

ESE 531: Digital Signal Processing

Lec 2: January 16, 2018

Discrete Time Signals and Systems



Lecture Outline

- ❑ Discrete Time Signals
- ❑ Signal Properties
- ❑ Discrete Time Systems

Discrete Time Signals



Signals

DEFINITION

Signal (n): A detectable physical quantity ... by which messages or information can be transmitted (Merriam-Webster)

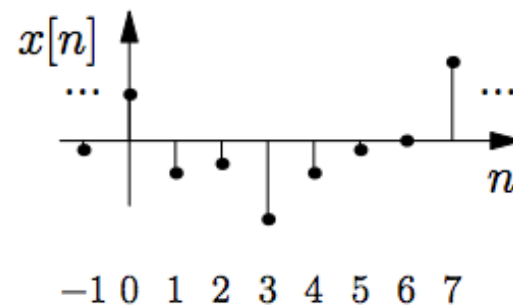
- ❑ Signals carry information
- ❑ Examples:
 - Speech signals transmit language via acoustic waves
 - Radar signals transmit the position and velocity of targets via electromagnetic waves
 - Electrophysiology signals transmit information about processes inside the body
 - Financial signals transmit information about events in the economy
- ❑ Signal processing systems manipulate the information carried by signals

Signals are Functions

DEFINITION

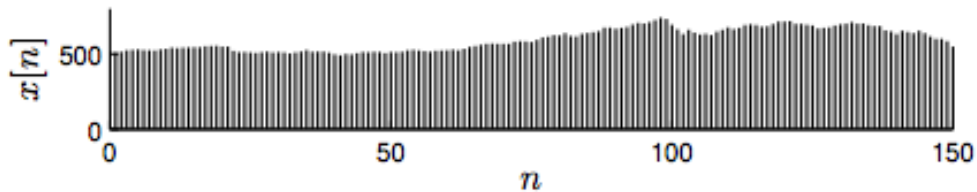
A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
 - Independent variable is an **integer**: $n \in \mathbb{Z}$ (will refer to as time)
 - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$

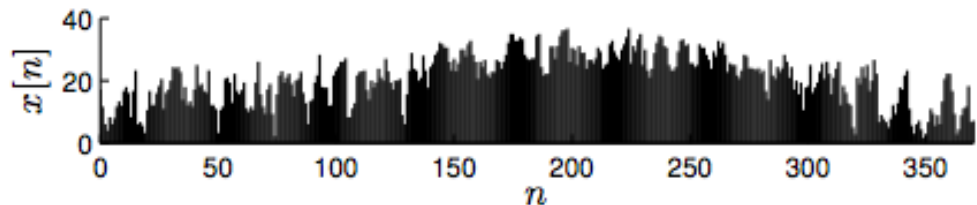


A Menagerie of Signals

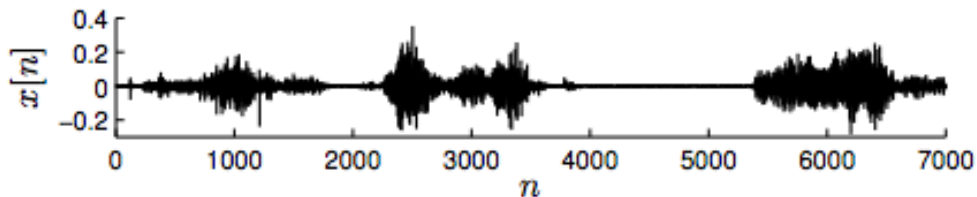
- Google Share daily share price for 5 months



- Temperature at Houston Intercontinental Airport in 2013 (Celcius)

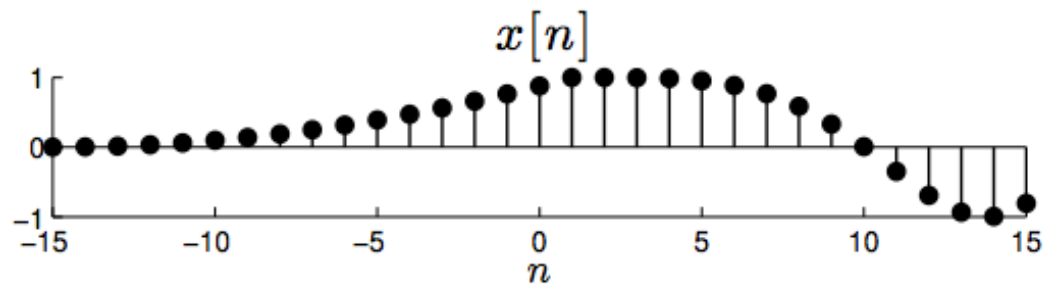


- Excerpt from Shakespeare's *Hamlet*

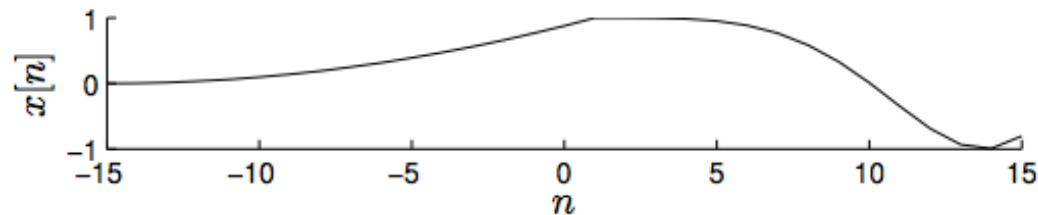


Plotting Signals Correctly

- In a discrete-time signal $x[n]$, the independent variable n is discrete (integer)
- To plot a discrete-time signal in a program like Matlab, you should use the stem or similar command and not the plot command
- Correct:



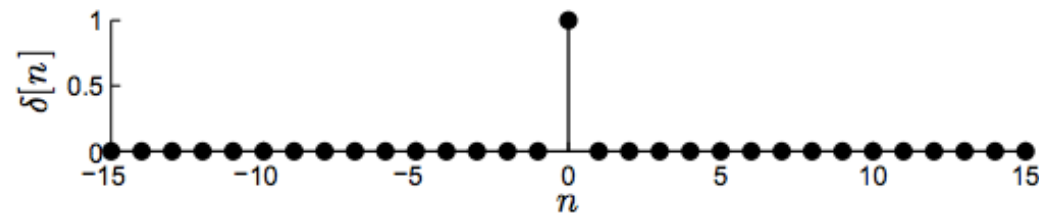
- Incorrect:



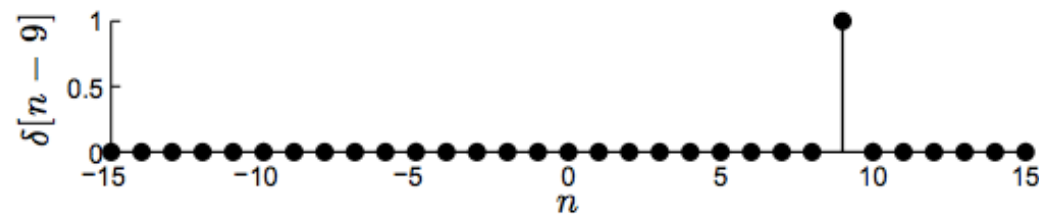
Unit Sample

DEFINITION

The **delta function** (aka unit impulse) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



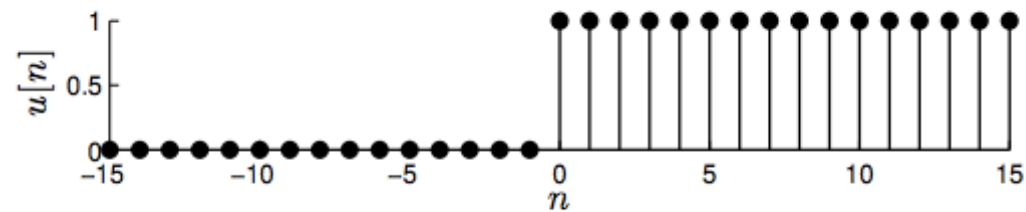
- The shifted delta function $\delta[n - m]$ peaks up at $n = m$; here $m = 9$



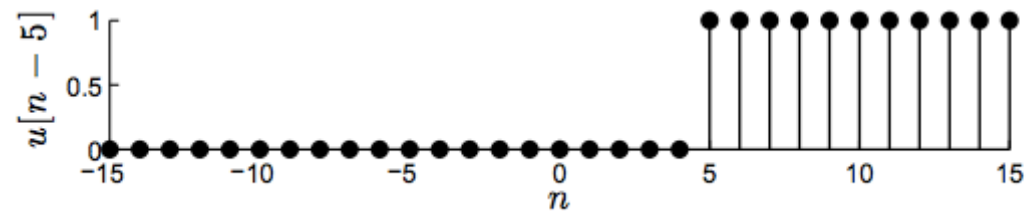
Unit Step

DEFINITION

The **unit step** $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



- The shifted unit step $u[n - m]$ jumps from 0 to 1 at $n = m$; here $m = 5$

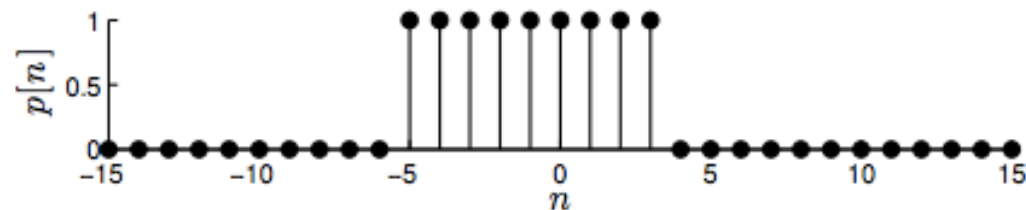


Unit Pulse

DEFINITION

The **unit pulse** (aka boxcar) $p[n] = \begin{cases} 0 & n < N_1 \\ 1 & N_1 \leq n \leq N_2 \\ 0 & n > N_2 \end{cases}$

- Ex: $p[n]$ for $N_1 = -5$ and $N_2 = 3$



- One of many different formulas for the unit pulse

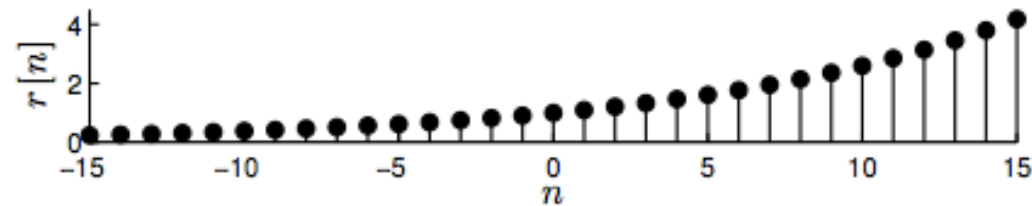
$$p[n] = u[n - N_1] - u[n - (N_2 + 1)]$$

Real Exponential

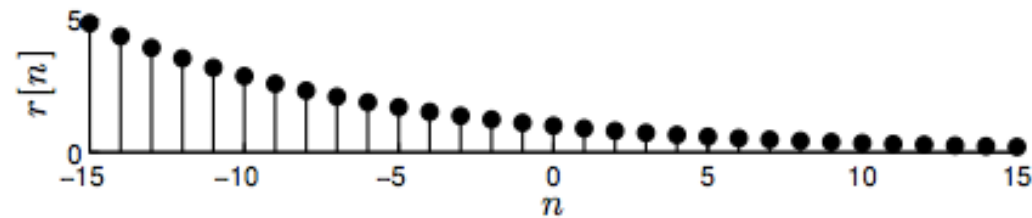
DEFINITION

The **real exponential** $r[n] = a^n$, $a \in \mathbb{R}$, $a \geq 0$

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$

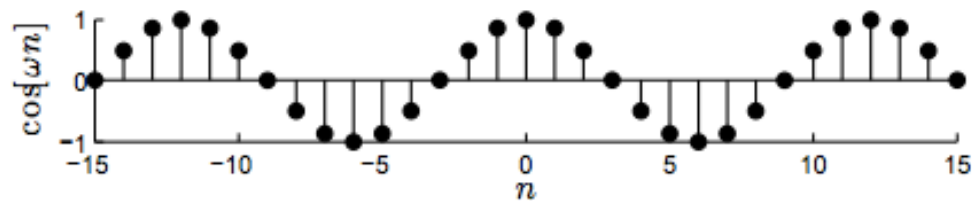


- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$

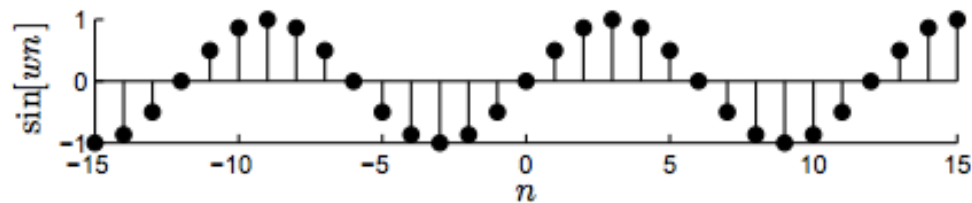


Sinusoids

- There are two natural real-valued sinusoids: $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$
- **Frequency:** ω (units: radians/sample)
- **Phase:** ϕ (units: radians)
- $\cos(\omega n)$

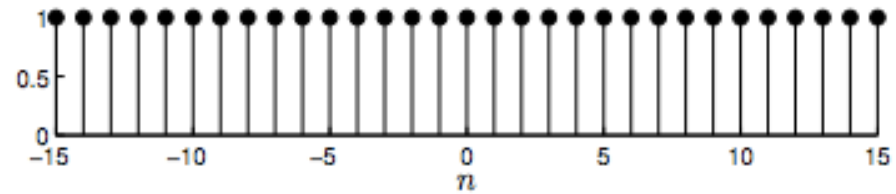


- $\sin(\omega n)$

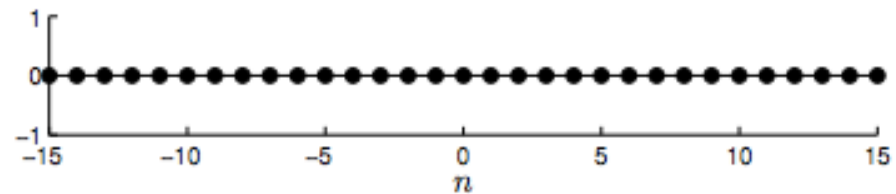


Sinusoid Examples

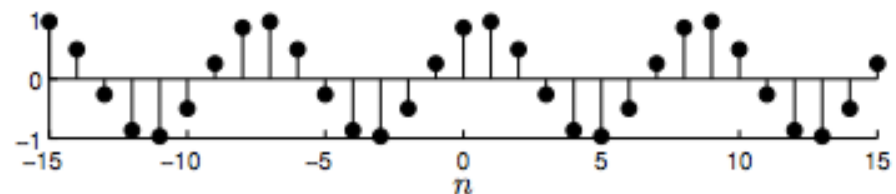
■ $\cos(0n)$



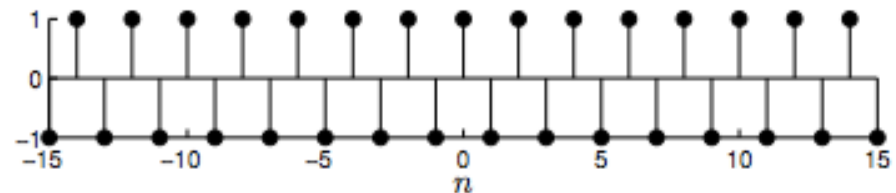
■ $\sin(0n)$



■ $\sin(\frac{\pi}{4}n + \frac{2\pi}{6})$



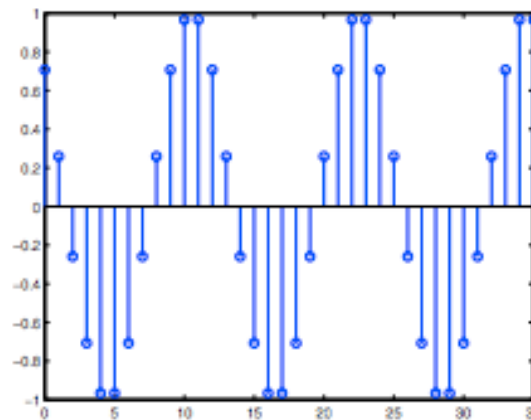
■ $\cos(\pi n)$



Sinusoid in Matlab

- It's easy to play around in Matlab to get comfortable with the properties of sinusoids

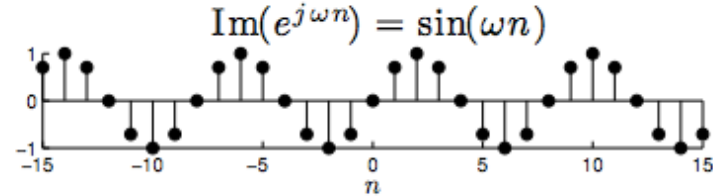
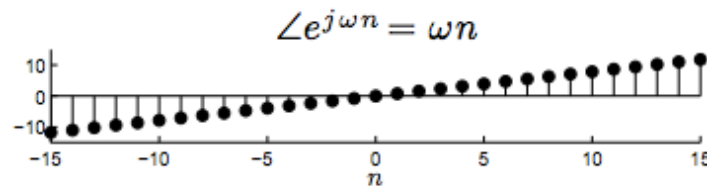
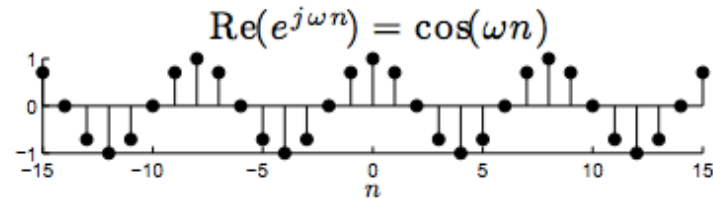
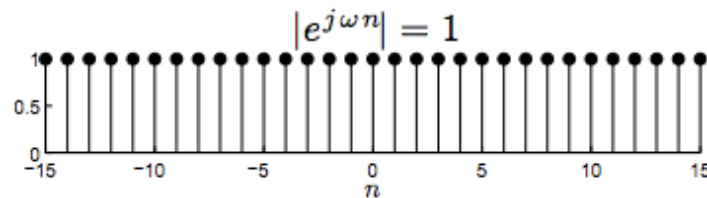
```
N=36;  
n=0:N-1;  
omega=pi/6;  
phi=pi/4;  
x=cos(omega*n+phi);  
stem(n,x)
```



Complex Sinusoid

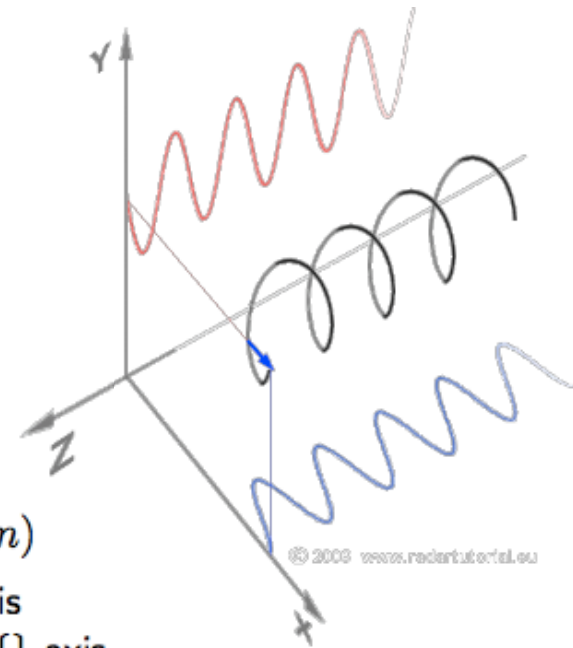
- The complex-valued sinusoid combines both the cos and sin terms (via Euler's identity)

$$e^{j(\omega n + \phi)} = \cos(\omega n + \phi) + j \sin(\omega n + \phi)$$



Complex Sinusoid as Helix

$$e^{j(\omega n + \phi)} = \cos(\omega n + \phi) + j \sin(\omega n + \phi)$$



- A complex sinusoid is a **helix** in 3D space $(\text{Re}\{\}, \text{Im}\{\}, n)$
 - **Real part** (cos term) is the projection onto the $\text{Re}\{\}$ axis
 - **Imaginary part** (sin term) is the projection onto the $\text{Im}\{\}$ axis
- Frequency ω determines rotation speed and direction of helix
 - $\omega > 0 \Rightarrow$ anticlockwise rotation
 - $\omega < 0 \Rightarrow$ clockwise rotation

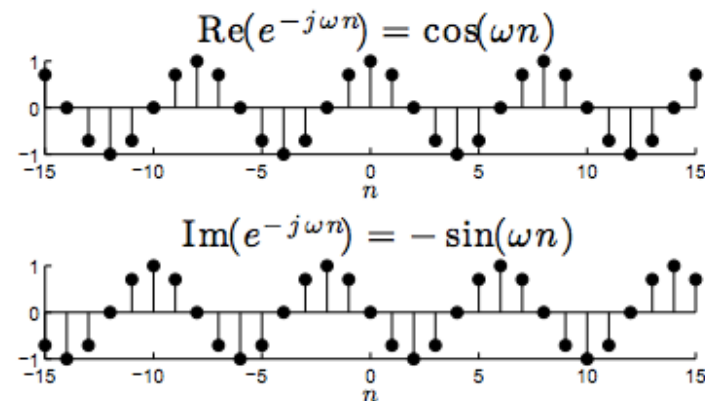
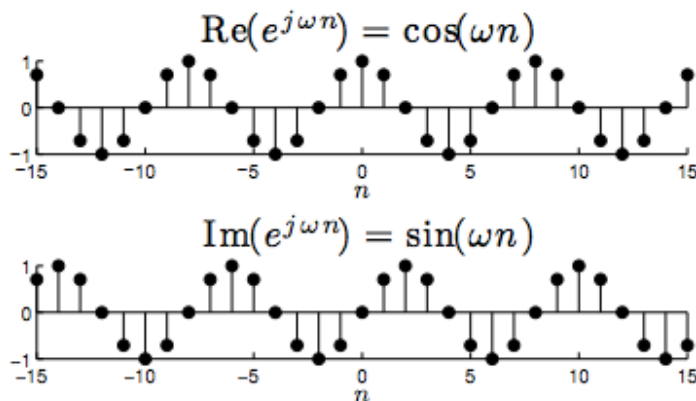
Animation: https://upload.wikimedia.org/wikipedia/commons/4/41/Rising_circular.gif

Negative Frequency

- Negative frequency is nothing to be afraid of! Consider a sinusoid with a negative frequency $-\omega$

$$e^{j(-\omega)n} = e^{-j\omega n} = \cos(-\omega n) + j \sin(-\omega n) = \cos(\omega n) - j \sin(\omega n)$$

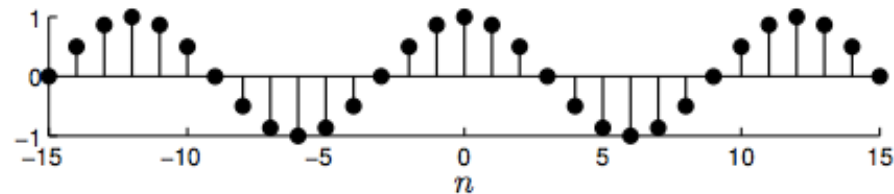
- Also note: $e^{j(-\omega)n} = e^{-j\omega n} = (e^{j\omega n})^*$
- Bottom line: negating the frequency is equivalent to complex conjugating a complex sinusoid, which flips the sign of the imaginary, sin term



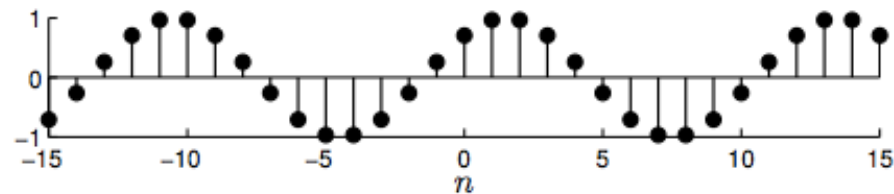
Phase of a Sinusoid

- ϕ is a (frequency independent) shift that is referenced to one period of oscillation

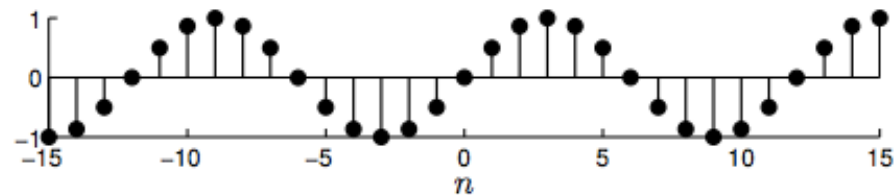
- $\cos\left(\frac{\pi}{6}n - 0\right)$



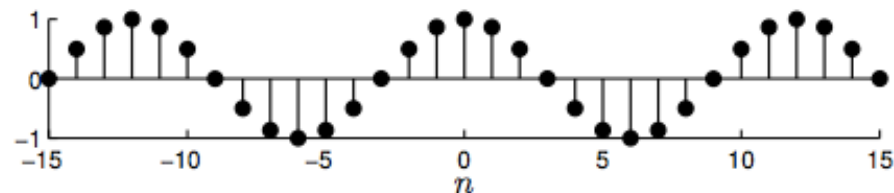
- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{4}\right)$



- $\cos\left(\frac{\pi}{6}n - \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{6}n\right)$



- $\cos\left(\frac{\pi}{6}n - 2\pi\right) = \cos\left(\frac{\pi}{6}n\right)$





Complex Exponentials

- Complex sinusoid $e^{j(\omega n + \phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Generalize to $e^{\text{General Complex Numbers}}$
- Consider the general complex number $z = |z| e^{j\omega}$, $z \in \mathbb{C}$
 - $|z|$ = magnitude of z
 - $\omega = \angle(z)$, phase angle of z
 - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**



Complex Exponentials

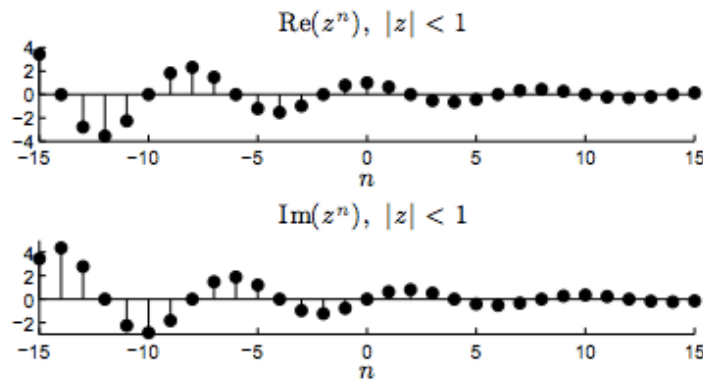
- Complex sinusoid $e^{j(\omega n + \phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Generalize to $e^{\text{General Complex Numbers}}$
- Consider the general complex number $z = |z| e^{j\omega}$, $z \in \mathbb{C}$
 - $|z|$ = magnitude of z
 - $\omega = \angle(z)$, phase angle of z
 - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**
- Now we have
$$z^n = (|z|e^{j\omega})^n = |z|^n (e^{j\omega})^n = |z|^n e^{j\omega n}$$
 - $|z|^n$ is a **real exponential** (a^n with $a = |z|$)
 - $e^{j\omega n}$ is a **complex sinusoid**

Complex Exponentials

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

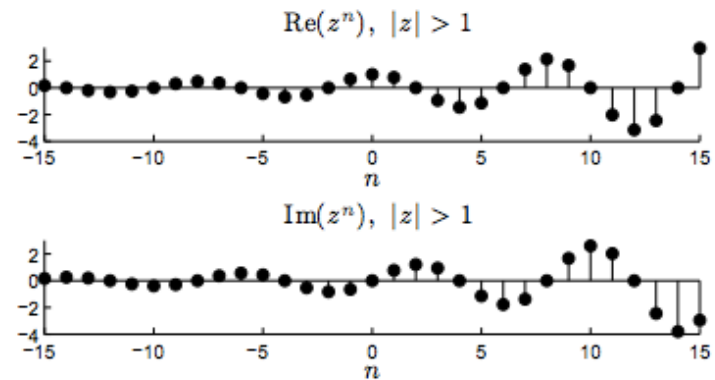
- $|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a **complex sinusoid**

$$|z| < 1$$



Bounded

$$|z| > 1$$

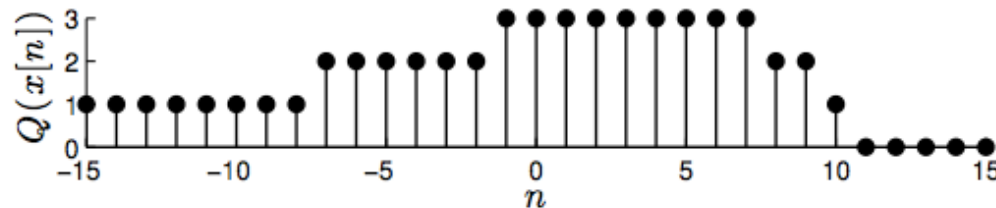


Unbounded

Digital Signals

■ Digital signals are a special sub-class of discrete-time signals

- Independent variable is still an integer: $n \in \mathbb{Z}$
- Dependent variable is from a **finite set of integers**: $x[n] \in \{0, 1, \dots, D - 1\}$
- Typically, choose $D = 2^q$ and represent each possible level of $x[n]$ as a digital code with q bits
- Ex: Digital signal with $q = 2$ bits $\Rightarrow D = 2^2 = 4$ levels

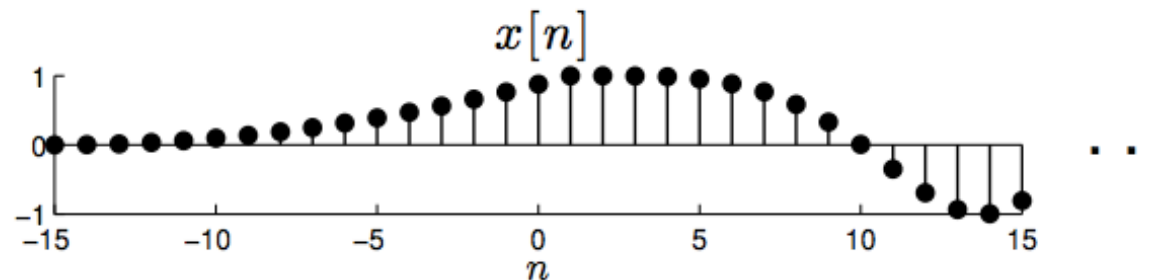


- Ex: Compact discs use $q = 16$ bits $\Rightarrow D = 2^{16} = 65536$ levels

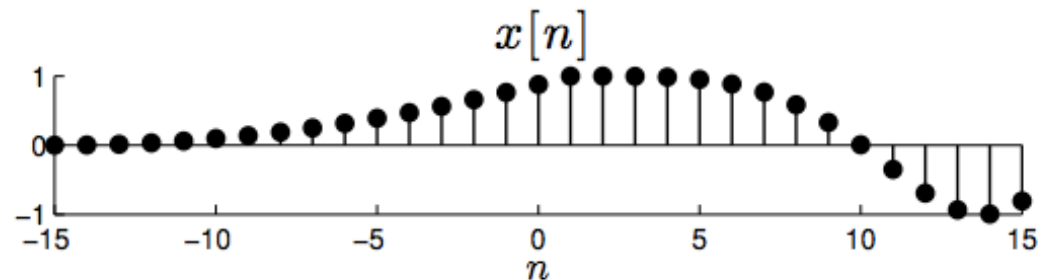
Signal Properties

Finite/Infinite Length Sequences

- An **infinite-length** discrete-time signal $x[n]$ is defined for all $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$



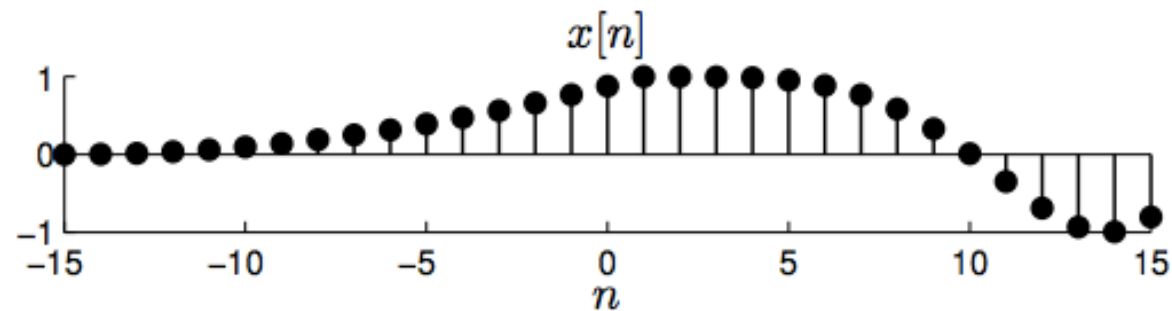
- A **finite-length** discrete-time signal $x[n]$ is defined only for a finite range of $N_1 \leq n \leq N_2$



- Important: a finite-length signal is undefined for $n < N_1$ and $n > N_2$

Windowing

- Converts a longer signal into a shorter one
$$y[n] = \begin{cases} x[n] & N_1 \leq n \leq N_2 \\ 0 & \text{otherwise} \end{cases}$$

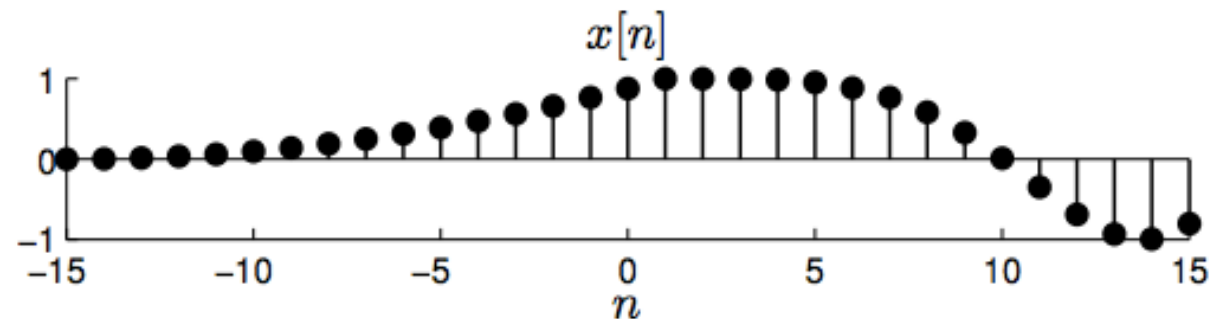


Zero Padding

- Converts a shorter signal into a longer one

- Say $x[n]$ is defined for $N_1 \leq n \leq N_2$

- Given $N_0 \leq N_1 \leq N_2 \leq N_3$
$$y[n] = \begin{cases} 0 & N_0 \leq n < N_1 \\ x[n] & N_1 \leq n \leq N_2 \\ 0 & N_2 < n \leq N_3 \end{cases}$$

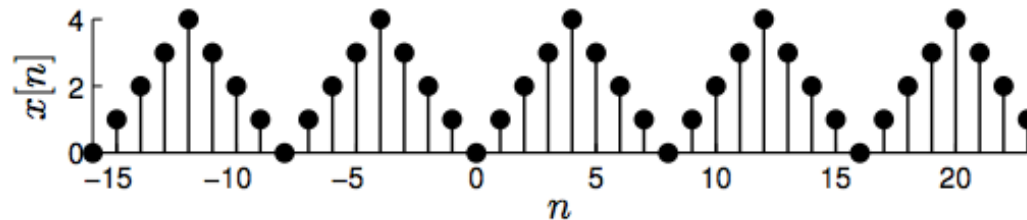


Periodic Signals

DEFINITION

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall m \in \mathbb{Z}$$



Notes:

- The period N must be an integer
- A periodic signal is infinite in length

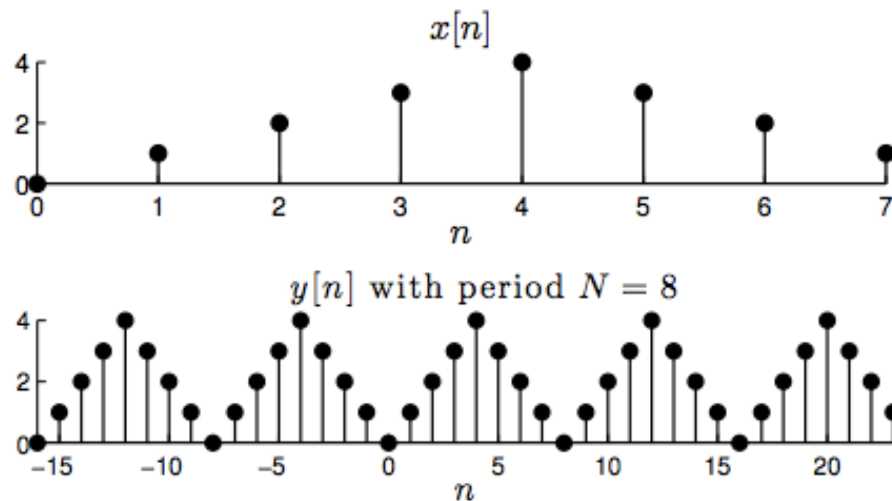
DEFINITION

A discrete-time signal is **aperiodic** if it is not periodic

Periodization

- Converts a finite-length signal into an infinite-length, periodic signal
- Given finite-length $x[n]$, replicate $x[n]$ periodically with period N

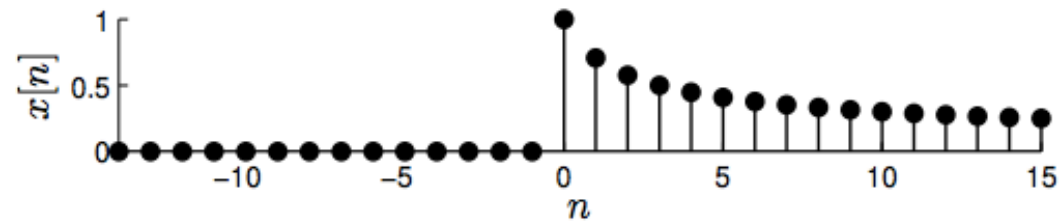
$$\begin{aligned}y[n] &= \sum_{m=-\infty}^{\infty} x[n - mN], \quad n \in \mathbb{Z} \\ &= \cdots + x[n + 2N] + x[n + N] + x[n] + x[n - N] + x[n - 2N] + \cdots\end{aligned}$$



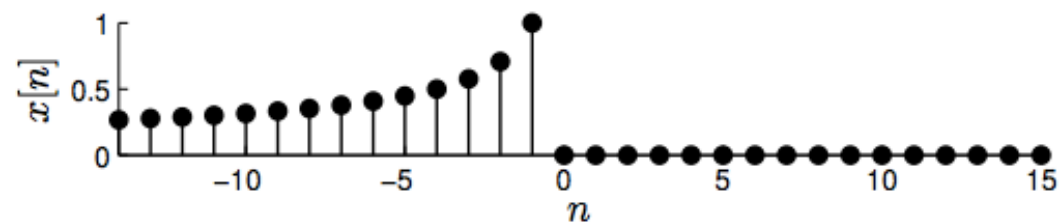
Causal Signals

DEFINITION

A signal $x[n]$ is **causal** if $x[n] = 0$ for all $n < 0$.



- A signal $x[n]$ is **anti-causal** if $x[n] = 0$ for all $n \geq 0$

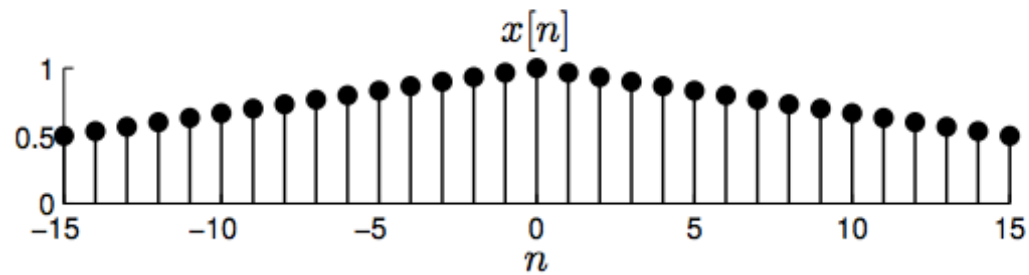


- A signal $x[n]$ is **acausal** if it is not causal

Even Signals

DEFINITION

A real signal $x[n]$ is **even** if $x[-n] = x[n]$

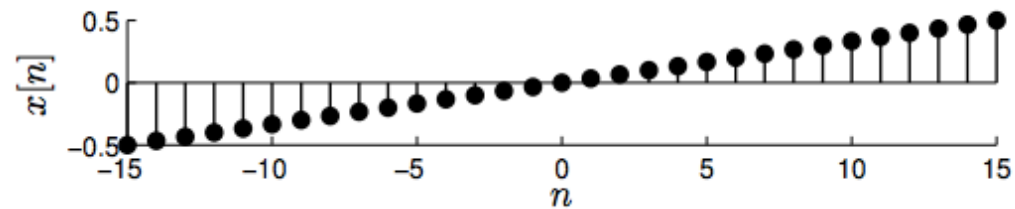


- Even signals are symmetrical around the point $n = 0$

Odd Signals

DEFINITION

A real signal $x[n]$ is **odd** if $x[-n] = -x[n]$



- Odd signals are anti-symmetrical around the point $n = 0$



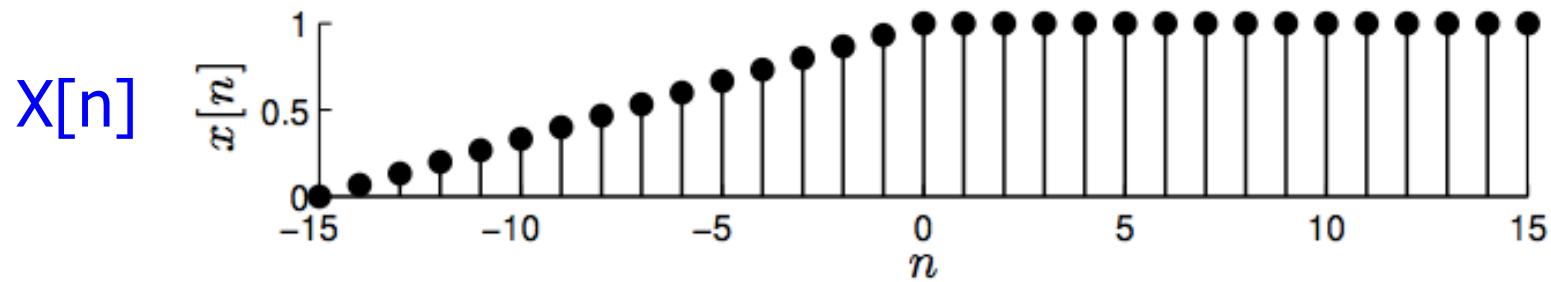
Signal Decomposition

- **Useful fact:** Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part
- Even part: $e[n] = \frac{1}{2} (x[n] + x[-n])$ (easy to verify that $e[n]$ is even)
- Odd part: $o[n] = \frac{1}{2} (x[n] - x[-n])$ (easy to verify that $o[n]$ is odd)
- **Decomposition** $x[n] = e[n] + o[n]$
- Verify the decomposition:

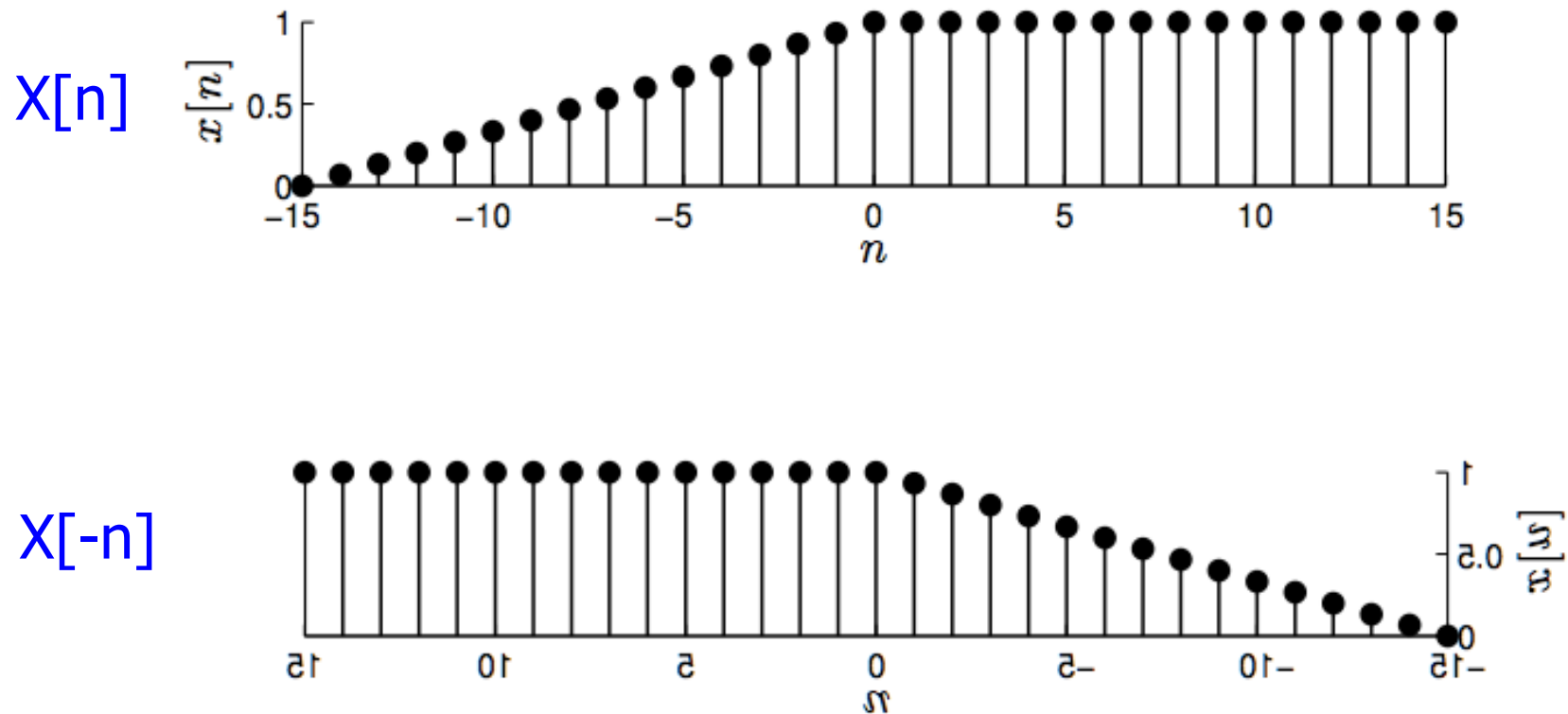
$$\begin{aligned} e[n] + o[n] &= \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n]) \\ &= \frac{1}{2}(x[n] + x[-n] + x[n] - x[-n]) \\ &= \frac{1}{2}(2x[n]) = x[n] \quad \checkmark \end{aligned}$$



Decomposition Example



Decomposition Example



Decomposition Example

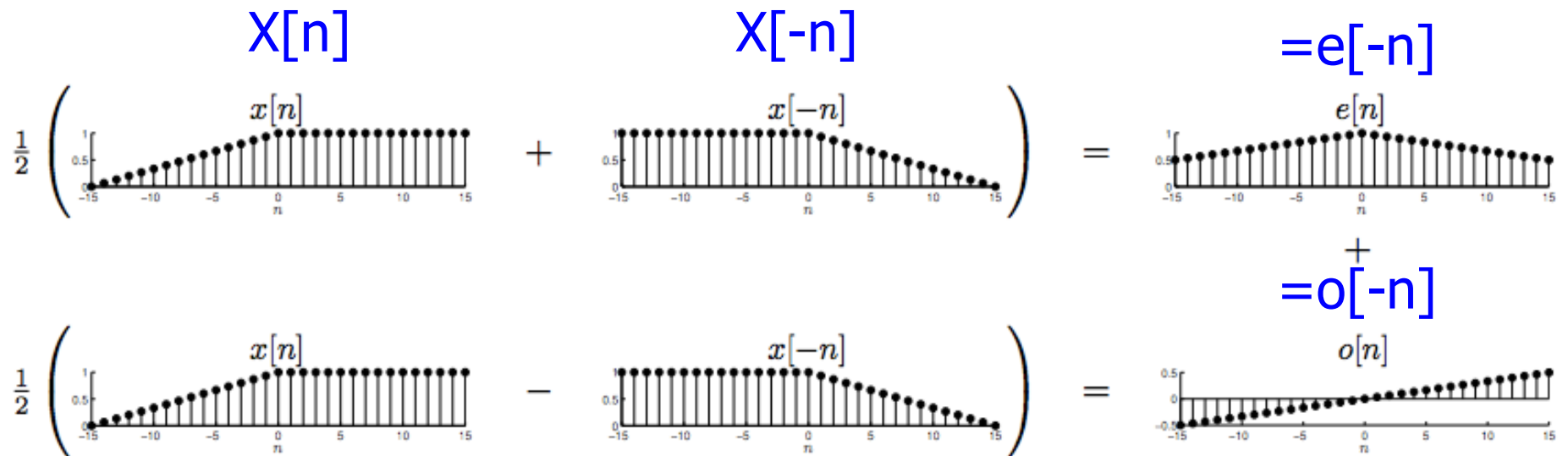
$X[n]$

$X[-n]$

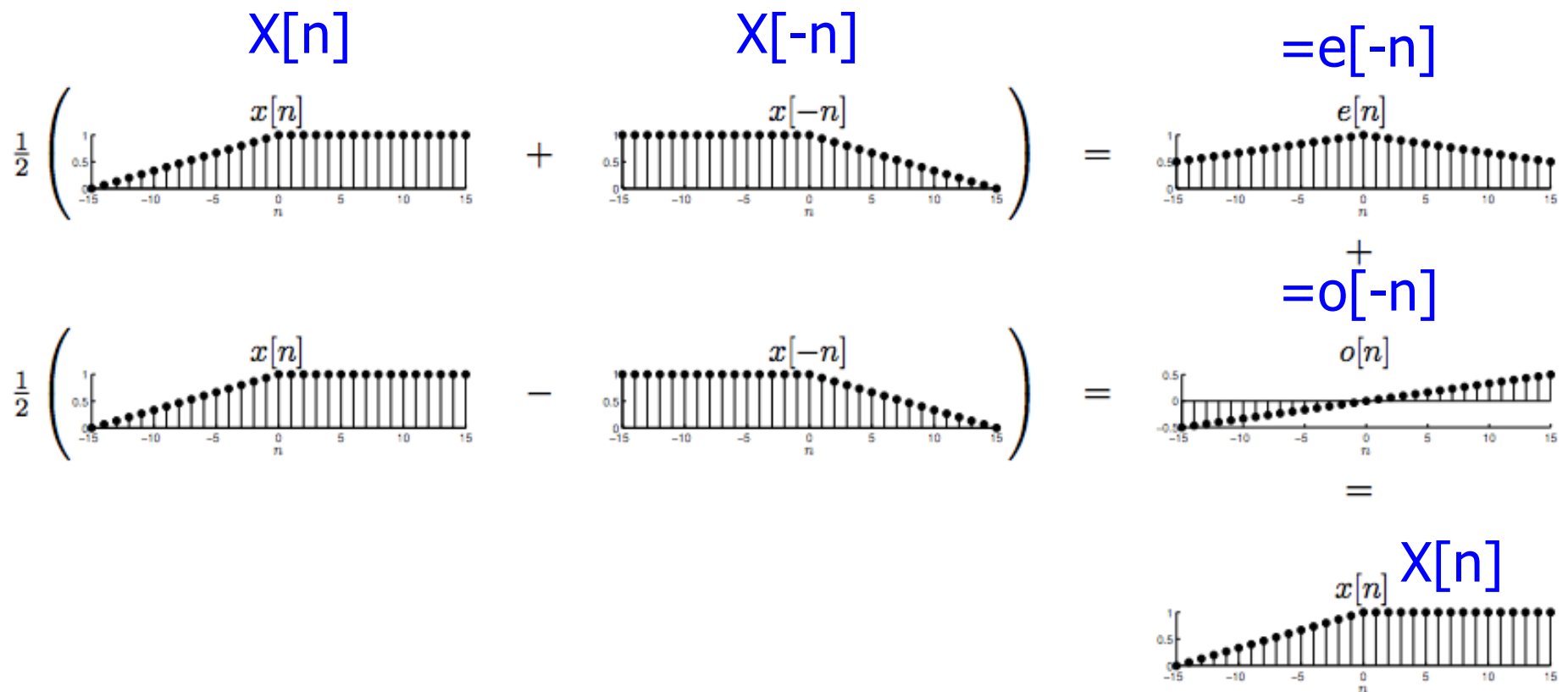
$$\frac{1}{2} \left(\begin{array}{c} x[n] \\ \text{Plot of } x[n] \text{ vs } n \end{array} + \begin{array}{c} x[-n] \\ \text{Plot of } x[-n] \text{ vs } n \end{array} \right) = e[-n]$$


$$\frac{1}{2} \left(\begin{array}{c} x[n] \\ \text{Plot of } x[n] \text{ vs } n \end{array} - \begin{array}{c} x[-n] \\ \text{Plot of } x[-n] \text{ vs } n \end{array} \right) = o[-n]$$


Decomposition Example



Decomposition Example





Discrete-Time Sinusoids

- Discrete-time sinusoids $e^{j(\omega n + \phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property #1: Aliasing
- Weird property #2: Aperiodicity



Property #1: Aliasing of Sinusoids

- Consider two sinusoids with two different frequencies

- $\omega \Rightarrow x_1[n] = e^{j(\omega n + \phi)}$
- $\omega + 2\pi \Rightarrow x_2[n] = e^{j((\omega + 2\pi)n + \phi)}$



Property #1: Aliasing of Sinusoids

- Consider two sinusoids with two different frequencies

- $\omega \Rightarrow x_1[n] = e^{j(\omega n + \phi)}$
- $\omega + 2\pi \Rightarrow x_2[n] = e^{j((\omega + 2\pi)n + \phi)}$

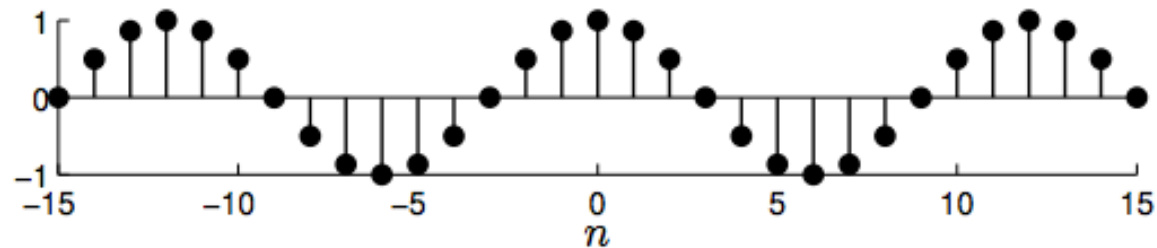
- But note that

$$x_2[n] = e^{j((\omega + 2\pi)n + \phi)} = e^{j(\omega n + \phi) + j2\pi n} = e^{j(\omega n + \phi)} e^{j2\pi n} = e^{j(\omega n + \phi)} = x_1[n]$$

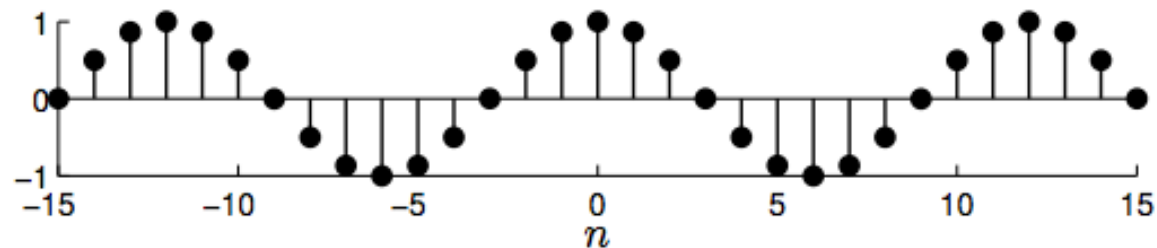
- The signals x_1 and x_2 have different frequencies but are **identical**!
- We say that x_1 and x_2 are aliases; this phenomenon is called **aliasing**
- Note: Any integer multiple of 2π will do; try with $x_3[n] = e^{j((\omega + 2\pi m)n + \phi)}$, $m \in \mathbb{Z}$

Aliasing Example

■ $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$

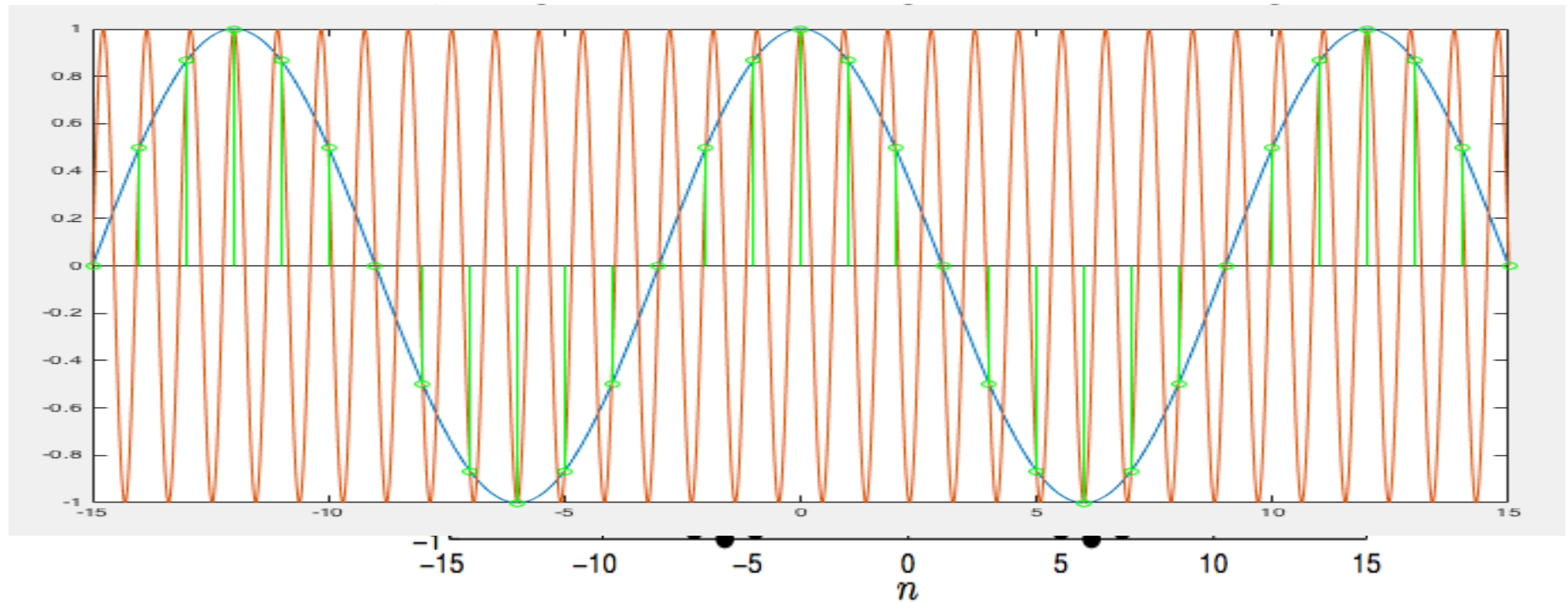


■ $x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$



Aliasing Example

■ $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$





Alias-Free Frequencies

- Since

$$x_3[n] = e^{j(\omega+2\pi m)n+\phi} = e^{j(\omega n+\phi)} = x_1[n] \quad \forall m \in \mathbb{Z}$$

the only frequencies that lead to unique (distinct) sinusoids lie in an interval of length 2π

- Two intervals are typically used in the signal processing literature (and in this course)
 - $0 \leq \omega < 2\pi$
 - $-\pi < \omega \leq \pi$



Which is higher in frequency?

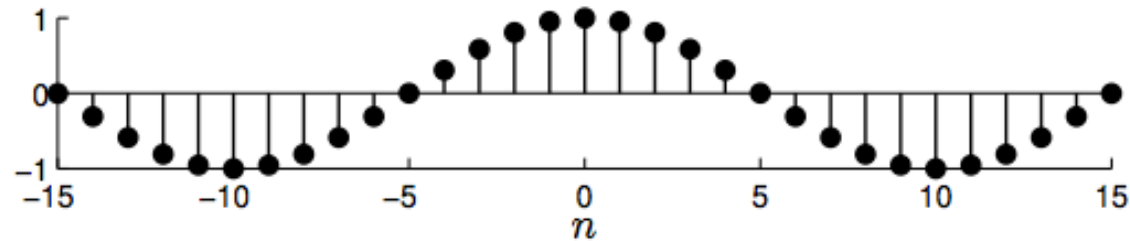
□ $\cos(\pi n)$ or $\cos(3\pi / 2n)$?

Low and High Frequencies

$$e^{j(\omega n + \phi)}$$

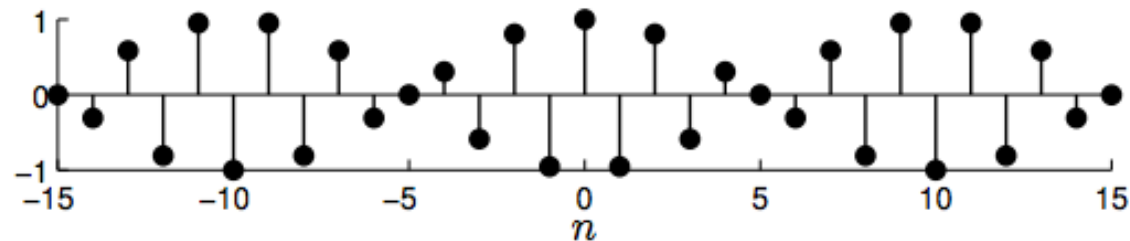
- **Low frequencies:** ω close to 0 or 2π rad

Ex: $\cos\left(\frac{\pi}{10}n\right)$

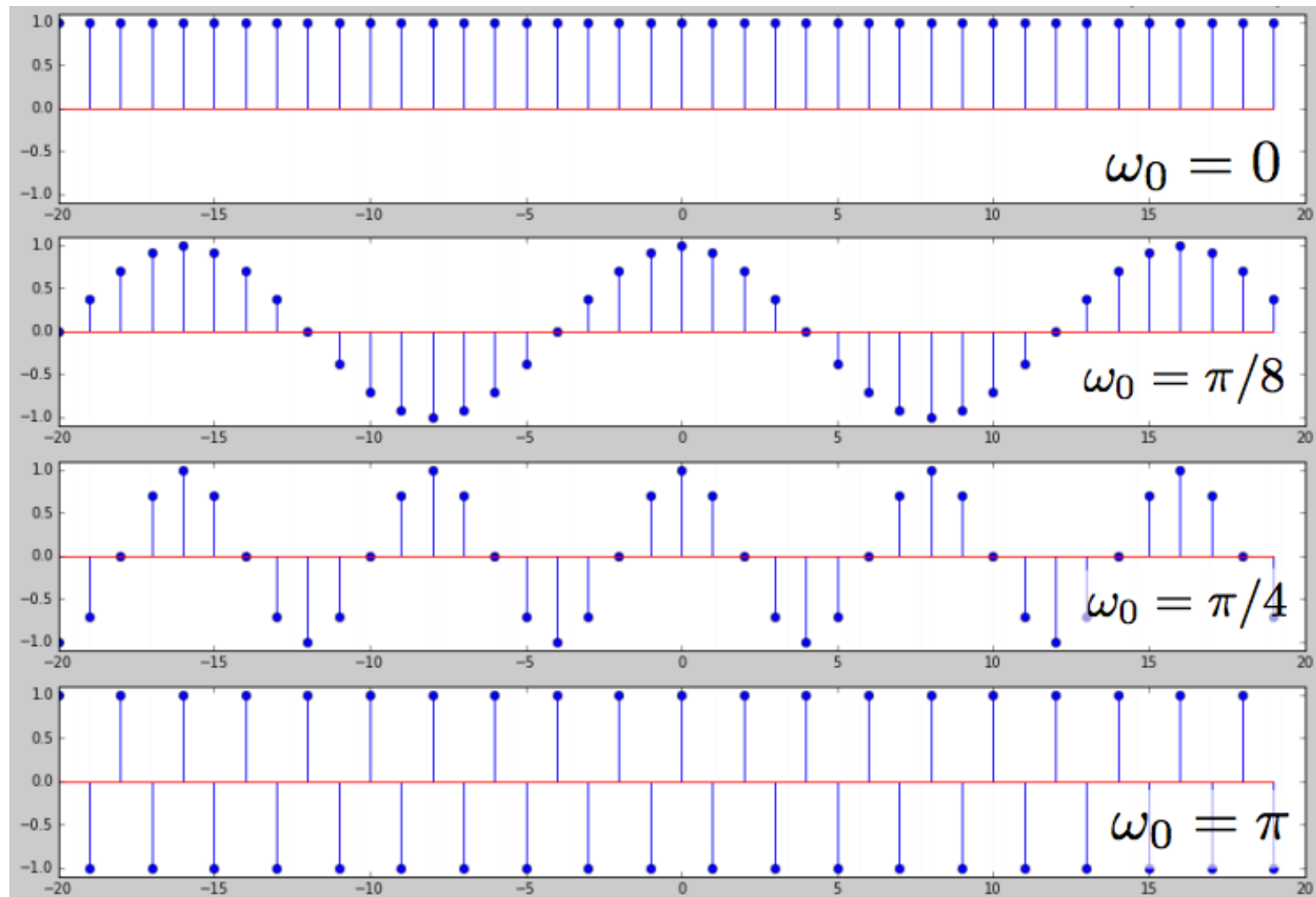


- **High frequencies:** ω close to π or $-\pi$ rad

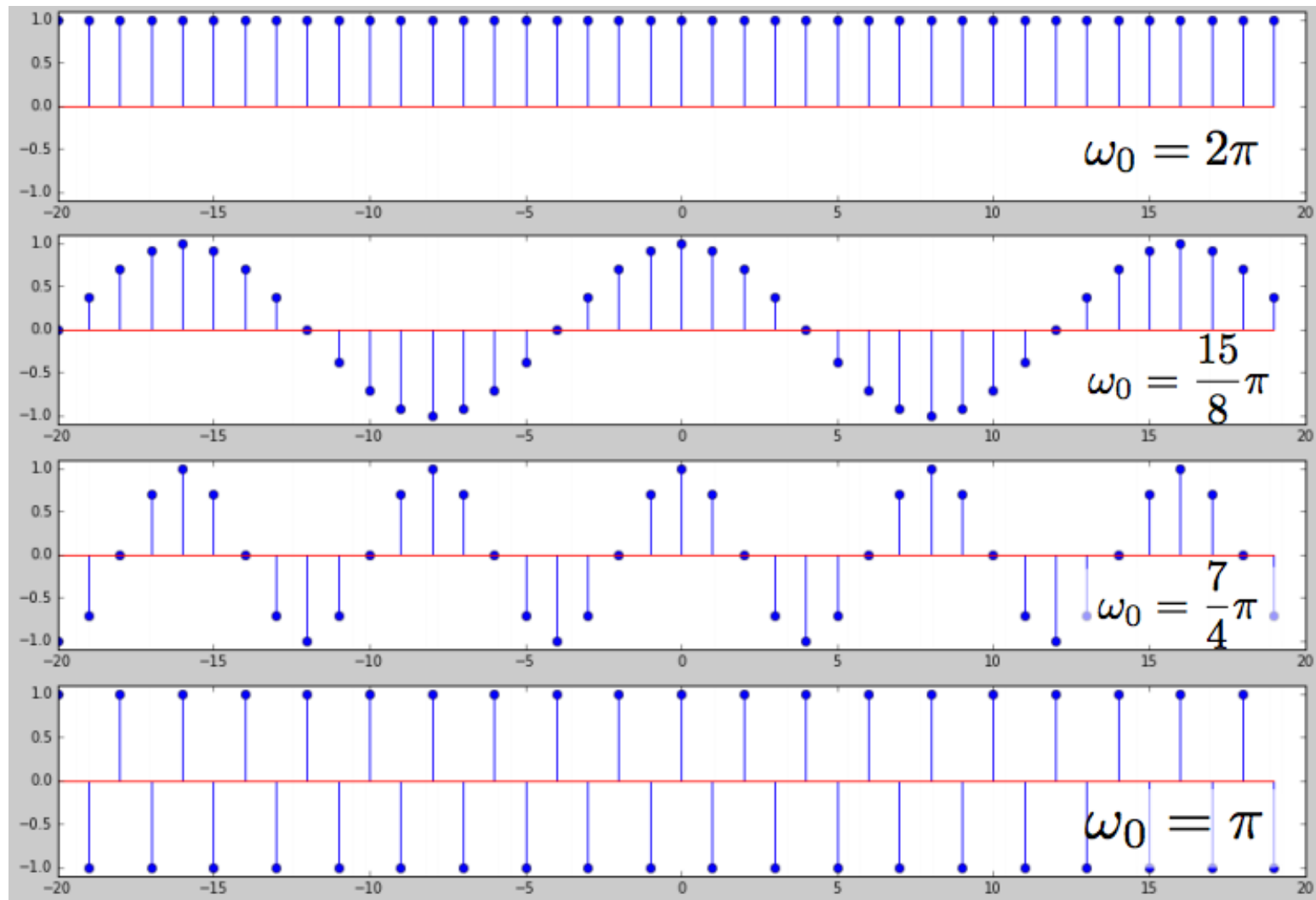
Ex: $\cos\left(\frac{9\pi}{10}n\right)$



Increasing Frequency



Decreasing Frequency





Property #2: Periodicity of Sinusoids

■ Consider $x_1[n] = e^{j(\omega n + \phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (harmonic frequency)

■ It is easy to show that x_1 is periodic with period N , since

$$x_1[n + N] = e^{j(\omega(n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} = e^{j(\omega n + \phi)} e^{j(\frac{2\pi k}{N} N)} = x_1[n] \checkmark$$

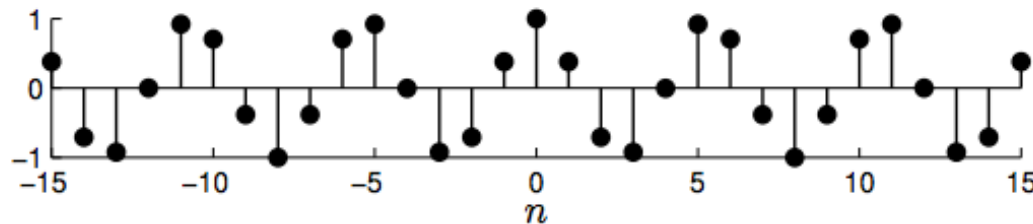
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- Ex: $x_1[n] = \cos(\frac{2\pi 3}{16}n)$, $N = 16$



- Note: x_1 is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer



Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)



Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)
- Is x_2 periodic?

$$x_2[n + N] = e^{j(\omega(n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}$$

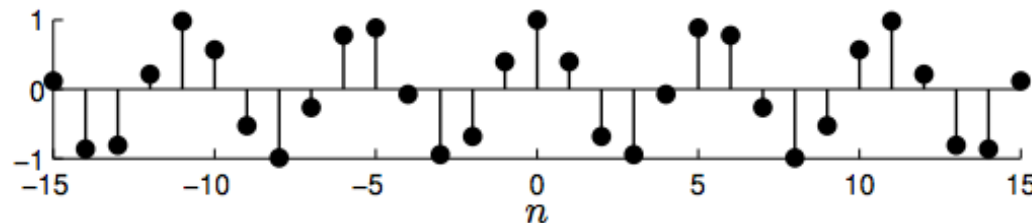
Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is x_2 periodic?

$$x_2[n + N] = e^{j(\omega(n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}$$

- Ex: $x_2[n] = \cos(1.16 n)$



- If its frequency ω is not harmonic, then a sinusoid oscillates but is not periodic!



Harmonic Sinusoids

$$e^{j(\omega n + \phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with **harmonic frequencies**

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

- Which means that
 - **Most** discrete-time sinusoids are **not** periodic!
 - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)



Periodic or not?

□ $\cos(5/7 \pi n)$

□ $\cos(\pi / 5n)$

□ What are N and k? (I.e How many samples is one period?)



Periodic or not?

- ❑ $\cos(5/7 \pi n)$
 - $N=14, k=5$
 - $\cos(5/14*2 \pi n)$
 - Repeats every $N=14$ samples
- ❑ $\cos(\pi / 5n)$
 - $N=10, k=1$
 - $\cos(1/10*2 \pi n)$
 - Repeats every $N=10$ samples



Periodic or not?

- ❑ $\cos(5/7 \pi n)$
 - $N=14, k=5$
 - $\cos(5/14*2 \pi n)$
 - Repeats every $N=14$ samples
- ❑ $\cos(\pi / 5n)$
 - $N=10, k=1$
 - $\cos(1/10*2 \pi n)$
 - Repeats every $N=10$ samples
- ❑ $\cos(5/7 \pi n) + \cos(\pi / 5n) ?$



Periodic or not?

□ $\cos(5/7 \pi n) + \cos(\pi / 5n) ?$

■ $N = \text{SCM}\{10, 14\} = 70$

■ $\cos(5/7 * \pi n) + \cos(\pi / 5n)$

■ $n = N = 70 \rightarrow \cos(5/7 * 70 \pi) + \cos(\pi / 5 * 70) = \cos(25 * 2 \pi) + \cos(7 * 2 \pi)$

Discrete-Time Systems

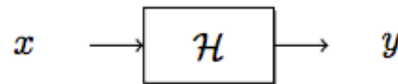


Discrete Time Systems

DEFINITION

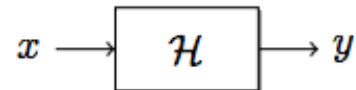
A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples:
 - A speech recognition system converts acoustic waves of speech into text
 - A radar system transforms the received radar pulse to estimate the position and velocity of targets
 - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
 - A 30 day moving average smooths out the day-to-day variability in a stock price

Signal Length and Systems



- Recall that there are two kinds of signals: infinite-length and finite-length
- Accordingly, we will consider two kinds of systems:
 - 1 Systems that transform an infinite-length-signal x into an infinite-length signal y
 - 2 Systems that transform a length- N signal x into a length- N signal y
(Such systems can also be used to process periodic signals with period N)
- For generality, we will assume that the input and output signals are complex valued



System Examples

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n + 2] \quad \forall n$$

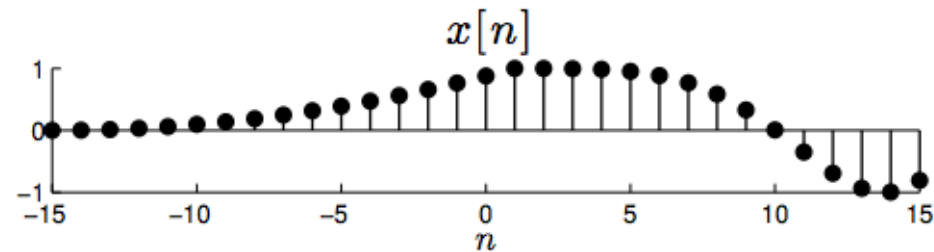
- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

System Examples



- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n - m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n - 1]) \quad \forall n$$

- Recursive average

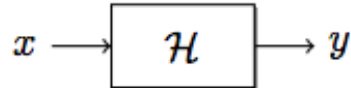
$$y[n] = x[n] + \alpha y[n - 1] \quad \forall n$$



System Properties

- ❑ Memoryless
- ❑ Linearity
- ❑ Time Invariance
- ❑ Causality
- ❑ BIBO Stability

Memoryless



- $y[n]$ depends only on $x[n]$
- Examples:
 - Ideal delay system (or shift system):
 - $y[n] = x[n-m]$ memoryless?
 - Square system:
 - $y[n] = (x[n])^2$ memoryless?

Linear Systems

DEFINITION

A system $\mathcal{H}$ is (zero-state) **linear** if it satisfies the following two properties:

1 Scaling

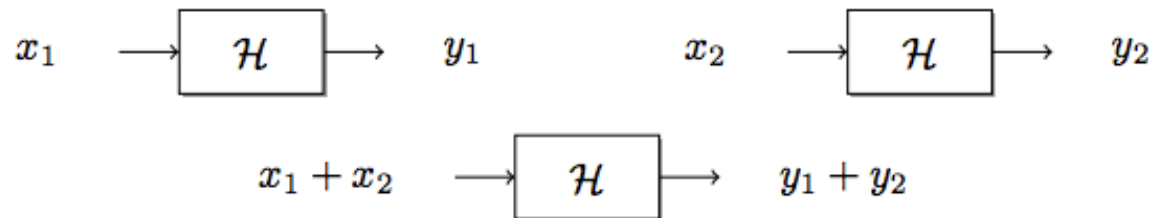
$$\mathcal{H}\{\alpha x\} = \alpha \mathcal{H}\{x\} \quad \forall \alpha \in \mathbb{C}$$



2 Additivity

If $y_1 = \mathcal{H}\{x_1\}$ and $y_2 = \mathcal{H}\{x_2\}$ then

$$\mathcal{H}\{x_1 + x_2\} = y_1 + y_2$$





Proving Linearity

- A system that is not linear is called **nonlinear**
- To prove that a system is linear, you must prove rigorously that it has **both** the scaling and additivity properties for **arbitrary** input signals
- To prove that a system is nonlinear, it is sufficient to exhibit a **counterexample**

Linearity Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- **Scaling:** (Strategy to prove – Scale input x by $\alpha \in \mathbb{C}$, compute output y via the formula at top, and verify that it is scaled as well)

- Let

$$x'[n] = \alpha x[n], \quad \alpha \in \mathbb{C}$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1])$$

Linearity Example: Moving Average

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- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}(\alpha x[n] + \alpha x[n-1]) = \alpha \left(\frac{1}{2}(x[n] + x[n-1]) \right) = \alpha y[n] \quad \checkmark$$

Linearity Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)

- Let

$$x'[n] = x_1[n] + x_2[n]$$

- Let $y'/y_1/y_2$ denote the output when $x'/x_1/x_2$ is input

Linearity Example: Moving Average

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- Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}(\{x_1[n] + x_2[n]\} + \{x_1[n-1] + x_2[n-1]\})$$

Linearity Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \frac{1}{2}(x[n] + x[n-1])$$

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- Let $y'/y_1/y_2$ denote the output when $x'/x_1/x_2$ is input

- Then

$$\begin{aligned} y'[n] &= \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}(\{x_1[n] + x_2[n]\} + \{x_1[n-1] + x_2[n-1]\}) \\ &= \frac{1}{2}(x_1[n] + x_1[n-1]) + \frac{1}{2}(x_2[n] + x_2[n-1]) = y_1[n] + y_2[n] \quad \checkmark \end{aligned}$$

Example: Squaring is Nonlinear



■ **Additivity:** Input two signals into the system and see what happens

- Let

$$y_1[n] = (x_1[n])^2, \quad y_2[n] = (x_2[n])^2$$

- Set

$$x'[n] = x_1[n] + x_2[n]$$

- Then

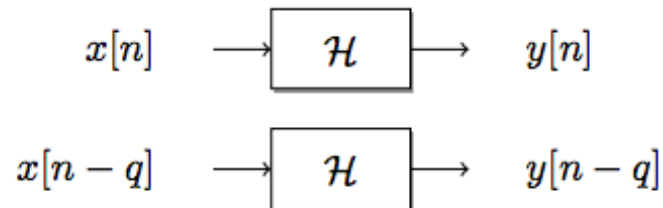
$$y'[n] = (x'[n])^2 = (x_1[n] + x_2[n])^2 = (x_1[n])^2 + 2x_1[n]x_2[n] + (x_2[n])^2 \neq y_1[n] + y_2[n]$$

- Nonlinear!

Time-Invariant Systems

DEFINITION

A system $\mathcal{H}$ processing infinite-length signals is **time-invariant** (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal



- Intuition: A time-invariant system behaves the same no matter when the input is applied
- A system that is not time-invariant is called **time-varying**

Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \frac{1}{2}(x[n] + x[n-1])$$

■ Let

$$x'[n] = x[n - q], \quad q \in \mathbb{Z}$$

■ Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

■ Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}(x[n-q] + x[n-q-1]) = y[n-q] \quad \checkmark$$

Example: Decimation



- This system is time-varying; demonstrate with a counter-example

- Let

$$x'[n] = x[n - 1]$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] = x'[2n] = x[2n - 1] \neq x[2(n - 1)] = y[n - 1]$$



Causal Systems

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- ❑ Forward difference system:
 - $y[n] = x[n+1] - x[n]$ causal?

- ❑ Backward difference system:
 - $y[n] = x[n] - x[n-1]$ causal?

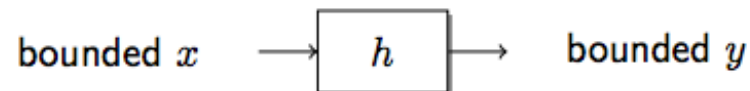
Stability

□ BIBO Stability

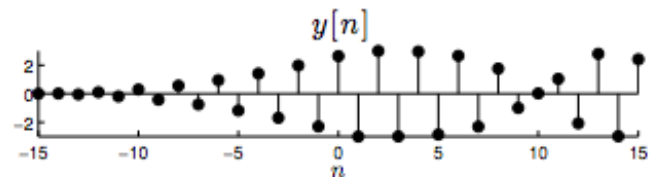
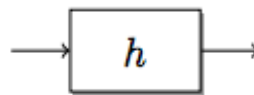
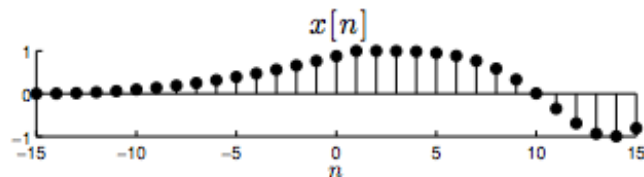
■ Bounded-input bounded-output Stability

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_{\infty} < \infty$ and $\|y\|_{\infty} < \infty$,
or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all n





System Properties - Summary

❑ Causality

- $y[n]$ only depends on $x[m]$ for $m \leq n$

❑ Linearity

- Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
 - $Ax_1[n] + Bx_2[n] \rightarrow Ay_1[n] + By_2[n]$

❑ Memoryless

- $y[n]$ depends only on $x[n]$

❑ Time Invariance

- Shifted input results in shifted output
 - $x[n-q] \rightarrow y[n-q]$

❑ BIBO Stability

- A bounded input results in a bounded output (ie. max signal value exists for output if max)



Examples

□ Causal? Linear? Time-invariant? Memoryless?
BIBO Stable?

□ Time Shift:

- $y[n] = x[n - m]$

□ Accumulator:

- $$y[n] = \sum_{k=-\infty}^n x[k]$$

□ Compressor ($M > 1$):

$$y[n] = x[Mn]$$

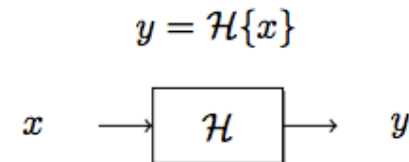
Big Ideas

□ Discrete Time Signals

- Unit impulse, unit step, exponential, sinusoids, complex sinusoids
- Can be finite length, infinite length
- Properties
 - Even, odd, causal
 - Periodicity and aliasing
 - Discrete frequency bounded!

□ Discrete Time Systems

- Transform one signal to another
- Properties
 - Linear, Time-invariance, memoryless, causality, BIBO stability





Admin

- ❑ Enroll in Piazza site:
 - piazza.com/upenn/spring2018/ese531
- ❑ HW 1 out after class