ESE 531: Digital Signal Processing

Lec 21: April 5, 2018

Fast Fourier Transform Pt 2

Lecture Outline

- DFT vs. DTFT
- FFT practice
- Chirp Transform Algorithm
- Circular convolution as linear convolution with aliasing

Discrete Fourier Transform

- The DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \quad \text{DFT, analysis}$$

- It is understood that,

$$\begin{aligned} x[n] &= 0 \quad \text{outside } 0 \le n \le N-1 \\ X[k] &= 0 \quad \text{outside } 0 \le k \le N-1 \end{aligned}$$

DFT vs. DTFT

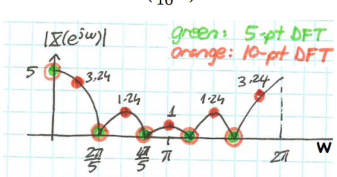
- The DFT are samples of the DTFT at N equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega = k\frac{2\pi}{N}} \qquad 0 \le k \le N-1$$

DFT vs DTFT

- Back to example

$$\begin{aligned} X[k] &= \sum_{n=0}^{4} W_{10}^{nk} \\ &= e^{-j\frac{4\pi}{10}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)} \end{aligned}$$



Fast Fourier Transform Algorithms

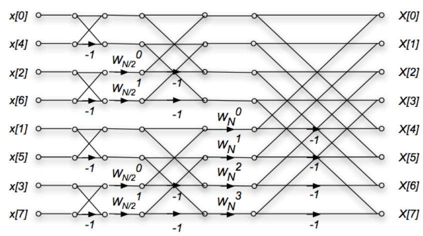
- We are interested in efficient computing methods for the DFT and inverse DFT:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad k = 0, \ldots, N-1 \\ x[n] &= \sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad n = 0, \ldots, N-1 \end{aligned}$$

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}.$$

Decimation-in-Time FFT

Combining all these stages, the diagram for the 8 sample DFT is:



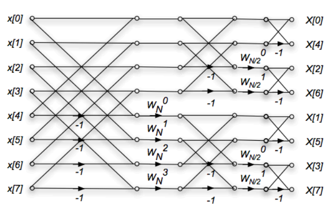
- $3=\log_2(N)=\log_2(8)$ stages
- $4=N/2=8/2$ multiplications in each stage
  - 1st stage has trivial multiplication

Decimation-in-Time FFT

- In general, there are $\log_2 N$ stages of decimation-in-time.
- Each stage requires N/2 complex multiplications, some of which are trivial.
- The total number of complex multiplications is $(N/2) \log_2 N$, or $O(N \log_2 N)$
- The order of the input to the decimation-in-time FFT algorithm must be permuted.
  - First stage: split into odd and even. Zero low-order bit (LSB) first
  - Next stage repeats with next zero-lower bit first.
  - Net effect is reversing the bit order of indexes

Decimation-in-Frequency FFT

The diagram for and 8-point decimation-in-frequency DFT is as follows



This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

Example 1:

A long *periodic* sequence x of period $N = 2^r$ (r is an integer) is to be convolved with a finite-length sequence h of length K.

(a) *Show* that the output y of this convolution (filtering) is *periodic*; what is its period?

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(b) Let $K = mN$ where m is an integer; N is large. How would you implement this convolution *efficiently*? Explain your analysis clearly.
Compare the *total number of multiplications* required in your scheme to that in the direct implementation of FIR filtering. (Consider the case $r = 10$, $m = 10$).

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Example 2:

A sequence $x = \{x[n], n = 0,1,\ldots,N-1\}$ is given; let $X(e^{j\omega})$ be its DTFT.

(a) Suppose $N = 10$. You want to evaluate both $X(e^{j2\pi 7/12})$ and $X(e^{j2\pi 3/8})$. The only computation you can perform is one DFT, on any one input sequence of your choice. Can you find the desired DTFT values? *(Show your analysis and explain clearly.)*

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(b) Suppose N is large. You want to obtain $X(e^{j\omega})$ at the following $2M$ frequencies:

$$\omega = \frac{2\pi}{M}m,\ \ m = 0,1,\ldots,M-1 \quad \text{and} \quad \omega = \frac{2\pi}{M}m + \frac{2\pi}{N},\ \ m = 0,1,\ldots,M-1.$$

Here $M = 2^{\mu} \ll N = 2^{\nu}$

A standard radix-2 FFT algorithm is available. You may execute the FFT algorithm *once* or *more than once*, and *multiplications* and *additions* outside of the FFT are *allowed*, if necessary.

You want to get the $2M$ DTFT values with as few *total multiplications* as possible (*including those in the FFT*). Give explicitly the best method you can find for this, with an estimate of the *total number of multiplications* needed in terms of M and N.

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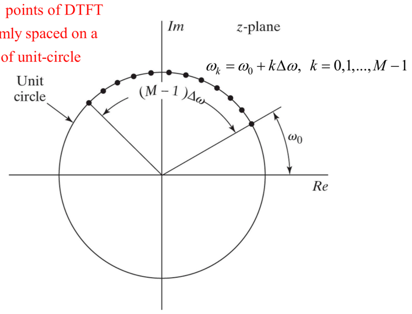
Does your result change if extra multiplications outside of FFTs are *not* allowed?

Chirp Transform Algorithm

- Uses convolution to evaluate the DFT
- This algorithm is not optimal in minimizing any measure of computational complexity, but it has been useful in a variety of applications, particularly when implemented in technologies that are well suited to doing convolution with a fixed, prespecified impulse response.
- The CTA is also more flexible than the FFT, since it can be used to compute *any* set of equally spaced samples of the Fourier transform on the unit circle.

Chirp Transform Algorithm

For M points of DTFT uniformly spaced on a sector of unit-circle



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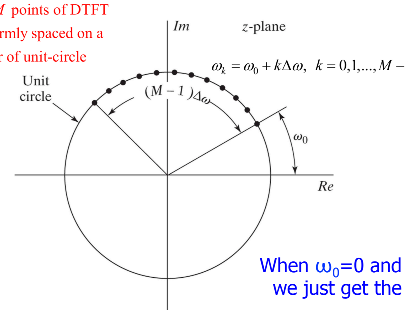
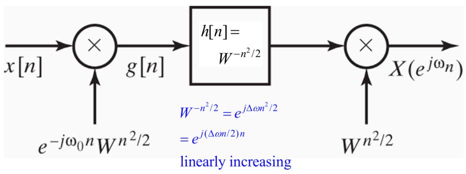


When ω_0=0 and M=N, we just get the DFT

Chirp Transform Algorithm

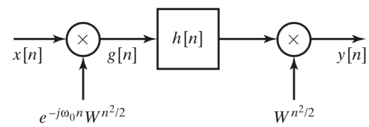
$W \triangleq e^{-j\Delta\omega}$



$W^{-n^2/2} = e^{j\Delta\omega n^2/2}$
$= e^{j(\Delta\omega n/2)n}$
linearly increasing
frequency "chirp"

FIR CTA

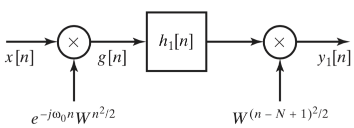
$$h[n] = \begin{cases} W^{-n^2/2}, & -(N-1) \le n \le M-1, \\ 0, & \text{otherwise,} \end{cases}$$



$$X(e^{j\omega_n}) = y[n], \qquad n = 0, 1, \ldots, M-1.$$

Causal FIR CTA

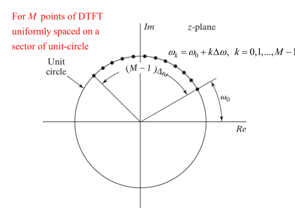
$$h_1[n] = \begin{cases} W^{-(n-N+1)^2/2}, & n = 0, 1, \ldots, M+N-2, \\ 0, & \text{otherwise.} \end{cases}$$



$$X(e^{j\omega_n}) = y_1[n+N-1], \qquad n = 0, 1, \ldots, M-1.$$

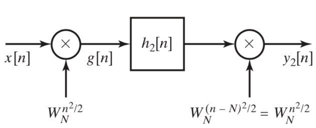
Example: Chirp Transform Parameters

- We have a finite-length sequence x[n] that is nonzero only on the interval n = 0, ..., 25, (Length N=26) and we wish to compute 16 samples of the DTFT $X(e^{j\omega})$ at the frequencies $\omega_k = 2\pi/27 + 2\pi k/1024$ for k = 0, . . . , 15.



Causal FIR CTA for DFT

$$h_2[n] = \begin{cases} W_N^{-n^2/2}, & n = 1, 2, \ldots, M+N-1, \\ 0, & \text{otherwise.} \end{cases}$$



$$X(e^{j2\pi n/N}) = y_2[n+N], \qquad n = 0, 1, \ldots, M-1.$$

Circular Convolution

- Circular Convolution:

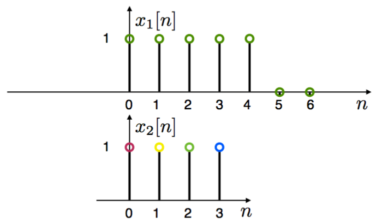
$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

For two signals of length N

Note: Circular convolution is commutative

$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$

Compute Circular Convolution Sum



$$x_1[n] \textcircled{\scriptsize N} x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

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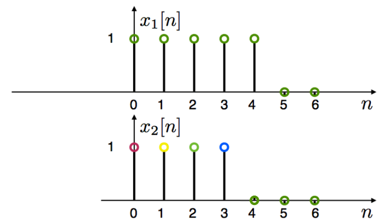

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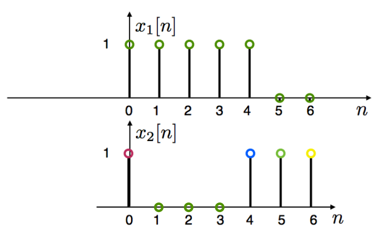

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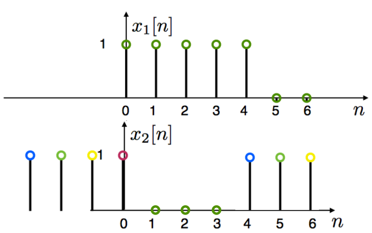
Compute Circular Convolution Sum

y[0]=2



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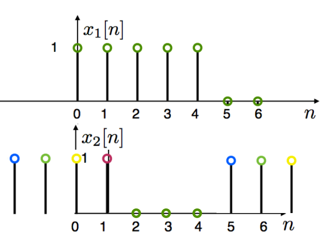

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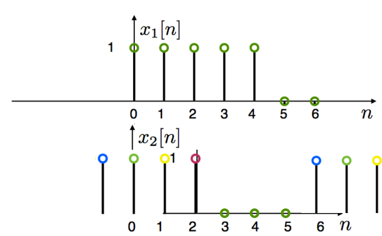
Compute Circular Convolution Sum

y[0]=2
y[1]=2
y[2]=3



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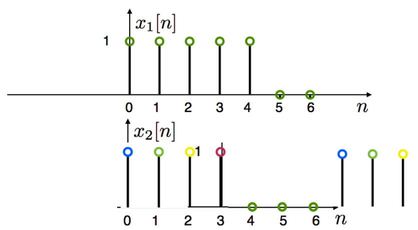
Compute Circular Convolution Sum

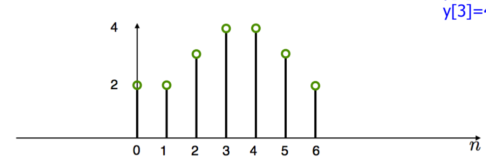
y[0]=2
y[1]=2
y[2]=3
y[3]=4



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

Result

y[0]=2
y[1]=2
y[2]=3
y[3]=4



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

Linear Convolution

- We start with two non-periodic sequences:

$$x[n] \quad 0 \le n \le L-1$$
$$h[n] \quad 0 \le n \le P-1$$

  - E.g. x[n] is a signal and h[n] a filter's impulse response
- We want to compute the linear convolution:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{L-1} x[m]h[n-m]$$

  - y[n] is nonzero for $0 \le n \le L+P-2$ with length M=L+P-1

Requires LP multiplications

Linear Convolution via Circular Convolution

- Zero-pad x[n] by P-1 zeros

$$x_{zp}[n] = \begin{cases} x[n] & 0 \le n \le L-1 \\ 0 & L \le n \le L+P-2 \end{cases}$$

- Zero-pad h[n] by L-1 zeros

$$h_{zp}[n] = \begin{cases} h[n] & 0 \le n \le P-1 \\ 0 & P \le n \le L+P-2 \end{cases}$$

- Now, both sequences are length M=L+P-1

Circular Conv. via Linear Conv. w/ Aliasing

- If the DTFT $X(e^{j\omega})$ of a sequence x[n] is sampled at N frequencies $\omega_k=2\pi k/N$, then the resulting sequence X[k] corresponds to the periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN].$$

- And $X[k] = \begin{cases} X(e^{j(2\pi k/N)}), & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases}$ is the DFT of one period given as

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases}$$

Circular Conv. via Linear Conv. w/ Aliasing

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases}$$

- If x[n] has length less than or equal to N, then $x_p[n]=x[n]$
- However if the length of x[n] is greater than N, this might not be true and we get aliasing in time
  - N-point convolution results in N-point sequence

Circular Conv. via Linear Conv. w/ Aliasing

- Given two N-point sequences ($x_1[n]$ and $x_2[n]$) and their N-point DFTs ($X_1[k]$ and $X_2[k]$)
- The N-point DFT of $x_3[n]=x_1[n]*x_2[n]$ is defined as

$$X_3[k] = X_3(e^{j(2\pi k/N)})$$

Circular Conv. via Linear Conv. w/ Aliasing

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- The N-point DFT of $x_3[n]=x_1[n]*x_2[n]$ is defined as

$$X_3[k] = X_3(e^{j(2\pi k/N)})$$

- And $X_3[k]=X_1[k]X_2[k]$, where the inverse DFT of $X_3[k]$ is

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n-rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise}, \end{cases}$$

Circular Conv. as Linear Conv. w/ Aliasing

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n-rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise}, \end{cases}$$

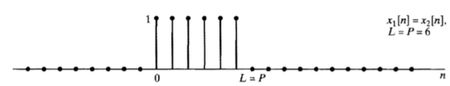
- Thus

$$x_{3_p}[n] = x_1[n] \textcircled{N} x_2[n]$$

- The N-point circular convolution is the sum of linear convolutions shifted in time by N

Example 1:

- Let



- The N=L=6-point circular convolution results in

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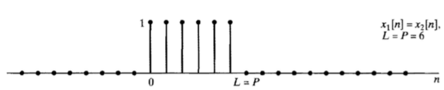

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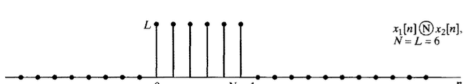

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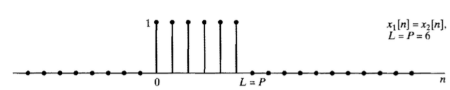

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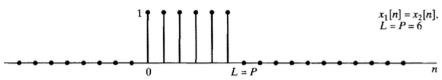

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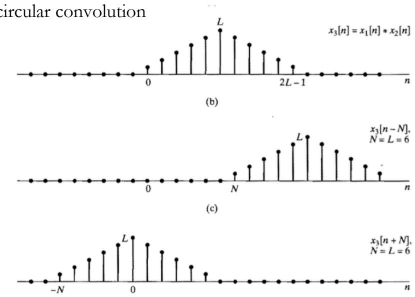
Example 1:

- The sum of N-shifted linear convolutions equals the N-point circular convolution



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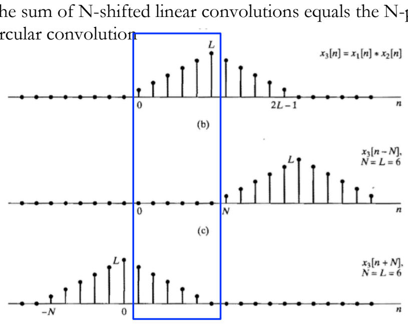

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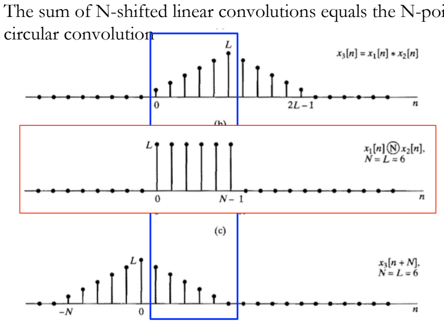

Example 1:

- If I want the circular convolution and linear convolution to be the same, what do I do?

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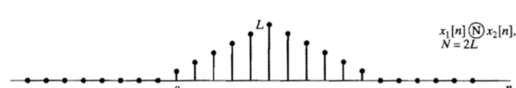
  - Take the N=2L-point circular convolution



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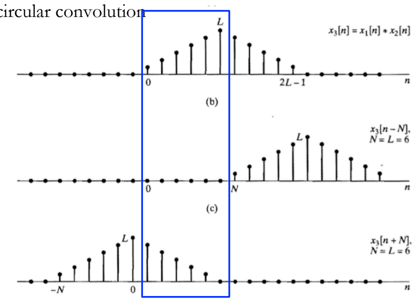
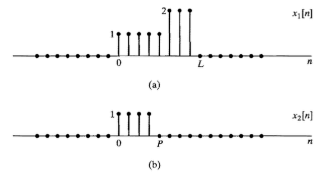


Example 2:

- Let



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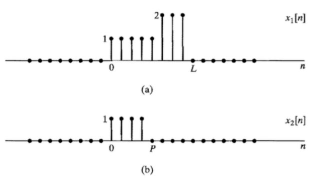
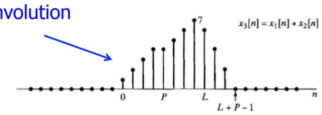
- Let



Linear convolution

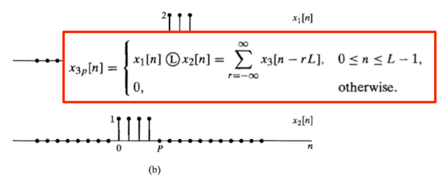


- What does the L-point circular convolution look like?

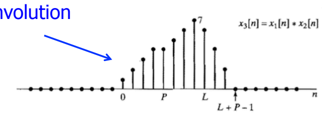
Example 2:

- Let

$$x_{3p}[n] = \begin{cases} x_1[n] \textcircled{L} x_2[n] = \sum_{r=-\infty}^{\infty} x_3[n-rL], & 0 \le n \le L-1, \\ 0, & \text{otherwise.} \end{cases}$$



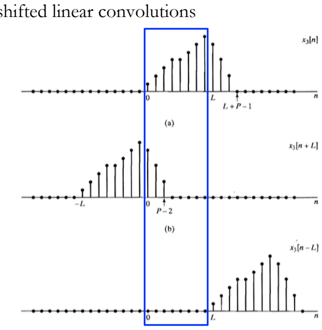
Linear convolution



- What does the L-point circular convolution look like?

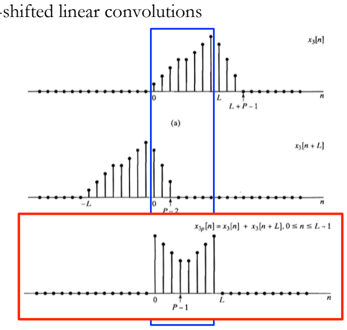
Example 2:

- The L-shifted linear convolutions



Example 2:

- The L-shifted linear convolutions



Big Ideas

- Discrete Fourier Transform (DFT)
  - For finite signals assumed to be zero outside of defined length
  - N-point DFT is sampled DTFT at N points
  - Useful properties allow easier linear convolution
- Fast Fourier Transform
  - Enable computation of an N-point DFT (or DFT^{-1}) with the order of just $N \cdot \log_2 N$ complex multiplications.
- Fast Convolution Methods
  - Use circular convolution (i.e DFT) to perform fast linear convolution
    - Overlap-Add, Overlap-Save
  - Circular convolution is linear convolution with aliasing
- Design DSP methods to minimize computations!

Admin

- Read adaptive filter reference for next lecture
- Tania W office hours next week moved to M 1-3:30pm (will post on Piazza)
- Project
  - Due 4/25