ESE 531: Digital Signal Processing

Lec 17: March 22, 2018

Optimal Filter Design

Optimal Filter Design

- Window method
  - Design Filters heuristically using windowed sinc functions
  - Choose order and window type
  - Check DTFT to see if filter specs are met
- Optimal design
  - Design a filter h[n] with $H(e^{j\omega})$
  - Approximate $H_d(e^{j\omega})$ with some optimality criteria - or satisfies specs.

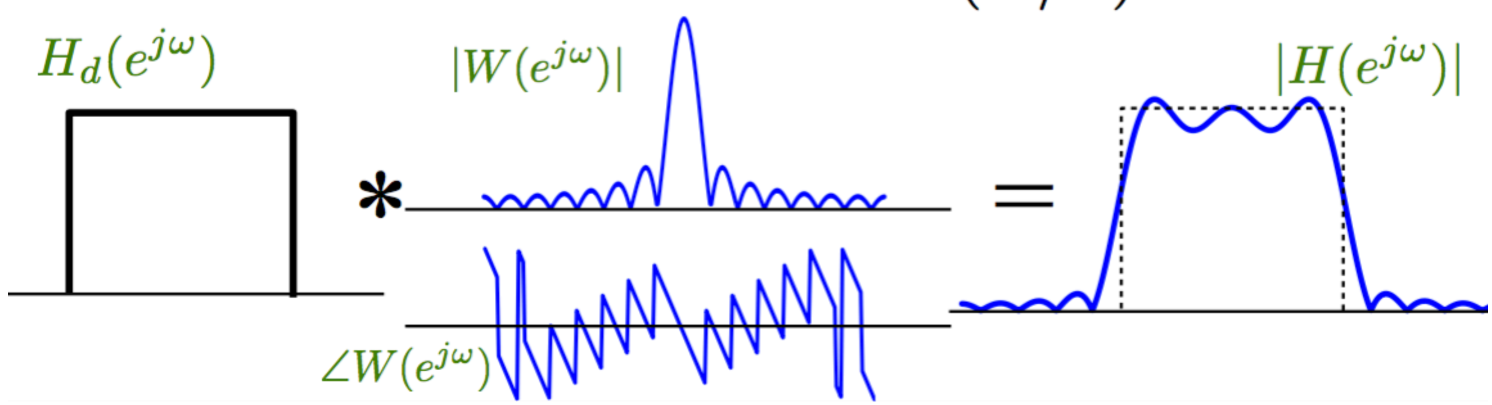
FIR Design by Windowing

- We already saw this before,

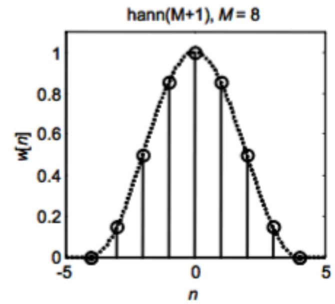
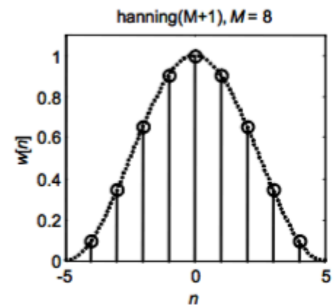
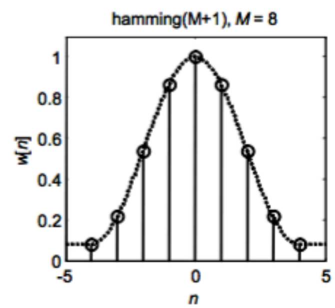
$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

- For Boxcar (rectangular) window

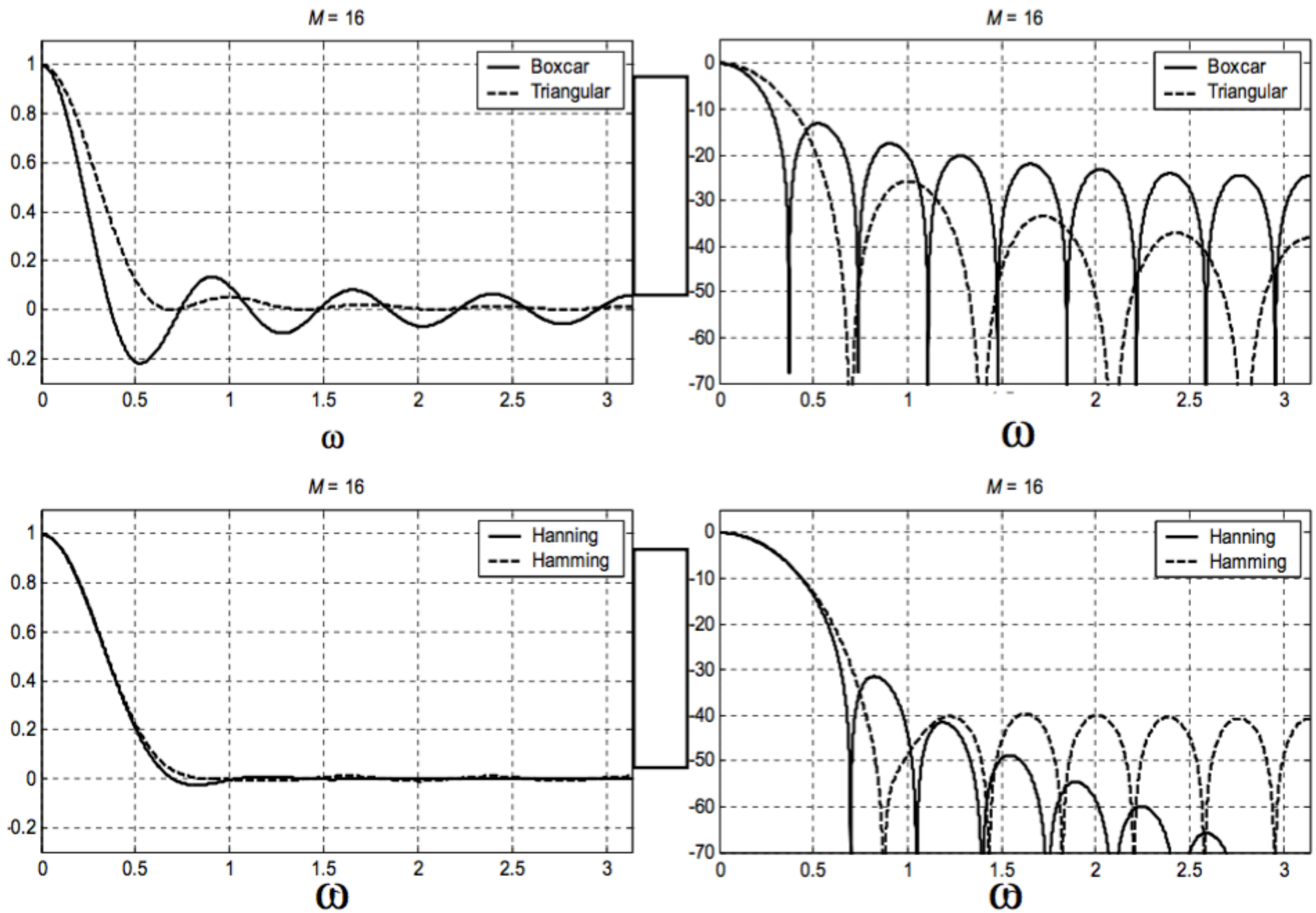
$$W(e^{j\omega}) = e^{-j\omega\frac{M}{2}} \frac{\sin(w(M+1)/2)}{\sin(w/2)}$$



Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n]=\begin{cases}\frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2\end{cases}$	`hann(M+1)`	
Hanning	$w[n]=\begin{cases}\frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2\end{cases}$	`hanning(M+1)`	
Hamming	$w[n]=\begin{cases}0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2\end{cases}$	`hamming(M+1)`	

Tradeoff – Ripple vs. Transition Width



Alternative Design through FFT

- To design order M filter:
- Over-Sample/discretize the frequency response at P points where P >> M (P=15M is good)

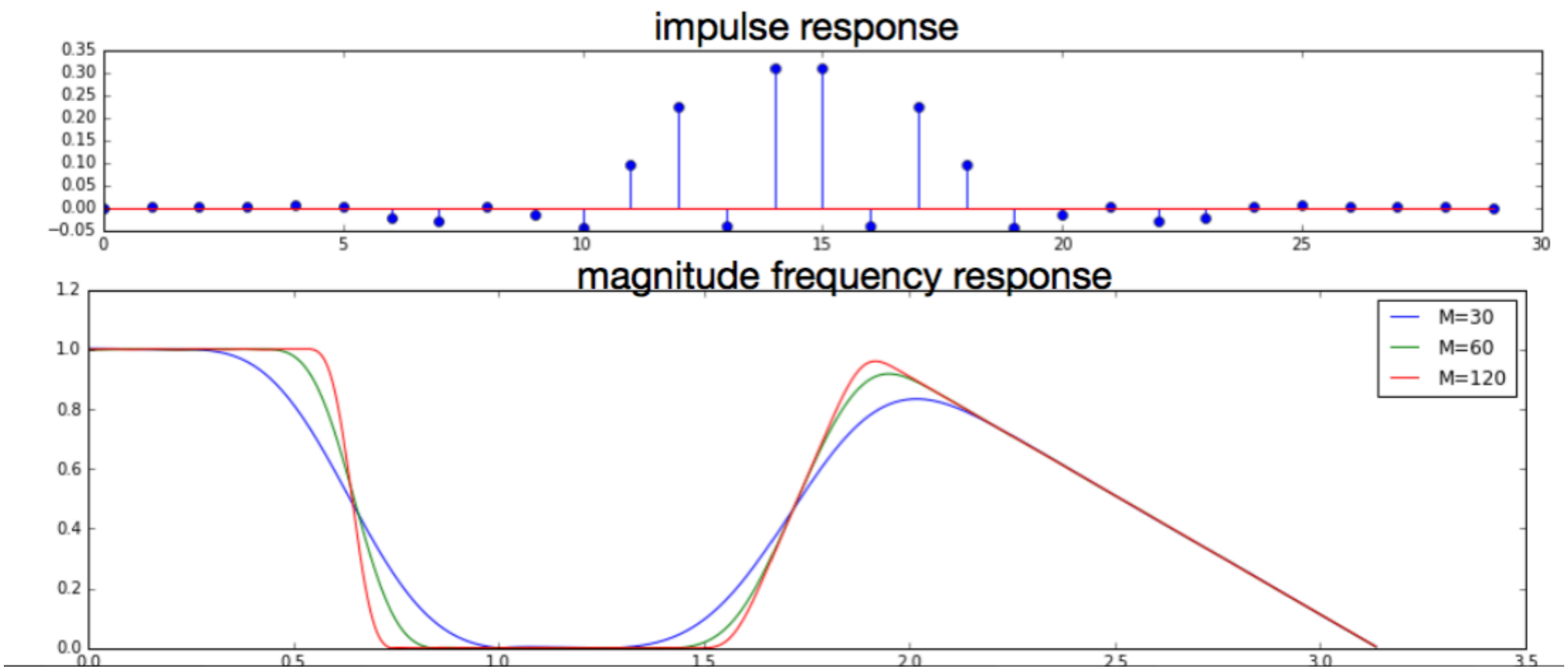
$$H_1(e^{j\omega_k}) = H_d(e^{j\omega_k})e^{-j\omega_k \frac{M}{2}}$$

- Sampled at: $\omega_k = k\frac{2\pi}{P} \qquad |k = [0, \cdots, P-1]$
- Compute $h_1[n] = \mathrm{IDFT}_P(H_1[k])$
- Apply M+1 length window:

$$h_w[n] = w[n]h_1[n]$$

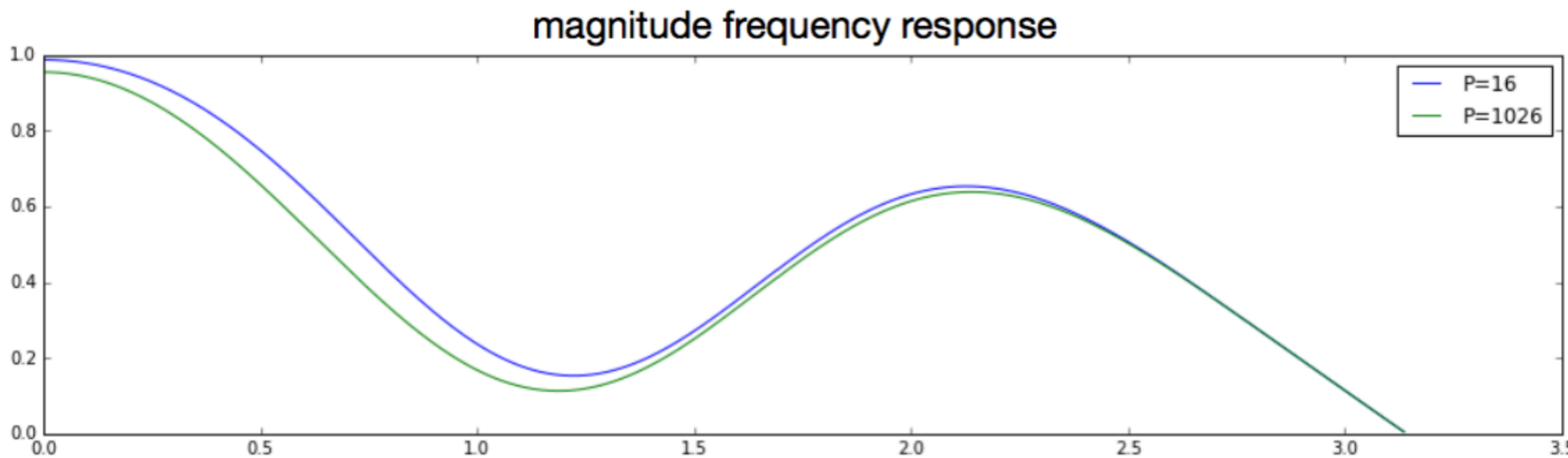
Example

- signal.firwin2(M+1,omega_vec/pi, amp_vec)
- taps1 = signal.firwin2(30, [0.0,0.2,0.21,0.5, 0.6, 1.0], [1.0, 1.0, 0.0,0.0,1.0,0.0])



Example

- For M+1=14
  - P = 16 and P = 1026



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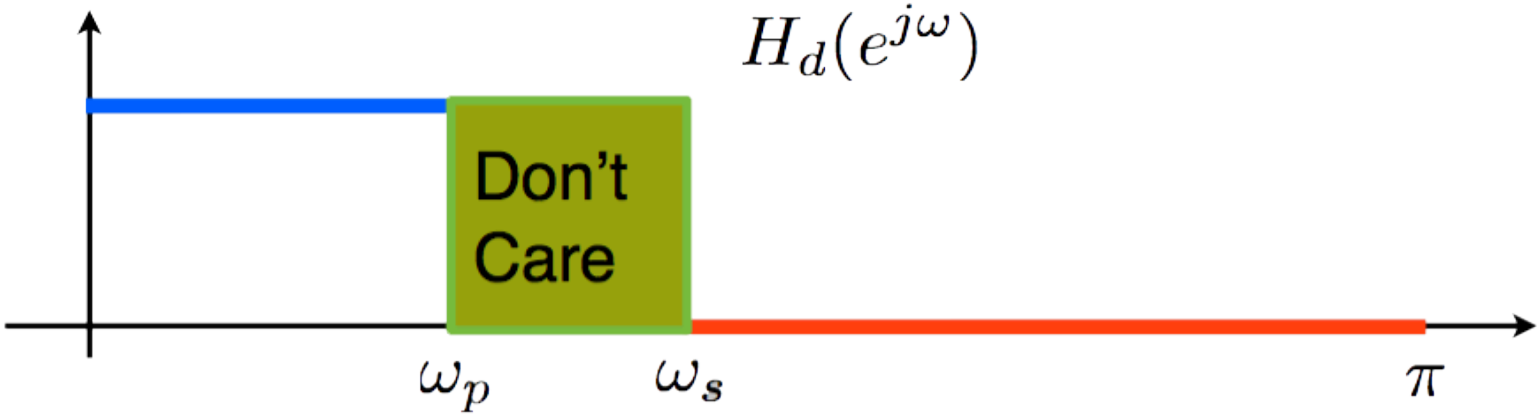
Optimality



- Least Squares:

$$\text{minimize} \quad \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- Variation: Weighted Least Squares:

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Optimality

- Chebychev Design (min-max)

$$\text{minimize}_{\omega\in\text{care}} \quad \max |H(e^{j\omega}) - H_d(e^{j\omega})|$$

  - Parks-McClellan algorithm - equi-ripple
  - Also known as Remez exchange algorithms (signal.remez)
  - Can also use convex optimization

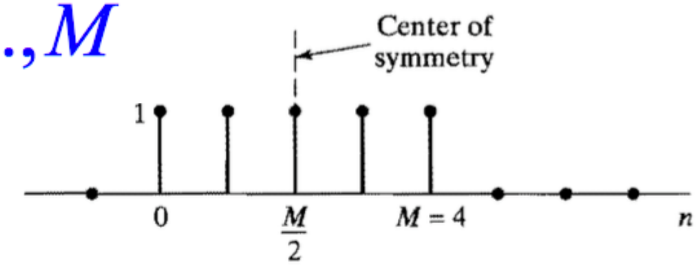
Parks-McClellan

- Assume designing a Type I GLP FIR filter:

FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right)$$

$$a[0] = h[M/2],$$

$$a[k] = 2h[(M/2)-k], \qquad k = 1,2,\ldots,M/2.$$

Parks-McClellan

- Assume designing a Type I GLP FIR filter:

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}.$$

$$A_e(e^{j\omega}) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n).$$

Parks-McClellan

- Reformulate

$$A_e(e^{j\omega}) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n).$$

- To fitting a polynomial

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos\omega)^k,$$

$$A_e(e^{j\omega}) = P(x)|_{x=\cos\omega},$$

$$P(x) = \sum_{k=0}^{L} a_k x^k.$$

Parks-McClellan

- Define approximation error function

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})],$$

- Apply min-max or Chebyshev criteria

$$\min_{\{h_e[n]:0\le n\le L\}} \left(\max_{\omega\in F} |E(\omega)| \right),$$

Parks-McClellan – Alternation Theorem

Alternation Theorem: Let F_P denote the closed subset consisting of the disjoint union of closed subsets of the real axis x. Furthermore,

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

is an r^{th}-order polynomial, and $D_P(x)$ denotes a given desired function of x that is continuous on F_P; $W_P(x)$ is a positive function, continuous on F_P, and

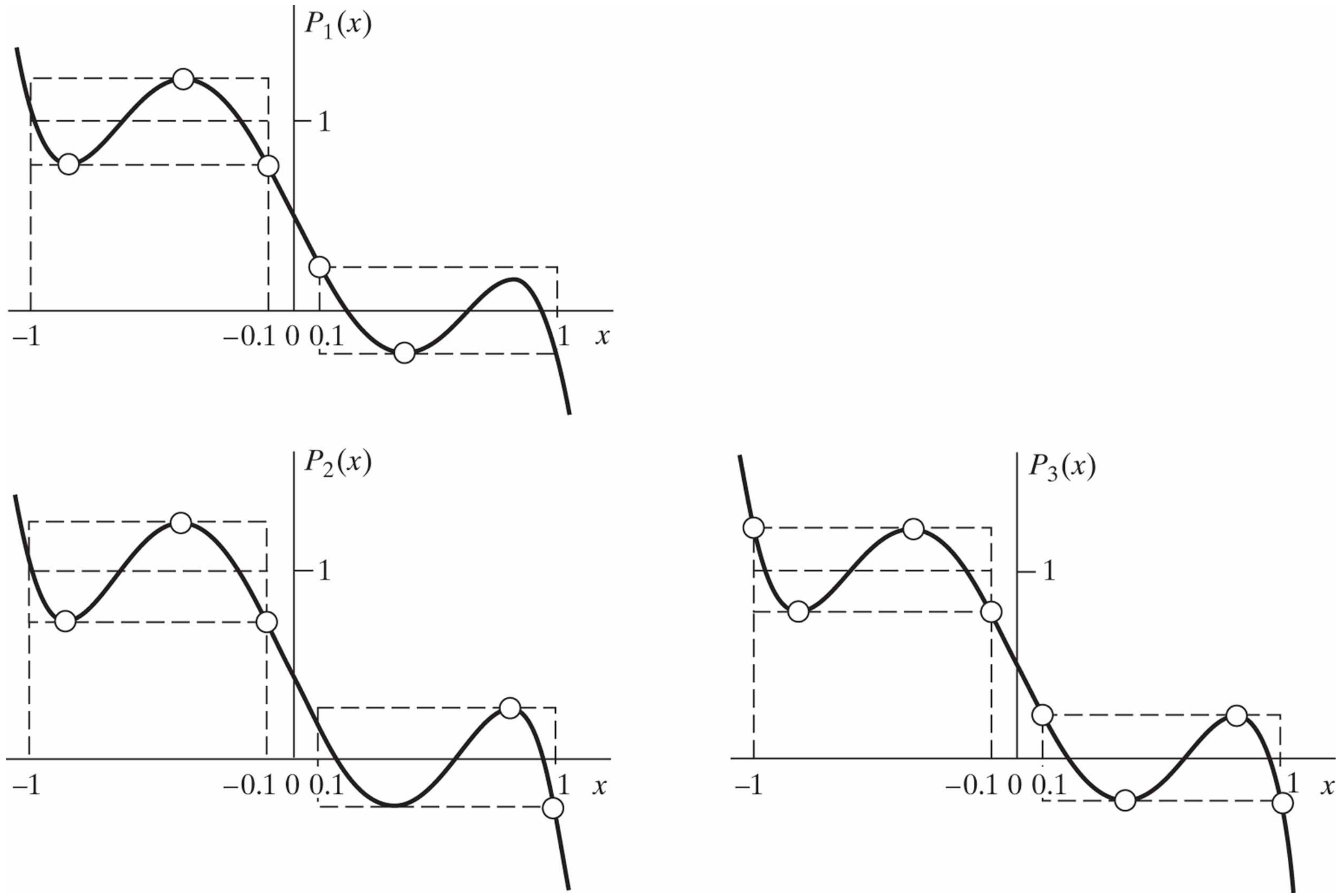
$$E_P(x) = W_P(x)[D_P(x) - P(x)]$$

is the weighted error. The maximum error is defined as

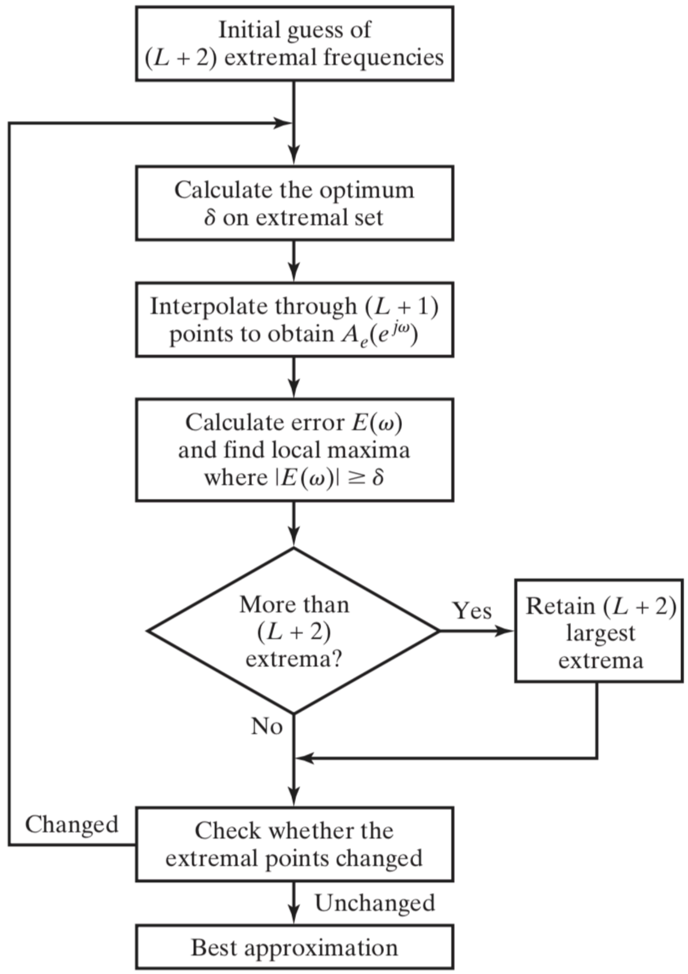
$$\|E\| = \max_{x \in F_P} |E_P(x)|.$$

A necessary and sufficient condition that $P(x)$ be the unique r^{th}-order polynomial that minimizes $\|E\|$ is that $E_P(x)$ exhibit *at least* $(r+2)$ alternations; i.e., there must exist at least $(r+2)$ values x_i in F_p such that $x_1 < x_2 < \cdots < x_{r+2}$ and such that $E_p(x_i) = -E_P(x_{i+1}) = \pm\|E\|$ for $i = 1, 2, \ldots, (r+1)$.

Alternation Theorem Example – 5th order



Parks–McClellan algorithm



Mathematical Optimization

(mathematical) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- $x = (x_1, \ldots, x_n)$: optimization variables
- $f_0 : \mathbf{R}^n \to \mathbf{R}$: objective function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$: constraint functions

optimal solution $x^\star$ has smallest value of f_0 among all vectors that satisfy the constraints

Examples

portfolio optimization

- variables: amounts invested in different assets
- constraints: budget, max./min. investment per asset, minimum return
- objective: overall risk or return variance

device sizing in electronic circuits

- variables: device widths and lengths
- constraints: manufacturing limits, timing requirements, maximum area
- objective: power consumption

data fitting

- variables: model parameters
- constraints: prior information, parameter limits
- objective: measure of misfit or prediction error

Solving Optimization Problems

general optimization problem

- very difficult to solve
- methods involve some compromise, *e.g.*, very long computation time, or not always finding the solution

exceptions: certain problem classes can be solved efficiently and reliably

- least-squares problems
- linear programming problems
- convex optimization problems

Linear Programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

solving linear programs

- no analytical formula for solution
- reliable and efficient algorithms and software
- computation time proportional to $n^2 m$ if $m \geq n$; less with structure
- a mature technology

using linear programming

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs ($e.g.$, problems involving ℓ_1- or ℓ_∞-norms, piecewise-linear functions)

Least-Squares Optimization

$$\text{minimize} \quad \|Ax - b\|_2^2$$

solving least-squares problems

- analytical solution: $x^\star = (A^T A)^{-1} A^T b$
- reliable and efficient algorithms and software
- computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k \times n}$); less if structured
- a mature technology

using least-squares

- least-squares problems are easy to recognize
- a few standard techniques increase flexibility (*e.g.*, including weights, adding regularization terms)

Convex Optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \dots, m \end{array}$$

- objective and constraint functions are convex:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

  if $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$

- includes least-squares problems and linear programs as special cases

Design Through Optimization

- Idea: Sample/discretize the frequency response

$$H(e^{j\omega}) \Rightarrow H(e^{j\omega_k})$$

- Sample points are fixed $\omega_k = k\frac{\pi}{P}$

$$-\pi \leq \omega_1 < \cdots < \omega_p \leq \pi$$

- M+1 is the filter order
- P >> M + 1 (rule of thumb P=15M)
- Yields a (good) approximation of the original problem

Example: Least Squares

- Target: Design M+1= 2N+1 filter
- First design non-causal $\tilde{H}(e^{j\omega})$ and hence $\tilde{h}[n]$

Example: Least Squares

- Target: Design M+1= 2N+1 filter
- First design non-causal $\tilde{H}(e^{j\omega})$ and hence $\tilde{h}[n]$
- Then, shift to make causal

$$h[n] = \tilde{h}[n - M/2]$$

$$H(e^{j\omega}) = e^{-j\frac{M}{2}}\tilde{H}(e^{j\omega})$$

Example: Least Squares

$$\tilde{h} = \left[\tilde{h}[-N], \tilde{h}[-N+1], \cdots, \tilde{h}[N]\right]^T$$

$$b = \left[H_d(e^{j\omega_1}), \cdots, H_d(e^{j\omega_P})\right]^T$$

$$A = \begin{bmatrix} e^{-j\omega_1(-N)} & \dots & e^{-j\omega_1(+N)} \\ \vdots & & \\ e^{-j\omega_P(-N)} & \dots & e^{-j\omega_P(+N)} \end{bmatrix}$$

$$\text{argmin}_{\tilde{h}} \quad ||A\tilde{h} - b||^2$$

Least-Squares

$$\text{argmin}_{\tilde{h}} \quad ||A\tilde{h} - b||^2$$

Solution:

$$\tilde{h} = (A^*A)^{-1}A^*b$$

- Result will generally be non-symmetric and complex valued.
- However, if $\tilde{H}(e^{j\omega})$ is real, $\tilde{h}[n]$ should have symmetry!

Design of Linear-Phase L.P Filter

- Suppose:
  - $\tilde{H}(e^{j\omega})$ is real-symmetric
  - M is even (M+1 taps)
- Then:
  - $\tilde{h}[n]$ is real-symmetric around midpoint
- So:

$$\begin{aligned}\tilde{H}(e^{j\omega}) &= \tilde{h}[0] + \tilde{h}[1]e^{-j\omega} + \tilde{h}[-1]e^{+j\omega} \\ &\quad + \tilde{h}[2]e^{-j2\omega} + \tilde{h}[-2]e^{+j2\omega} \ldots \\ &= \tilde{h}[0] + 2\cos(\omega)\tilde{h}[1] + 2\cos(2\omega)\tilde{h}[2] + \cdots\end{aligned}$$

Reminder: FIR GLP: Type I – Example, M=4

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$

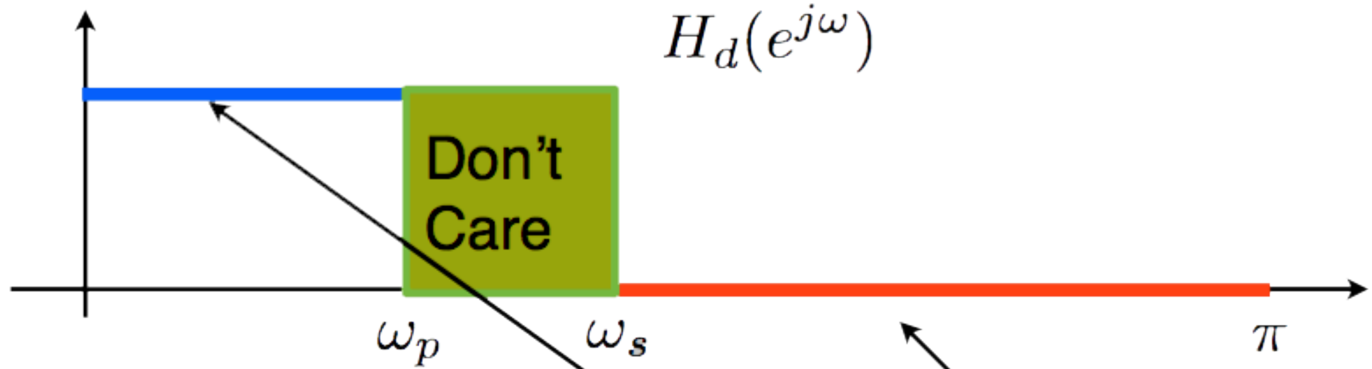
Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$

$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega}\right]$$

$$= \underbrace{\left[2h[0]\cos(2\omega) + 2h[1]\cos(\omega) + h[2]\right]}_{A(\omega)\ (even)} \; e^{-j2\omega}$$

Least-Squares Linear Phase Filter



Given M, ω_P, ω_S find the best LS filter:

$$A = \begin{bmatrix} 1 & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_p) \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix}$$

$$b = [1, 1, \cdots, 1, 0, 0, \cdots, 0]^T$$

Least-Squares Linear Phase Filter

Given M, ω_P, ω_s find the best LS filter:

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$$\tilde{h}_+ = [\tilde{h}[0], \cdots, \tilde{h}[\frac{M}{2}]]^T = (A^*A)^{-1}A^*b$$

$$\tilde{h} = \begin{cases} \tilde{h}_+[n] & n \geq 0 \\ \tilde{h}_+[-n] & n < 0 \end{cases}$$

$$h[n] = \tilde{h}[n - M/2]$$

Extension:

- LS has no preference for pass band or stop band
- Use weighting of LS to change ratio

want to solve the discrete version of:

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

where W(ω) is δp in the pass band and δs in stop band

Similarly: W(ω) is 1 in the pass band and δp/δs in stop band

Weighted Least-Squares

$$\text{argmin}_{\tilde{h}_+} \quad (A\tilde{h}_+ - b)^* W^2 (A\tilde{h}_+ - b)$$

Solution:

$$\tilde{h}_+ = (A^* W^2 A)^{-1} W^2 A^* b$$

$$W = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \cdots & & & \\ & & & \frac{\delta_p}{\delta_s} & & \\ & & & & \cdots & \\ 0 & & & & & \frac{\delta_p}{\delta_s} \end{bmatrix}$$

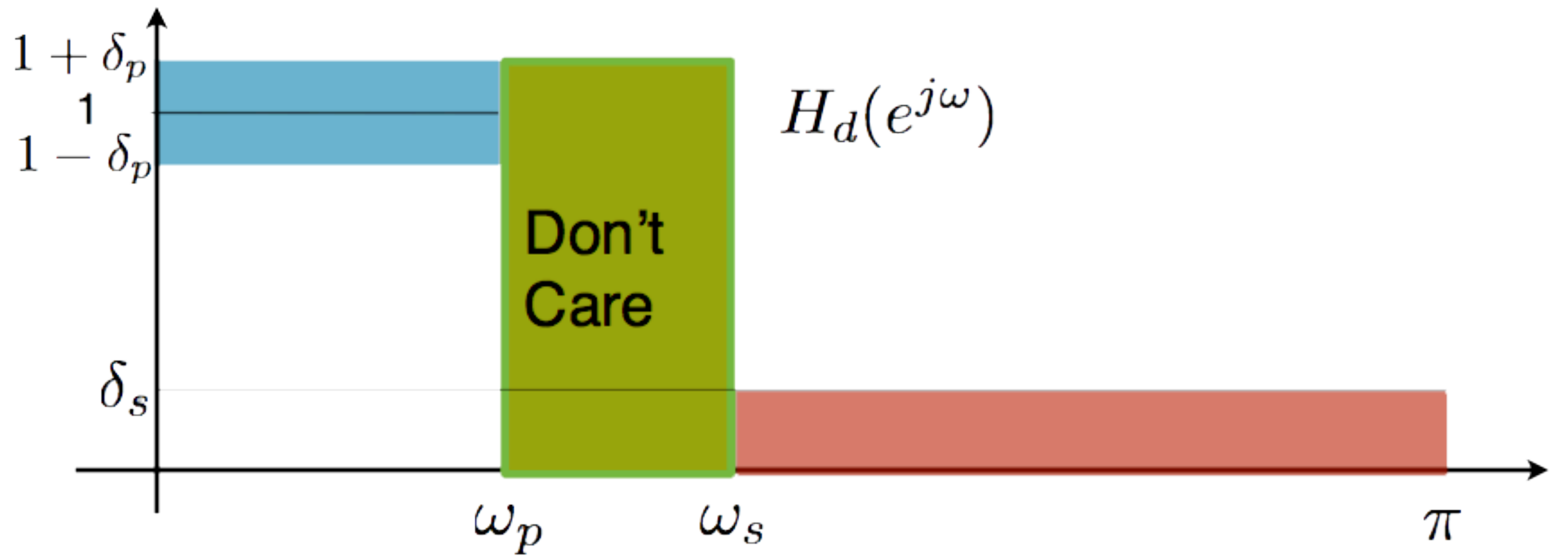
Optimality

- Chebychev Design (min-max)

$$\text{minimize}_{\omega \in \text{care}} \quad \max |H(e^{j\omega}) - H_d(e^{j\omega})|$$

  - Parks-McClellan algorithm - equiripple
  - Also known as Remez exchange algorithms (signal.remez)
  - Can also use optimization

Specifications



- Filter specifications are given in terms of boundaries

Min-Max Filter Design

- Constraints:
  - min-max pass-band ripple

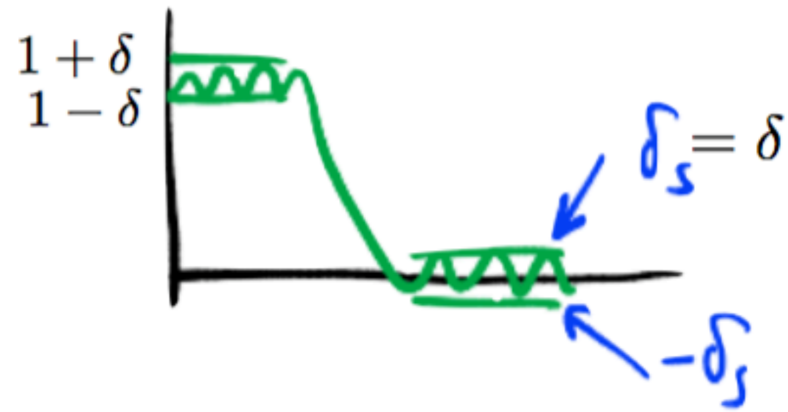
$$1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p, \qquad 0 \leq w \leq \omega_p$$

  - min-max stop-band ripple

$$|H(e^{j\omega})| \leq \delta_s, \qquad \omega_s \leq w \leq \pi$$

Min-Max Ripple Design

- Recall, $\tilde{H}(e^{j\omega})$ is symmetric and real
- Given ω_p, ω_s, M, find δ, $\tilde{h}_+$



$$\begin{aligned} &\text{minimize} \quad \delta \\ &\text{Subject to :} \\ &\quad 1-\delta \le \tilde{H}(e^{j\omega_k}) \le 1+\delta \qquad 0 \le \omega_k \le \omega_p \\ &\quad -\delta \le \tilde{H}(e^{j\omega_k}) \le \delta \qquad \omega_s \le \omega_k \le \pi \\ &\quad \delta > 0 \end{aligned}$$

- Formulation is a linear program with solution δ, $\tilde{h}_+$
- A well studied class of problems with good solvers

Min-Max Ripple via LP

$$\text{minimize} \quad \delta$$

$$\text{subject to :}$$

$$1 - \delta \preceq A_p \tilde{h}_+ \preceq 1 + \delta$$

$$-\delta \preceq A_s \tilde{h}_+ \preceq \delta$$

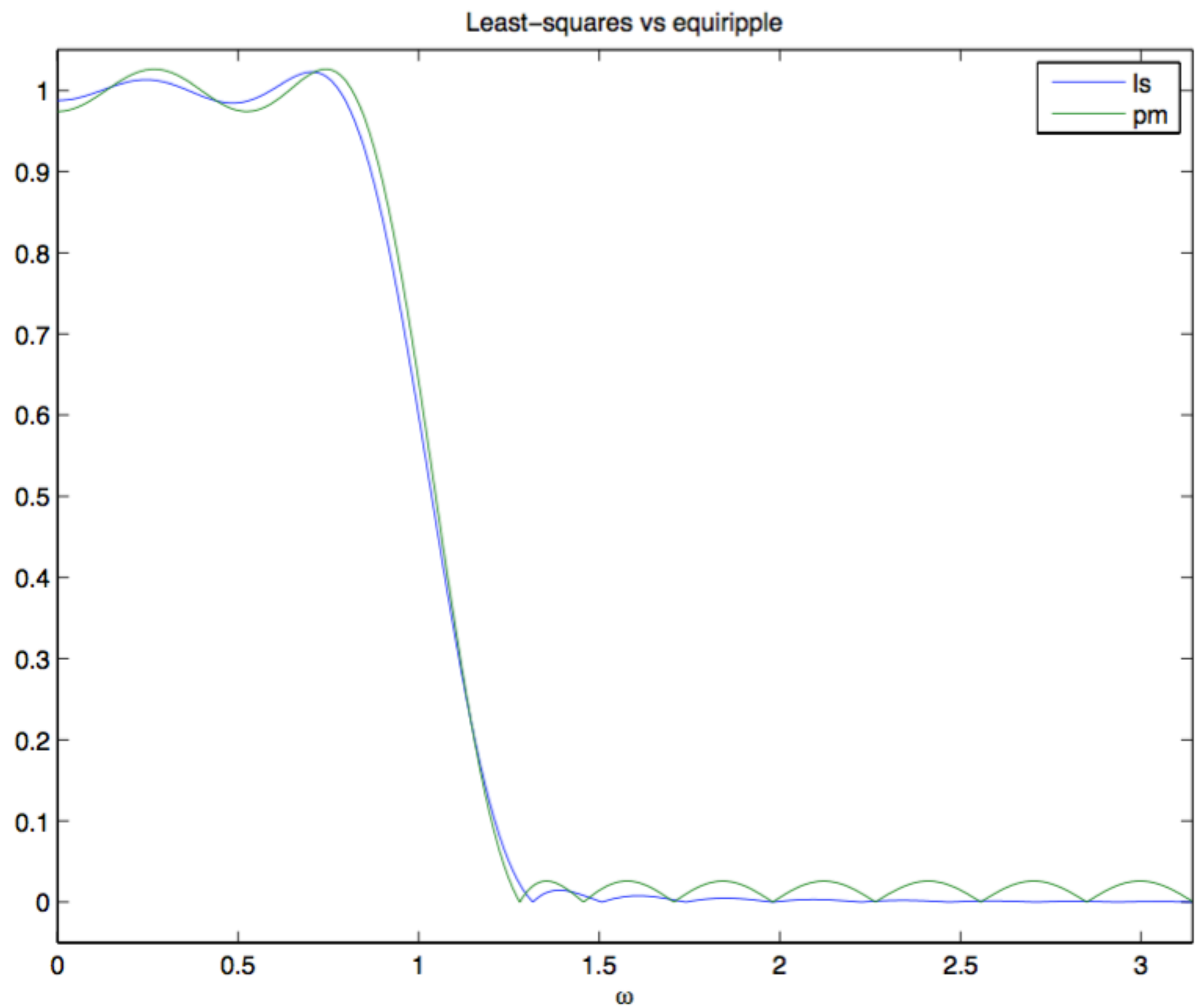
$$\delta > 0$$

$$A_p = \begin{bmatrix} 1 & 2\cos(\omega_1) & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ & \vdots & & \\ 1 & 2\cos(\omega_p) & \cdots & 2\cos(\frac{M}{2}\omega_p) \end{bmatrix}$$

$$A_s = \begin{bmatrix} 1 & 2\cos(\omega_s) & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ & \vdots & & \\ 1 & 2\cos(\omega_P) & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix}$$

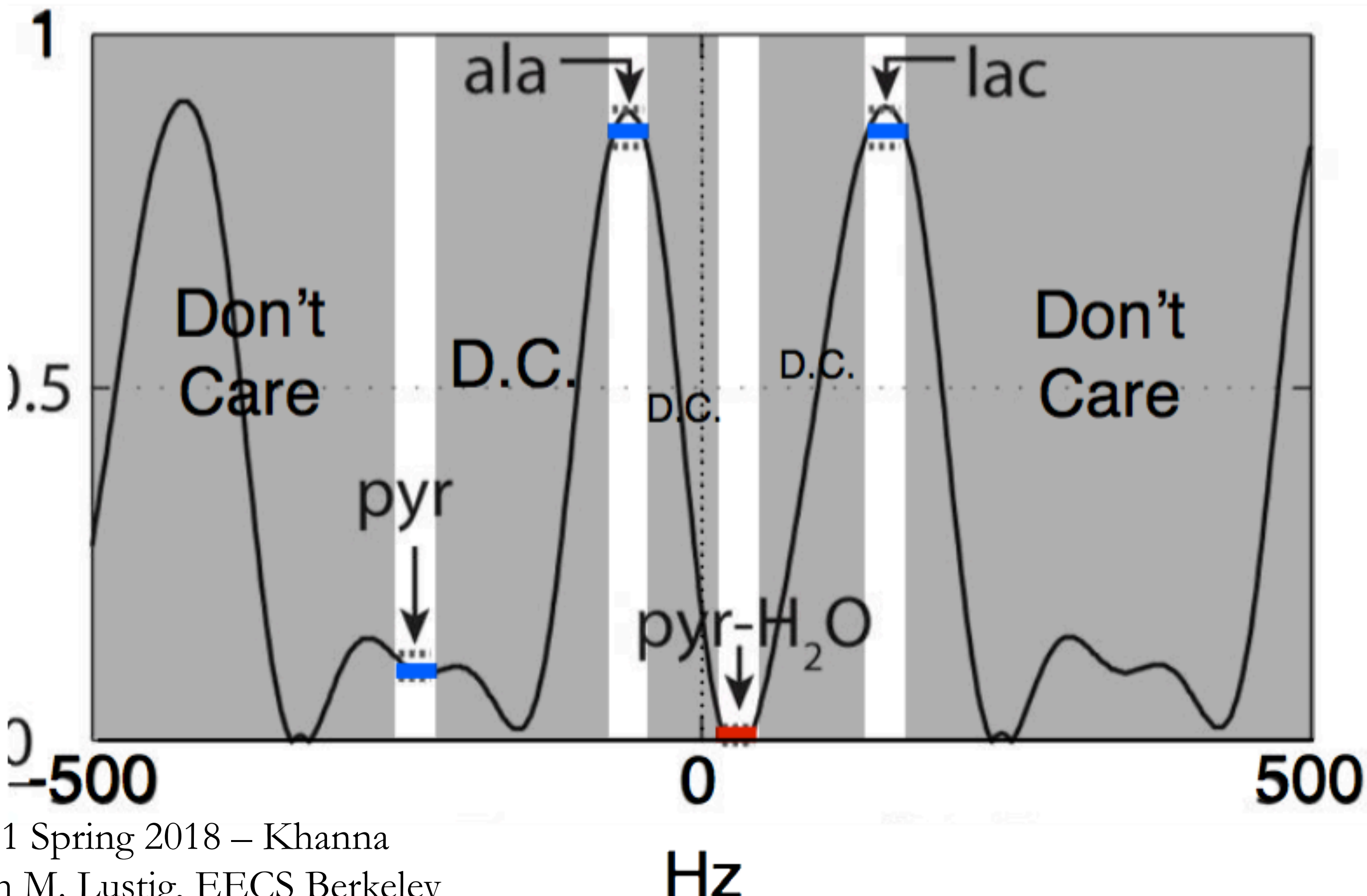
capital P

Least-Squares v.s. Min-Max



Example of Complex Filter

- Larson et. al, "Multiband Excitation Pulses for Hyperpolarized 13C Dynamic Chemical Shift Imaging" JMR 2008;194(1):121-127
- Need to design 11 taps filter with following frequency response:

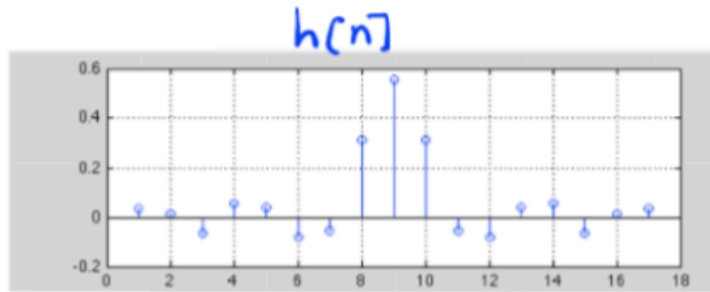


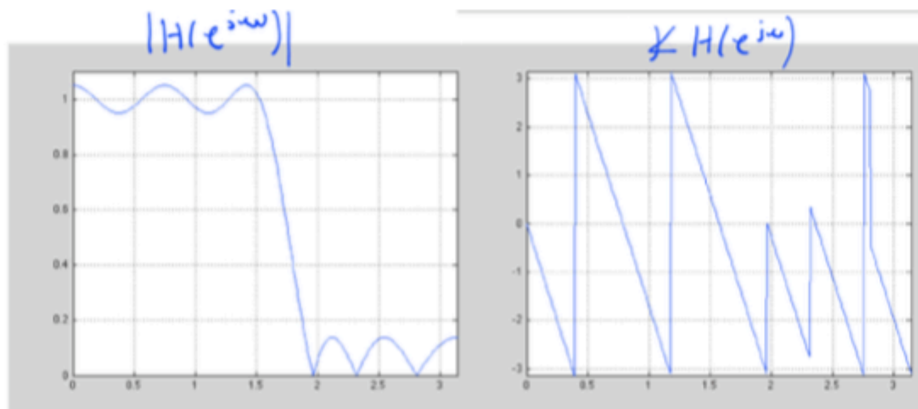
Convex Optimization

- Many tools and Solvers
- Tools:
  - CVX (Matlab) http://cvxr.com/cvx/
  - CVXOPT, CVXMOD (Python)
- Engines:
  - Sedumi (Free)
  - MOSEK (commercial)

Using CVX (in Matlab)

```
M = 16;
wp = 0.5*pi;
ws = 0.6*pi;
MM = 15*M;
w = linspace(0,pi,MM);

idxp = find(w <=wp);
idxs = find(w >=ws);

Ap = [ones(length(idxp),1) 2*cos(kron(w(idxp)',
[1:M/2]))];
As = [ones(length(idxs),1) 2*cos(kron(w(idxs)',
[1:M/2]))];

% optimization
cvx_begin
   variable hh(M/2+1,1);
   variable d(1,1);

   minimize(d)
   subject to
      Ap*hh <=1+d;
      Ap*hh >=1-d;
      As*hh <  d;
      As*hh > -d;
      ds>0;
cvx_end
h = [hh(end:-1:1) ; hh(2:end)];
```





Admin

- HW 7 due tomorrow @ midnight