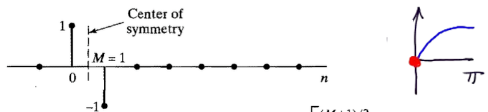




FIR GLP: Type

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin\omega k\right]$$





$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right]$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M}$$
$$= z^{-M}H(z^{-1}).$$

If z_0 is a zero then z_0^{-1} is also a zero.

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

If z_0 is a zero then z_0^{-1} is also a zero.

- If h[n] is real,

If z_0 is a zero then z_0^{*} is also a zero.

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})(1-r^{-1}e^{j\theta}z^{-1})(1-r^{-1}e^{-j\theta}z^{-1})$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})(1-r^{-1}e^{j\theta}z^{-1})(1-r^{-1}e^{-j\theta}z^{-1})$$

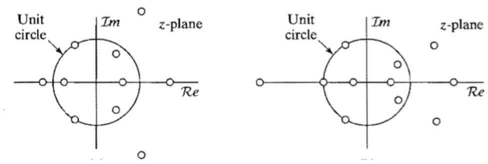
- If zero is on unit circle (r=1)

$$(1-e^{j\theta}z^{-1})(1-e^{-j\theta}z^{-1}).$$

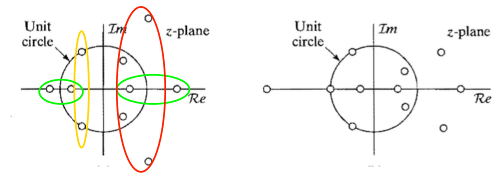
- If zero is real and not on unit circle (θ =0)

$$(1\pm rz^{-1})(1\pm r^{-1}z^{-1}).$$

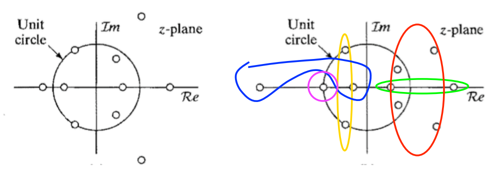
Zeros of GLP System – Type I and II



Zeros of GLP System – Type I and II



Zeros of GLP System – Type I and II



Zeros of GLP System – Type II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M}$$
$$= z^{-M}H(z^{-1}).$$

Zeros of GLP System – Type II

- FIR GLP System Function

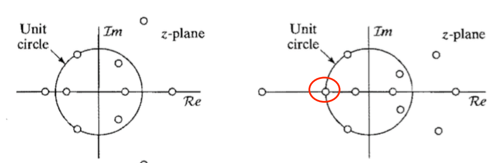
$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M}$$
$$= z^{-M}H(z^{-1}).$$

Consider $z=-1$: $H(-1)=(-1)^{-M}H(-1)$

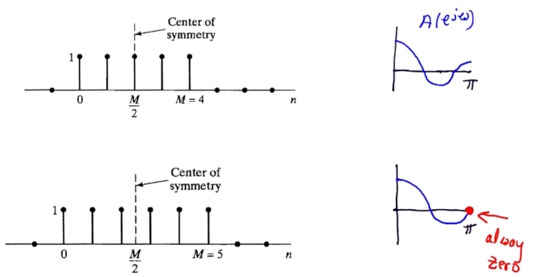
$\Rightarrow$ for M odd, $z=-1$ must be a zero (Type II)

Zeros of GLP System – Type I and II

Type I | Type II



FIR GLP: Type I and II



Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

If z_0 is a zero then z_0^{-1} is also a zero.

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

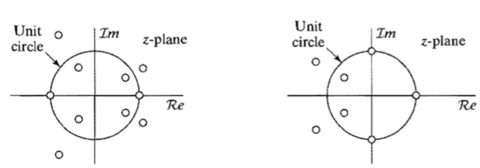
- If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

- If zero is real and not on unit circle (θ =0)

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

Zeros of GLP System – Type III and IV



Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = -z^{-M} H(z^{-1}).$$

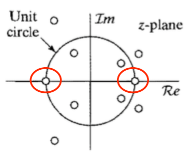
$H(1) = -H(1) \Rightarrow z = 1$ must be a zero

$H(-1) = (-1)^{-M+1} H(-1)$

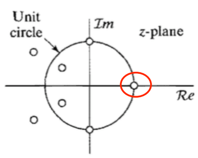
$\Rightarrow$ for M even, $z = -1$ must be a zero (Type III)

Zeros of GLP System – Type III and IV

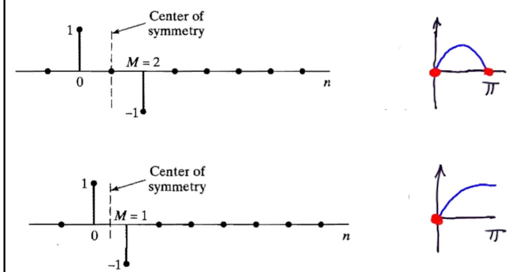
Type III



Type IV



FIR GLP: Type III and IV



GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

$$H(z) = H_{\min}(z) H_{uc}(z) H_{\max}(z)$$

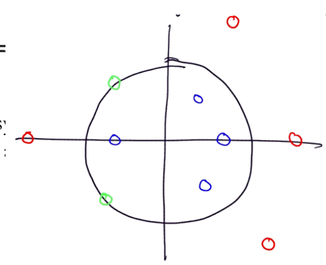
- A min phase system, system containing only zeros on unit circle, and max phase system

GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

$$H(z) =$$

- A min phase s on unit circle,



Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
- All Pass Systems
  - Used for delay compensation
- Minimum Phase Systems
  - Can compensate for magnitude distortion
  - Minimum energy-delay property
- Generalized Linear Phase Systems
  - Useful for design of causal FIR filters

Midterm Exam

- Midterm – 3/13
  - During class
    - Starts at exactly 4:30pm, ends at exactly 5:50pm (80 minutes)
  - Location DRLB A8
  - Old exam posted on previous year's website
  - Covers Lec 1- 13
  - Closed book, one page cheat sheet allowed
  - Calculators allowed, no smart phones
  - Review Session TBD (likely 3/12)
  - Tania office hours moved to Monday (3/12)

Admin

- HW 6
  - Out now
  - Due Friday