# ESE 531: Digital Signal Processing

Lec 18: March 26, 2018

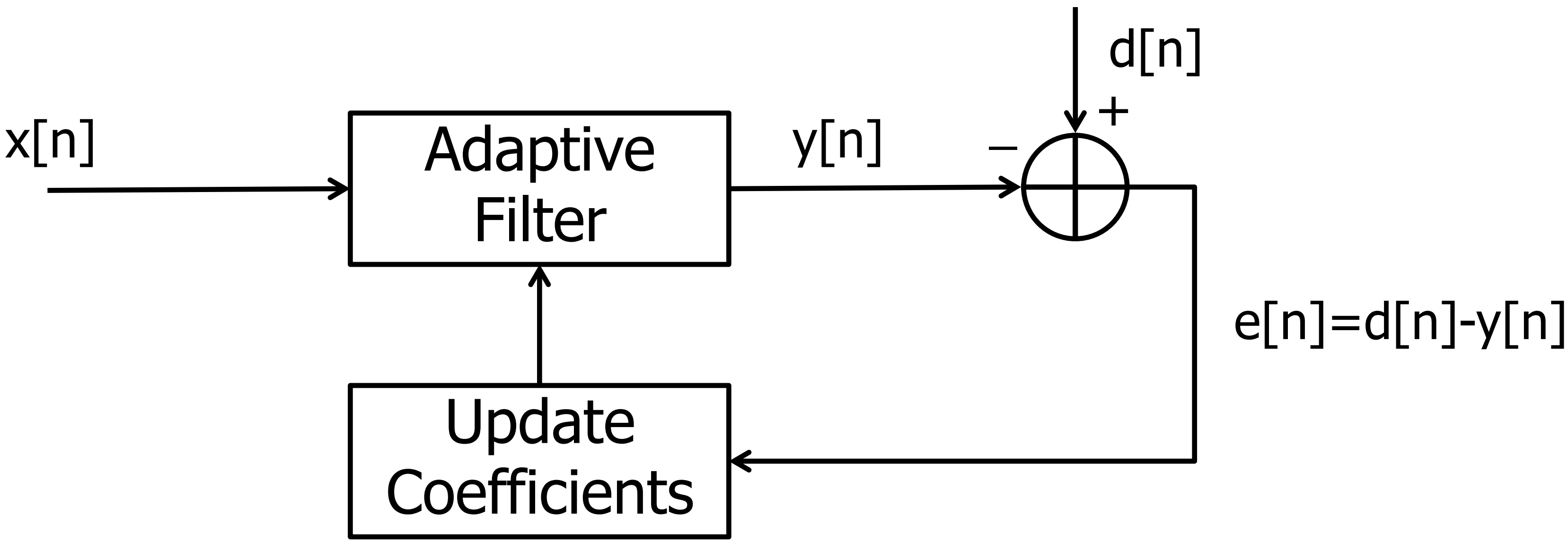
Discrete Fourier Transform

# Today

- Adaptive filtering
  - Blind equalization setup
- Discrete Fourier Series
- Discrete Fourier Transform (DFT)
- DFT Properties
- Circular Convolution

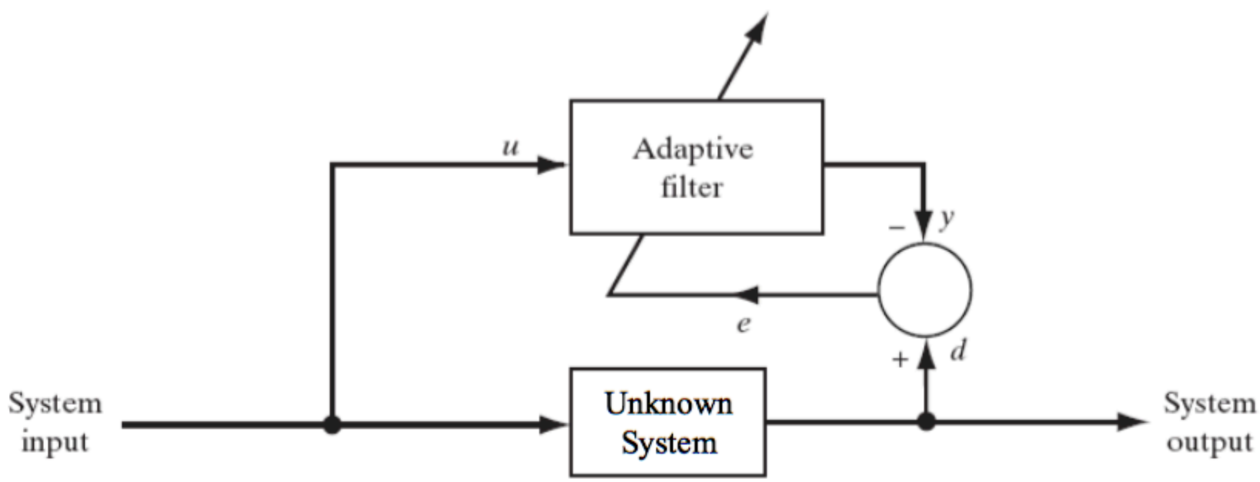
# Adaptive Filters

- An adaptive filter is an adjustable filter that processes in time
  - It adapts…



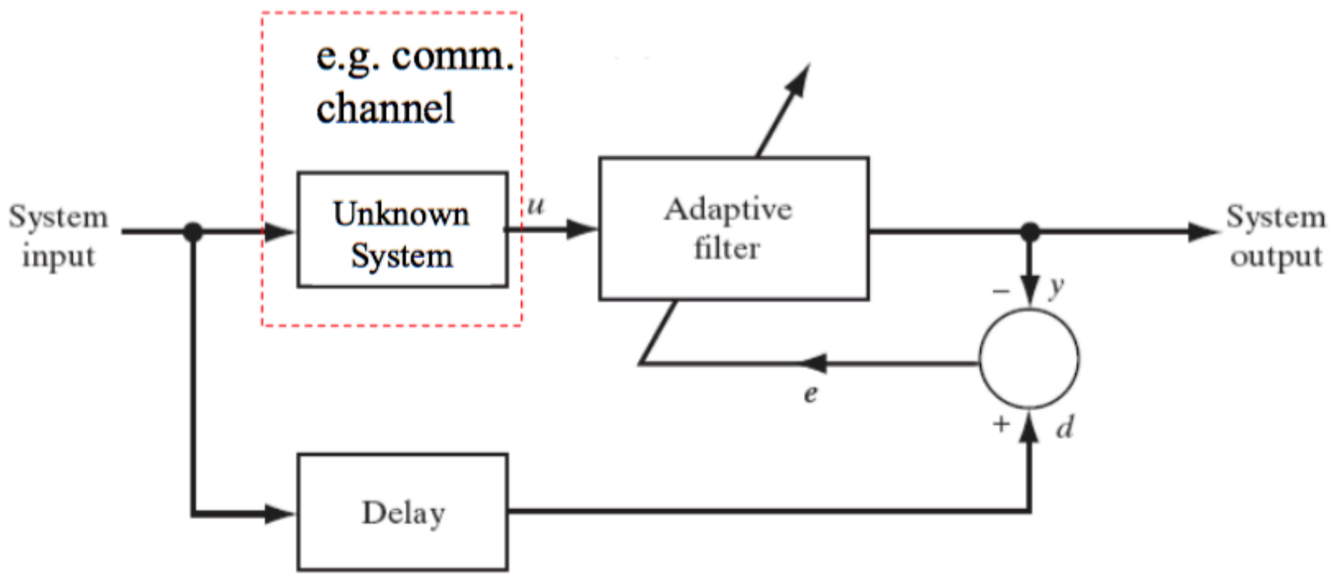
# Adaptive Filter Applications

- System Identification



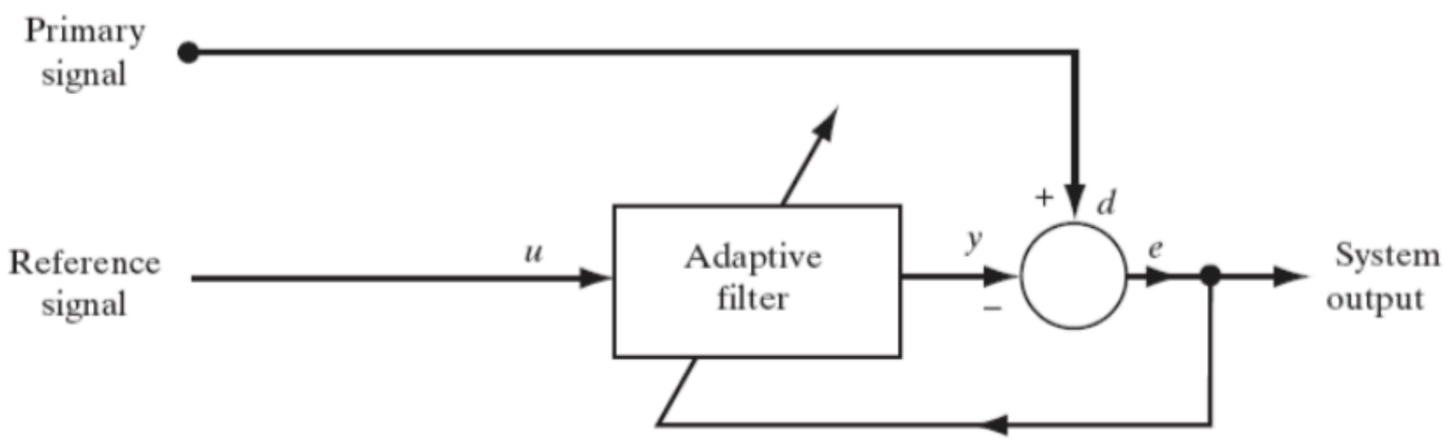
# Adaptive Filter Applications

- Identification of inverse system



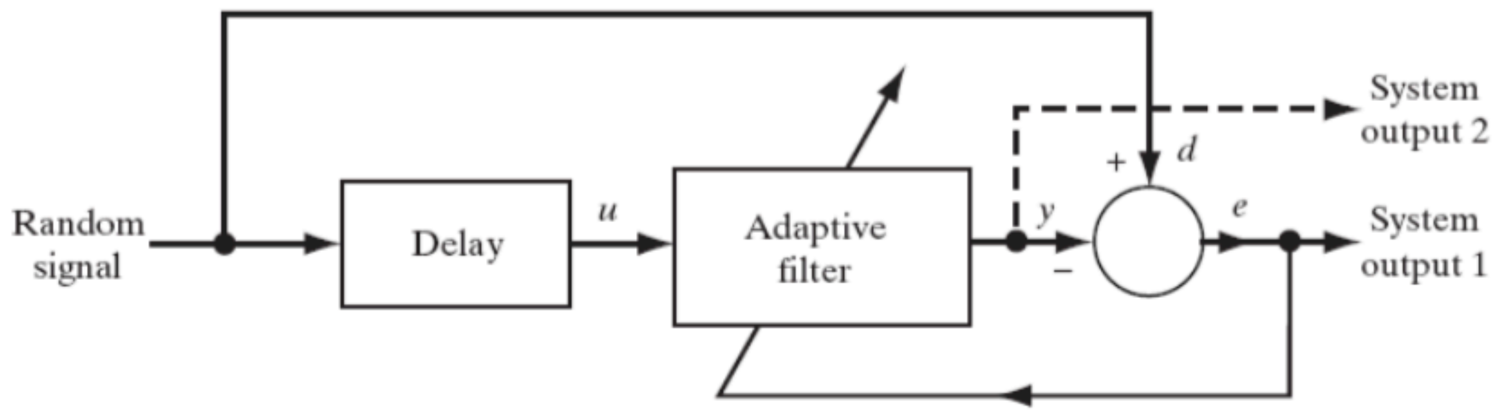
# Adaptive Filter Applications

- Adaptive Interference Cancellation



# Adaptive Filter Applications

- Adaptive Prediction



# Discrete Fourier Series

# Reminder: Eigenvalue (DTFT)

- $x[n]=e^{j\omega n}$

$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$=\sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k]$$

$$=e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$=H(e^{j\omega})e^{j\omega n}$$

$$\boxed{H(e^{j\omega})=\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}$$

- Describes the change in amplitude and phase of signal at frequency $\omega$
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase

# Discrete Fourier Series

- Definition:
  - Consider N-periodic signal:

$$\tilde{x}[n+N] = \tilde{x}[n] \quad \forall n$$

  - Frequency-domain also periodic in N:

$$\tilde{X}[k+N] = \tilde{X}[k] \quad \forall k$$

  - "~" indicates periodic signal/spectrum

# Discrete Fourier Series

- Define:

$$W_N \triangleq e^{-j2\pi/N}$$

- DFS:

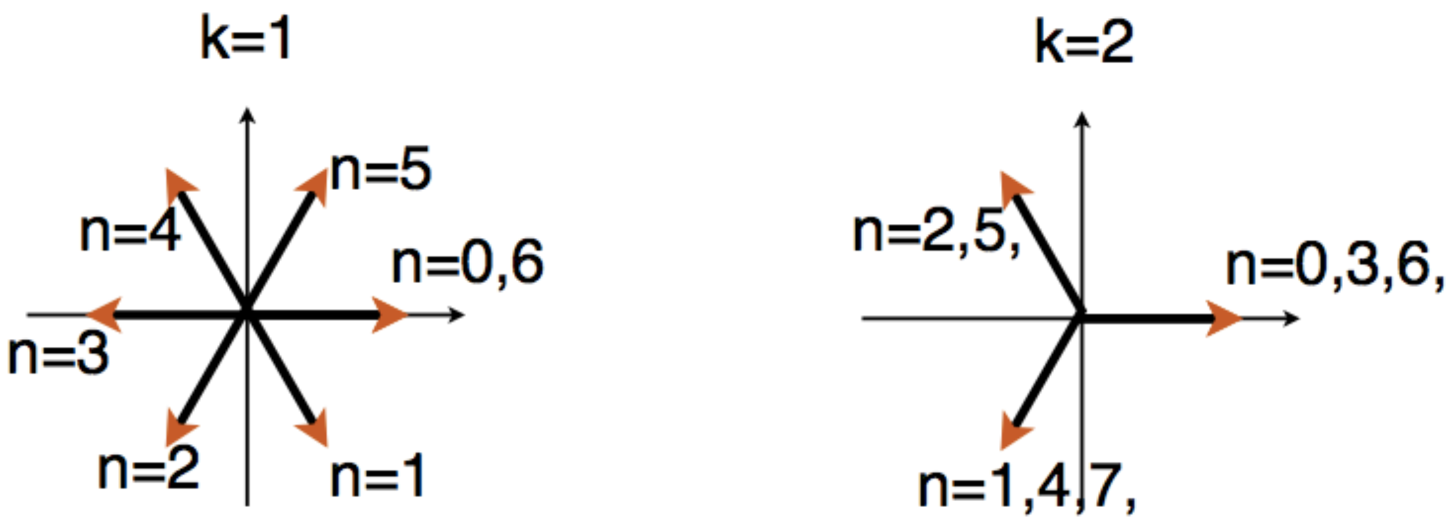
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn}$$

# Discrete Fourier Series

$$W_N \triangleq e^{-j2\pi/N}$$

- Properties of WN:
  - $W_N^{0} = W_N^{N} = W_N^{2N} = \ldots = 1$
  - $W_N^{k+r} = W_N^{k} W_N^{r}$ and, $W_N^{k+N} = W_N^{k}$
- Example: $W_N^{kn}$ (N=6)



# Discrete Fourier Transform

- By convention, work with one period:

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X[k] \triangleq \begin{cases} \tilde{X}[k] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Same, but different!

# Discrete Fourier Transform

- The DFT

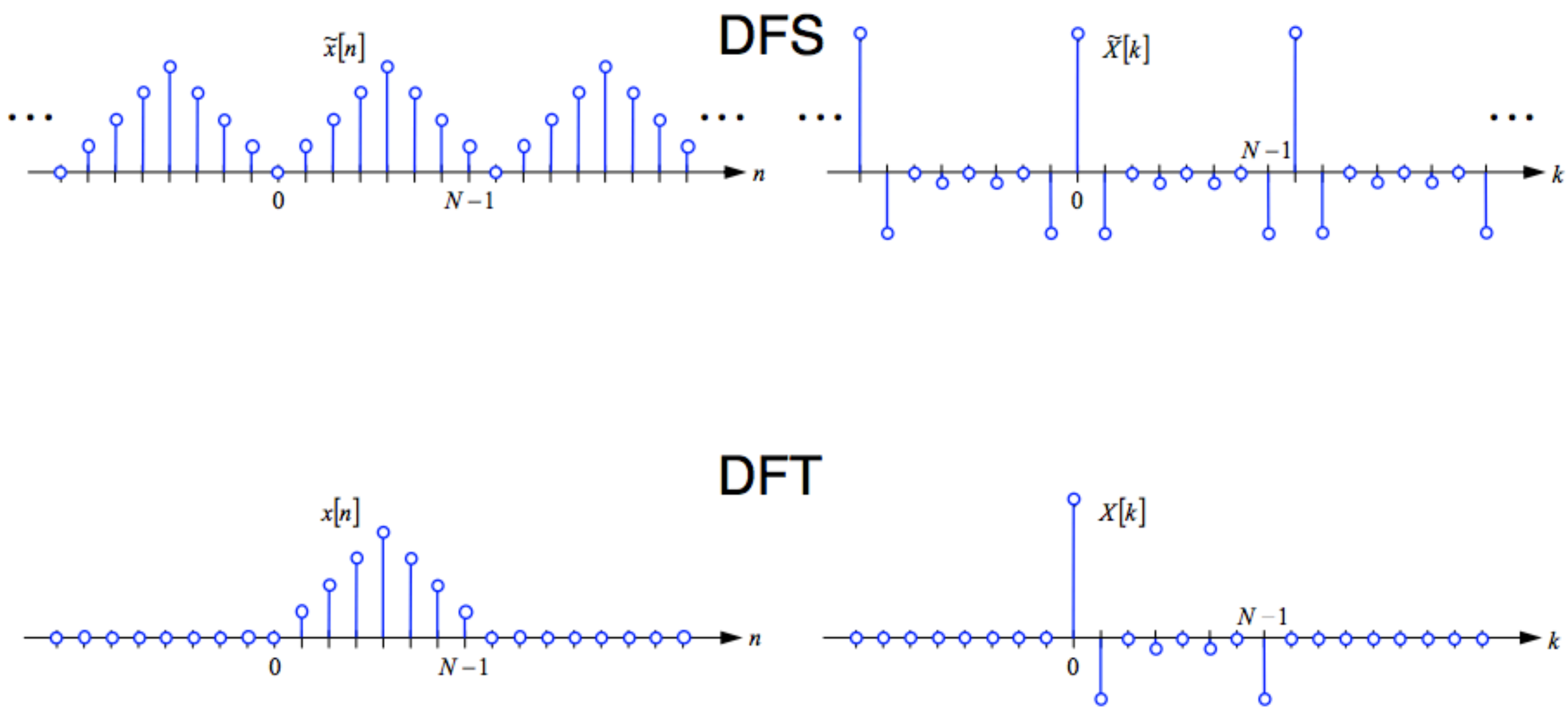
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$$ **Inverse DFT, synthesis**

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$ **DFT, analysis**

- It is understood that,

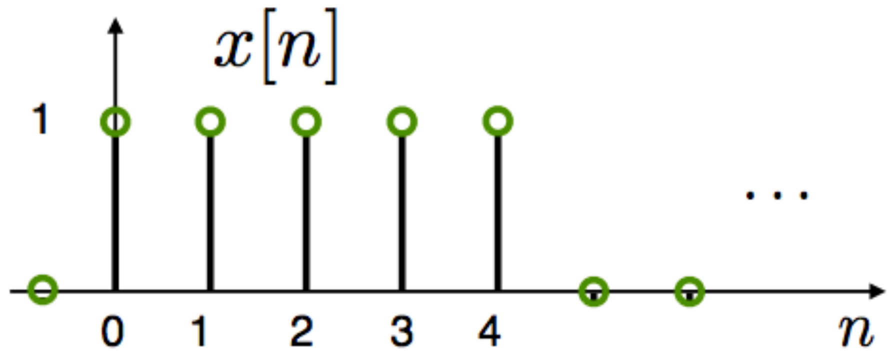
$$x[n] = 0 \quad \text{outside } 0 \le n \le N-1$$
$$X[k] = 0 \quad \text{outside } 0 \le k \le N-1$$

# DFS vs. DFT



# Example

$$W_N \triangleq e^{-j2\pi/N}$$



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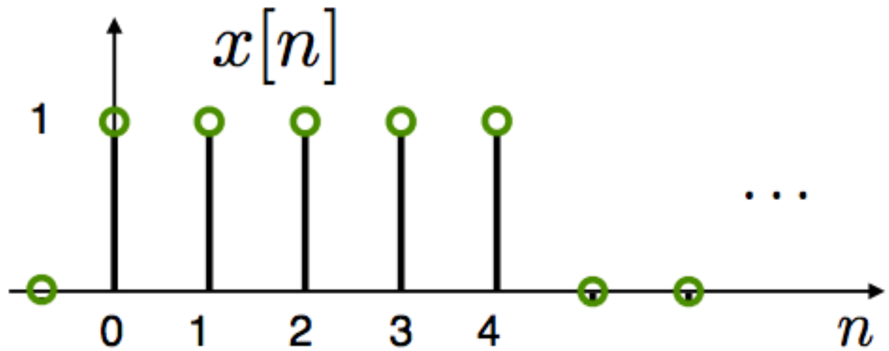


Take N=5

$$X[k] = \begin{cases} \sum_{n=0}^{4} W_5^{nk} & k = 0,1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

$$= 5\delta[k]$$

"5-point DFT"

# Discrete Fourier Series

$$W_N \triangleq e^{-j2\pi/N}$$

- Properties of WN:
  - $W_N^0 = W_N^N = W_N^{2N} = \ldots = 1$
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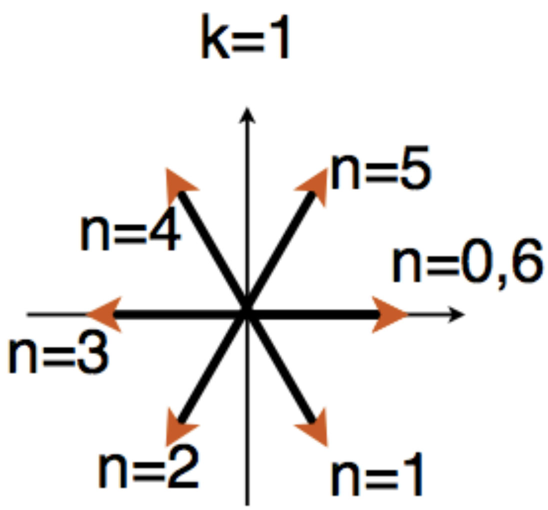


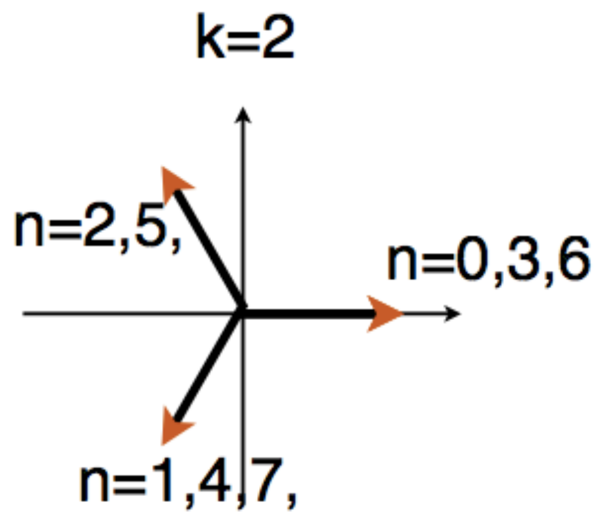

# Example

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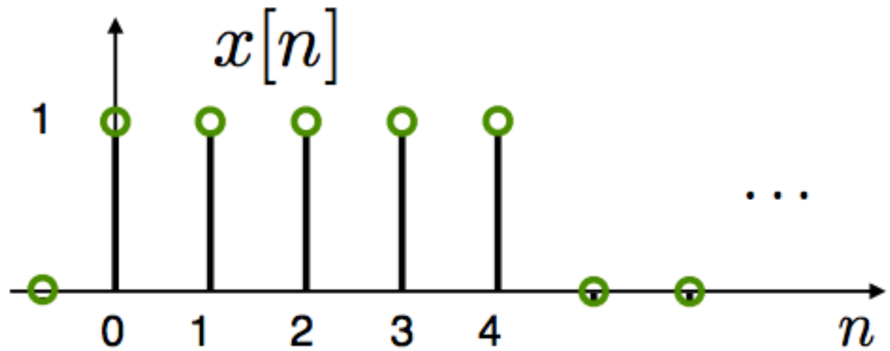


- Take N=5

$$
\begin{aligned}
X[k] &= \begin{cases} \sum_{n=0}^{4} W_5^{nk} & k = 0,1,2,3,4 \\ 0 & \text{otherwise} \end{cases} \\
&= 5\delta[k]
\end{aligned}
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"5-point DFT"

# Example

$$W_N \triangleq e^{-j2\pi/N}$$

- Q: What if we take N=10?
- A: $X[k] = \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.

# Example

$W_N \triangleq e^{-j2\pi/N}$

- Q: What if we take N=10?
- A: $X[k] = \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.


$$X[k] = \begin{cases} \sum_{n=0}^{4} W_{10}^{nk} & k = 0, 1, 2, \cdots, 9 \\ 0 & \text{otherwise} \end{cases}$$

"10-point DFT"

# Example

- Now, sum from n=0 to 9

$$X[k] = \sum_{n=0}^{9} W_{10}^{nk}$$

# Example

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$$
\begin{aligned}
X[k] &= \sum_{n=0}^{9} W_{10}^{nk} \\
&= \sum_{n=0}^{4} W_{10}^{nk} \\
&= e^{-j\frac{4\pi}{10}k} \frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}
\end{aligned}
$$

"10-point DFT"

# DFT vs. DTFT

- For finite sequences of length N:
  - The N-point DFT of x[n] is:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} \qquad 0 \leq k \leq N-1$$

  - The DTFT of x[n] is:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \qquad -\infty < \omega < \infty$$

# DFT vs. DTFT

- The DFT are samples of the DTFT at N equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega = k\frac{2\pi}{N}} \qquad 0 \leq k \leq N-1$$

# DFT vs DTFT

- Back to example

$$X[k] = \sum_{n=0}^{4} W_{10}^{nk}$$

$$= e^{-j\frac{4\pi}{10}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}$$

"10-point DFT"

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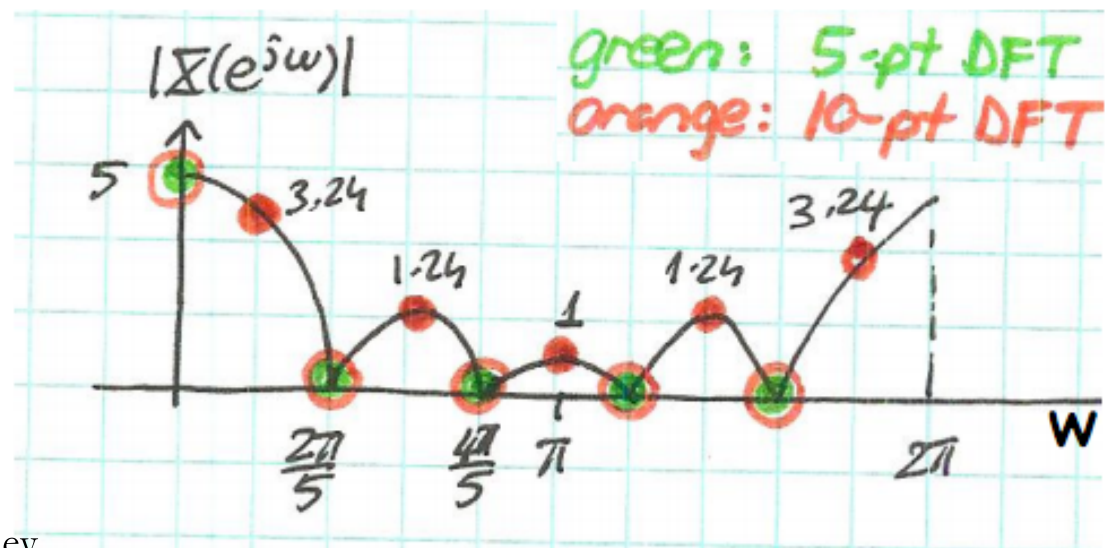

# DFT vs DTFT

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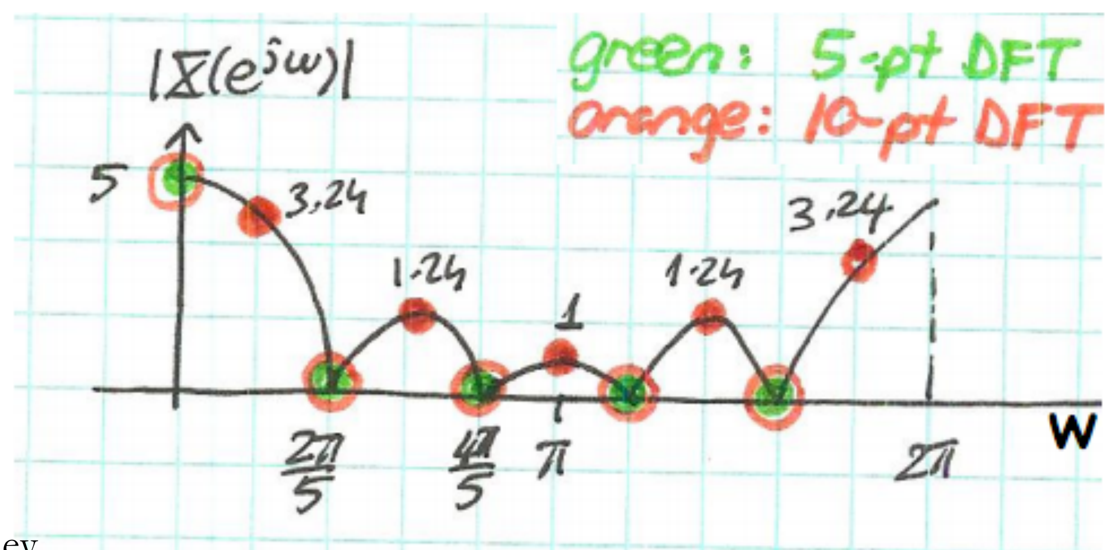
Use fftshift to center around dc



# DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$N \cdot x^*[n] = N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^*$$

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&= N \left(\frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}\right)^*
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# DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

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&= \sum_{k=0}^{N-1} X^*[k] W_N^{kn} \\
&= \mathcal{DFT} \{X^*[k]\}.
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$$\mathcal{DFT}^{-1}\left\{X[k]\right\} = \frac{1}{N}\left(\mathcal{DFT}\left\{X^*[k]\right\}\right)^*$$

- Implement IDFT by:
  - Take complex conjugate
  - Take DFT
  - Multiply by 1/N
  - Take complex conjugate

# DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

**DFT:**

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

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**IDFT:**

$$\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \begin{pmatrix} W_N^{-00} & \cdots & W_N^{-0k} & \cdots & W_N^{-0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-n0} & \cdots & W_N^{-nk} & \cdots & W_N^{-n(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)0} & \cdots & W_N^{-(N-1)k} & \cdots & W_N^{-(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix}$$

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$N^2$ complex multiples

# DFT as Matrix Operator

- Can write compactly as

$$\begin{aligned} \mathbf{X} &= \mathbf{W}_N \, \mathbf{x} \\ \mathbf{x} &= \frac{1}{N} \mathbf{W}_N^* \, \mathbf{X} \end{aligned}$$

# Properties of the DFT

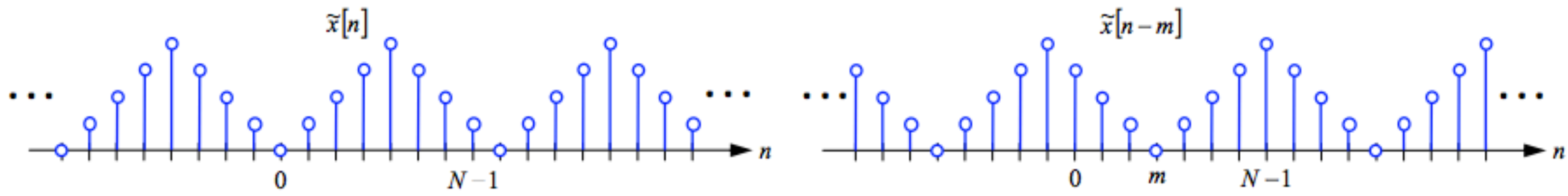
- Properties of DFT inherited from DFS
- Linearity

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

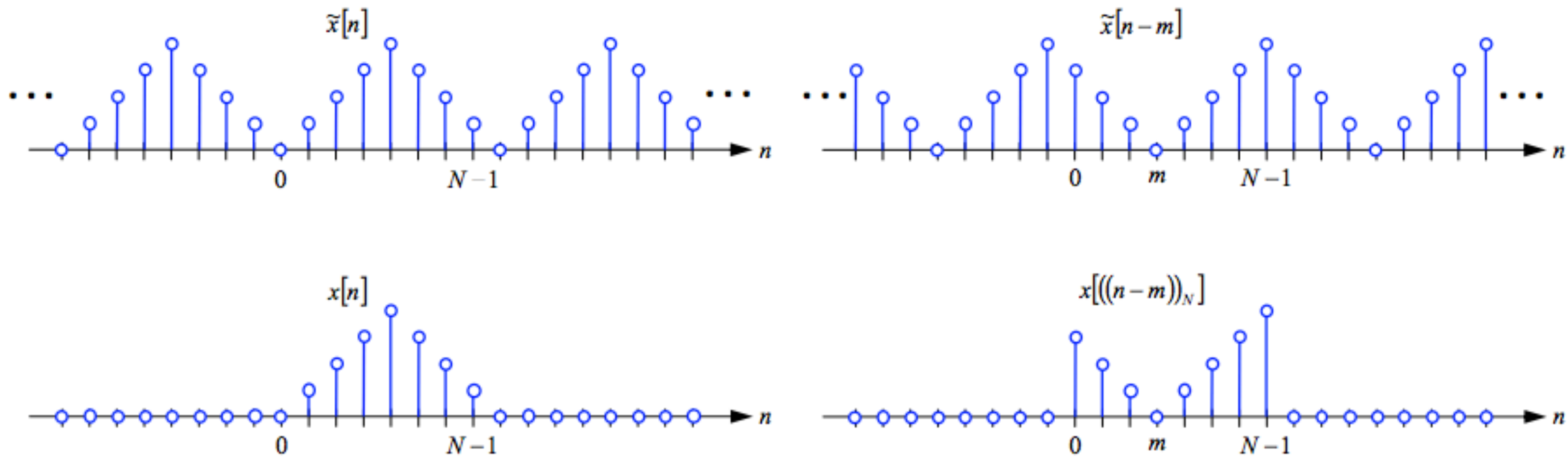
- Circular Time Shift

$$x[((n-m))_N] \leftrightarrow X[k]e^{-j(2\pi/N)km} = X[k]W_N^{km}$$

# Circular Shift



# Circular Shift



# Properties of DFT

- Circular frequency shift

$$x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- Complex Conjugation

$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

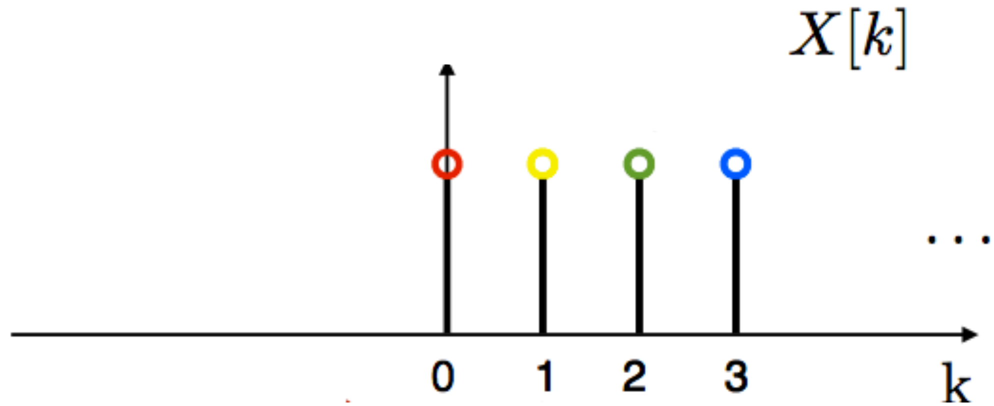
- Conjugate Symmetry for Real Signals

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

# Example: Conjugate Symmetry

4-point DFT

–Symmetry

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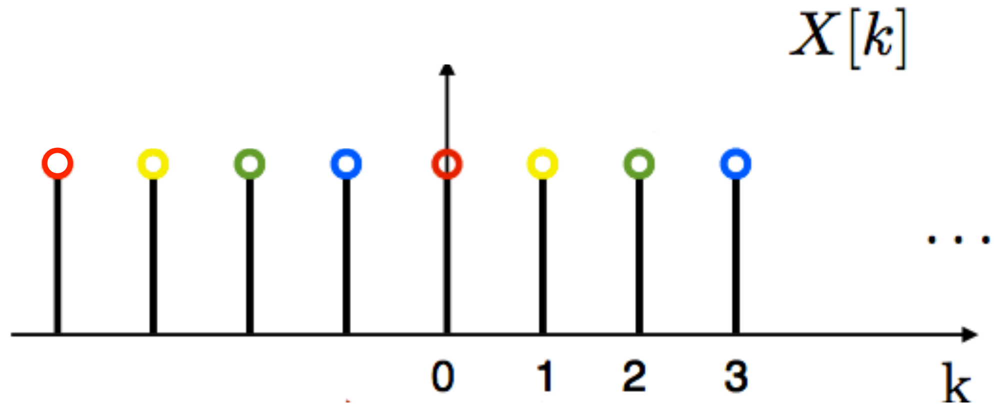
# Example: Conjugate Symmetry

4-point DFT

–Symmetry

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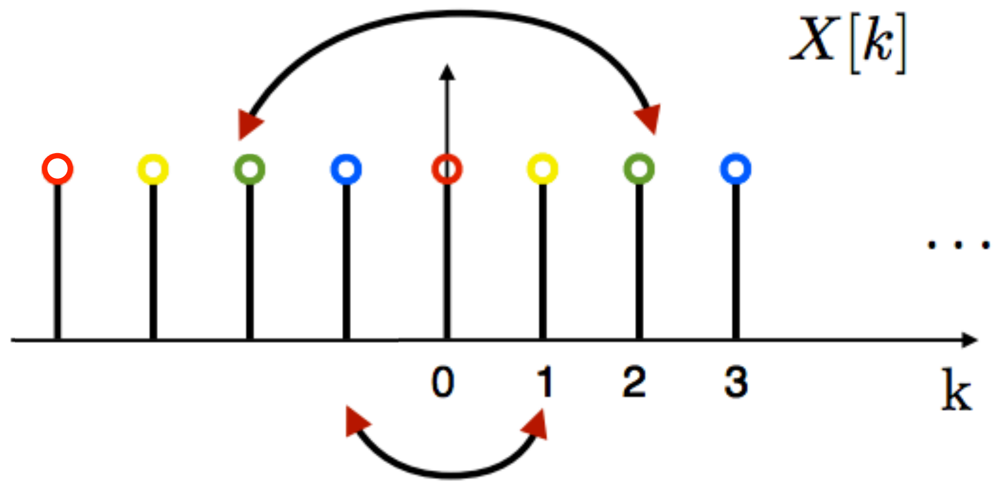
# Example: Conjugate Symmetry

- 4-point DFT
  - Symmetry

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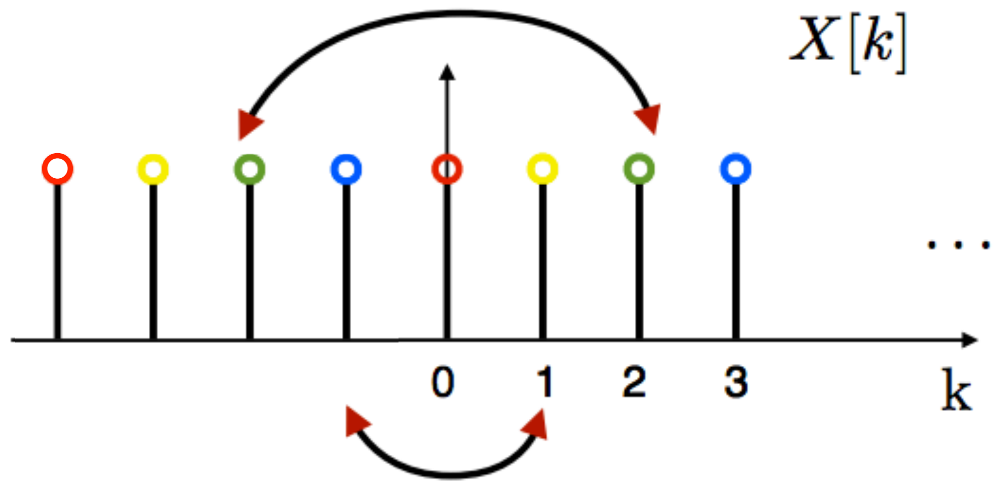
# Example: Conjugate Symmetry

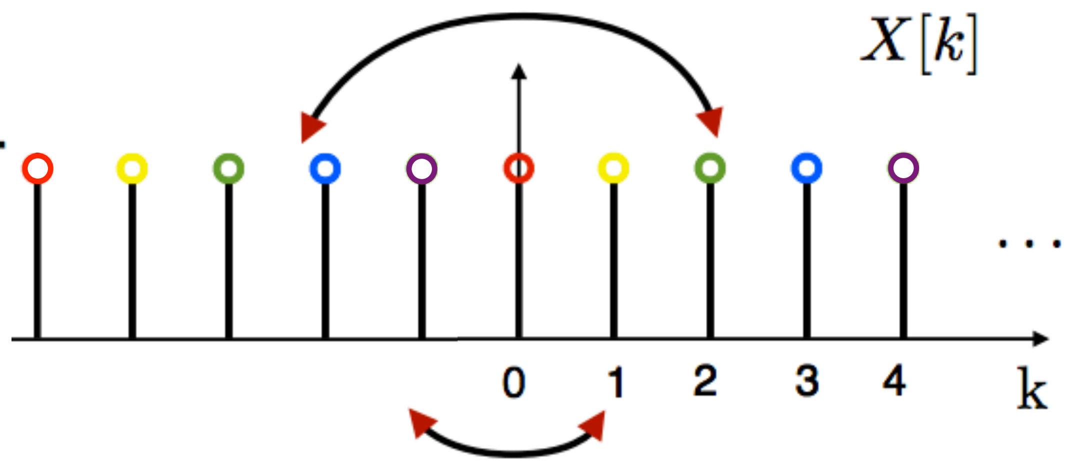
4-point DFT

–Symmetry




5-point DFT

–Symmetry

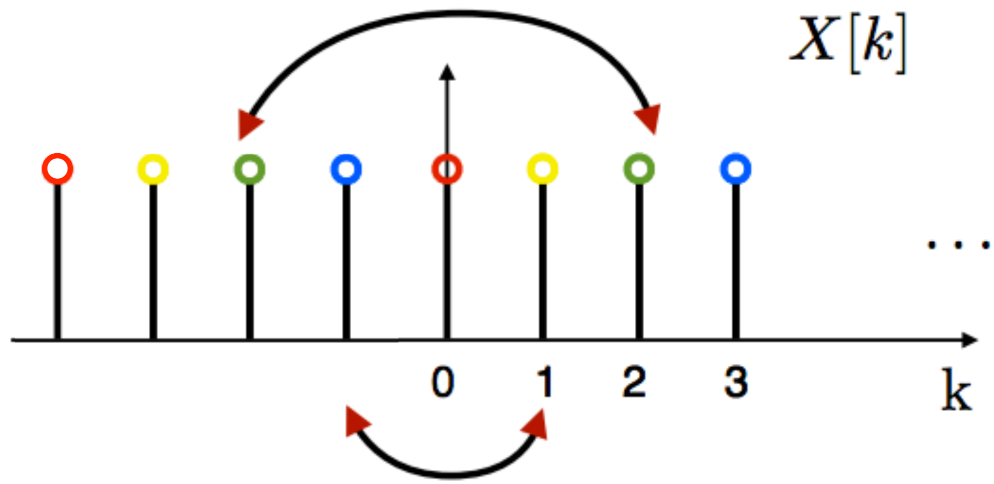



$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

# Example: Conjugate Symmetry

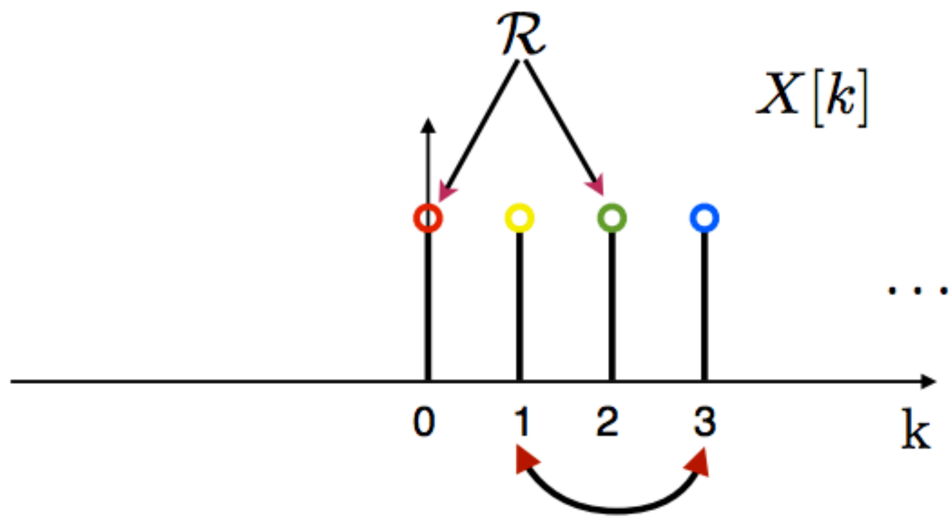
4-point DFT

–Symmetry




4-point DFT

–Symmetry

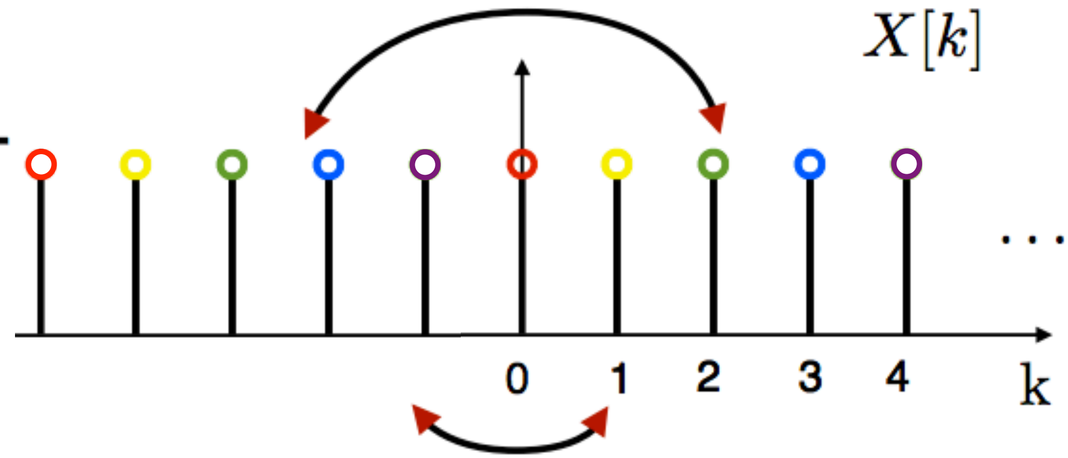



$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

# Example

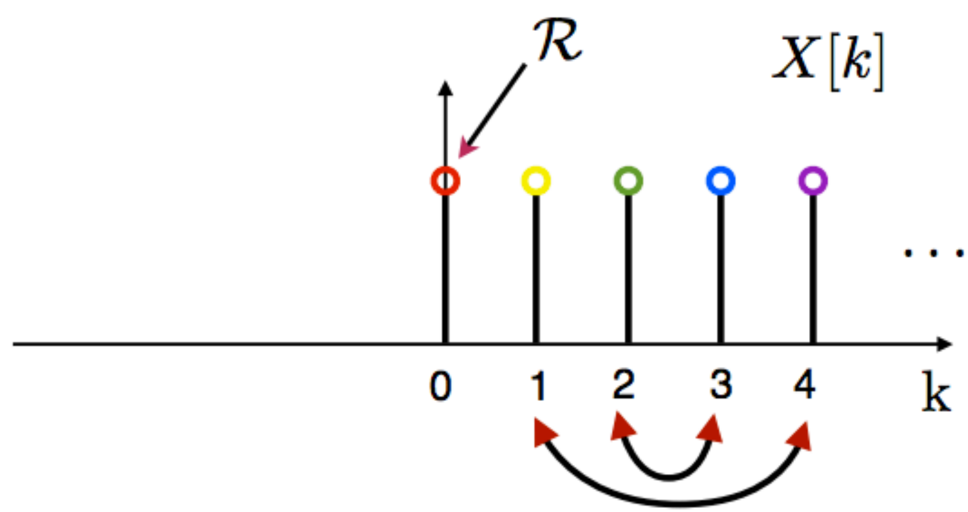
5-point DFT

–Symmetry




5-point DFT

–Symmetry



# Properties of the DFS/DFT

| Discrete Fourier Series | | | Discrete Fourier Transform | | |
|---|---|---|---|---|---|
| **Property** | ***N*-periodic sequence** | ***N*-periodic DFS** | **Property** | ***N*-point sequence** | ***N*-point DFT** |
| | $\tilde{x}[n]$ <br> $\tilde{x}_1[n],\ \tilde{x}_2[n]$ | $\tilde{X}[k]$ <br> $\tilde{X}_1[k],\ \tilde{X}_2[k]$ | | $x[n]$ <br> $x_1[n],\ x_2[n]$ | $X[k]$ <br> $X_1[k],\ X_2[k]$ |
| Linearity | $a\tilde{x}_1[n]+b\tilde{x}_2[n]$ | $a\tilde{X}_1[k]+b\tilde{X}_2[k]$ | Linearity | $ax_1[n]+bx_2[n]$ | $aX_1[k]+bX_2[k]$ |
| Duality | $\tilde{X}[n]$ | $N\,\tilde{x}[-k]$ | Duality | $X[n]$ | $N\,x[((-k))_N]$ |
| Time Shift | $\tilde{x}[n-m]$ | $W_N^{km}\tilde{X}[k]$ | Circular Time Shift | $x[((n-m))_N]$ | $W_N^{km}X[k]$ |
| Frequency Shift | $W_N^{-ln}\tilde{x}[n]$ | $\tilde{X}[k-l]$ | Circular Frequency Shift | $W_N^{-ln}x[n]$ | $X[((k-l))_N]$ |
| Periodic Convolution | $\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ | $\tilde{X}_1[k]\tilde{X}_2[k]$ | Circular Convolution | $\sum_{m=0}^{N-1}x_1[m]x_2[((n-m))_N]$ | $X_1[k]X_2[k]$ |
| Multiplication | $\tilde{x}_1[n]\tilde{x}_2[n]$ | $\frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}_1[l]\tilde{X}_2[k-l]$ | Multiplication | $x_1[n]x_2[n]$ | $\frac{1}{N}\sum_{l=0}^{N-1}X_1[l]X_2[((k-l))_N]$ |
| Complex Conjugation | $\tilde{x}^*[n]$ | $\tilde{X}^*[-k]$ | Complex Conjugation | $x^*[n]$ | $X^*[((-k))_N]$ |

# Properties (Continued)

| | | | | | |
|---|---|---|---|---|---|
| Time-Reversal and Complex Conjugation | $\tilde{x}^*[-n]$ | $\tilde{X}^*[k]$ | Time-Reversal and Complex Conjugation | $x^*[((-n))_N]$ | $X^*[k]$ |
| Real Part | $\mathrm{Re}\{\tilde{x}[n]\}$ | $\tilde{X}_{ep}[k]=\frac{1}{2}\left(\tilde{X}[k]+\tilde{X}^*[-k]\right)$ | Real Part | $\mathrm{Re}\{x[n]\}$ | $X_{ep}[k]=\frac{1}{2}\left(X[k]+X^*[((-k))_N]\right)$ |
| Imaginary Part | $j\,\mathrm{Im}\{\tilde{x}[n]\}$ | $\tilde{X}_{op}[k]=\frac{1}{2}\left(\tilde{X}[k]-\tilde{X}^*[-k]\right)$ | Imaginary Part | $j\,\mathrm{Im}\{x[n]\}$ | $X_{op}[k]=\frac{1}{2}\left(X[k]-X^*[((-k))_N]\right)$ |
| Even Part | $\tilde{x}_{ep}[n]=\frac{1}{2}\left(\tilde{x}[n]+\tilde{x}^*[-n]\right)$ | $\mathrm{Re}\{\tilde{X}[k]\}$ | Even Part | $x_{ep}[n]=\frac{1}{2}\left(x[n]+x^*[((-n))_N]\right)$ | $\mathrm{Re}\{X[k]\}$ |
| Odd Part | $\tilde{x}_{op}[n]=\frac{1}{2}\left(\tilde{x}[n]-\tilde{x}^*[-n]\right)$ | $j\,\mathrm{Im}\{\tilde{X}[k]\}$ | Odd Part | $x_{op}[n]=\frac{1}{2}\left(x[n]-x^*[((-n))_N]\right)$ | $j\,\mathrm{Im}\{X[k]\}$ |
| Symmetry for Real Sequence | $\tilde{x}[n]=\tilde{x}^*[n]$ | $\tilde{X}[k]=\tilde{X}^*[-k]$ <br> $\begin{cases}\mathrm{Re}\{\tilde{X}[k]\}=\mathrm{Re}\{\tilde{X}[-k]\}\\ \mathrm{Im}\{\tilde{X}[k]\}=-\mathrm{Im}\{\tilde{X}[-k]\}\end{cases}$ <br> $\begin{cases}\lvert\tilde{X}[k]\rvert=\lvert\tilde{X}[-k]\rvert\\ \angle\tilde{X}[k]=-\angle\tilde{X}[-k]\end{cases}$ | Symmetry for Real Sequence | $x[n]=x^*[n]$ | $X[k]=X^*[((-k))_N]$ <br> $\begin{cases}\mathrm{Re}\{X[k]\}=\mathrm{Re}\{X[((-k))_N]\}\\ \mathrm{Im}\{X[k]\}=-\mathrm{Im}\{X[((-k))_N]\}\end{cases}$ <br> $\begin{cases}\lvert X[k]\rvert=\lvert X[((-k))_N]\rvert\\ \angle X[k]=-\angle X[((-k))_N]\end{cases}$ |
| Parseval's Identity | $\sum_{n=0}^{N-1}\tilde{x}_1[n]\tilde{x}_2^*[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}_1[k]\tilde{X}_2^*[k]$ <br> $\sum_{n=0}^{N-1}\lvert\tilde{x}[n]\rvert^2=\frac{1}{N}\sum_{k=0}^{N-1}\lvert\tilde{X}[k]\rvert^2$ | | Parseval's Identity | $\sum_{n=0}^{N-1}x_1[n]x_2^*[n]=\frac{1}{N}\sum_{k=0}^{N-1}X_1[k]X_2^*[k]$ <br> $\sum_{n=0}^{N-1}\lvert x[n]\rvert^2=\frac{1}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2$ | |

# Duality

$$\text{If } x \xrightarrow{DFT} X, \text{ then } \{X[n]\}_{n=0}^{N-1} \xrightarrow{DFT} N\,\{x[((-k))_N]\}_{k=0}^{N-1}$$

# Duality

$$\text{If } x \xrightarrow{DFT} X, \text{ then } \{X[n]\}_{n=0}^{N-1} \xrightarrow{DFT} N\{x[((-k))_N]\}_{k=0}^{N-1}$$

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k],$$

$$\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k].$$

# Proof of Duality

$$\text{DFT of } \{x[n]\}_{n=0}^{N-1} \text{ is } \quad X[k] = \sum_{p=0}^{N-1} x[p] e^{-j\frac{2\pi}{N}kp}; \qquad k \le 0 \le N-1$$

$$\text{DFT of } \{X[n]\}_{n=0}^{N-1} \text{ is } \quad \sum_{n=0}^{N-1} \underbrace{\sum_{p=0}^{N-1} x[p] e^{-j\frac{2\pi}{N}pn}}_{X[n]} e^{-j\frac{2\pi}{N}kn}, \qquad k \le 0 \le N-1$$

$$= \sum_{p=0}^{N-1} x[p] \underbrace{\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(p+k)n}}_{\substack{N \text{ for } ((p+k))_N = 0, \\ 0 \text{ otherwise}}}$$

$$((p+k))_N = 0 \text{ for } 0 \le p \ \& \ k \le N-1 \Rightarrow p = ((-k))_N$$

$$p = -k + mN = ((-k))_N + rN + mN = ((-k))_N \quad \textit{because } 0 \le p \le N-1$$

$$\therefore \text{ DFT of } \{X[n]\}_{n=0}^{N-1} \text{ is } N\{x[((-k))_N]\}_{k=0}^{N-1}$$

# Circular Convolution

- Circular Convolution:

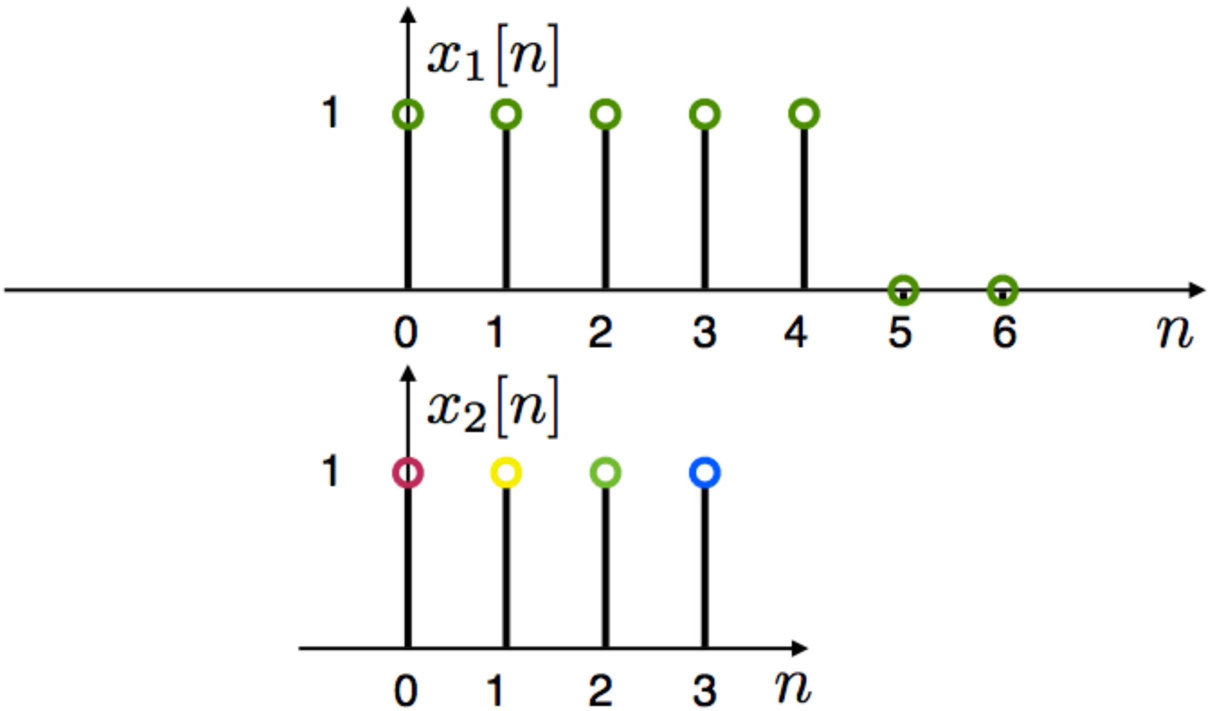
$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

For two signals of length N

Note: Circular convolution is commutative

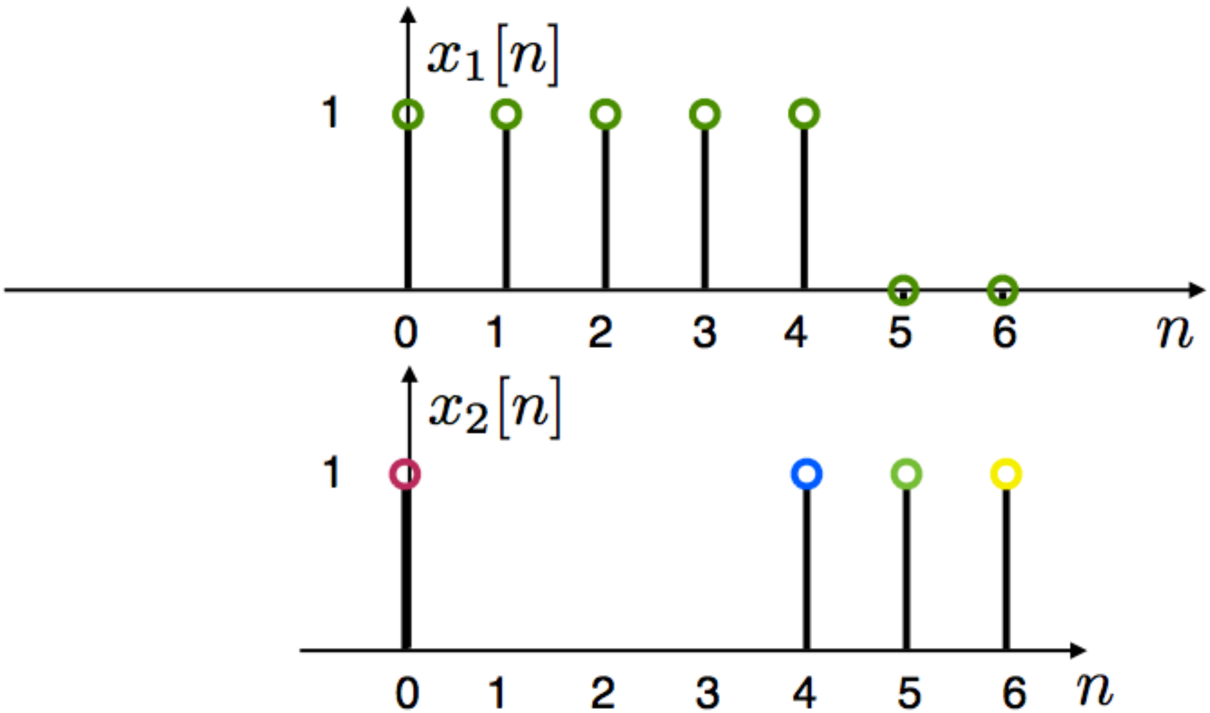
$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$

# Compute Circular Convolution Sum




$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

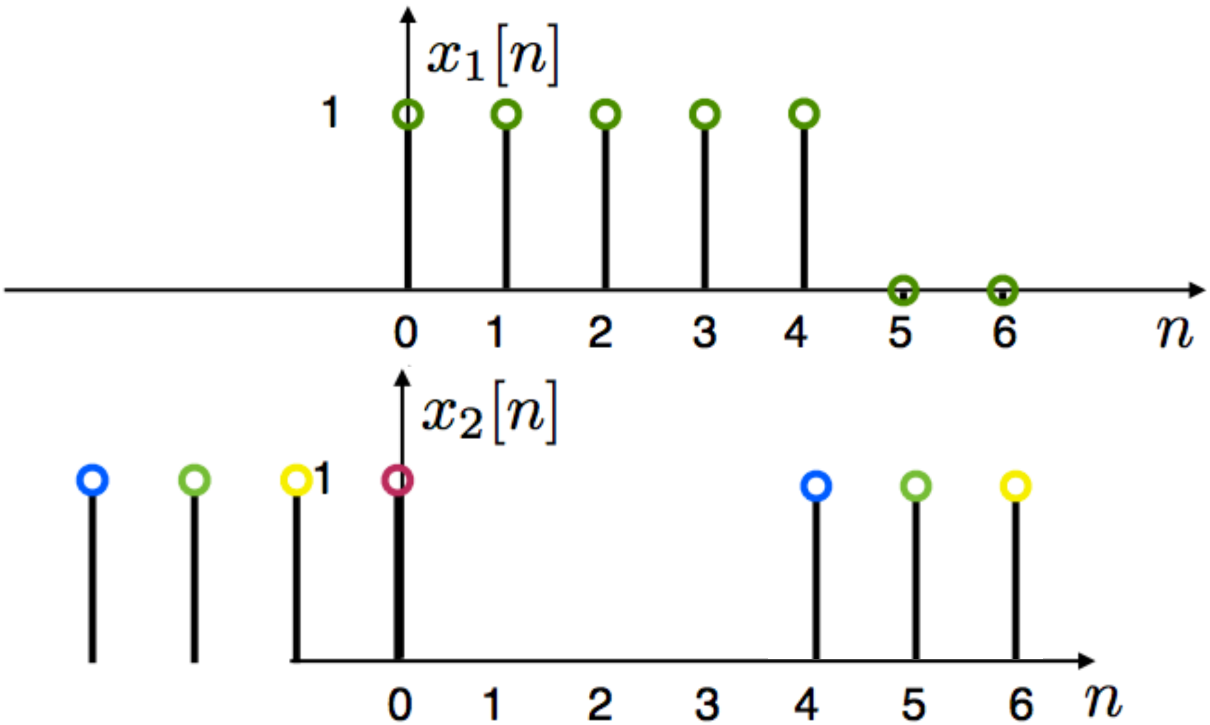
# Compute Circular Convolution Sum




$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

# Compute Circular Convolution Sum

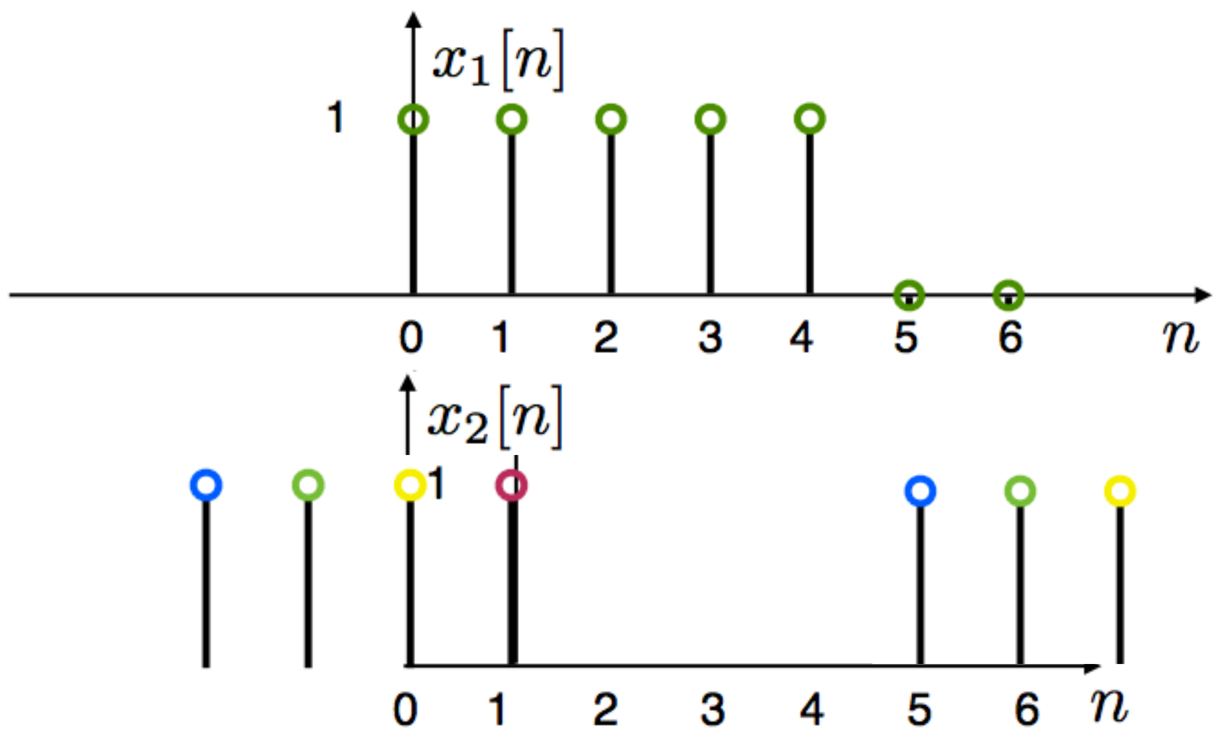
y[0]=2




$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

# Compute Circular Convolution Sum

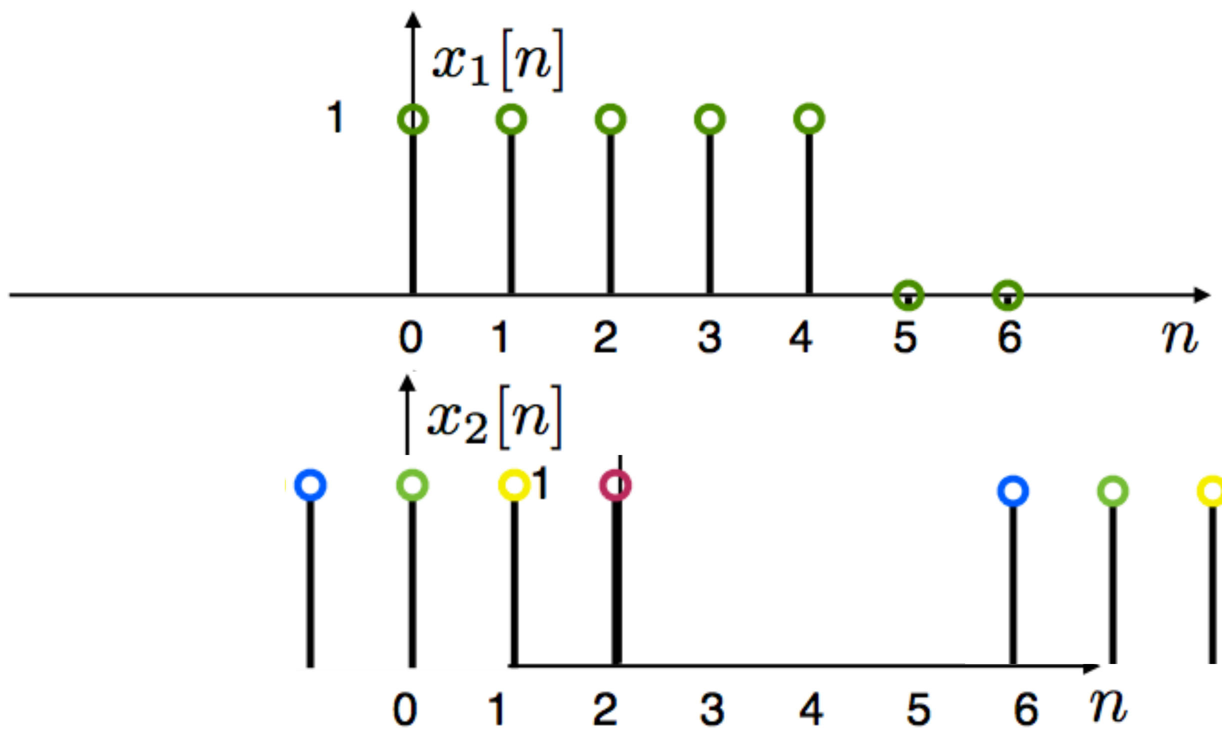
y[0]=2
y[1]=2




$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

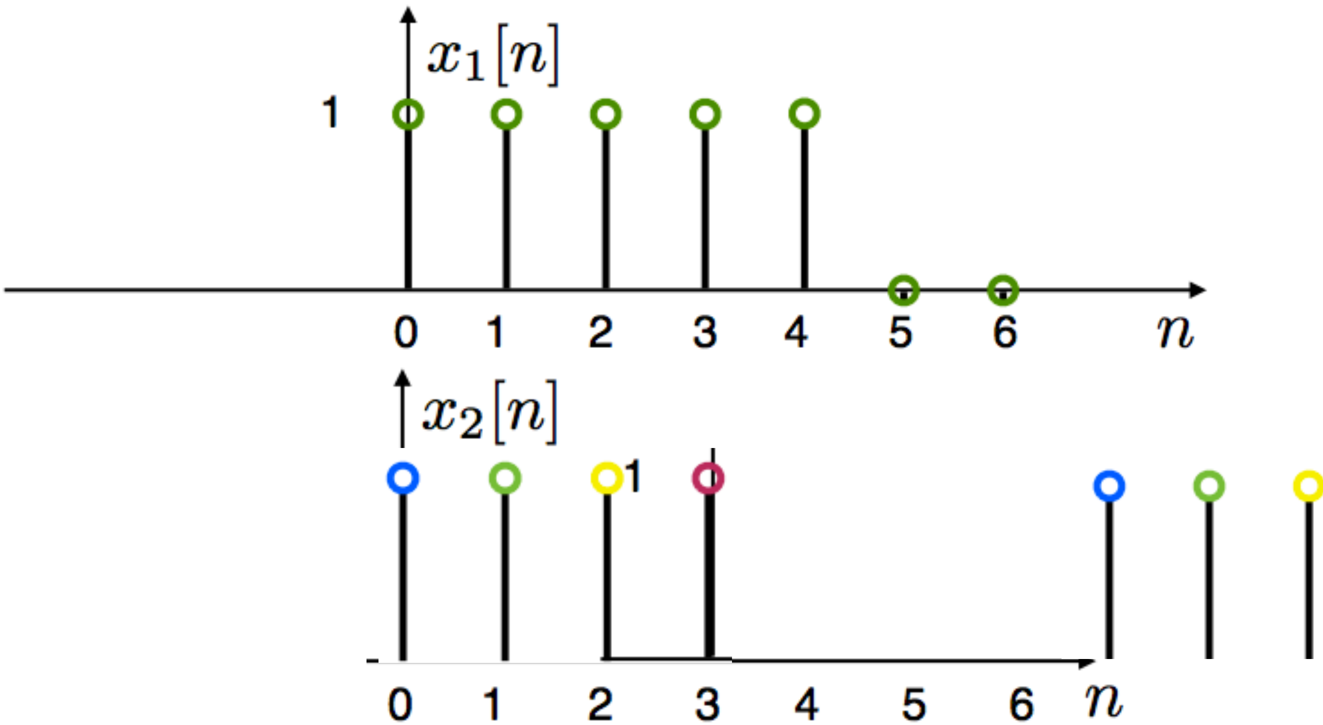
# Compute Circular Convolution Sum

y[0]=2
y[1]=2
y[2]=3




$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

# Compute Circular Convolution Sum

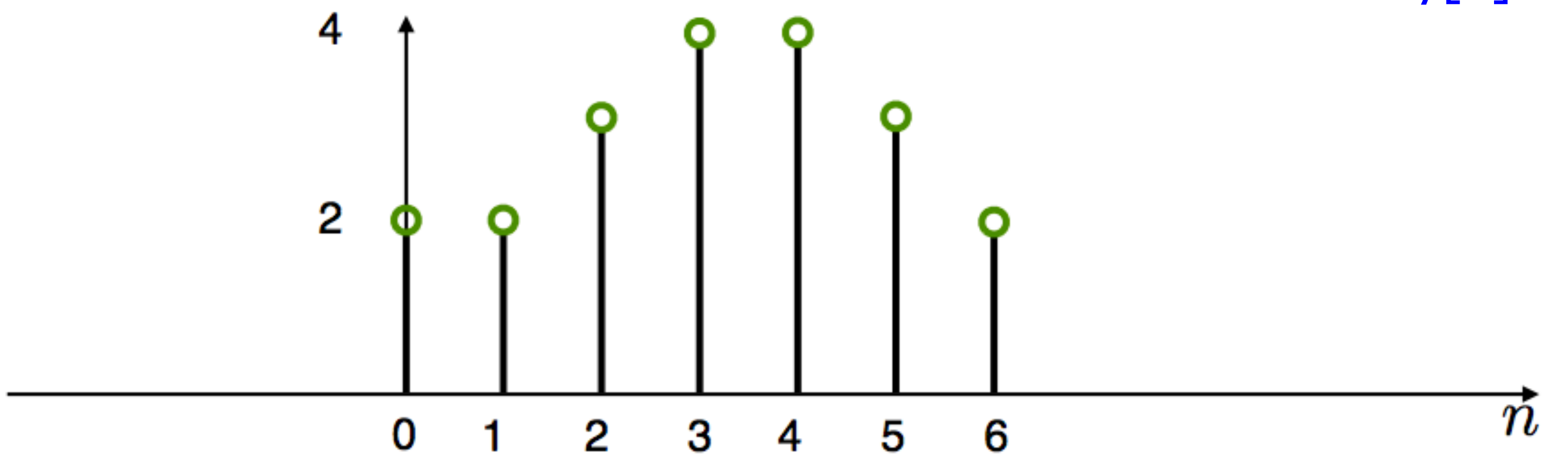



y[0]=2
y[1]=2
y[2]=3
y[3]=4

$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

# Result

y[0]=2
y[1]=2
y[2]=3
y[3]=4




$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

# Circular Convolution

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

  - Very useful!! (for linear convolutions with DFT)

# Multiplication

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledN X_2[k]$$

# Linear Convolution

- Next....
  - Using DFT, circular convolution is easy
  - But, linear convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Use DFT to do linear convolution

# Big Ideas

- Adaptive filtering
  - Use LMS algorithm to update filter coefficients for applications like system ID, channel equalization, and signal prediction
- Discrete Fourier Transform (DFT)
  - For finite signals assumed to be zero outside of defined length
  - N-point DFT is sampled DTFT at N points
  - Useful properties allow easier linear convolution
- DFT Properties
  - Inherited from DFS, but circular operations!

# Admin

- HW 7 out now
  - Due Friday

- Project posted Friday
  - Work in groups of up to 2
    - Can work alone if you want
    - Use Piazza to find partners