ESE 531: Digital Signal Processing

Lec 16: March 20, 2018

Design of FIR Filters

Linear Filter Design

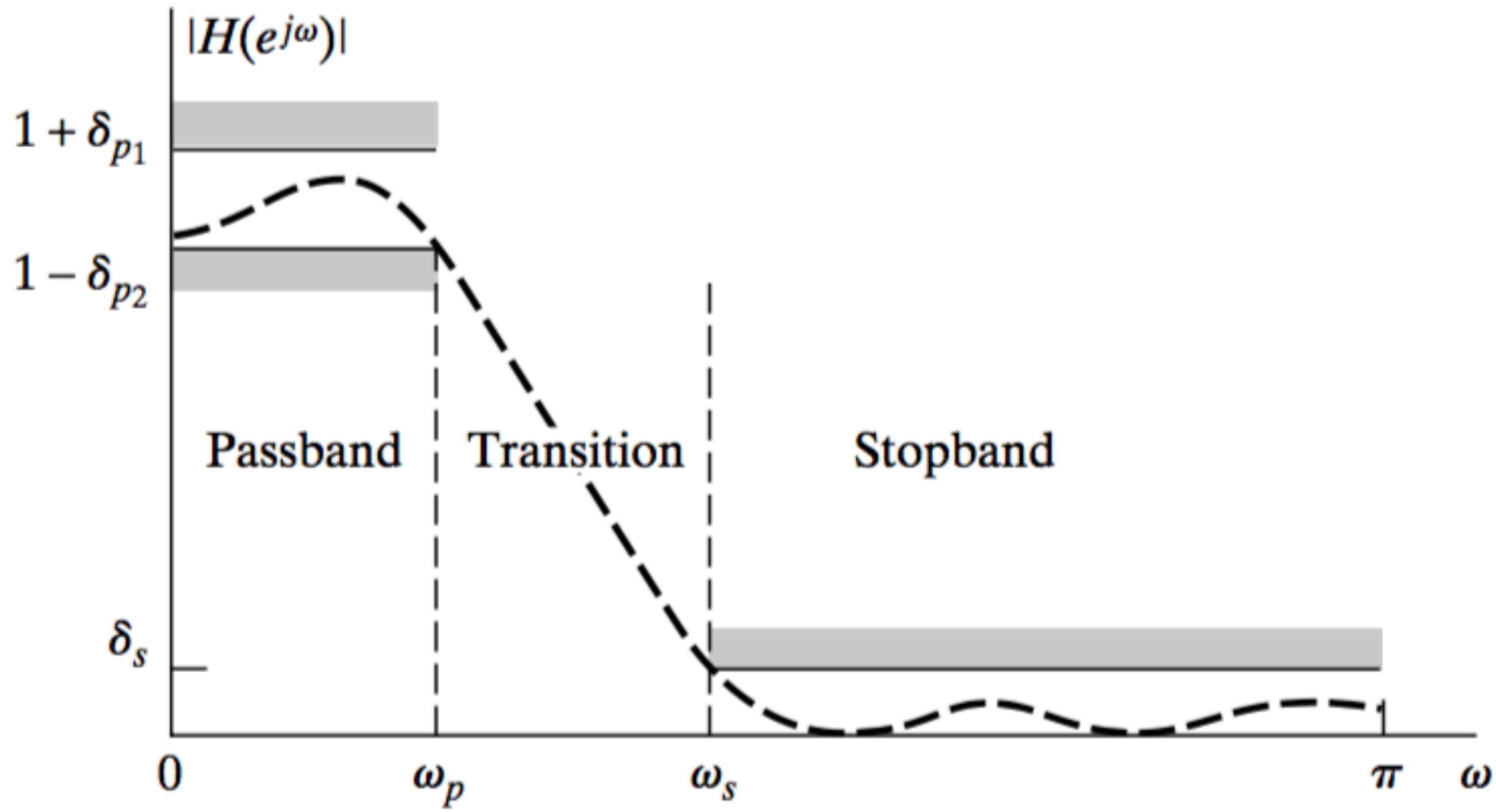
- Used to be an art
  - Now, lots of tools to design optimal filters
- For DSP there are two common classes
  - Infinite impulse response IIR
  - Finite impulse response FIR
- Both classes use finite order of parameters for design
- Today we will focus on FIR designs

What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
  - Both non-linear and linear phase

Filter Specifications



Impulse Invariance

- Let,

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi / T$$

$$h[n] = Th_c(nT)$$

- If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T} \right), \qquad |\omega| < \pi$$

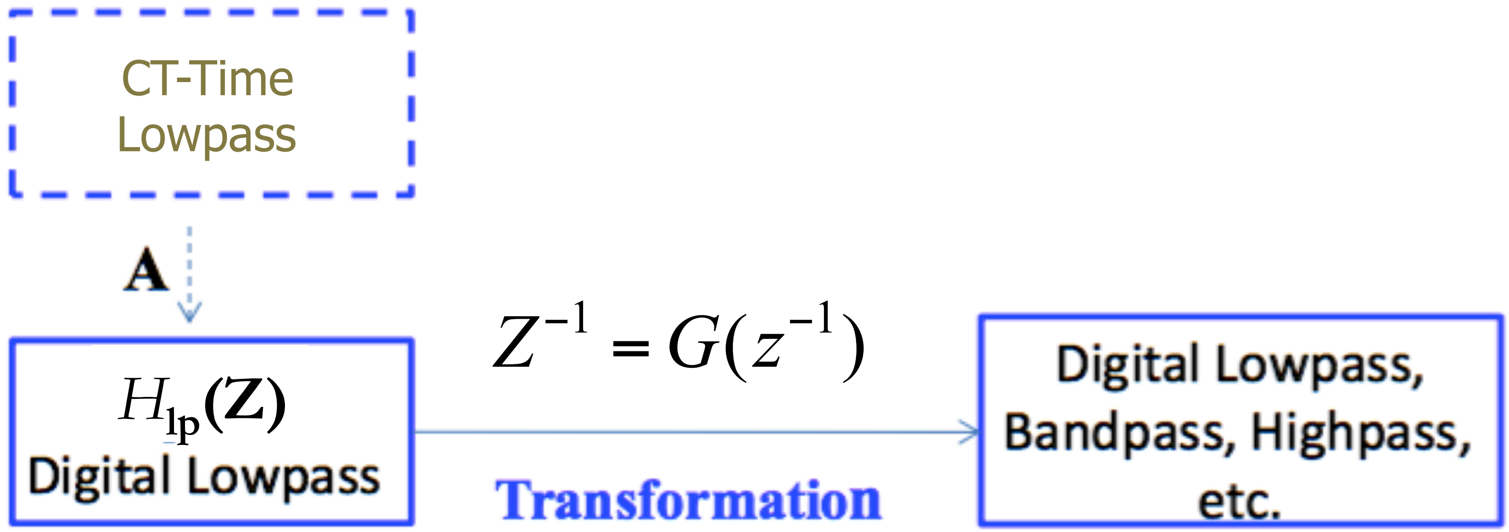
Bilinear Transformation

- The technique uses an algebraic transformation between the variables *s* and *z* that maps the entire jΩ-axis in the s-plane to one revolution of the unit circle in the *z*-plane.

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right).$$

Transformation of DT Filters



- Z – complex variable for the LP filter
- z – complex variable for the transformed filter

- Map Z-plane→z-plane with transformation G

CT Filters

- Butterworth
  - Monotonic in pass and stop bands
- Chebyshev, Type I
  - Equiripple in pass band and monotonic in stop band
- Chebyshev, Type II
  - Monotonic in pass band and equiripple in pass band
- Elliptic
  - Equiripple in pass and stop bands

- Appendix B in textbook

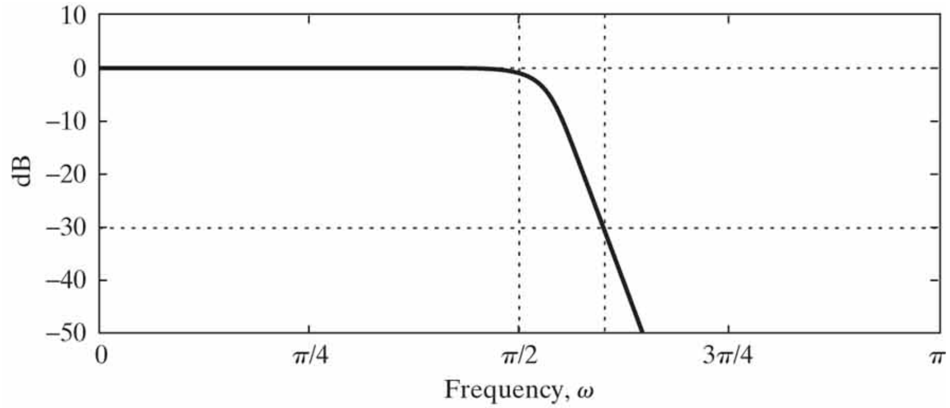
Design Comparison

- Design specifications
  - passband edge frequency $\omega_p = 0.5\pi$
  - stopband edge frequency $\omega_s = 0.6\pi$
  - maximum passband gain = 0 dB
  - minimum passband gain = -0.3dB
  - maximum stopband gain =-30dB

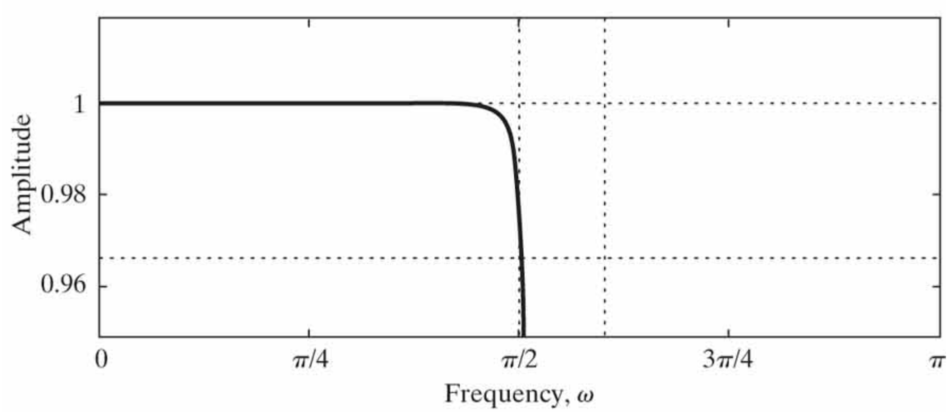
- Use Bilinear transformation to design DT low pass filter for each type

Butterworth

- Butterworth
  - Monotonic in pass and stop bands



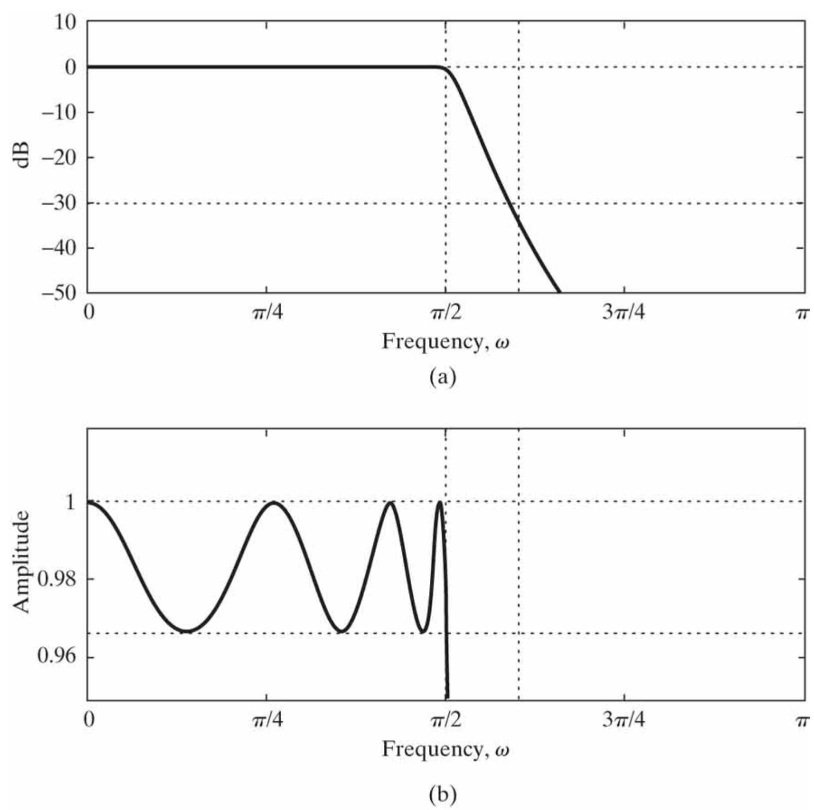
(a)



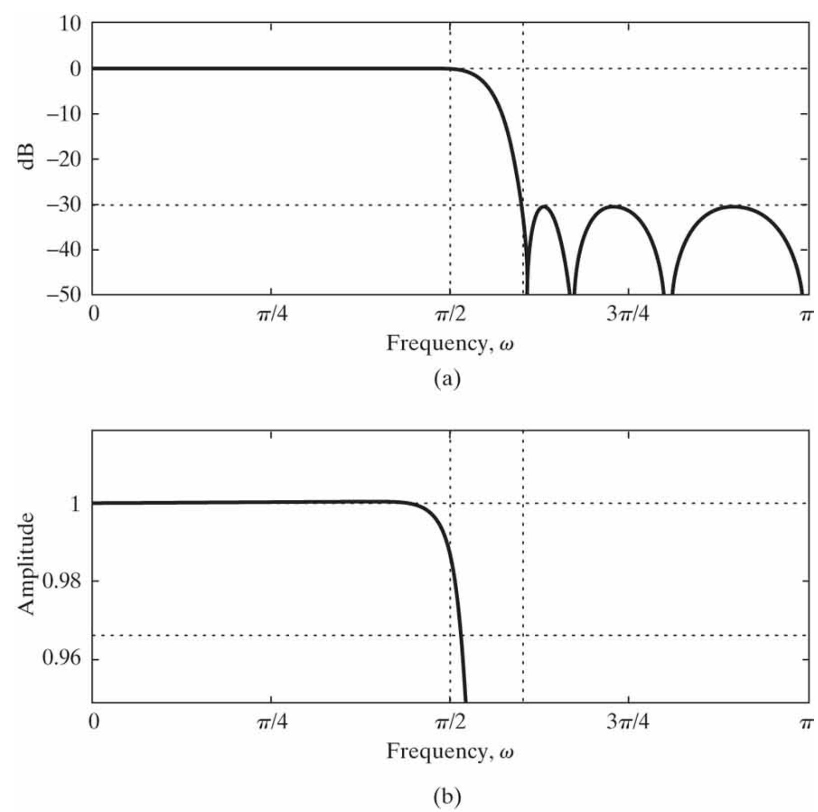
(b)

Chebyshev

- Type I
  - Equiripple in pass band and monotonic in stop band

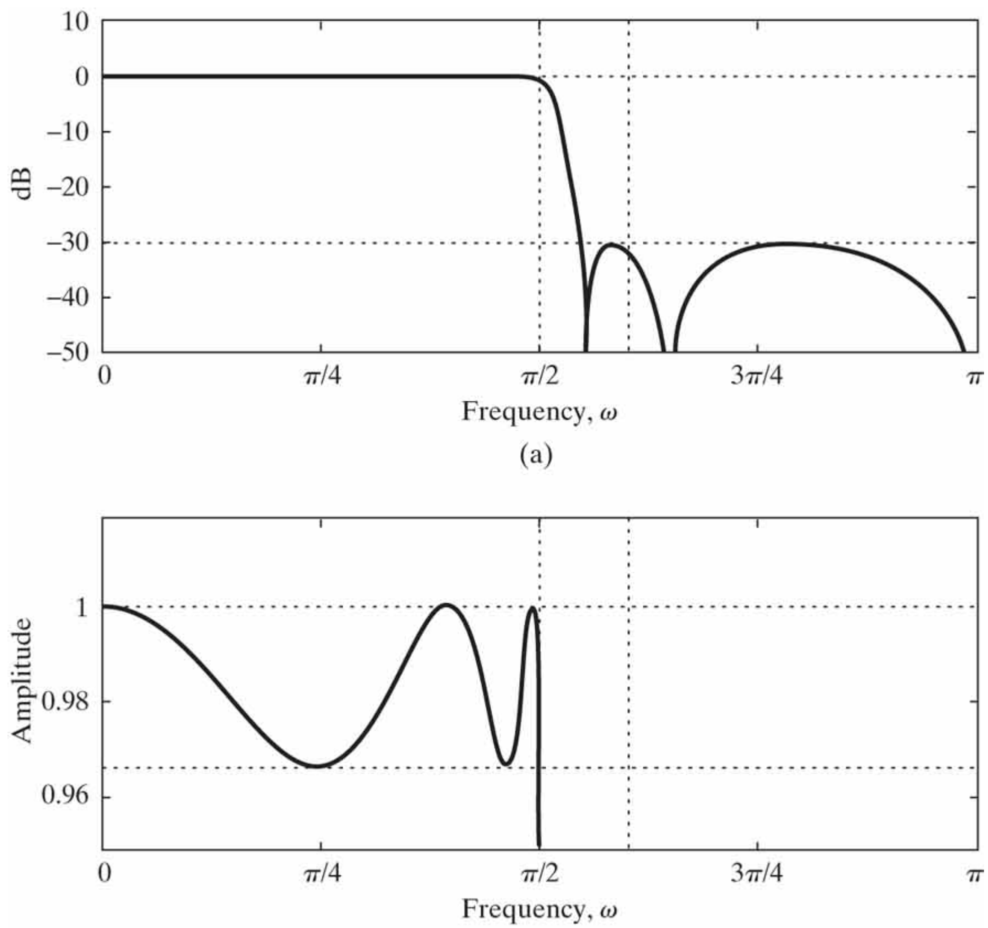


- Type II
  - Monotonic in pass band and equiripple in pass band



Elliptic

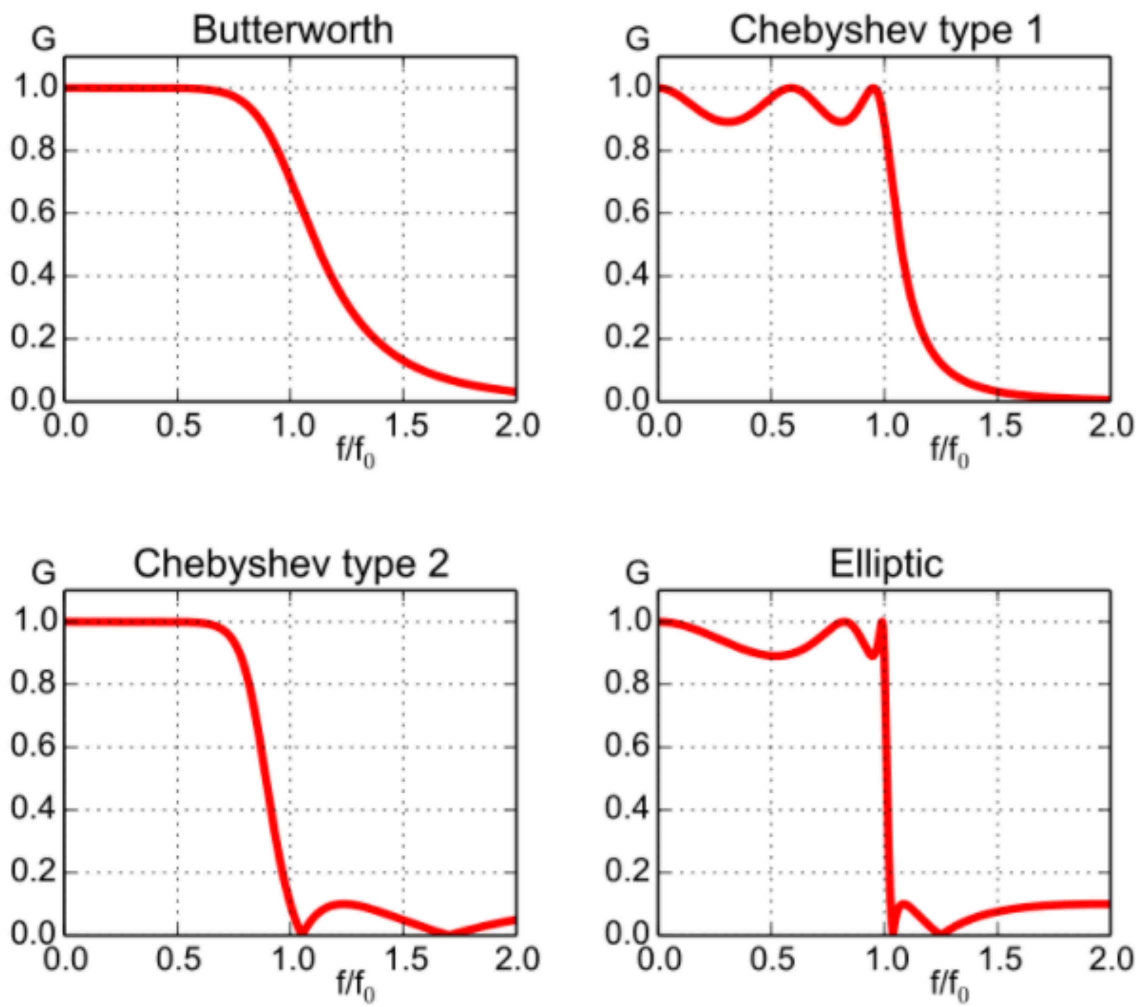
- Elliptic
  - Equiripple in pass and stop bands



(a)

(b)

Comparisons

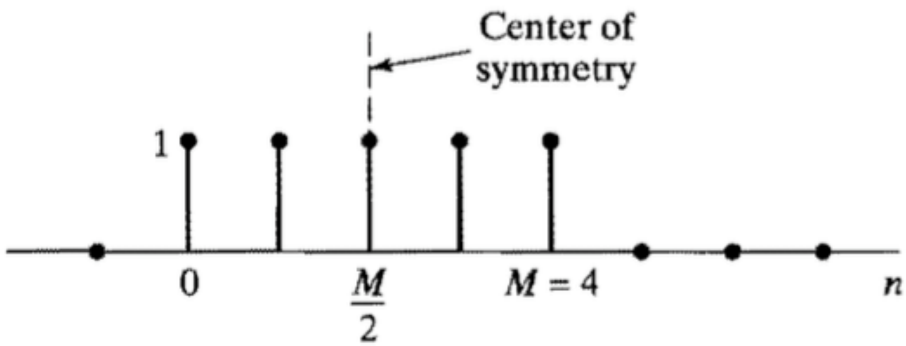


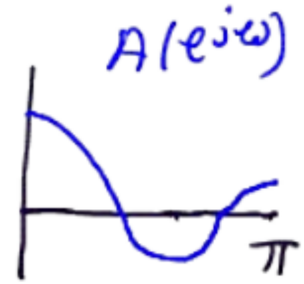
What is a Linear Filter?

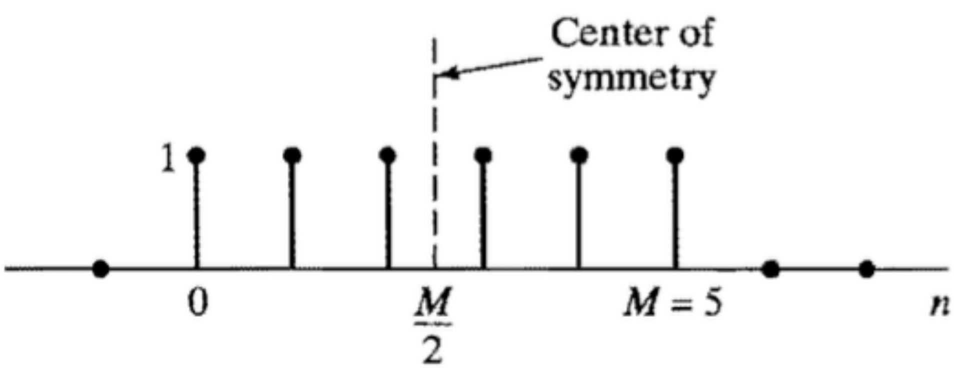
- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

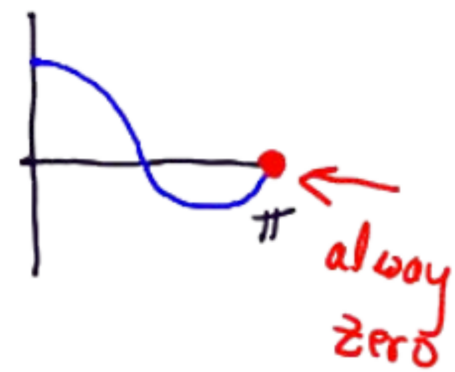
- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
  - Both non-linear and linear phase

FIR GLP: Type I and II

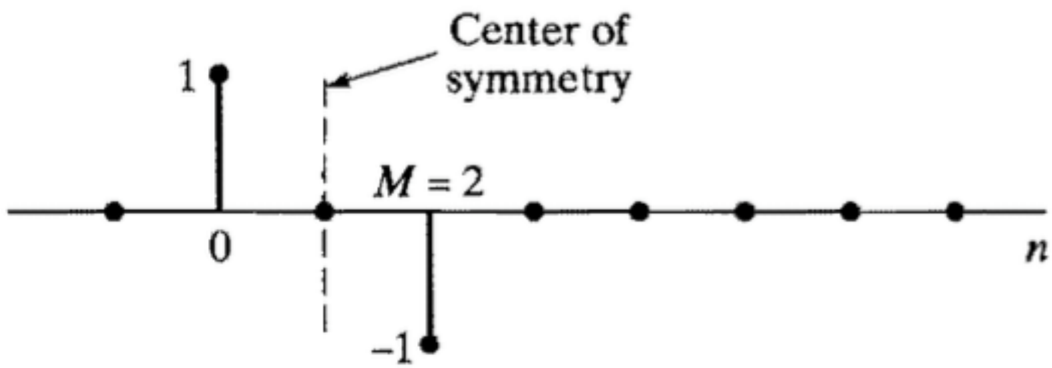


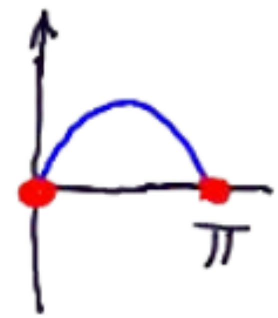


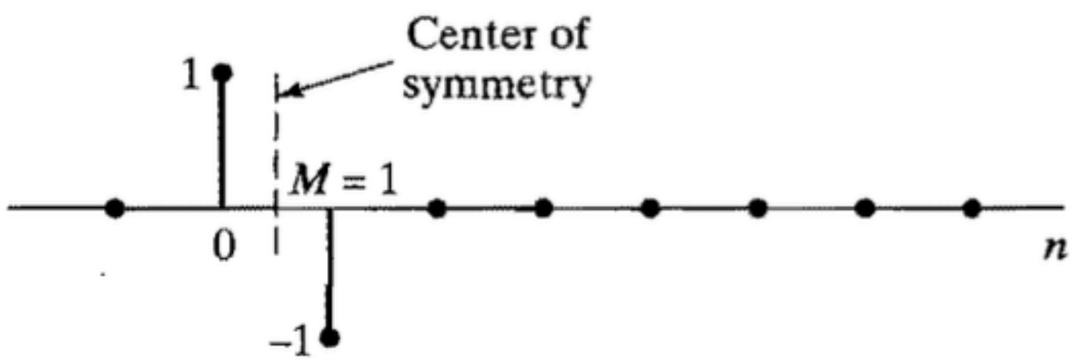


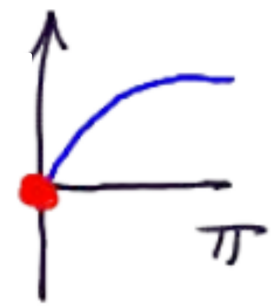


FIR GLP: Type III and IV









FIR Design by Windowing

- Given desired frequency response, $H_d(e^{j\omega})$, find an impulse response

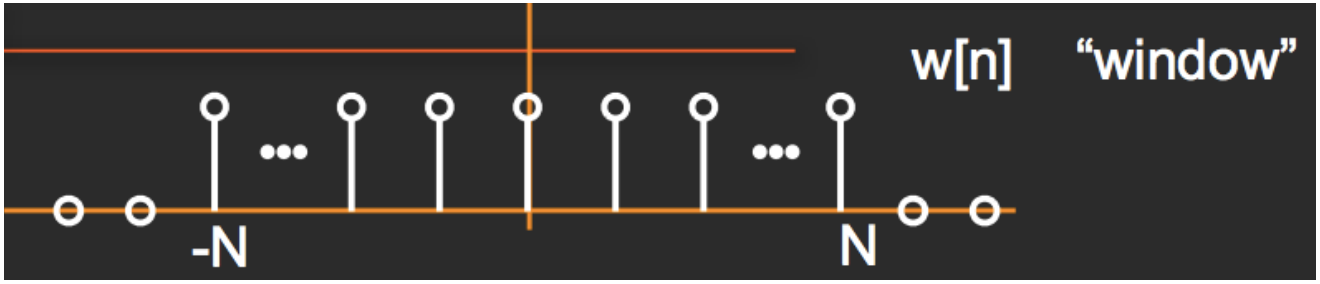
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

ideal

- Obtain the M^{th} order causal FIR filter by truncating/windowing it

$$h[n] = \left\{ \begin{array}{ll} h_d[n]w[n] & 0 \le n \le M \\ 0 & \text{otherwise} \end{array} \right\}$$

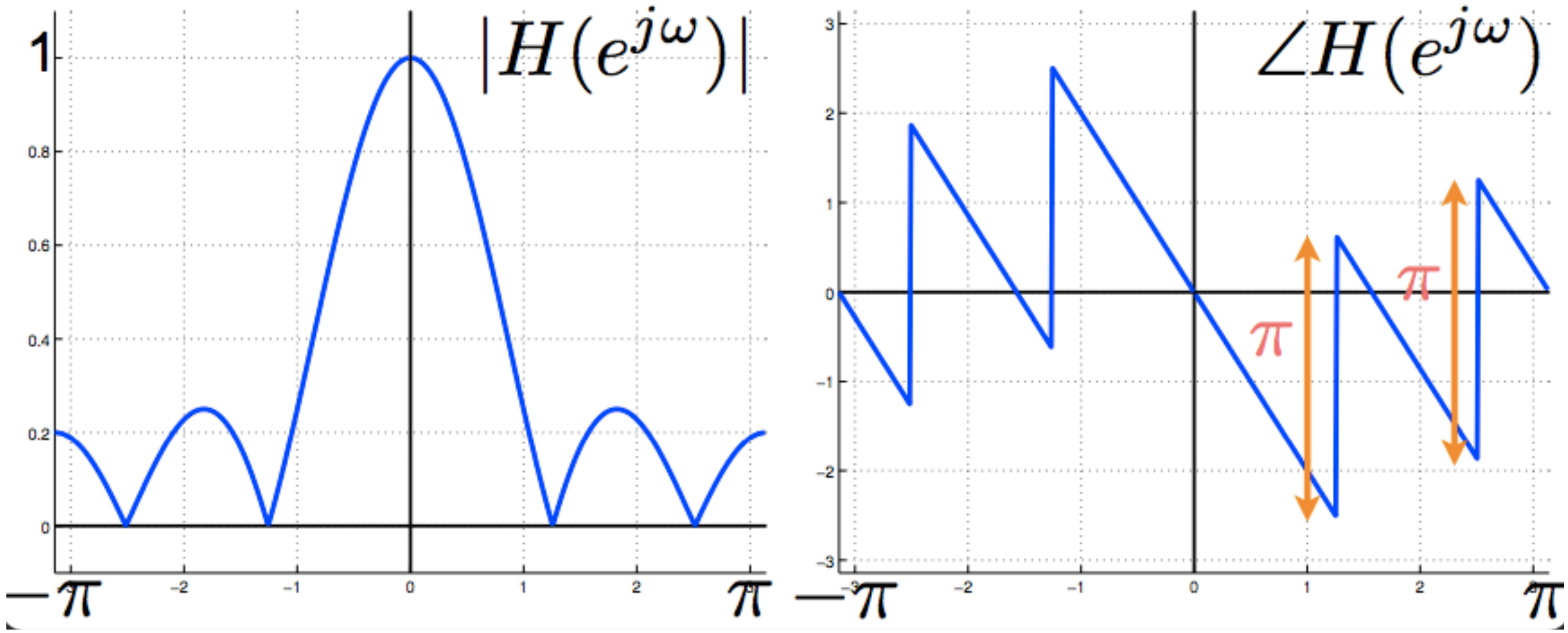
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$\frac{1}{M+1} w[n-M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average



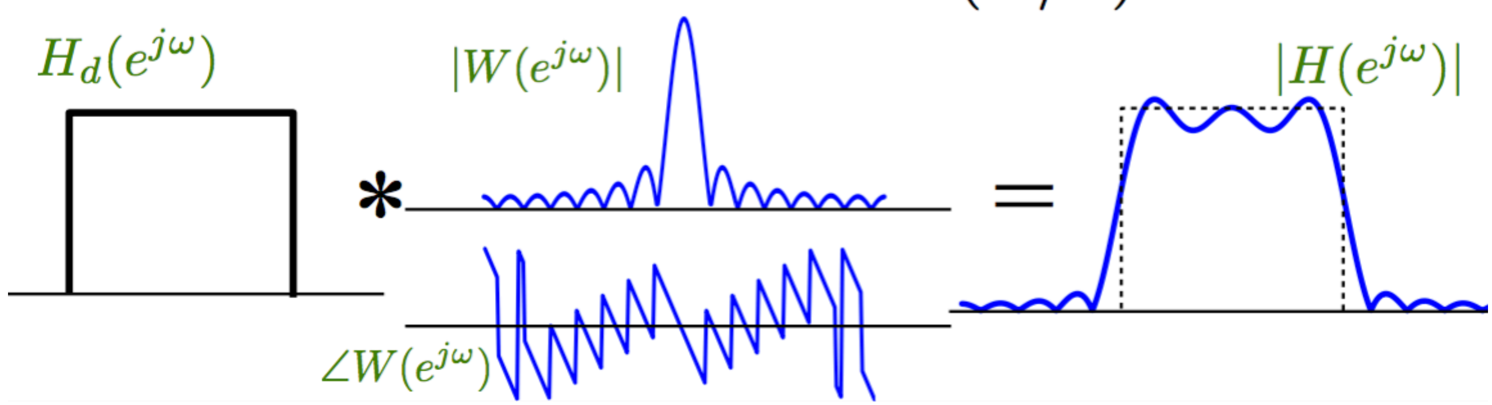
FIR Design by Windowing

- We already saw this before,

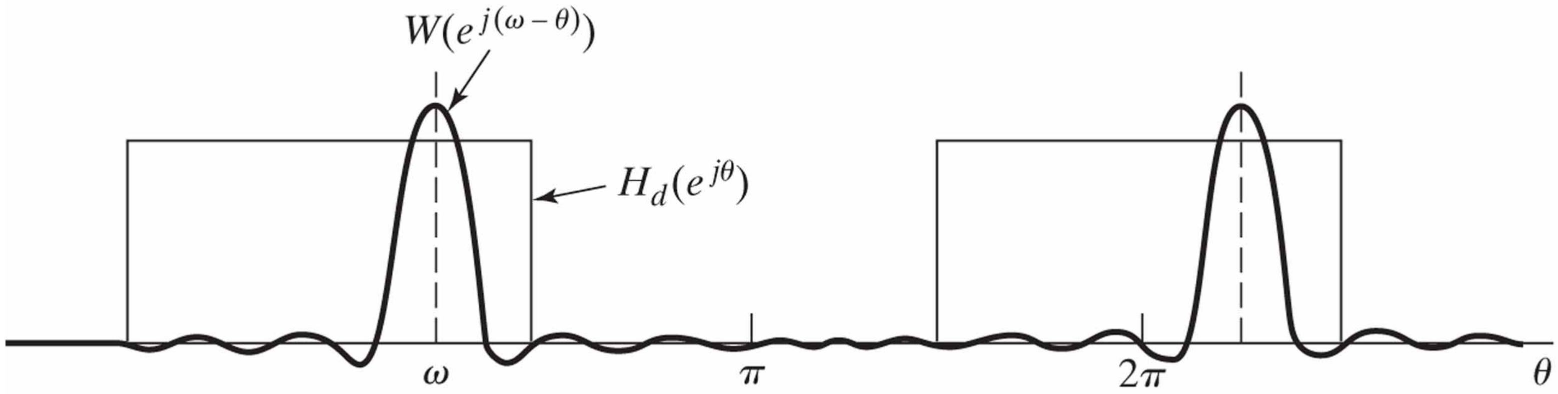
$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

- For Boxcar (rectangular) window

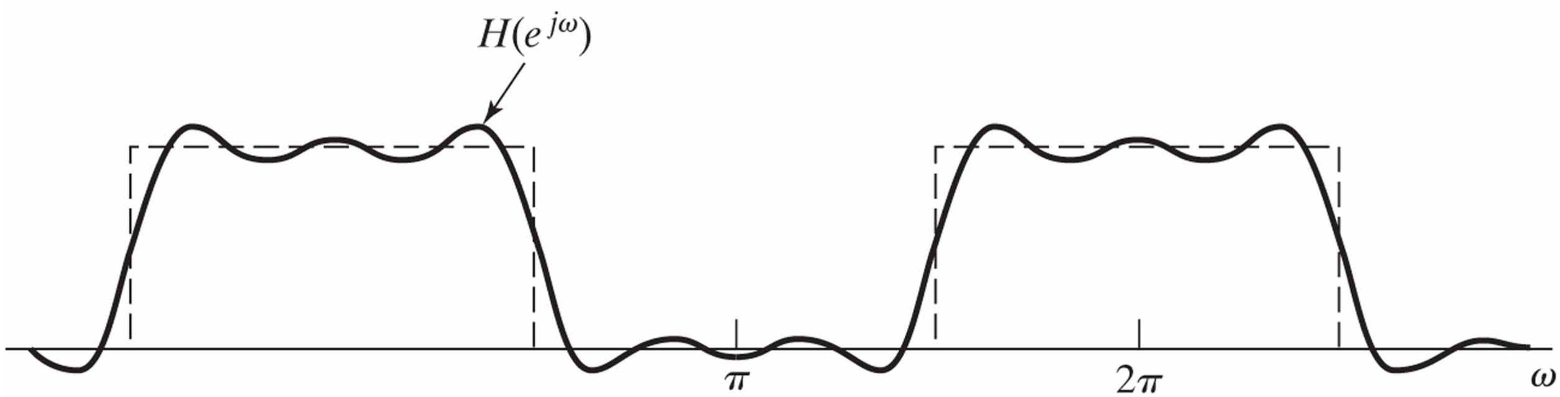
$$W(e^{j\omega}) = e^{-j\omega\frac{M}{2}} \frac{\sin(w(M+1)/2)}{\sin(w/2)}$$



FIR Design by Windowing

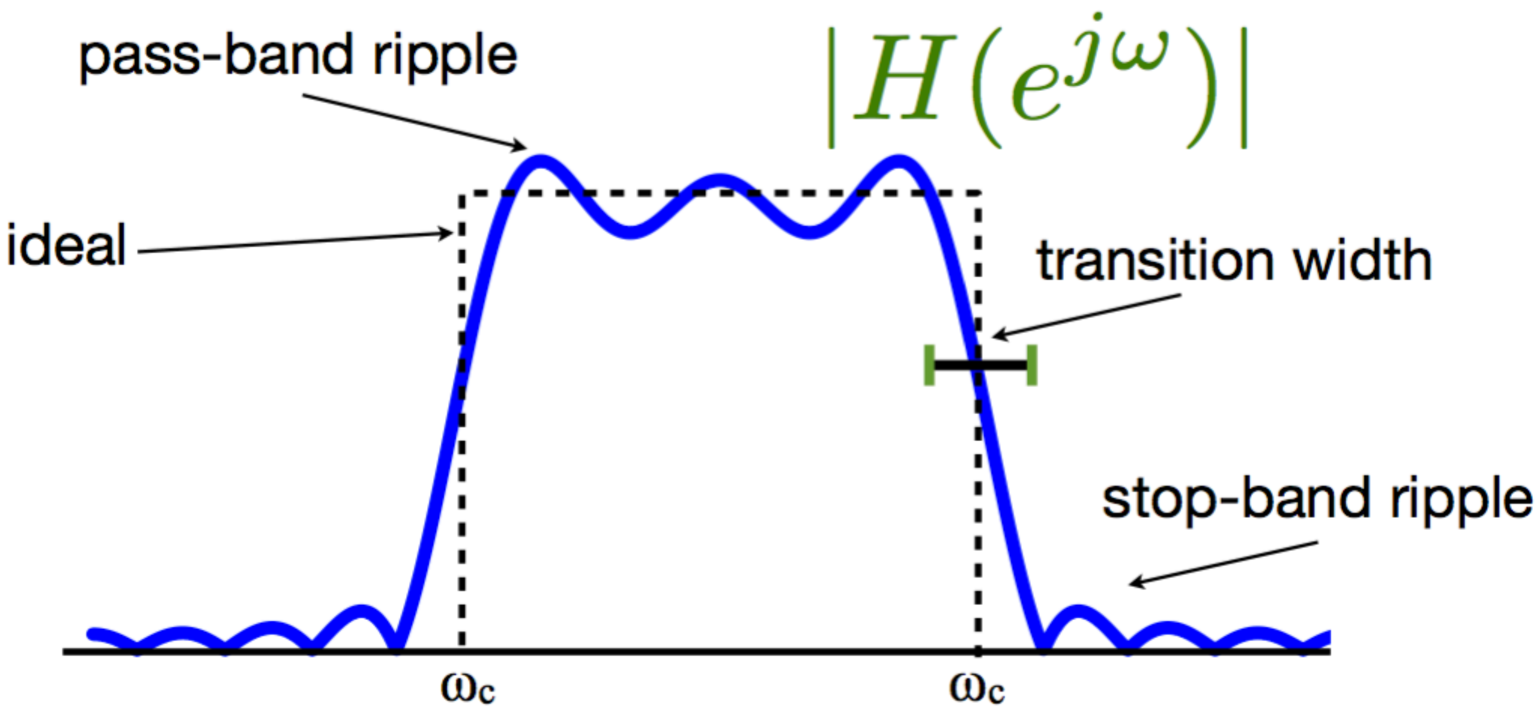


(a)

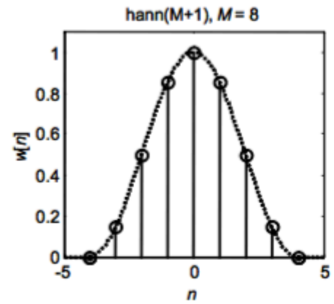
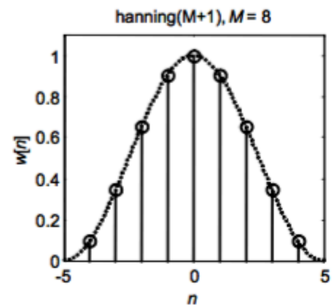
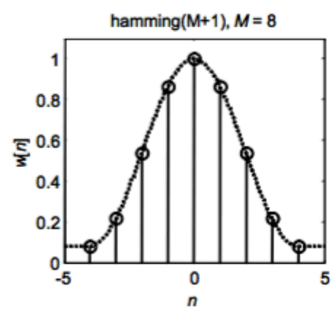


(b)

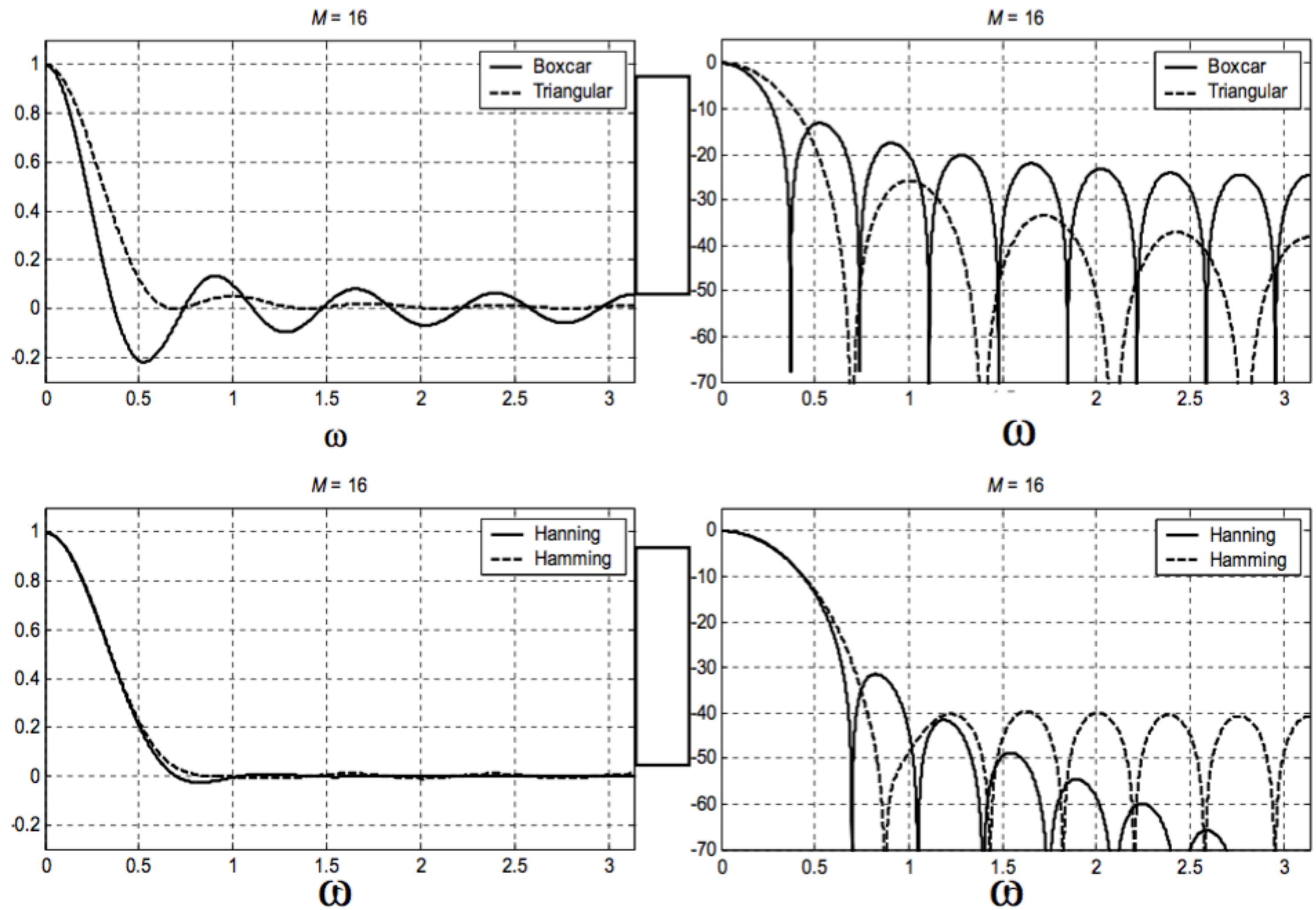
FIR Design by Windowing



Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hamming(M+1)`	

Tradeoff – Ripple vs. Transition Width



Kaiser Window

- Near optimal window quantified as the window maximally concentrated around $\omega=0$

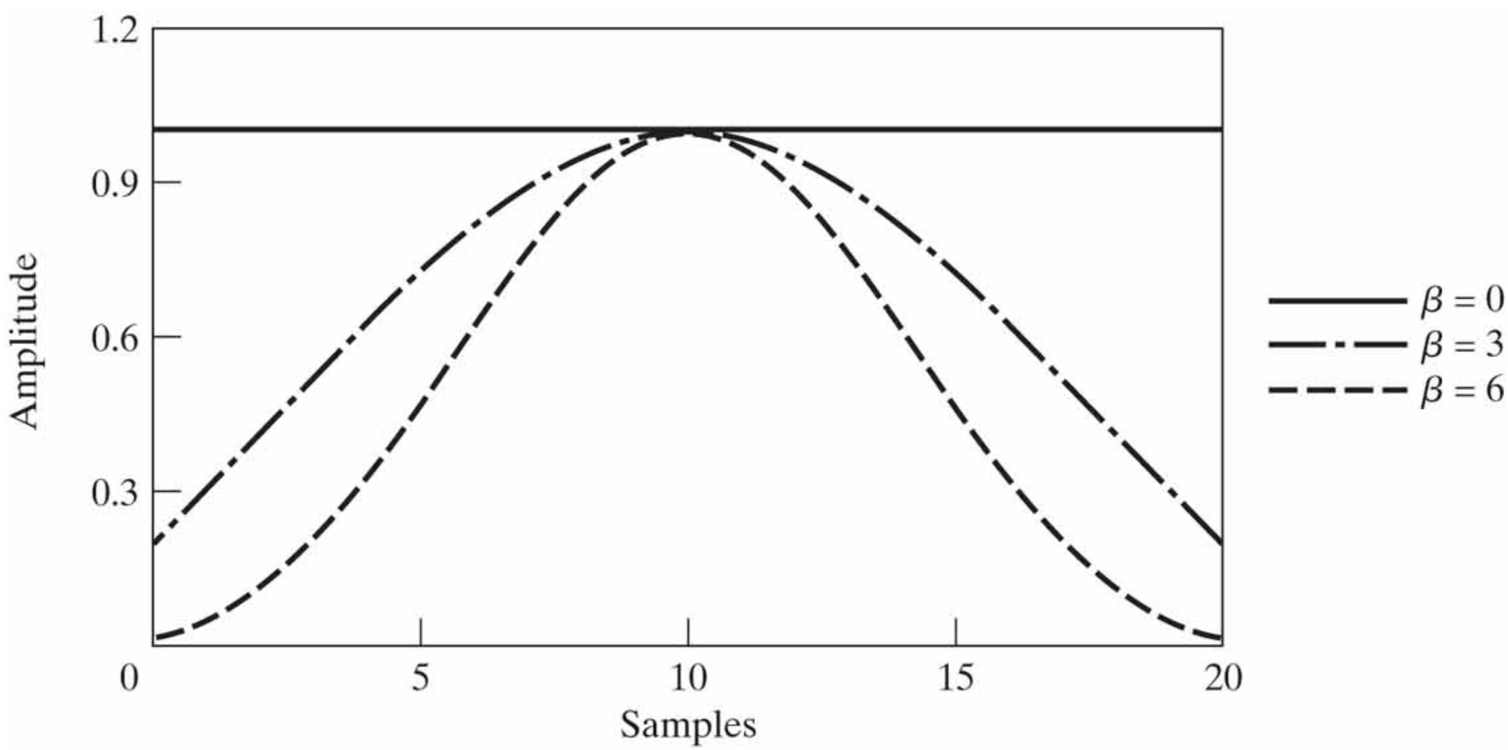
$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise,} \end{cases}$$

- Two parameters – M and β

- $\alpha=M/2$
- $I_0(x)$ – zeroth order Bessel function of the first kind

Kaiser Window

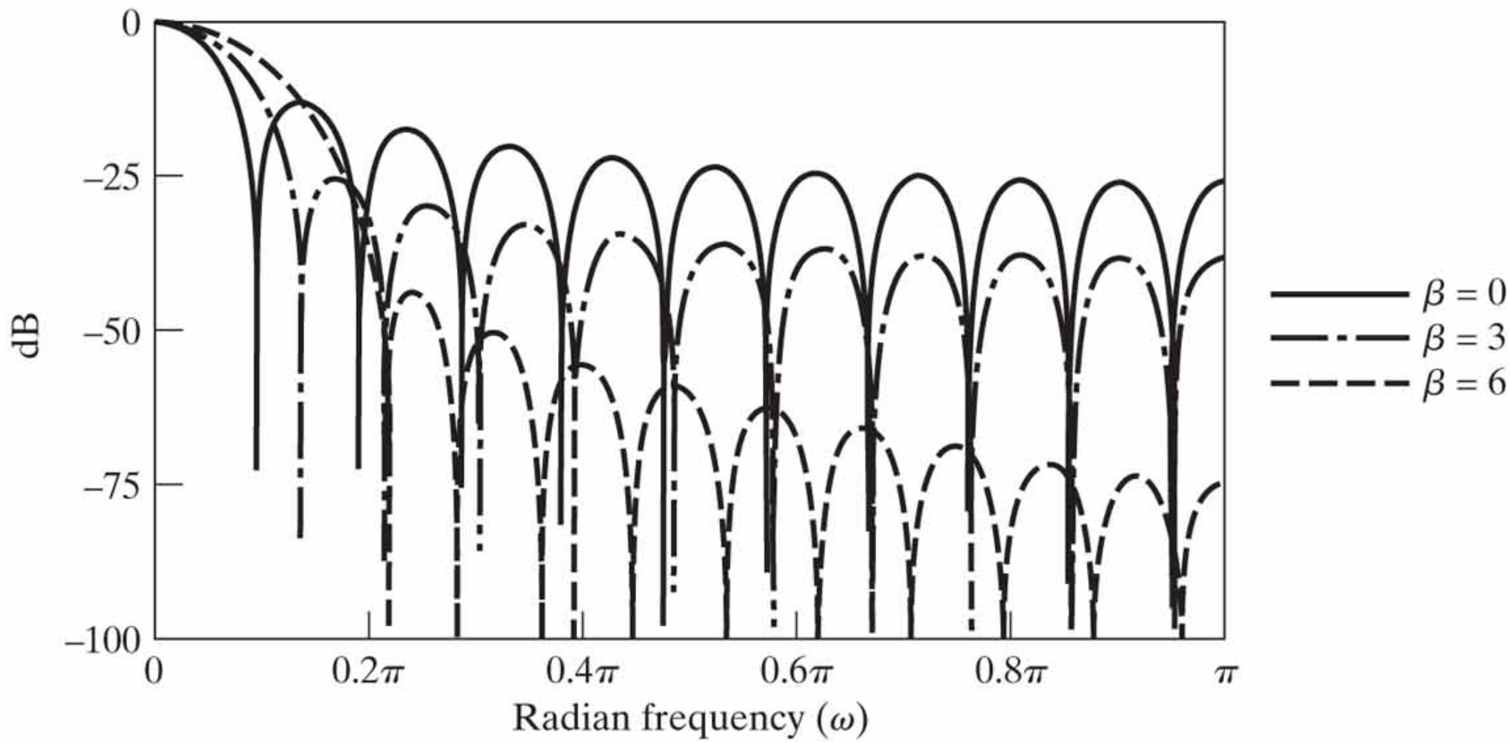
- M=20



(a)

Kaiser Window

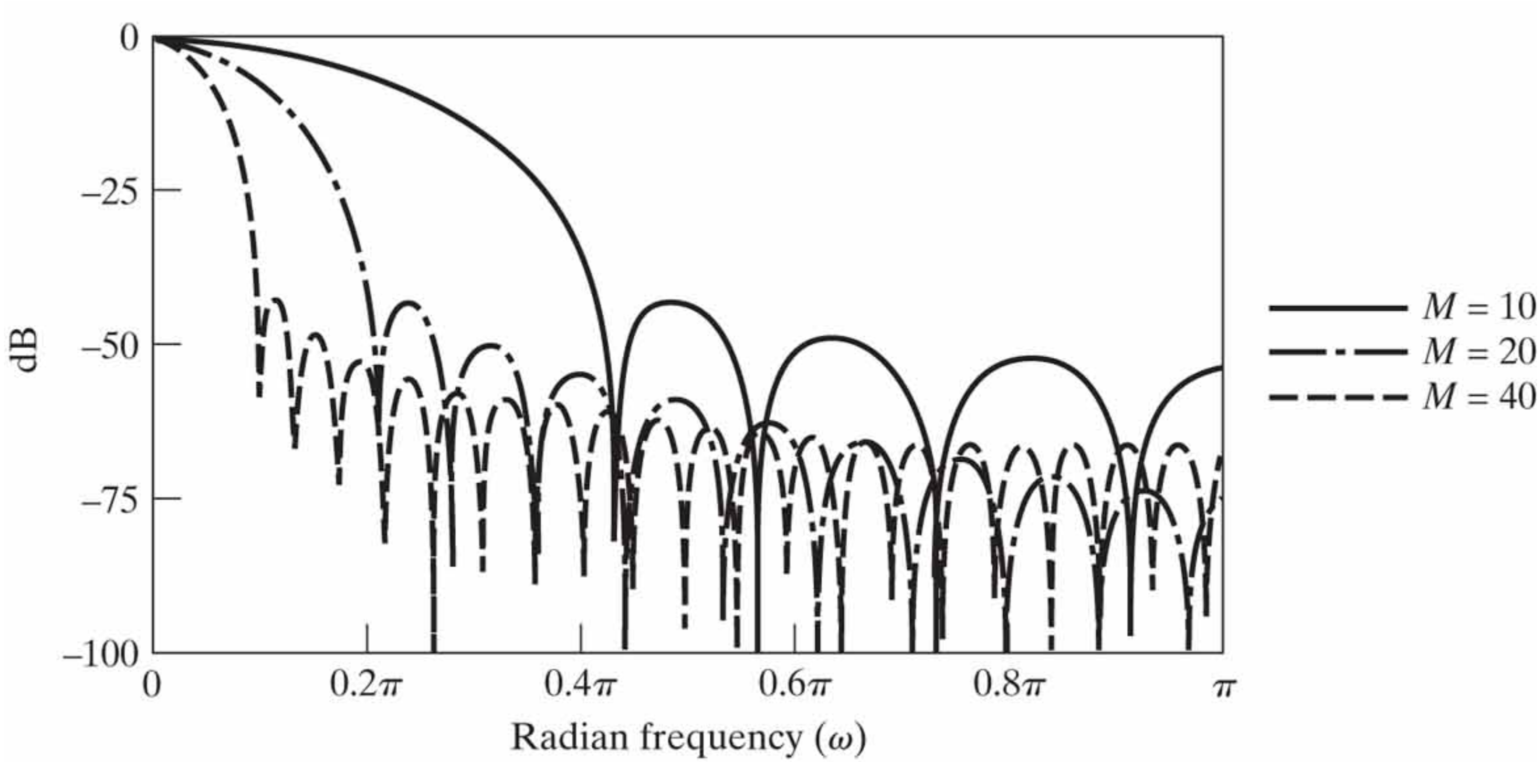
- M=20



(b)

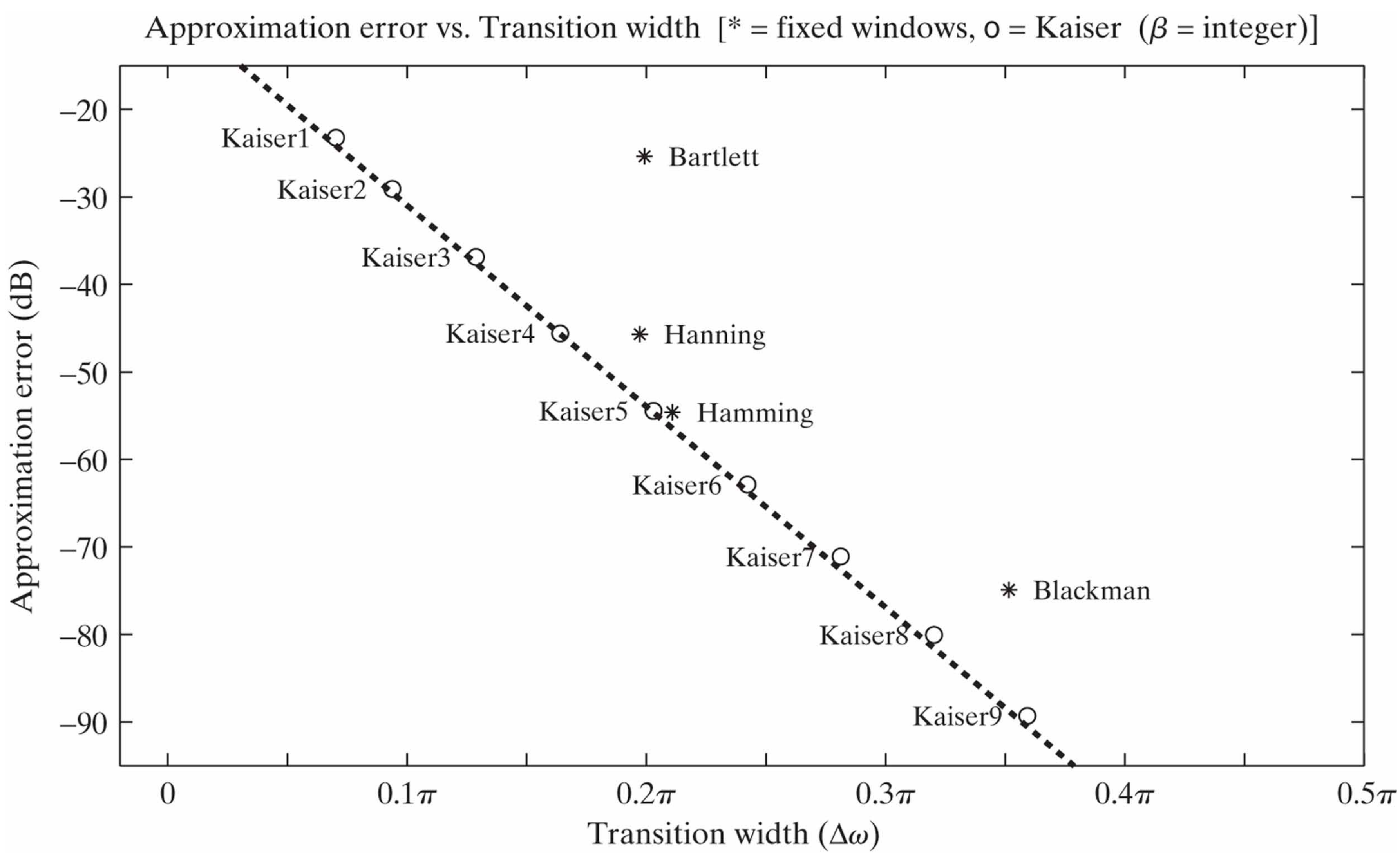
Kaiser Window

- $\beta = 6$



(c)

Approximation Error



FIR Filter Design

- Choose a desired frequency response $H_d(e^{j\omega})$
  - non causal (zero-delay), and infinite imp. response
  - If derived from C.T, choose T and use:

$$H_d(e^{j\omega}) = H_c(j\frac{\Omega}{T})$$

- Window:
  - Length M+1 ⇔ affects transition width
  - Type of window ⇔ transition-width/ ripple

FIR Filter Design

- Choose a desired frequency response $H_d(e^{j\omega})$
  - non causal (zero-delay), and infinite imp. response
  - If derived from C.T, choose T and use:

$$H_d(e^{j\omega}) = H_c(j\frac{\Omega}{T})$$

- Window:
  - Length M+1 ⇔ affects transition width
  - Type of window ⇔ transition-width/ ripple
  - Modulate to shift impulse response
    - Force causality

$$H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$

FIR Filter Design

- Determine truncated impulse response $h_1[n]$

$$h_1[n] = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}e^{j\omega n} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- Apply window

$$h_w[n] = w[n]h_1[n]$$

- Check:
  - Compute $H_w(e^{j\omega})$, if does not meet specs increase M or change window

Example: FIR Low-Pass Filter Design

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Choose M ⇒ Window length and set

$$H_1(e^{j\omega}) = H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$

$$h_1[n] = \begin{cases} \frac{\sin(\omega_c(n-M/2))}{\pi(n-M/2)} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}(n-M/2))$$

Example: FIR Low-Pass Filter Design

- The result is a windowed sinc function

$$h_w[n] = w[n]h_1[n]$$

$$\frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}(n - M/2))$$

- High Pass Design:
  - Design low pass
  - Transform to $h_w[n](-1)^n$

- General bandpass
  - Transform to $2h_w[n]\cos(\omega_0 n)$ or $2h_w[n]\sin(\omega_0 n)$

Lowpass Filter w/ Kaiser Window

- Filter specs:
  - passband edge frequency $\omega_{\mathrm{p}} = 0.4\pi$
  - stopband edge frequency $\omega_{\mathrm{s}} = 0.6\pi$
  - Max ripple (δ) = .001

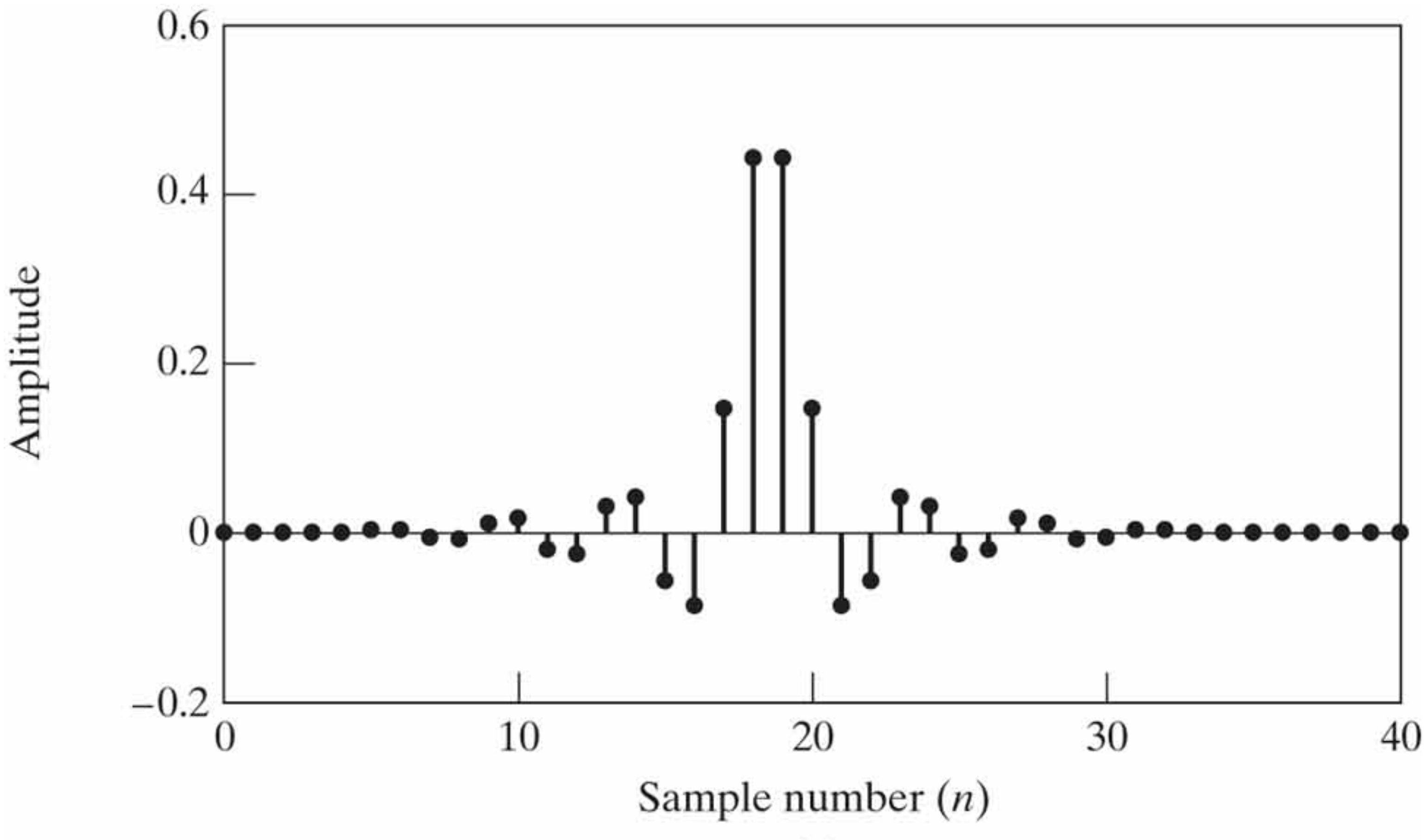
Lowpass Filter w/ Kaiser Window

- Filter specs:
  - passband edge frequency $\omega_p = 0.4\pi$
  - stopband edge frequency $\omega_s = 0.6\pi$
  - Max ripple (δ) = .001

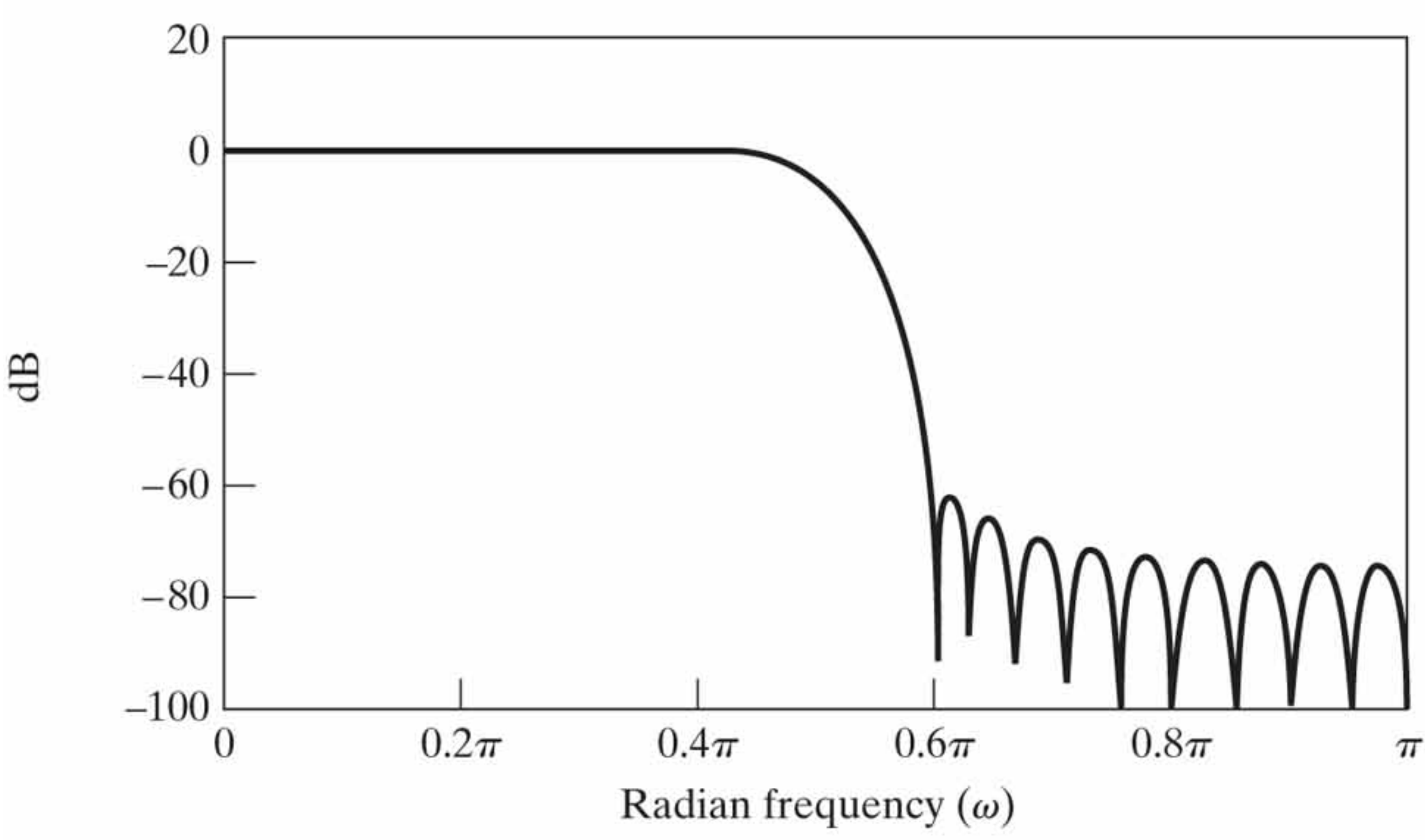
$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50, \\ 0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \le A \le 50, \\ 0.0, & A < 21. \end{cases}$$

$$M = \frac{A - 8}{2.285\Delta\omega}.$$

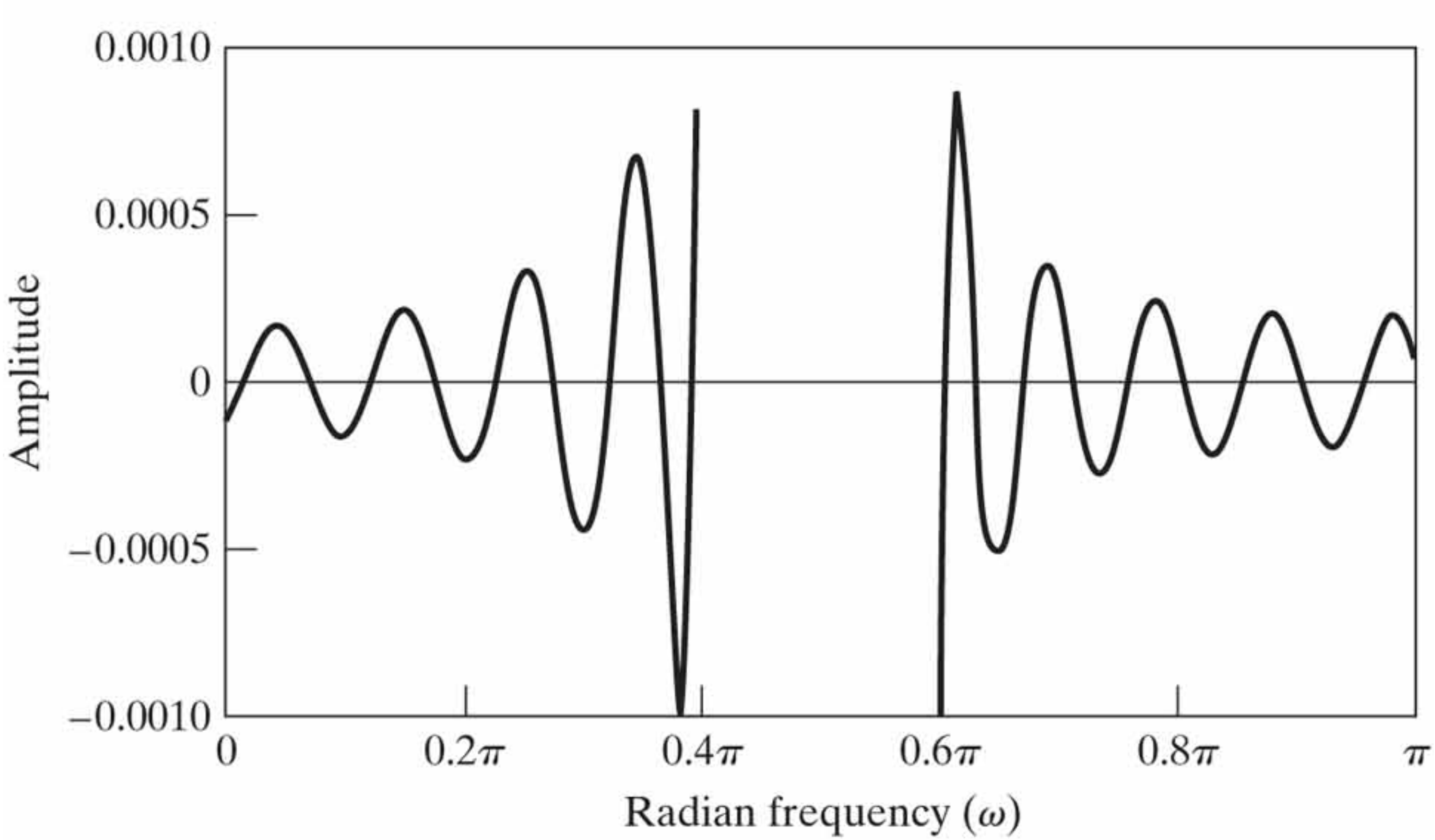
Lowpass Filter w/ Kaiser Window



Lowpass Filter w/ Kaiser Window



Lowpass Filter w/ Kaiser Window

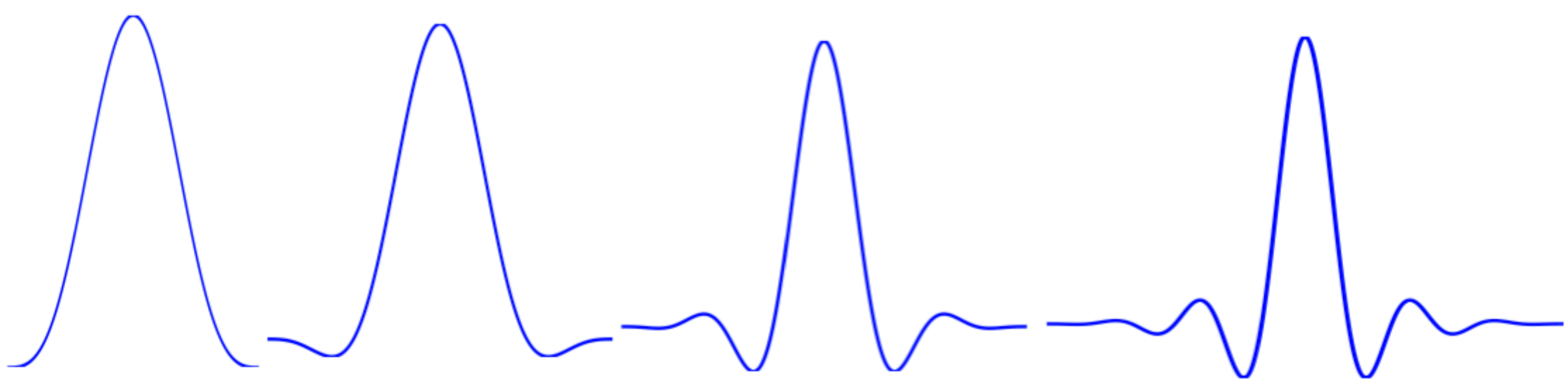


$$E_A(\omega) = \begin{cases} 1 - A_e(e^{j\omega}), & 0 \le \omega \le \omega_p, \\ 0 - A_e(e^{j\omega}), & \omega_s \le \omega \le \pi. \end{cases}$$

Characterization of Filter Shape

Time-Bandwidth Product, a unitless measure

$T(BW) = (M+1)\omega/2\pi$ ⇒ also, total # of zero crossings

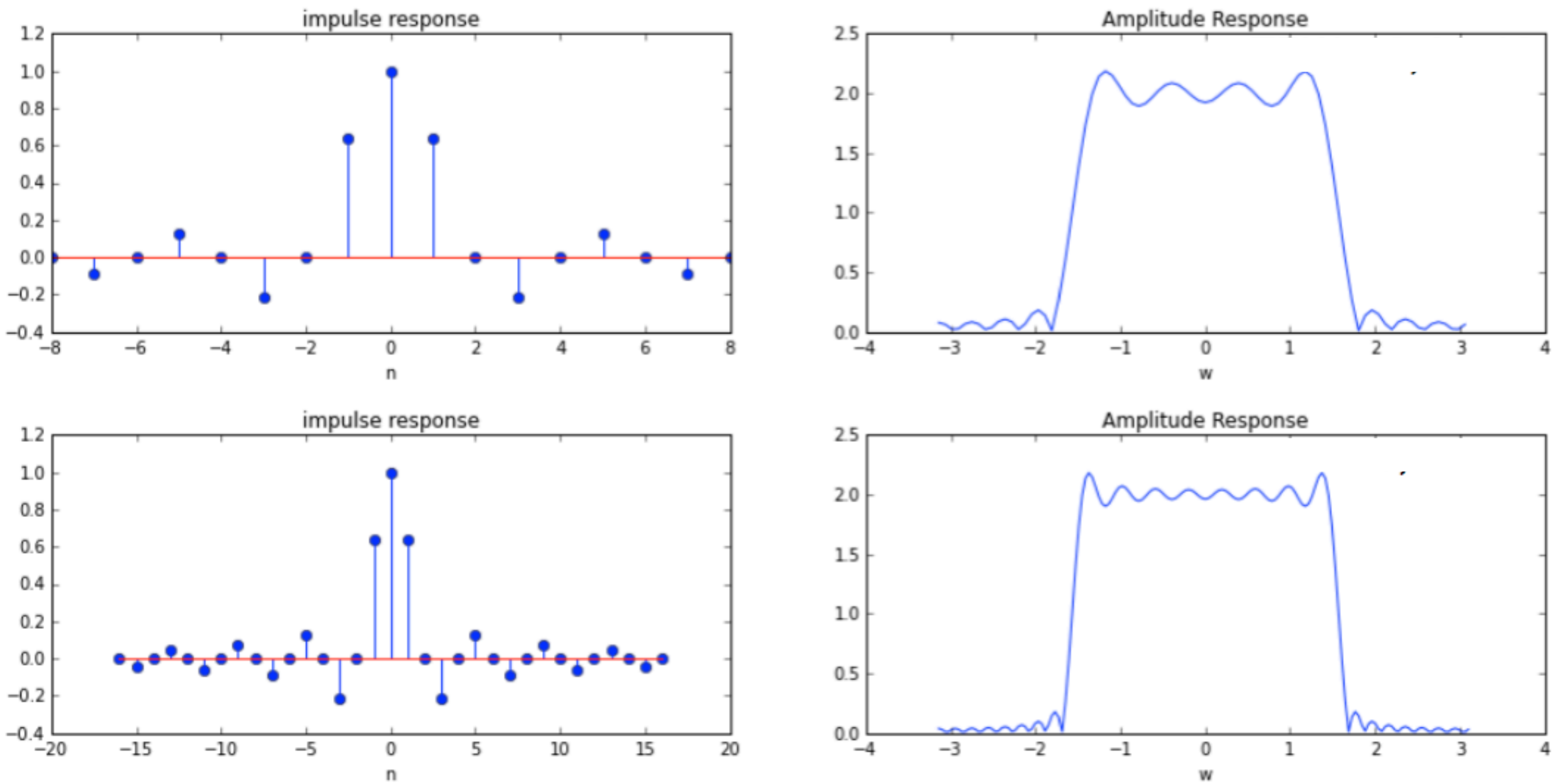


TBW=2 TBW=4 TBW=8 TBW=12

Larger TBW ⇒ More of the "sinc" function

hence, frequency response looks more like a rect function

Time Bandwidth Product



Alternative Design through FFT

- To design order M filter:
- Over-Sample/discretize the frequency response at P points where P >> M (P=15M is good)

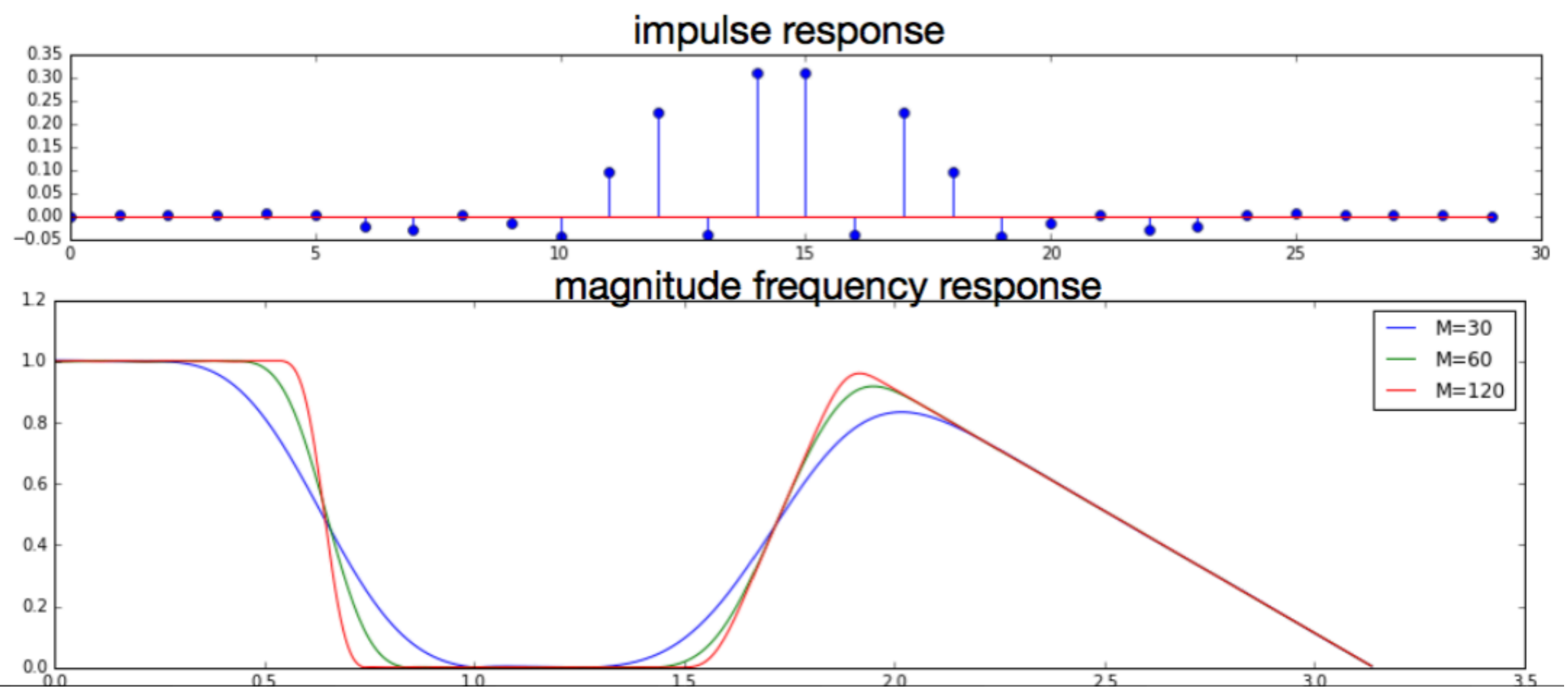
$$H_1(e^{j\omega_k}) = H_d(e^{j\omega_k})e^{-j\omega_k \frac{M}{2}}$$

- Sampled at: $\omega_k = k\frac{2\pi}{P} \qquad |k = [0, \cdots, P-1]$
- Compute $h_1[n] = \mathrm{IDFT}_P(H_1[k])$
- Apply M+1 length window:

$$h_w[n] = w[n]h_1[n]$$

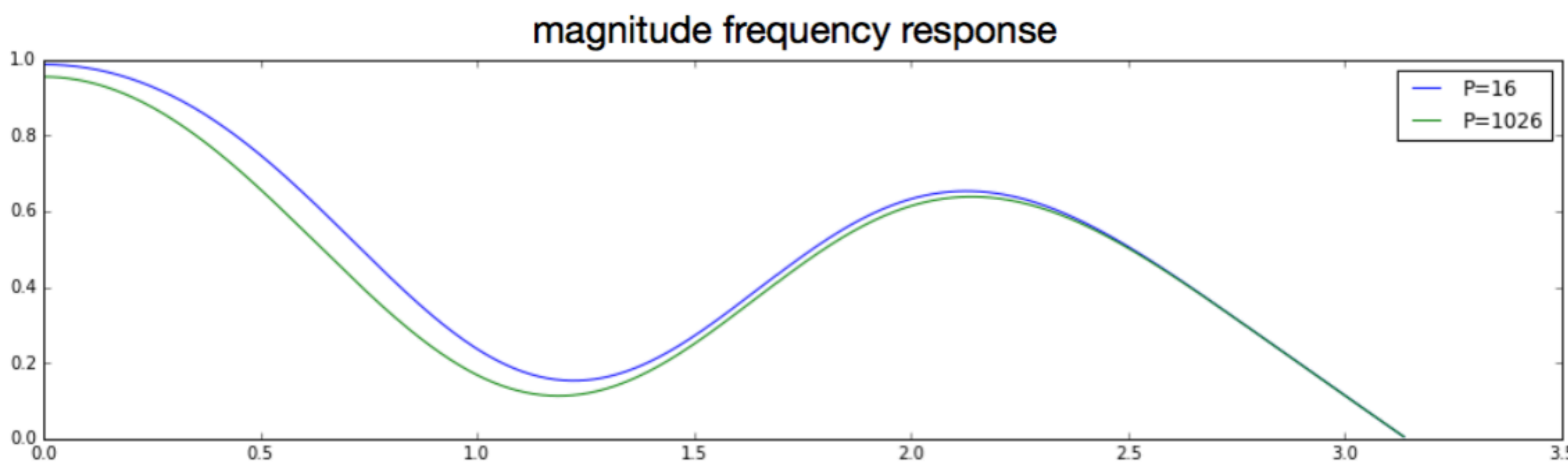
Example

- signal.firwin2(M+1,omega_vec/pi, amp_vec)
- taps1 = signal.firwin2(30, [0.0,0.2,0.21,0.5, 0.6, 1.0], [1.0, 1.0, 0.0,0.0,1.0,0.0])



Example

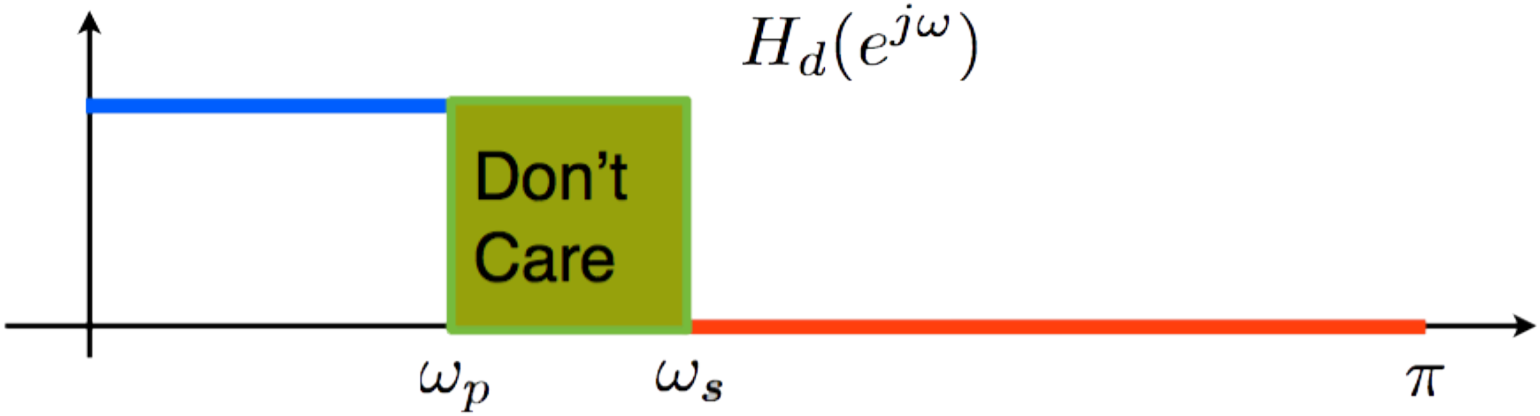
- For M+1=14
  - P = 16 and P = 1026



Optimal Filter Design

- Window method
  - Design Filters heuristically using windowed sinc functions
- Optimal design
  - Design a filter h[n] with $H(e^{j\omega})$
  - Approximate $H_d(e^{j\omega})$ with some optimality criteria - or satisfies specs.

Optimality



- Least Squares:

$$\text{minimize} \quad \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- Variation: Weighted Least Squares:

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Optimality

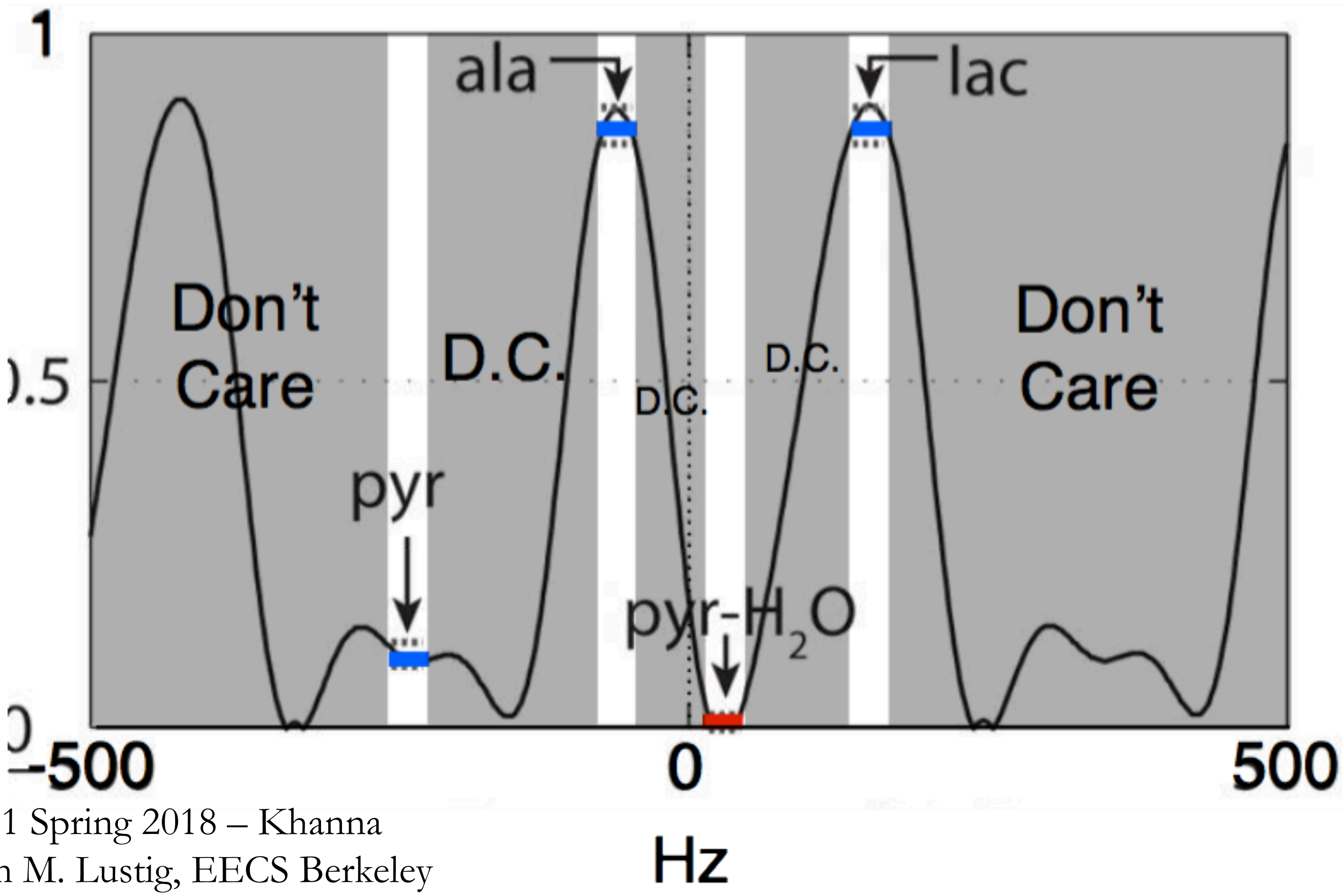
- Chebychev Design (min-max)

$$\text{minimize}_{\omega \in \text{care}} \quad \max |H(e^{j\omega}) - H_d(e^{j\omega})|$$

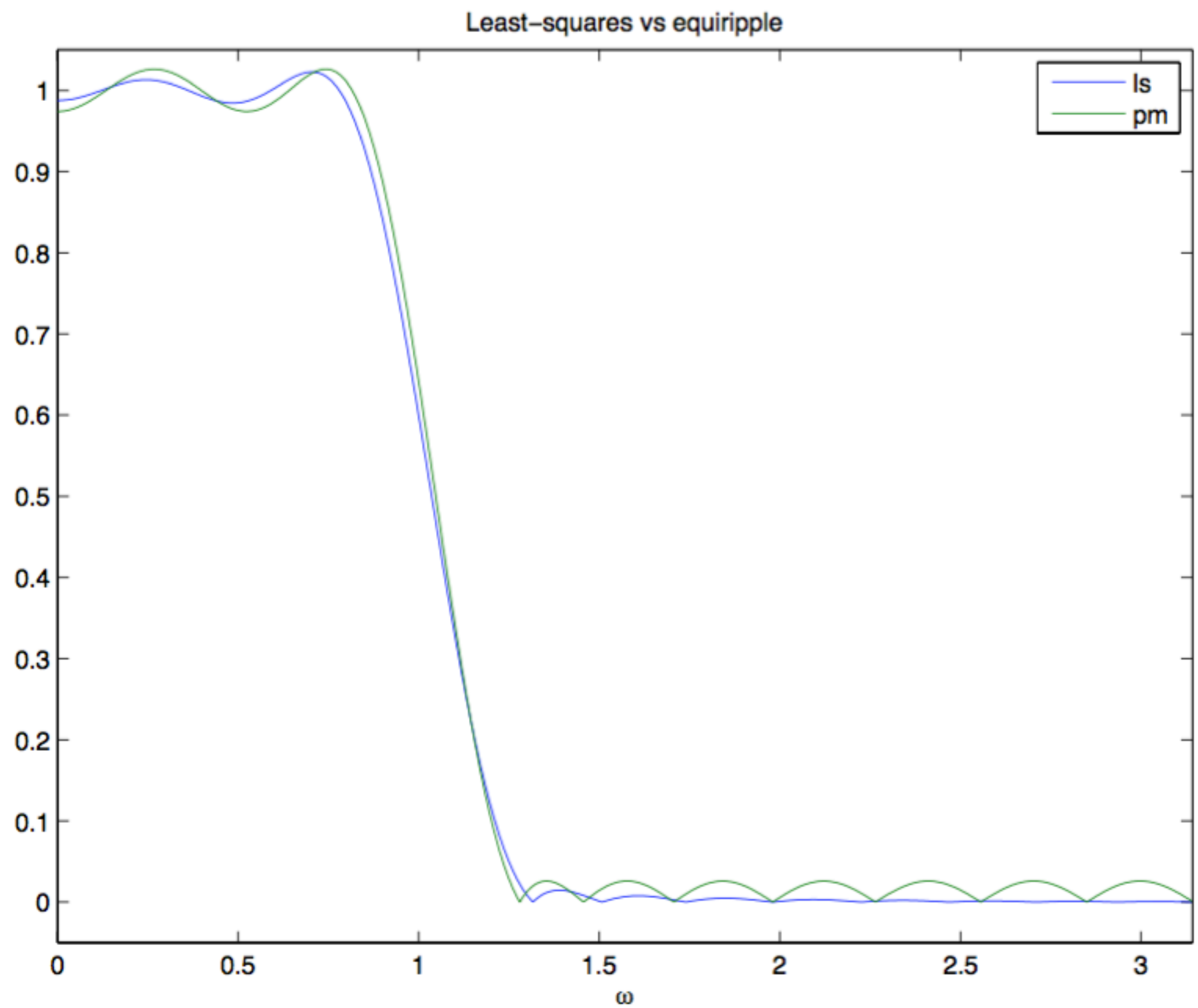
  - Parks-McClellan algorithm - equiripple
  - Also known as Remez exchange algorithms (signal.remez)
  - Can also use convex optimization

Example of Complex Filter

- Larson et. al, "Multiband Excitation Pulses for Hyperpolarized 13C Dynamic Chemical Shift Imaging" JMR 2008;194(1):121-127
- Need to design 11 taps filter with following frequency response:



Least-Squares v.s. Min-Max



Admin

- HW 7
  - Out now
  - Due Friday 3/23

- Pick up midterm