ESE 531: Digital Signal Processing

Lec 23: April 19, 2018

Wavelet Transform, Compressive Sensing

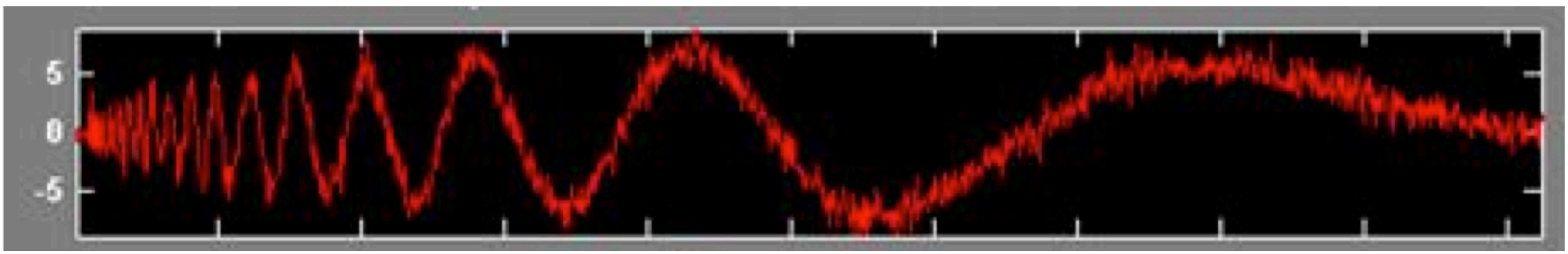
Previously

- Today
  - Wavelet Transform
  - Compressive Sampling/Sensing

Wavelet Transform

Motivation

- Some signals obviously have spectral characteristics that vary with time



Criticism of Fourier Spectrum

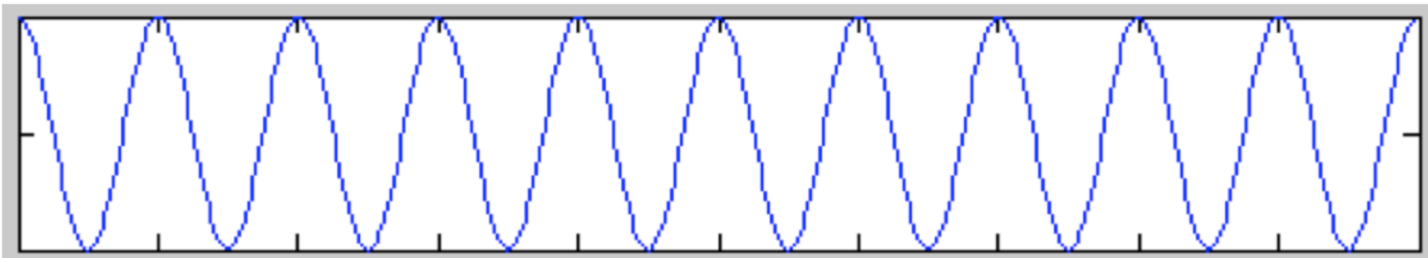
- It's giving you the spectrum of the 'whole time-series'

- Which is OK if the time-series is stationary. But what if its not?

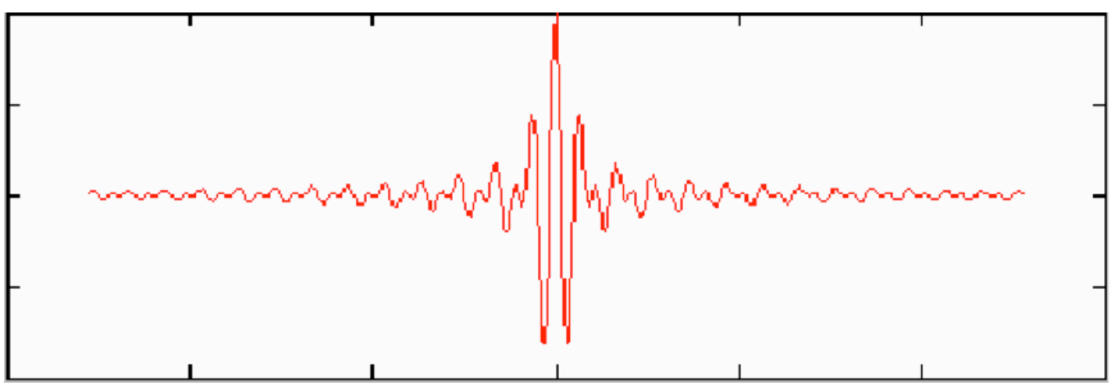
- We need a technique that can "march along" a time series and that is capable of:
  - Analyzing spectral content in different places
  - Detecting sharp changes in spectral character

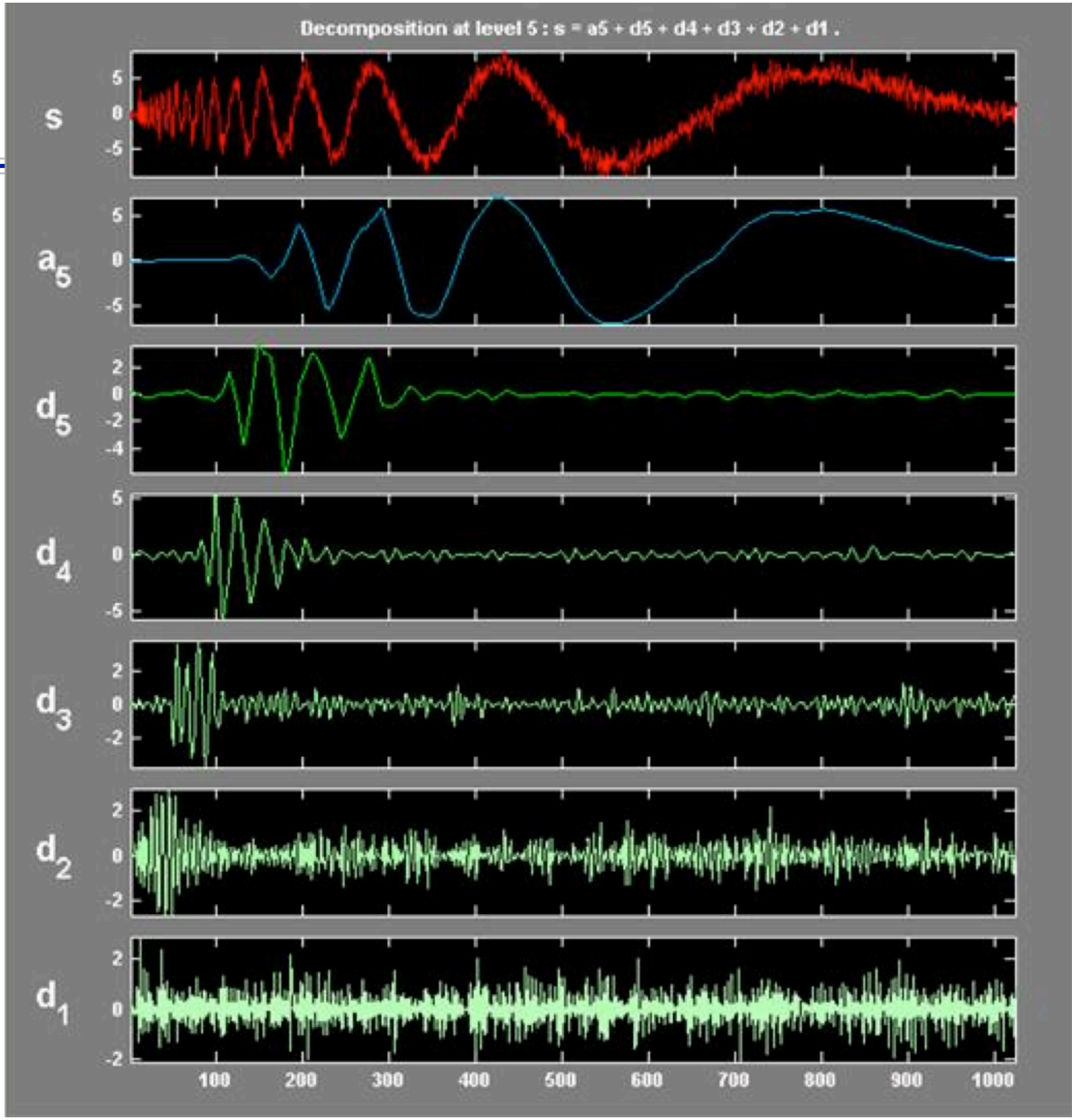
Fourier vs. Wavelet

- Fourier Analysis is based on an indefinitely long cosine wave of a specific frequency



- Wavelet Analysis is based on an short duration wavelet of a specific center frequency




Decomposition at level 5 : s = a5 + d5 + d4 + d3 + d2 + d1 .
s
a5
d5
d4
d3
d2
d1
100
200
300
400
500
600
700
800
900
1000

Wavelet Transform

- All wavelet derived from *mother* wavelet

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

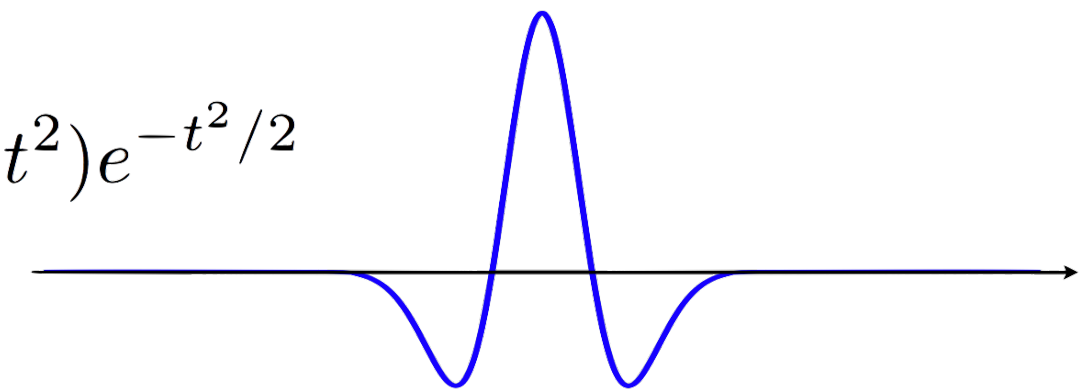
- Must satisfy

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1 \quad \Rightarrow \text{unit norm}$$

Examples of Wavelets

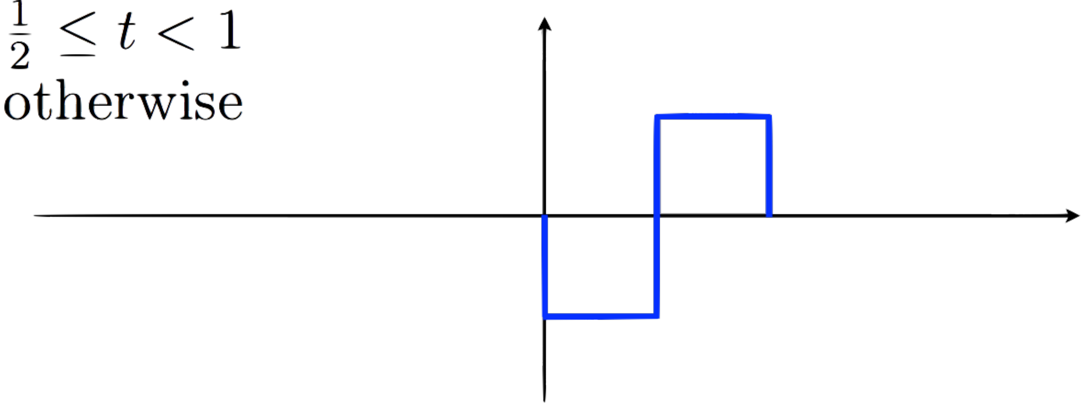
Mexican Hat

$$\Psi(t) = (1 - t^2)e^{-t^2/2}$$



Haar

$$\Psi(t) = \begin{cases} -1 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Wavelet – Scaled and Shifted

normalization

shift in time

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\psi\!\left(\frac{t-\tau}{s}\right)$$

change in scale:
big s means long
wavelength

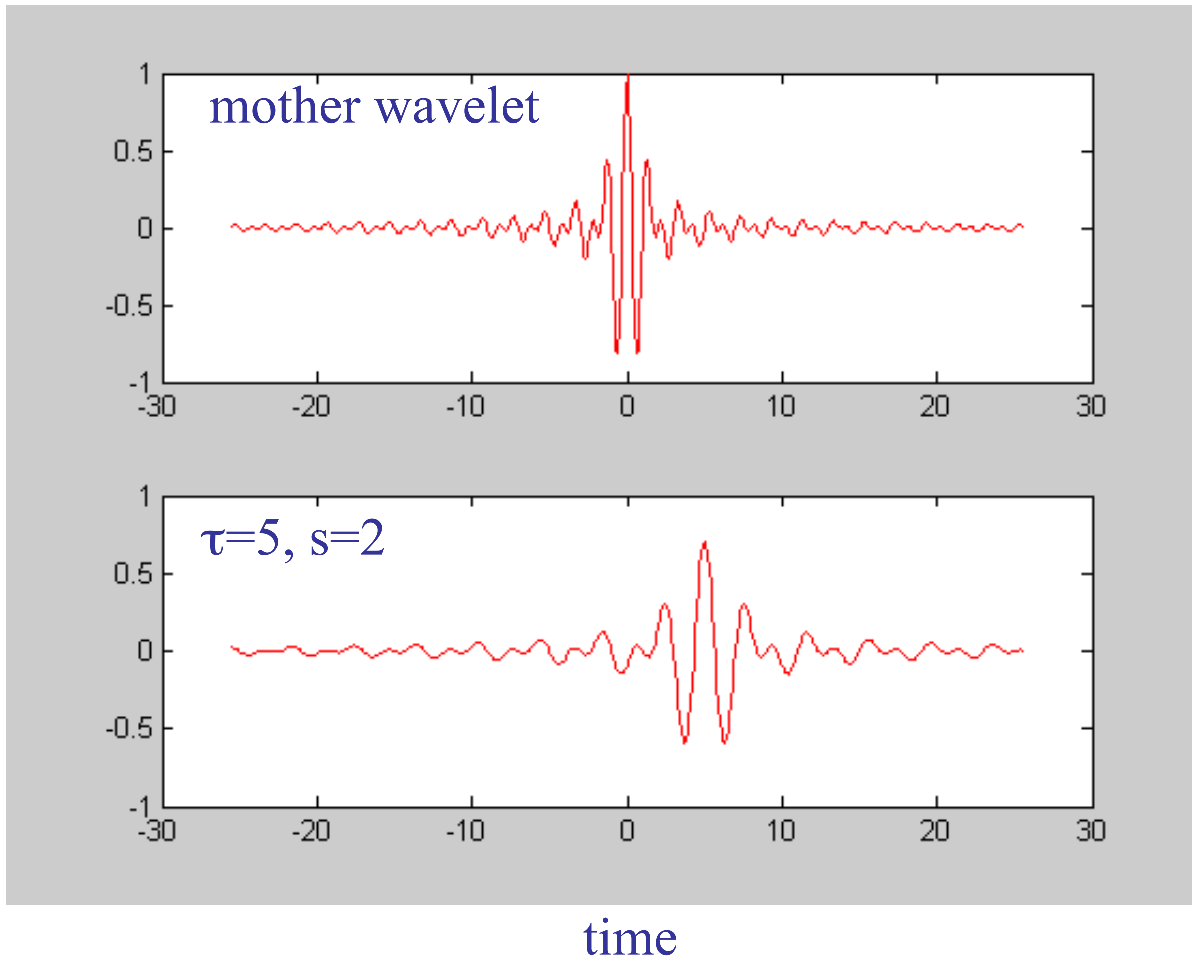
wavelet with
scale, s and time, τ

Mother wavelet

Shannon Wavelet

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)$$

- Ψ(t) = 2 sinc(2t) – sinc(t)



Wavelet Transform

time-series

$$\gamma(s,t) = \int f(t)\Psi_{s,\tau}(t)dt$$

coefficient of wavelet
with
scale, s and time, τ

wavelet with
scale, s, and shift, τ

Inverse Wavelet Transform

- Build up a time-series as sum of wavelets of different scales, s, and positions, t

$$f(t) = \int\int \gamma(s, \tau)\psi_{s,\tau}(t)\,d\tau\,ds$$

time-series

coefficients of wavelets

wavelet with scale, s and time, τ

Wavelet Transform

- Determining the wavelet coefficients for a fixed scale, *s*, can be thought of as a filtering operation

$$\gamma(s,t) = \int f(t)\Psi_{s,\tau}(t)dt$$

$$\gamma_s(t) = \int f(t)\Psi_s(t)\,dt = f(t) * \Psi_s(t)$$

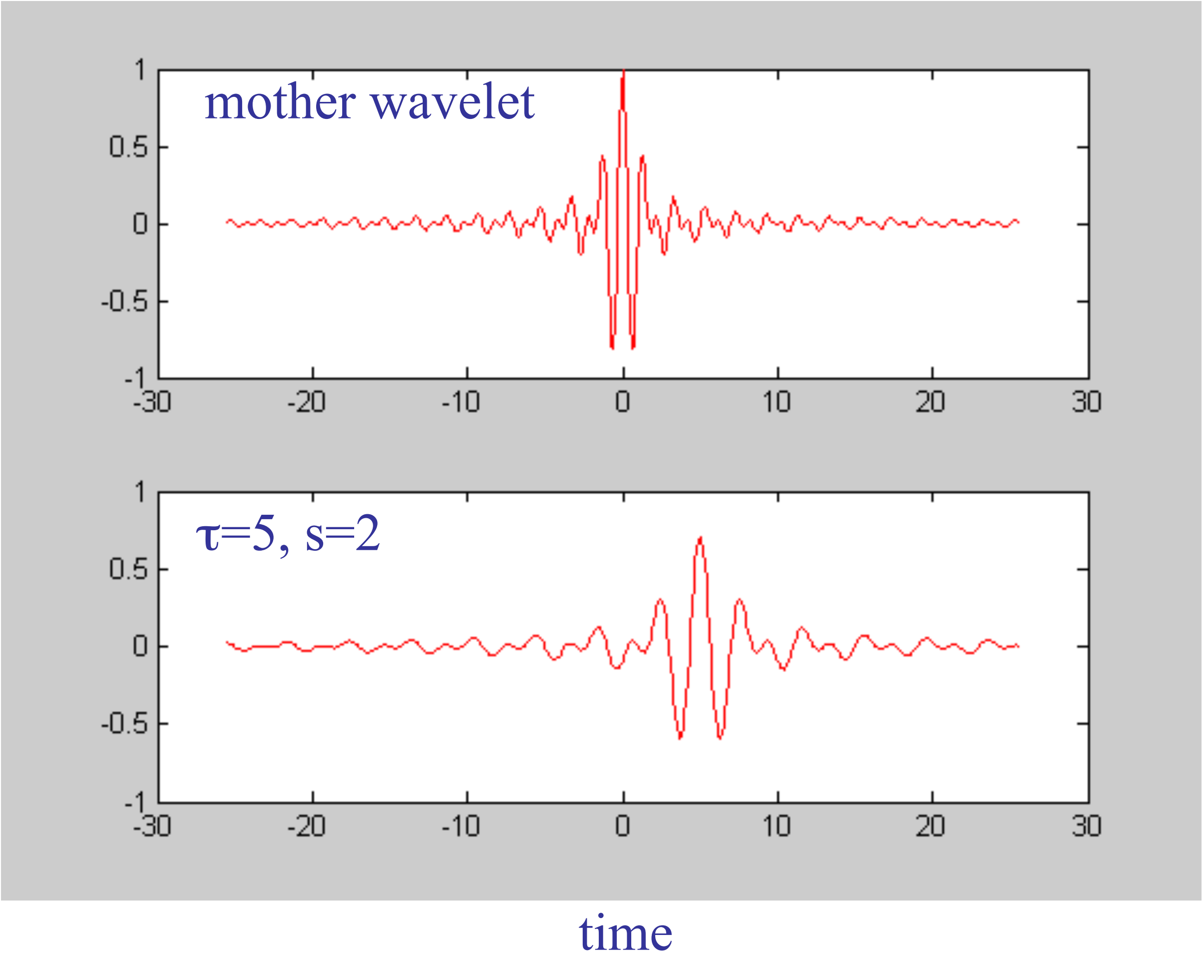
- where

$$\Psi_s(t) = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

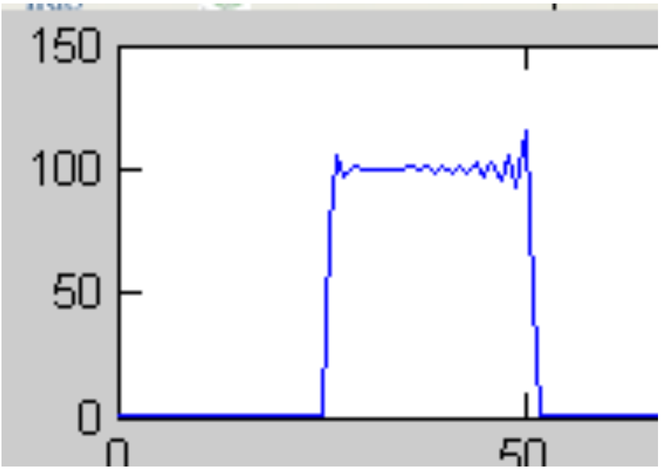
Shannon Wavelet

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)$$

- $\Psi(t) = 2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)$



Fourier spectrum of Shannon Wavelet



frequency, ω

- Wavelet coefficients are a result of bandpass filtering

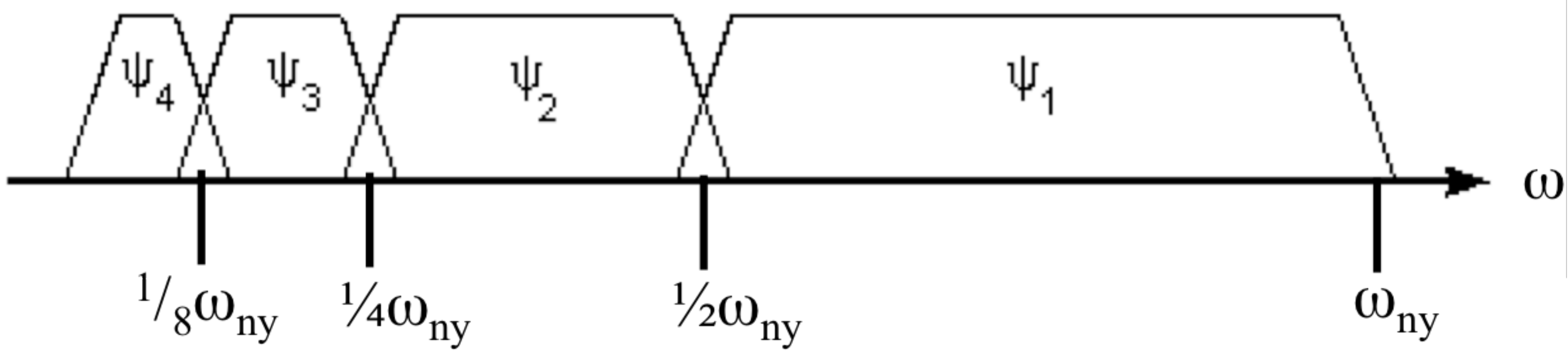
Discrete wavelets:

- Scale wavelets only by powers of 2
  - $s_j=2^j$
- And shifting by multiples of s_j for each successive scale
  - $\tau_{j,k} = 2^j k$
- Then $\gamma(s_j, \tau_{j,k}) = \gamma_{jk}$
  - where $j = 1, 2, \ldots\infty$, $k = -\infty\ldots -2, -1, 0, 1, 2, \ldots\infty$

$$\gamma_{j,k} = \frac{1}{\sqrt{2^j}} \int f(t)\Psi\left(\frac{t-k2^j}{2^j}\right)dt$$

Discrete Wavelet Transform

- The factor of two scaling means that the spectra of the wavelets divide up the frequency scale into octaves (frequency doubling intervals)



Discrete Wavelet Transform

- As we saw previously, the coefficients of Ψ is just the band-pass filtered time-series, where Ψ is the wavelet, now viewed as the impulse response of a bandpass filter.

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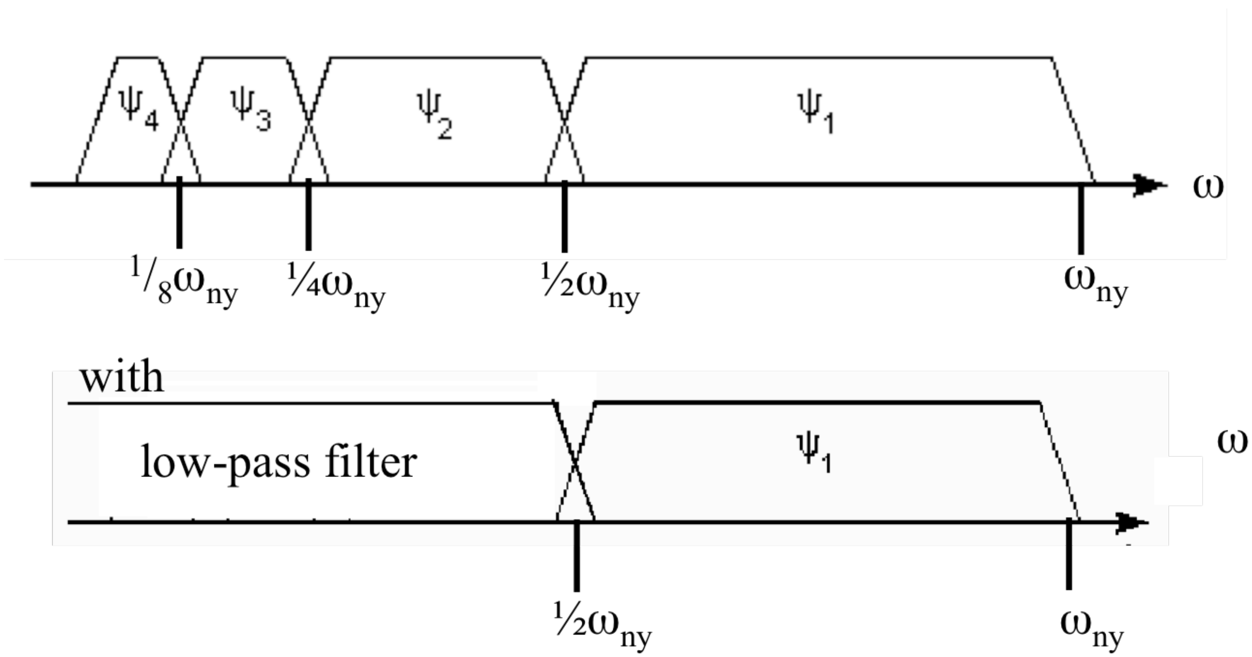
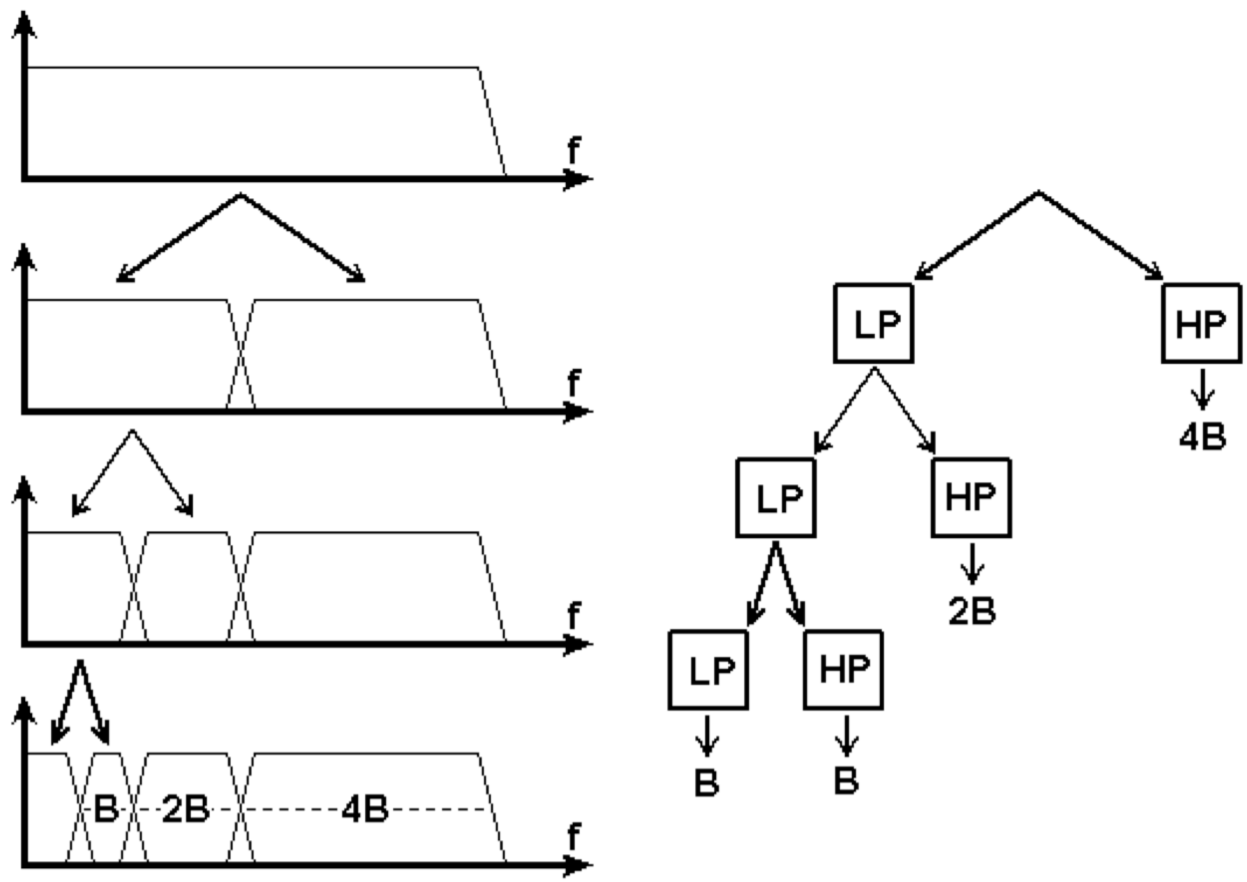
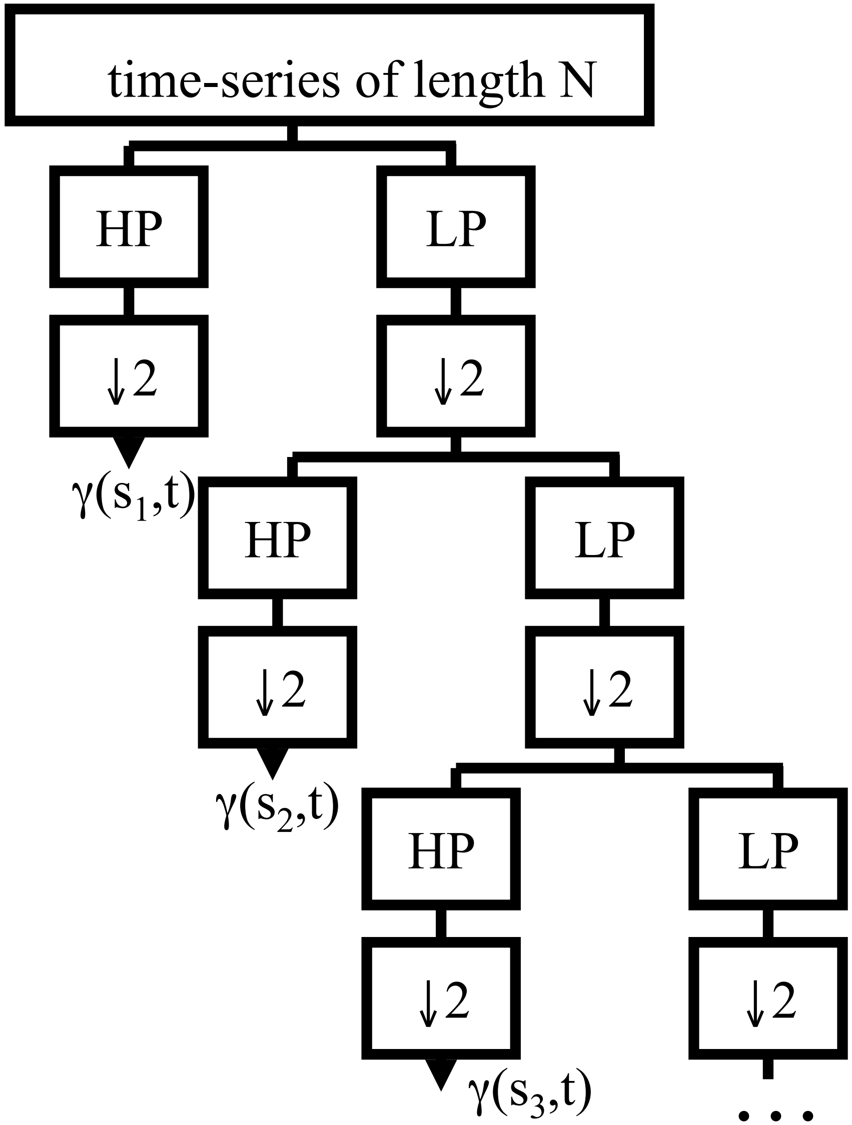


Digital Wavelet as Multirate Filter Bank

- Repeat recursively!



Digital Wavelet as Multirate Filter Bank

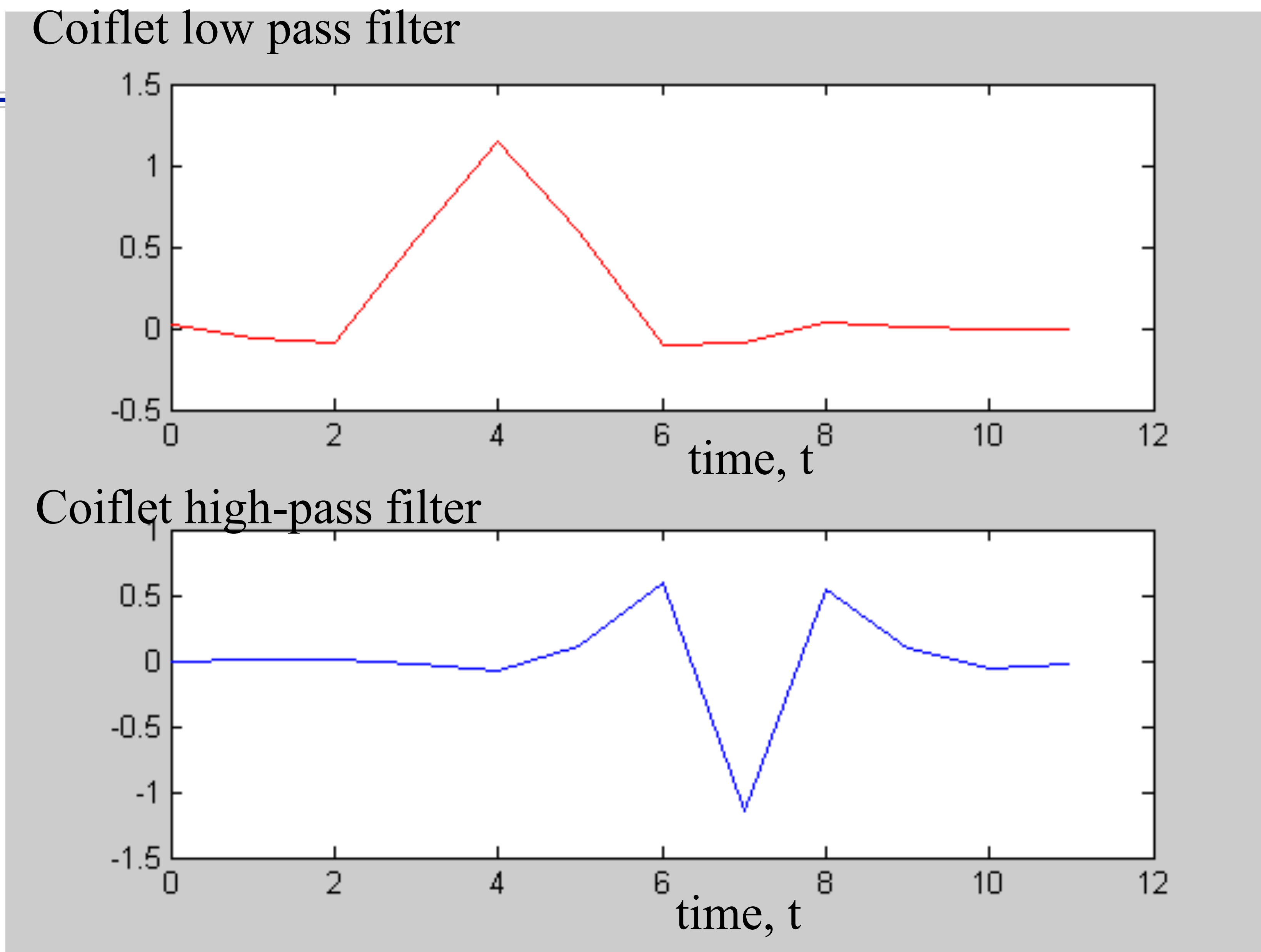


$\gamma(s_1,t)$: N/2 coefficients

$\gamma(s_2,t)$: N/4 coefficients

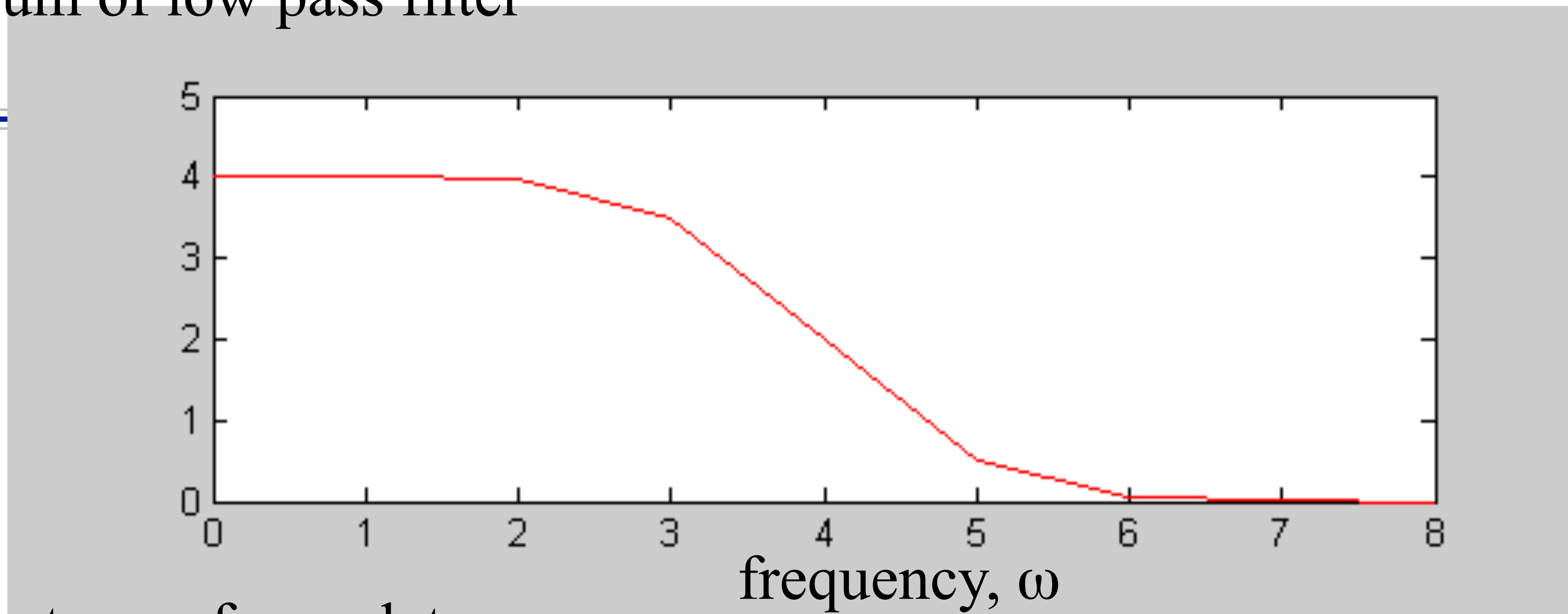
$\gamma(s_2,t)$: N/8 coefficients

Total: N coefficients

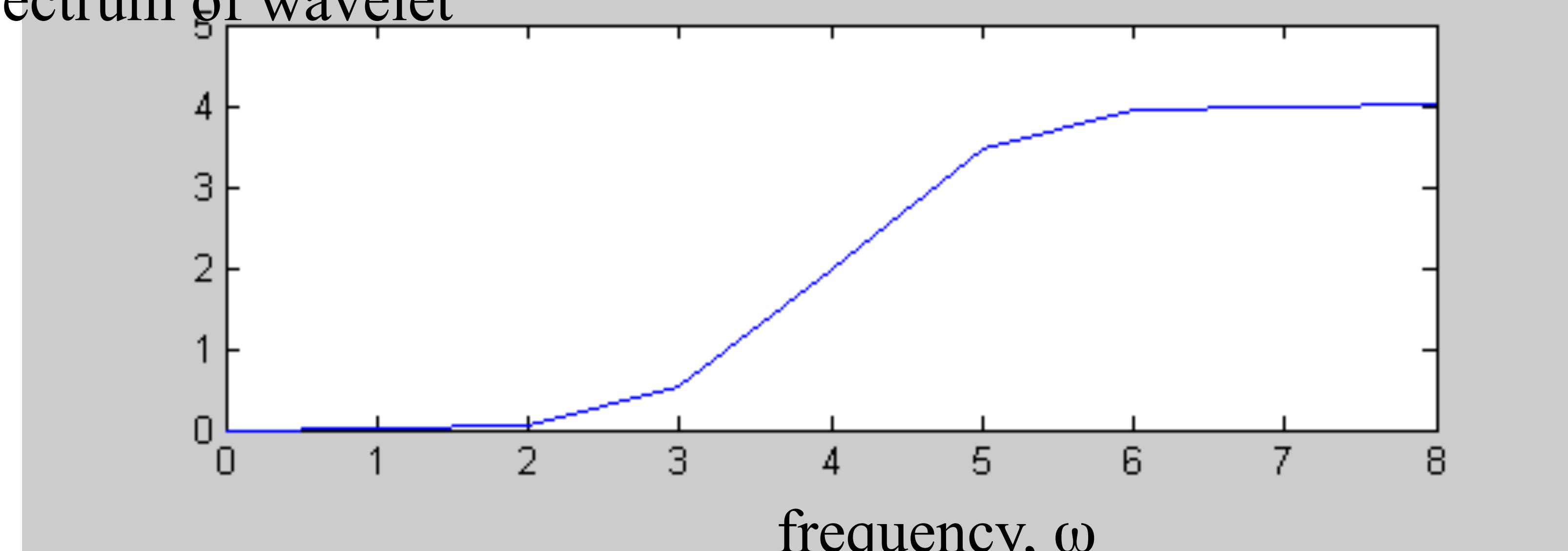


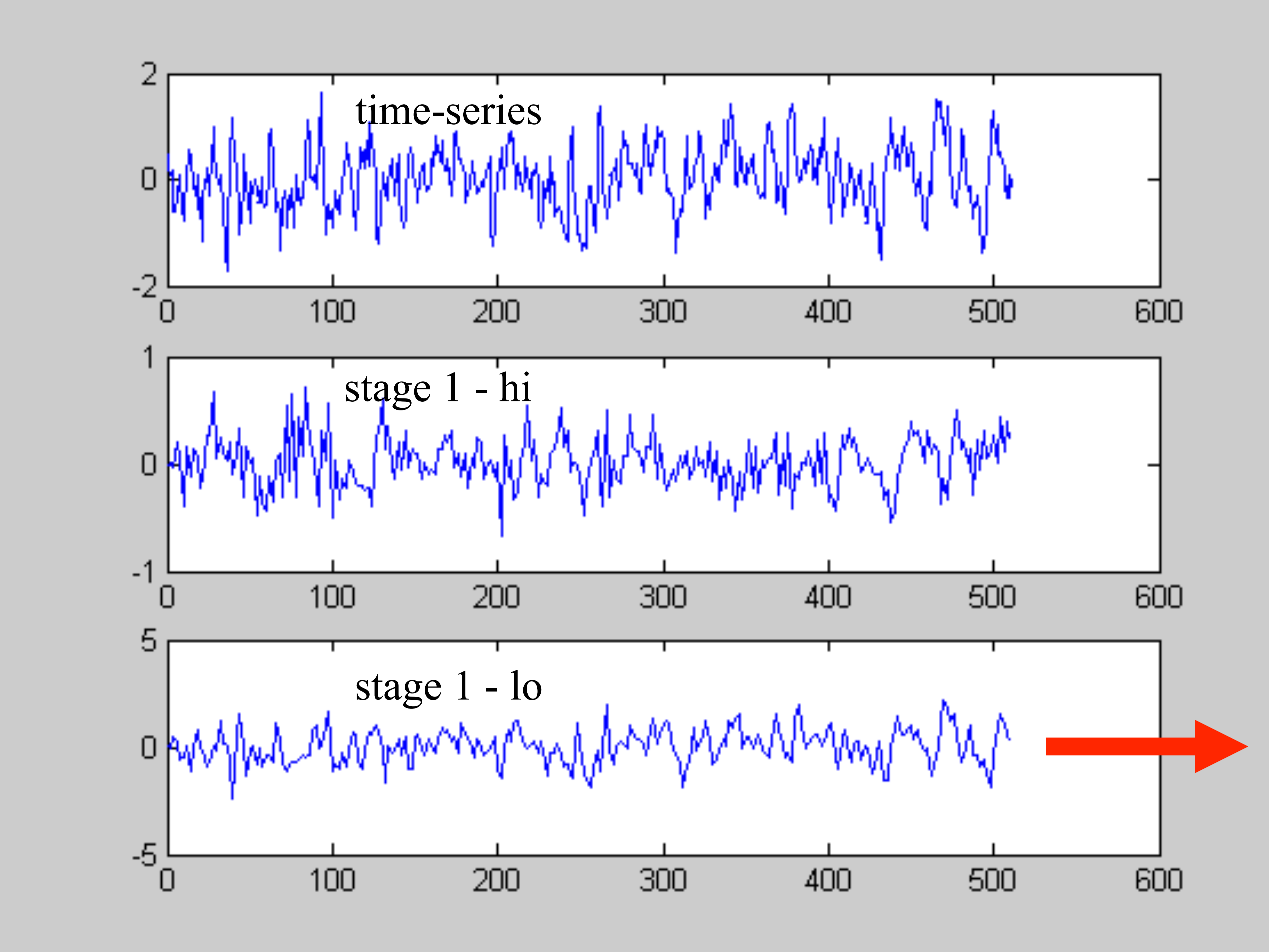
From http://en.wikipedia.org/wiki/Coiflet

Spectrum of low pass filter

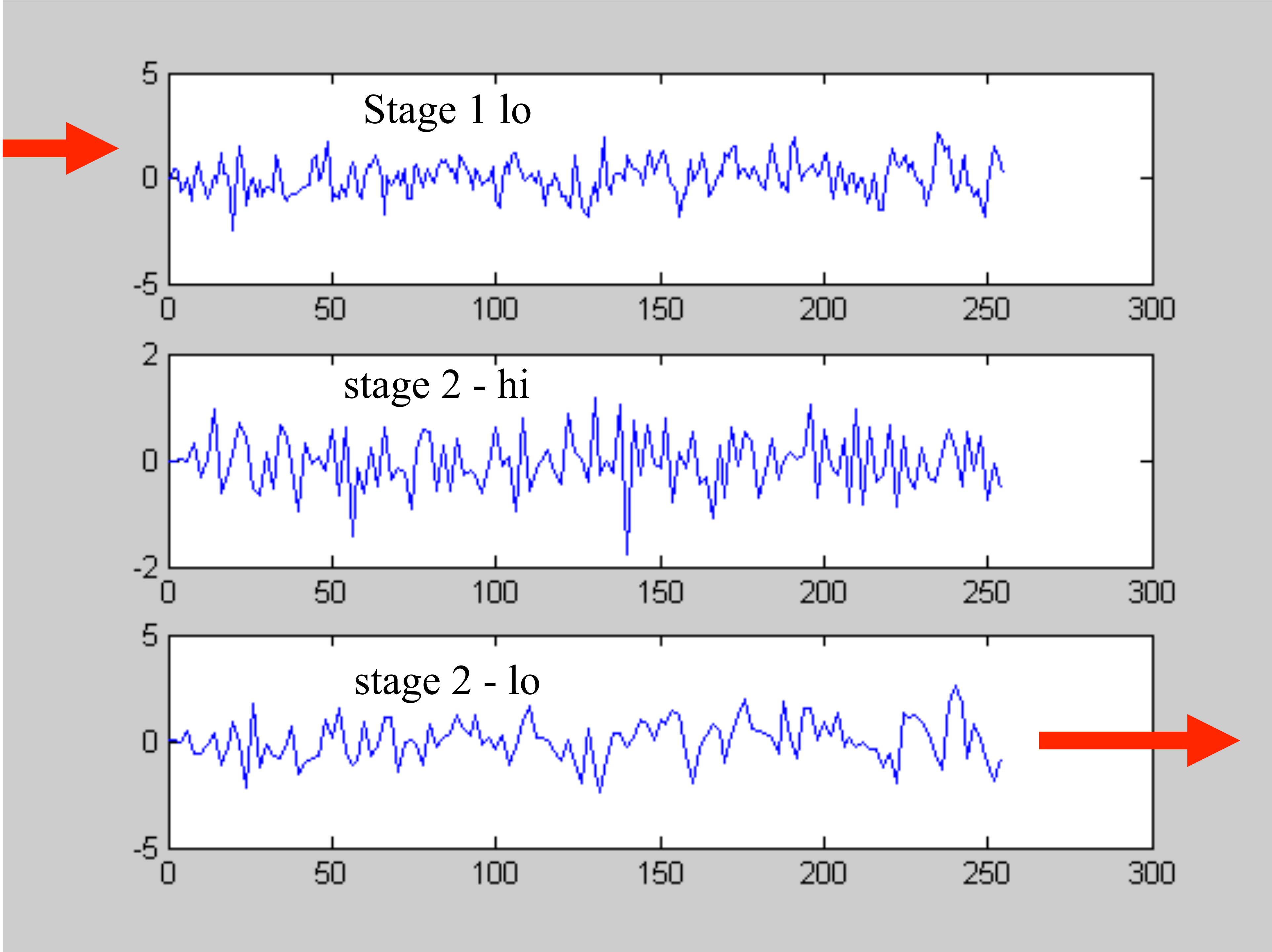


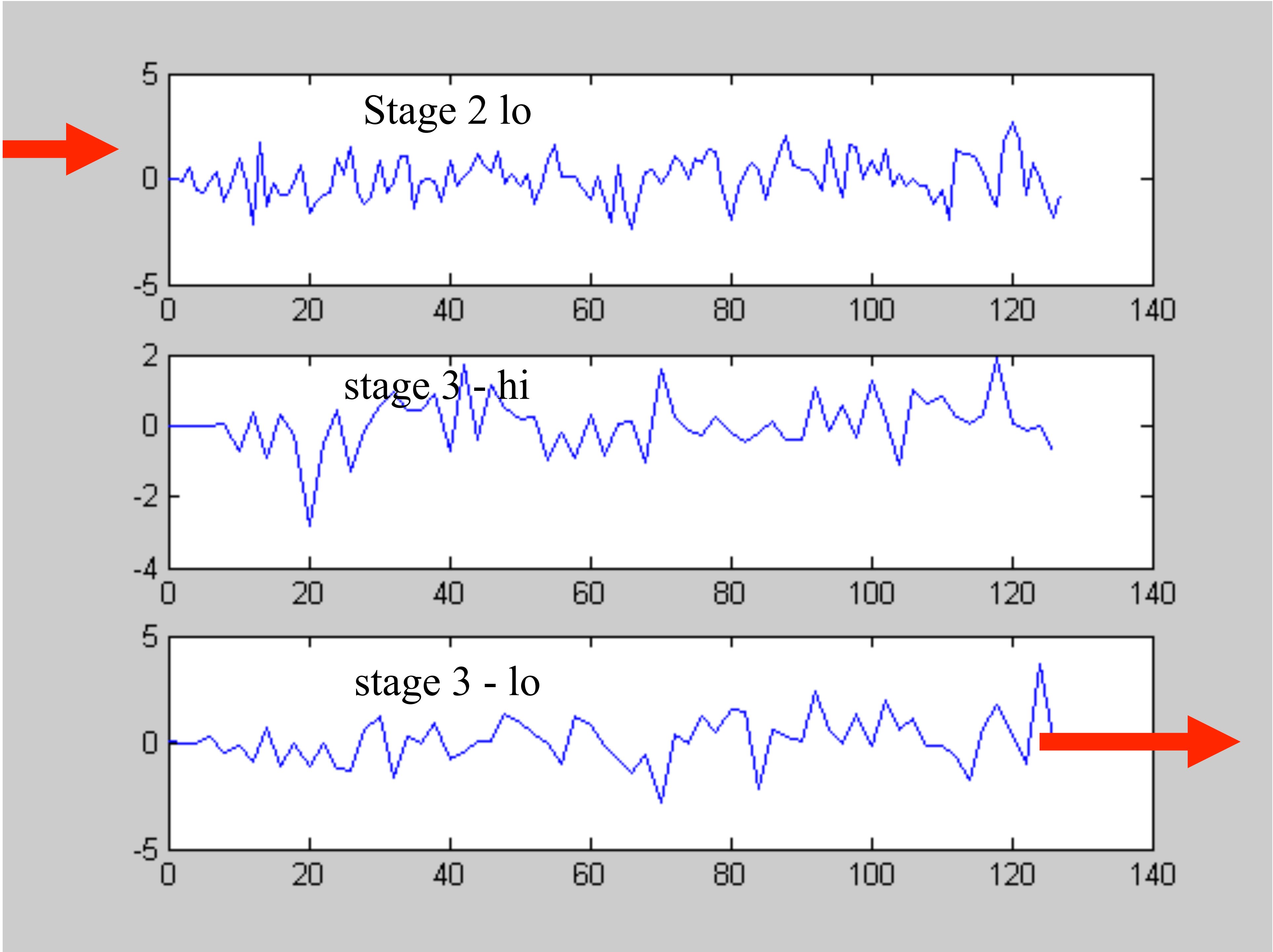
Spectrum of wavelet



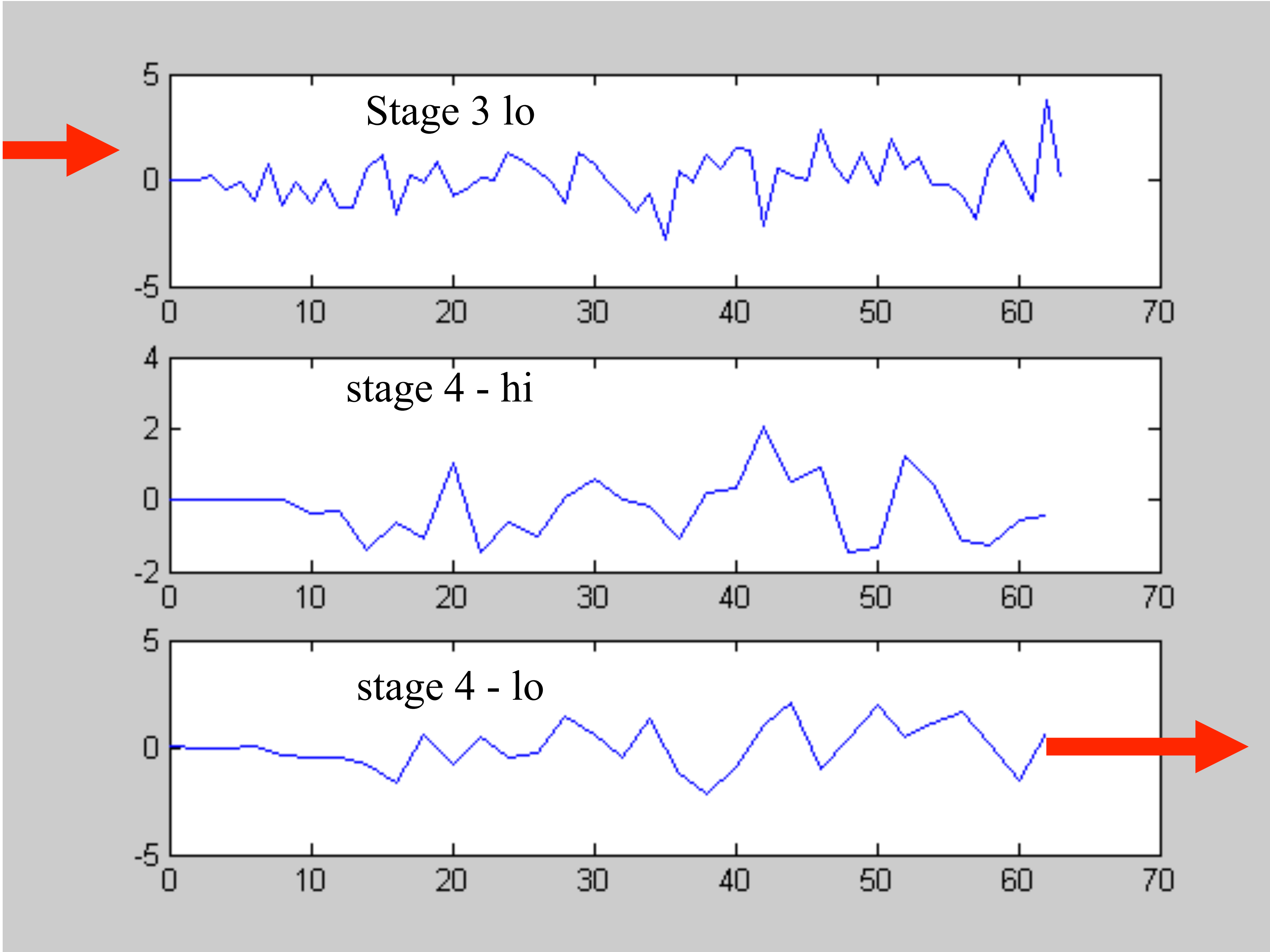


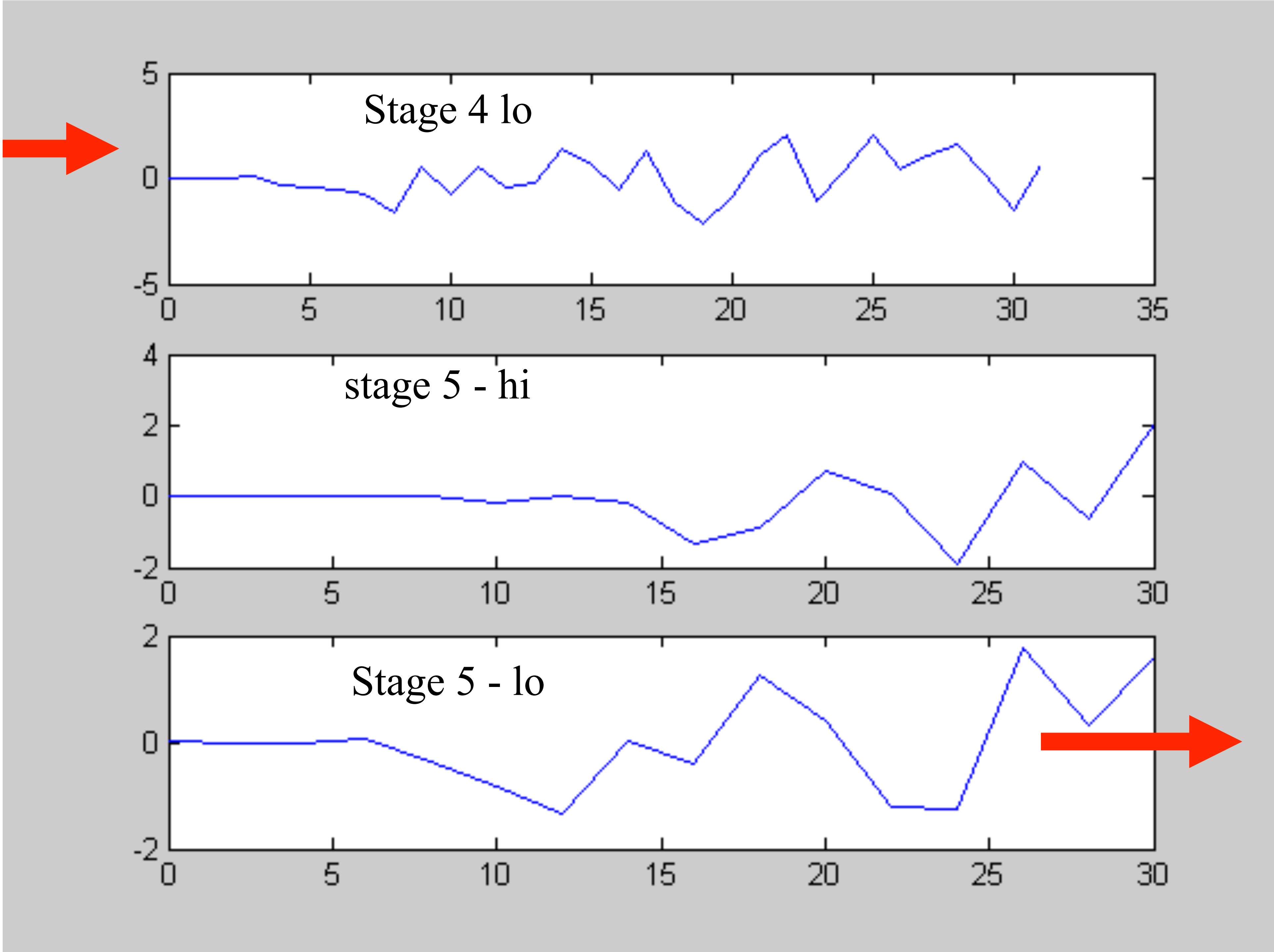
time-series
2
0
-2
0
100
200
300
400
500
600
stage 1 - hi
1
0
-1
0
100
200
300
400
500
600
stage 1 - lo
5
0
-5
0
100
200
300
400
500
600

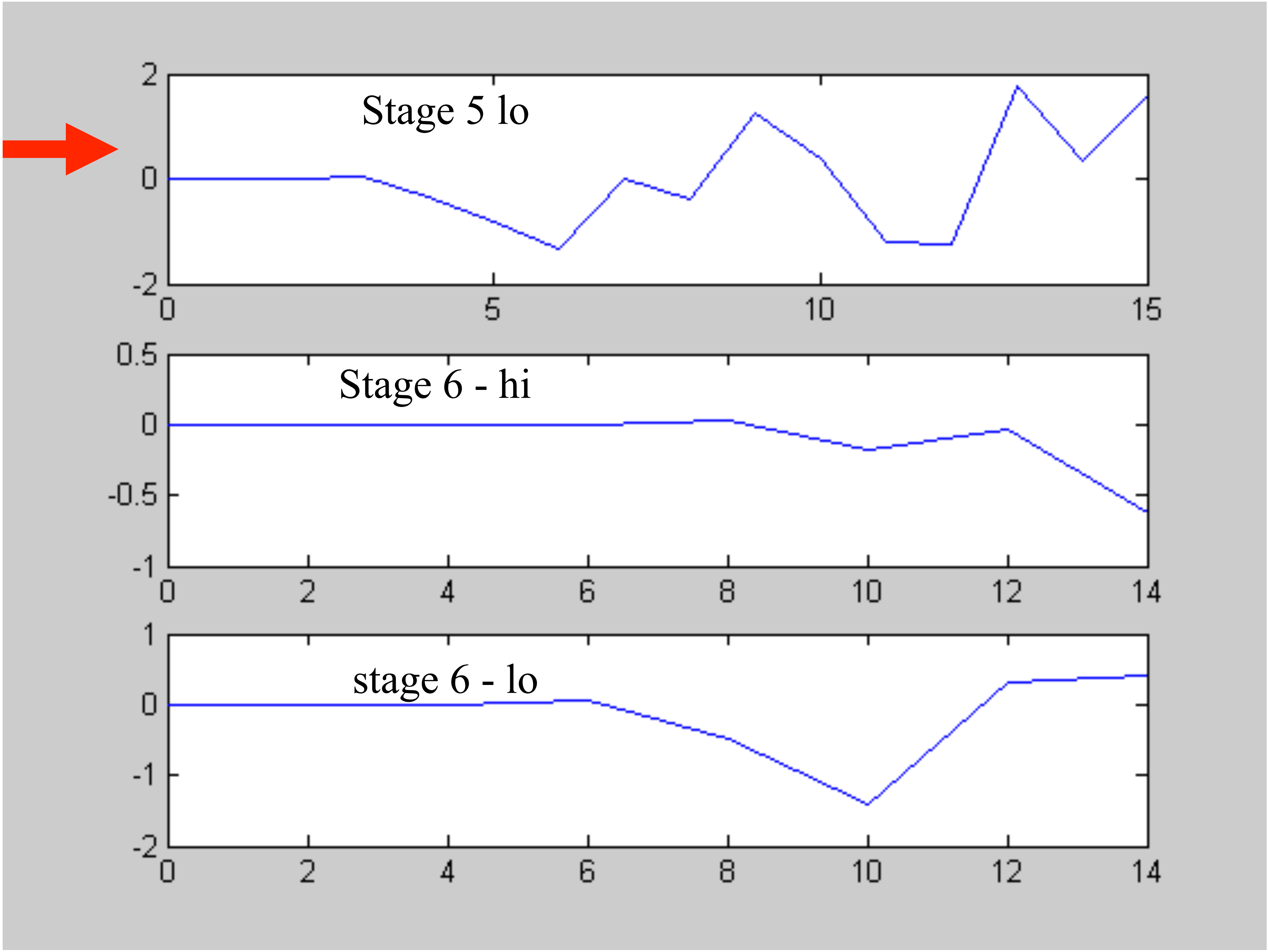

Stage 1 lo
stage 2 - hi
stage 2 - lo



Stage 2 lo
stage 3 - hi
stage 3 - lo
5
0
-5
2
-2
-4
0
20
40
60
80
100
120
140

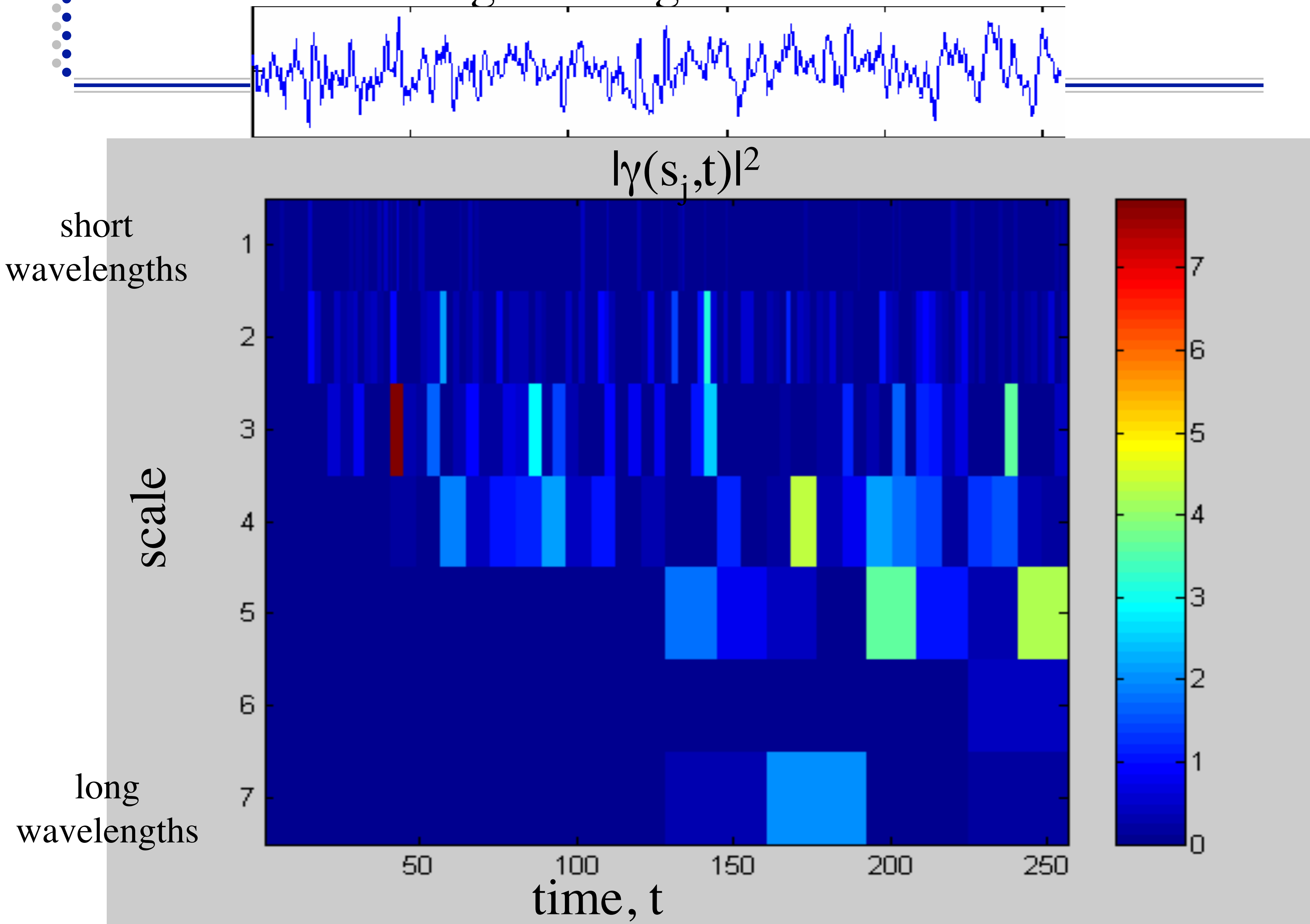

Stage 3 lo
stage 4 - hi
stage 4 - lo
5
0
-5
4
2
0
-2
0
10
20
30
40
50
60
70


Stage 4 lo
stage 5 - hi
Stage 5 - lo
5
0
-5
0
5
10
15
20
25
30
35
4
2
0
-2
0
5
10
15
20
25
30
2
0
-2
0
5
10
15
20
25
30



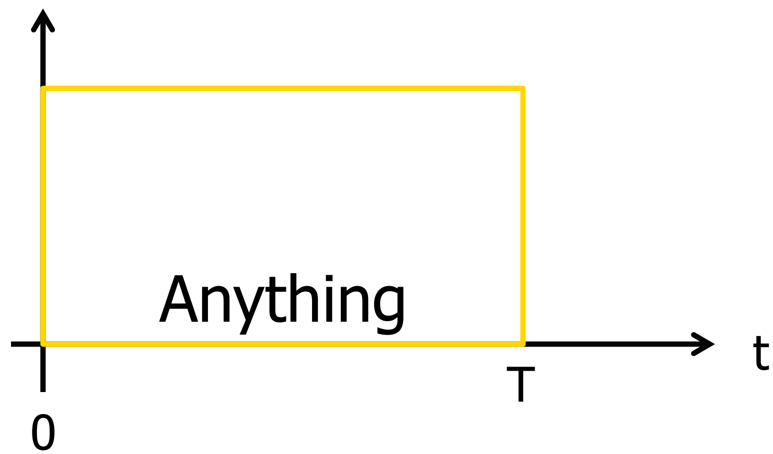
Stage 5 lo
Stage 6 - hi
stage 6 - lo

Putting it all together …



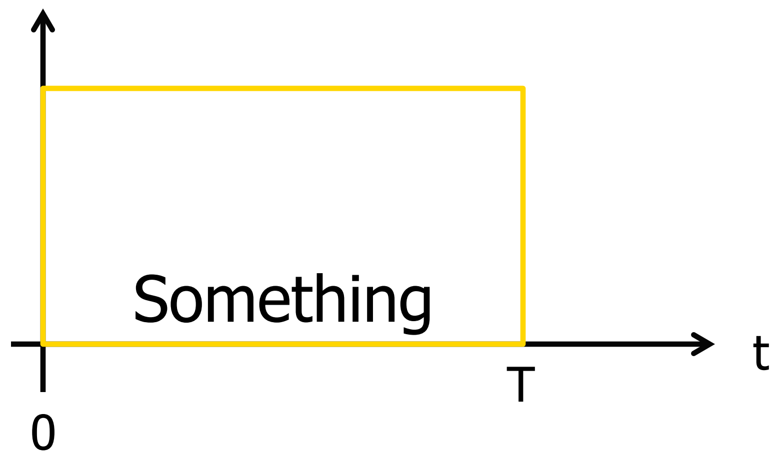
Compressive Sampling

Compressive Sampling



- What is the rate you need to sample at?
  - At least Nyquist

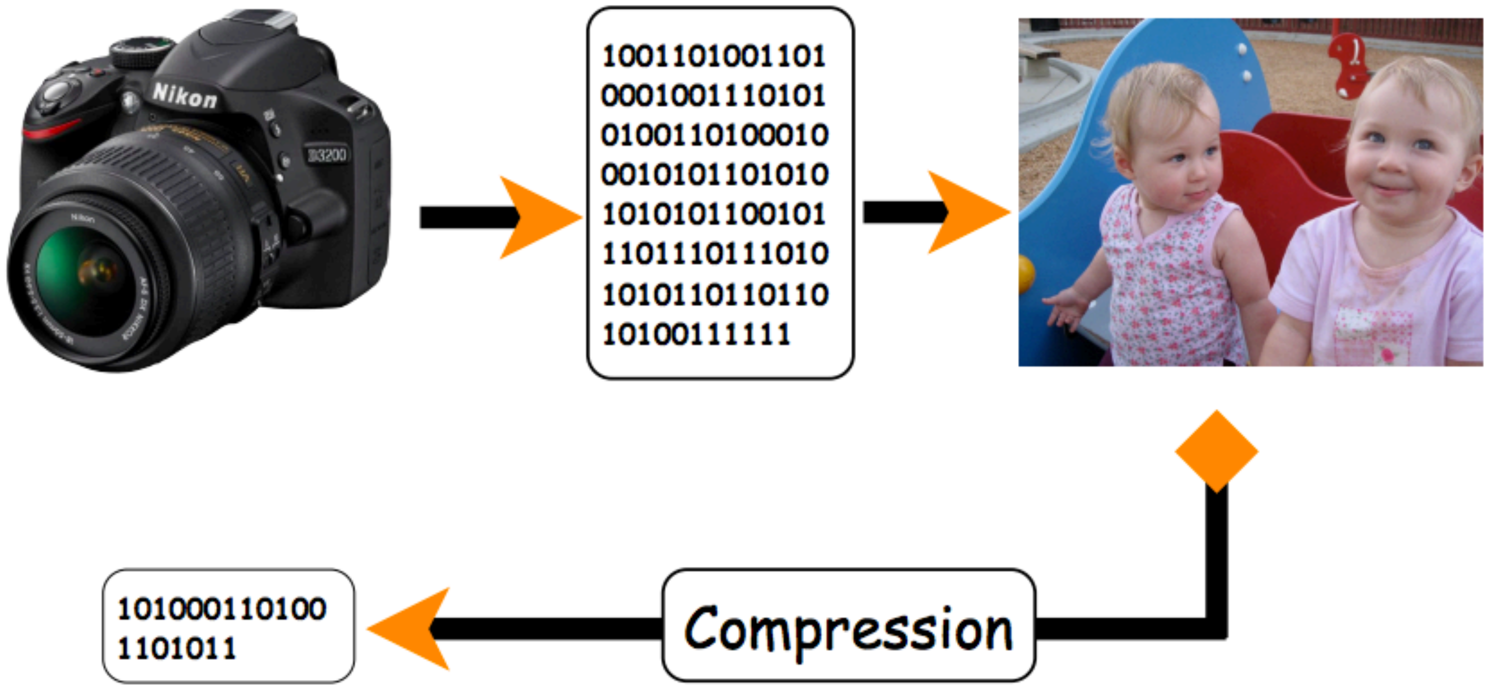
Compressive Sampling



- What is the rate you need to sample at?
  - Maybe less than Nyquist…

First: Compression

- Standard approach
  - First collect, then compress
    - Throw away unnecessary data



First: Compression

- Examples
  - Audio – 10x
    - Raw audio: 44.1kHz, 16bit, stereo = 1378 Kbit/sec
    - MP3: 44.1kHz, 16 bit, stereo = 128 Kbit/sec
  - Images – 22x
    - Raw image (RGB): 24bit/pixel
    - JPEG: 1280x960, normal = 1.09bit/pixel
  - Videos – 75x
    - Raw Video: (480x360)p/frame x 24b/p x 24frames/s + 44.1kHz x 16b x 2 = 98,578 Kbit/s
    - MPEG4: 1300 Kbit/s

First: Compression

- Almost all compression algorithm use transform coding
  - mp3: DCT
  - JPEG: DCT
  - JPEG2000: Wavelet
  - MPEG: DCT & time-difference

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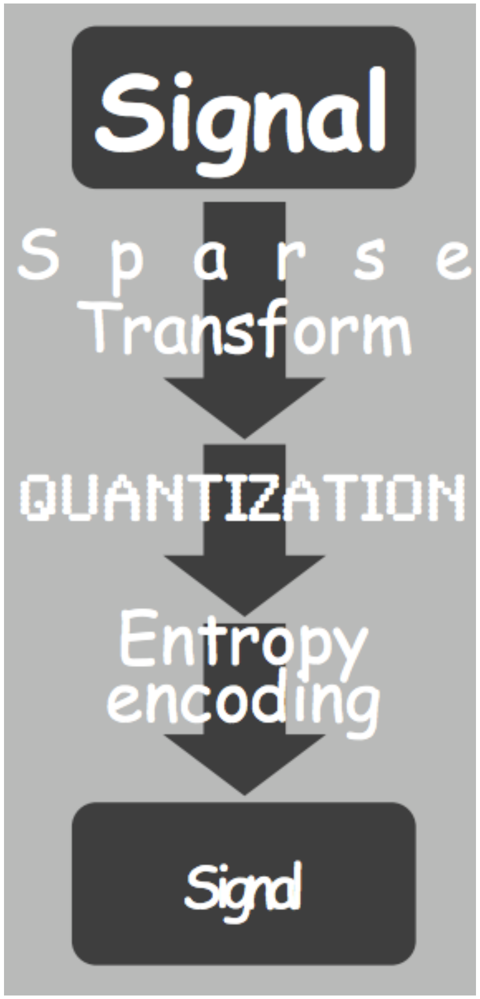
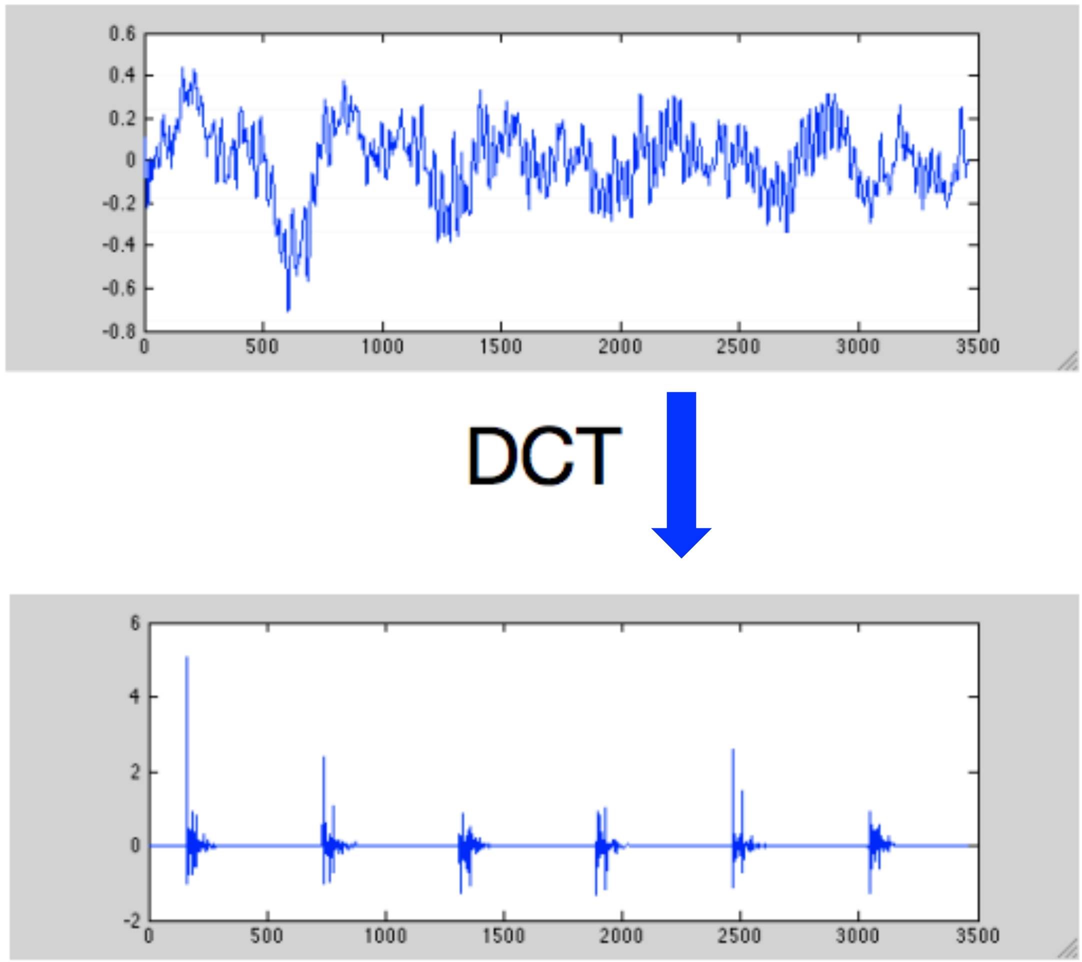
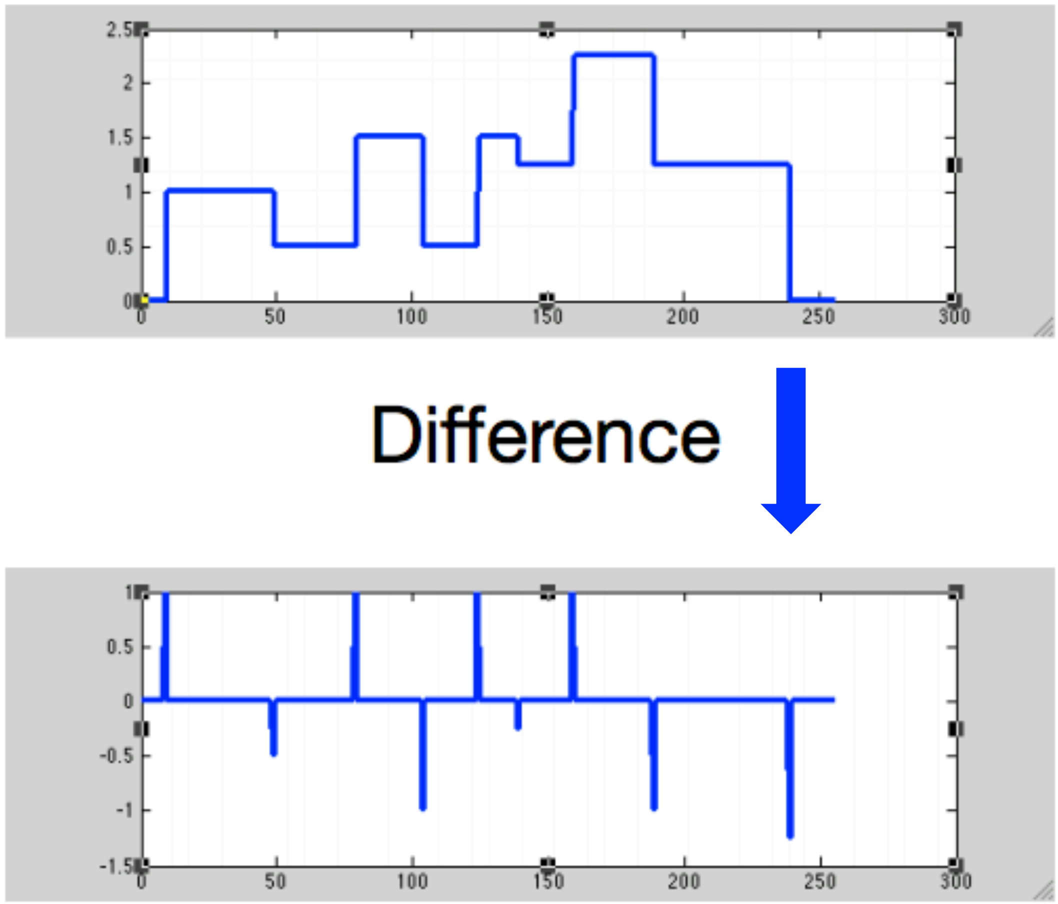


Sparse Transform

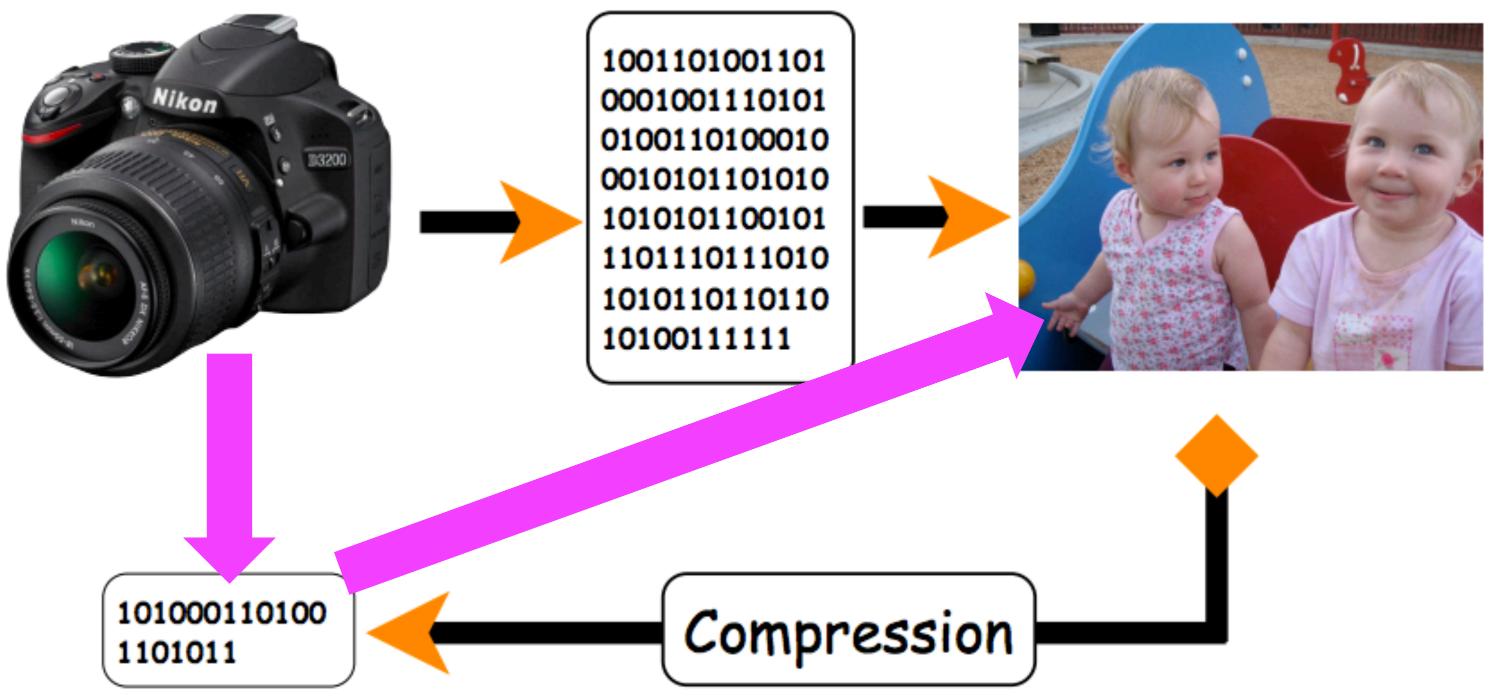


Sparse Transform



Compressive Sensing/Sampling

- Standard approach
  - First collect, then compress
    - Throw away unnecessary data



Compressive Sampling

- Sample at lower than the Nyquist rate and still accurately recover the signal, and in some cases exactly recover

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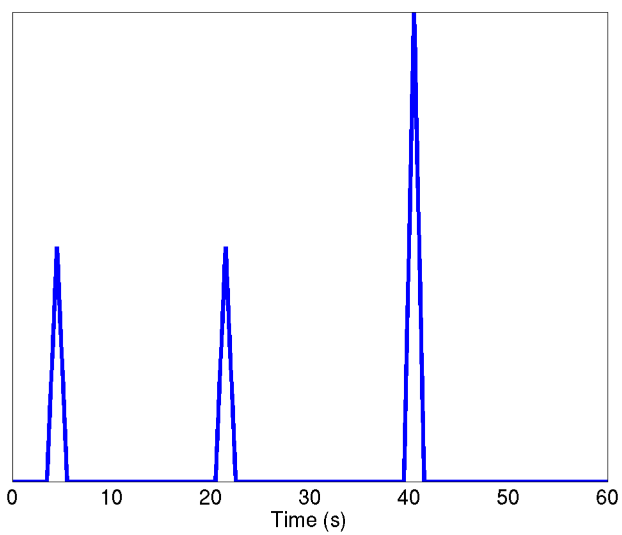
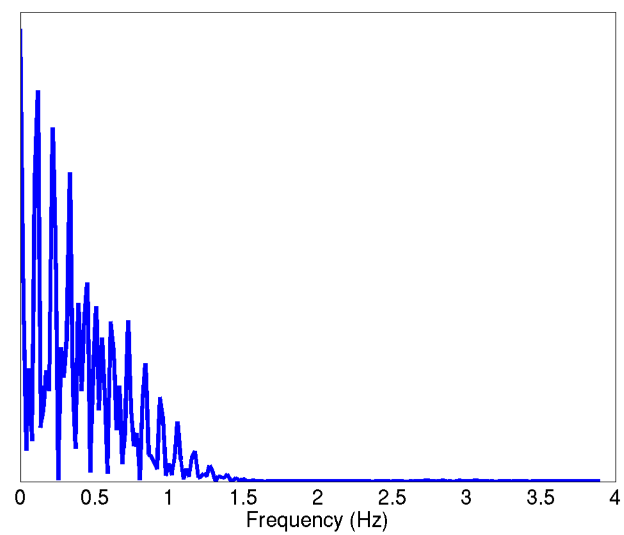
Sparse signal in time



Frequency spectrum



Compressive Sampling

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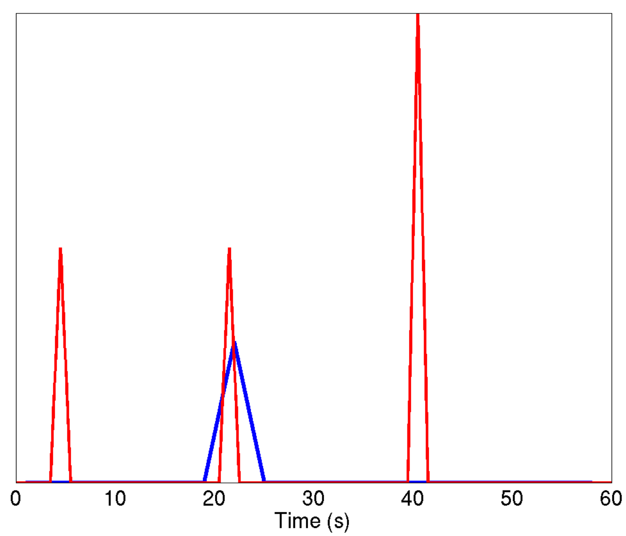
Undersampled in time



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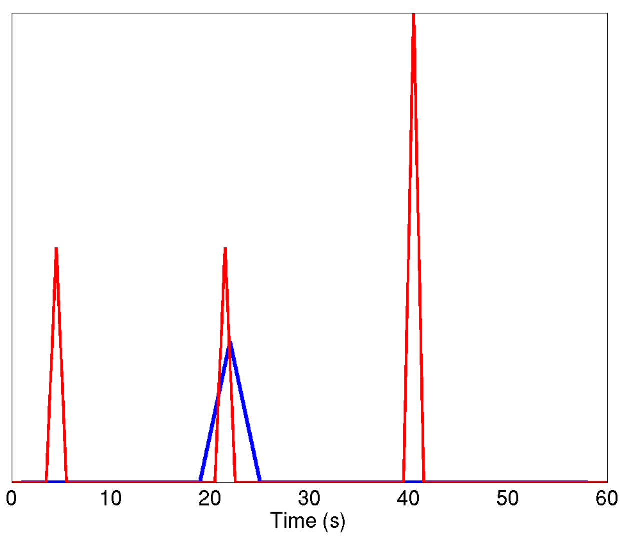

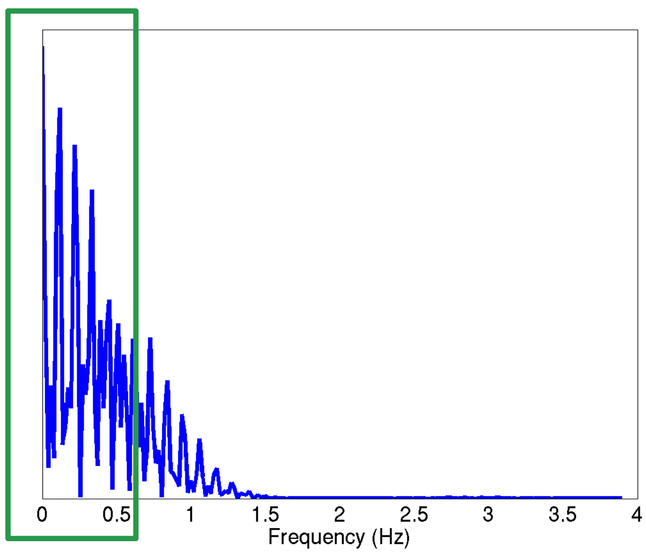

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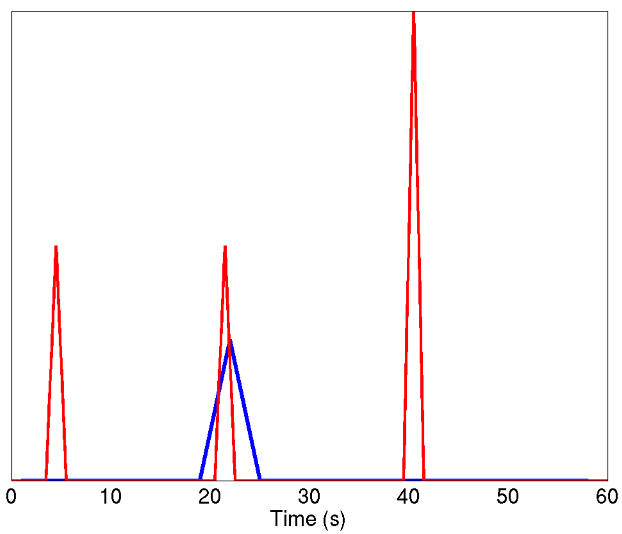
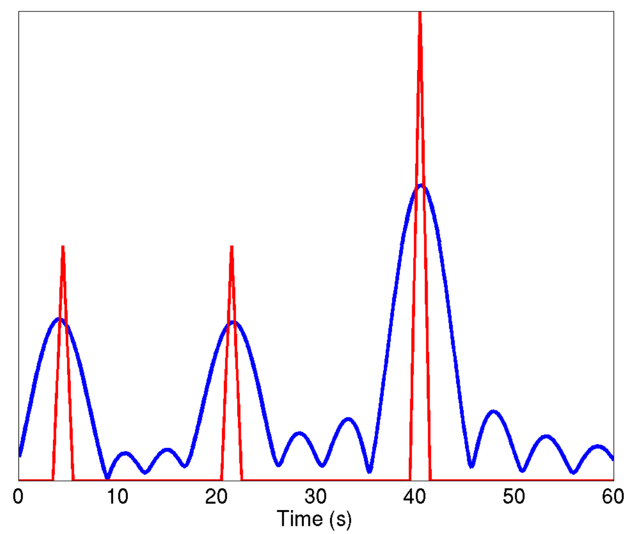
Undersampled in time



Undersampled in frequency
(reconstructed in time with IFFT)



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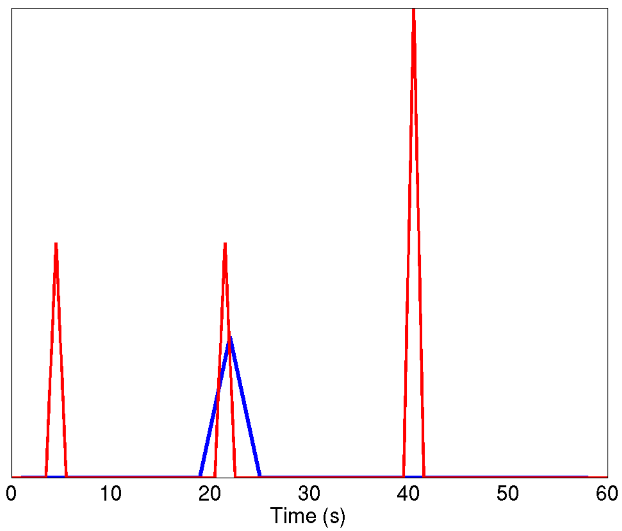

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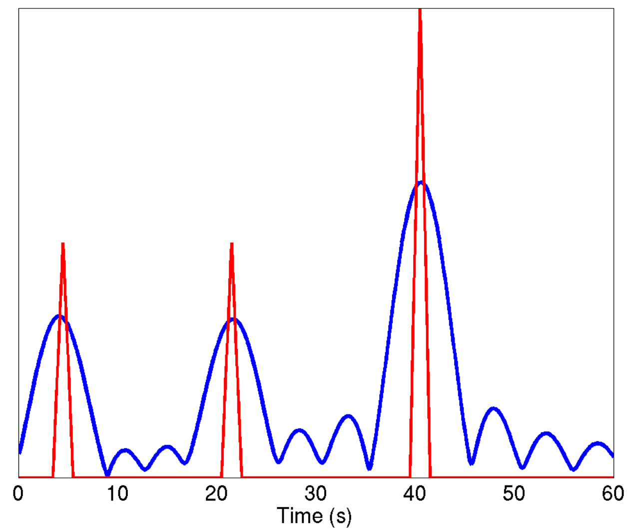
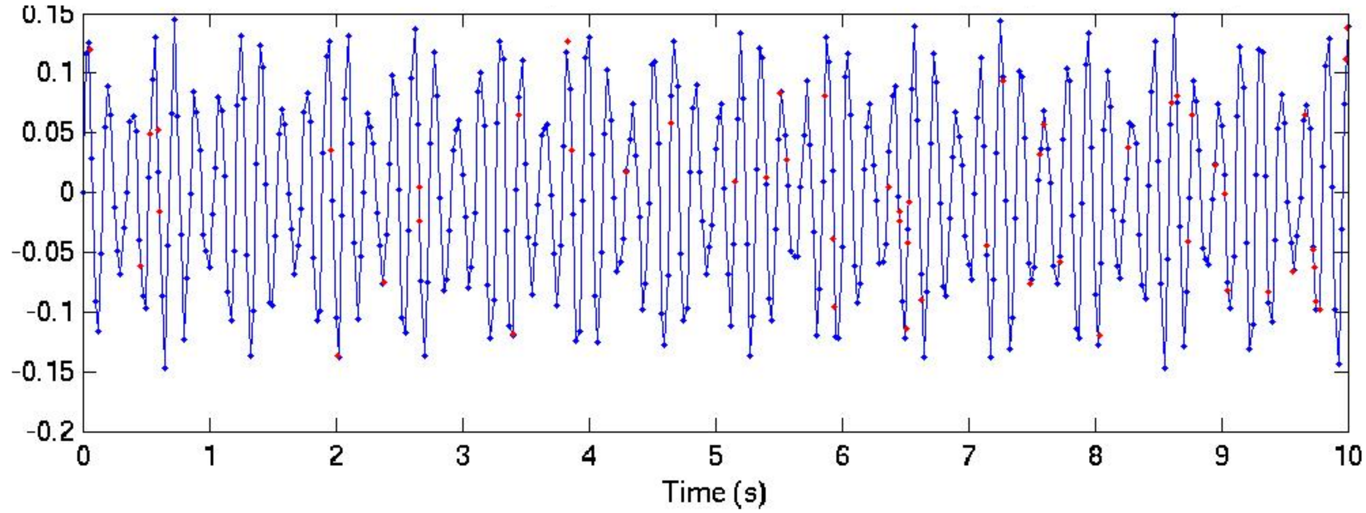
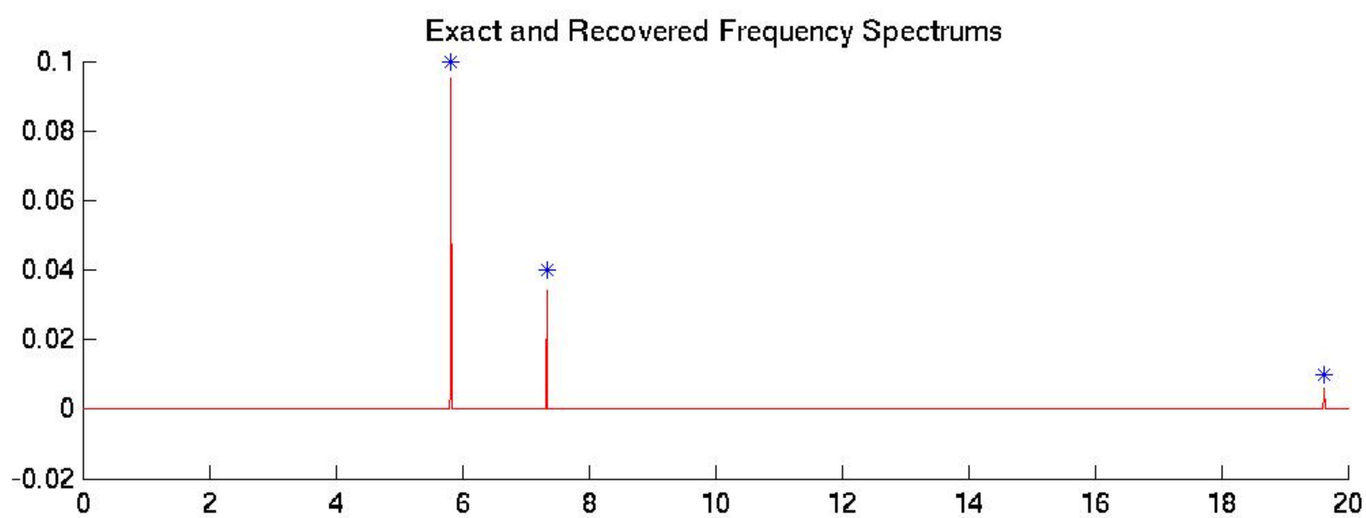


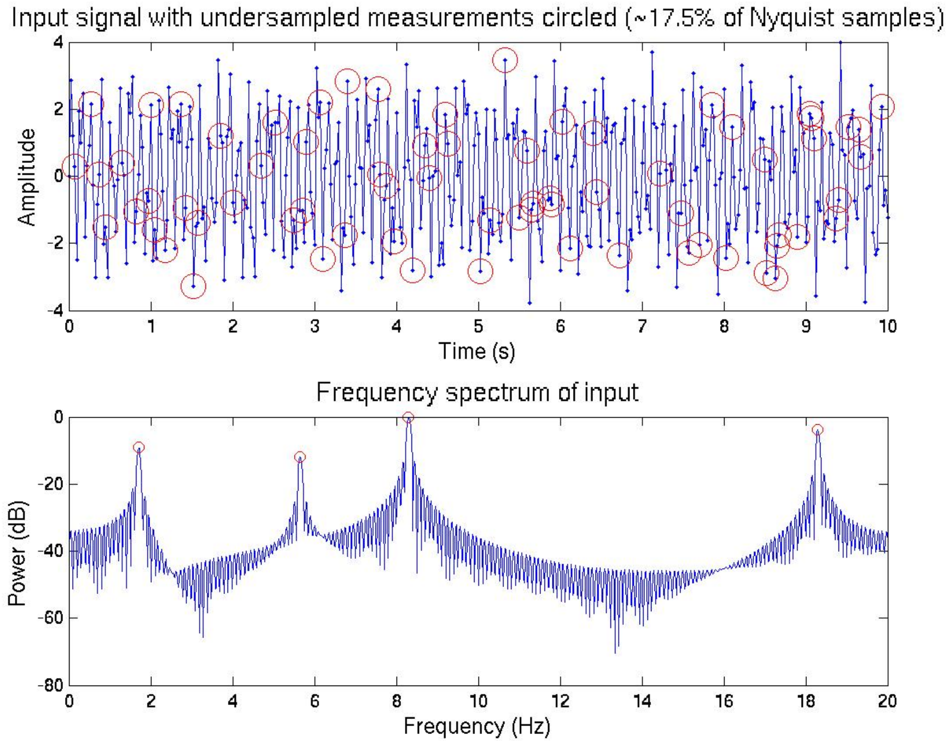
Requires sparsity and incoherent sampling

Compressive Sampling: Simple Example





Compressive Sampling

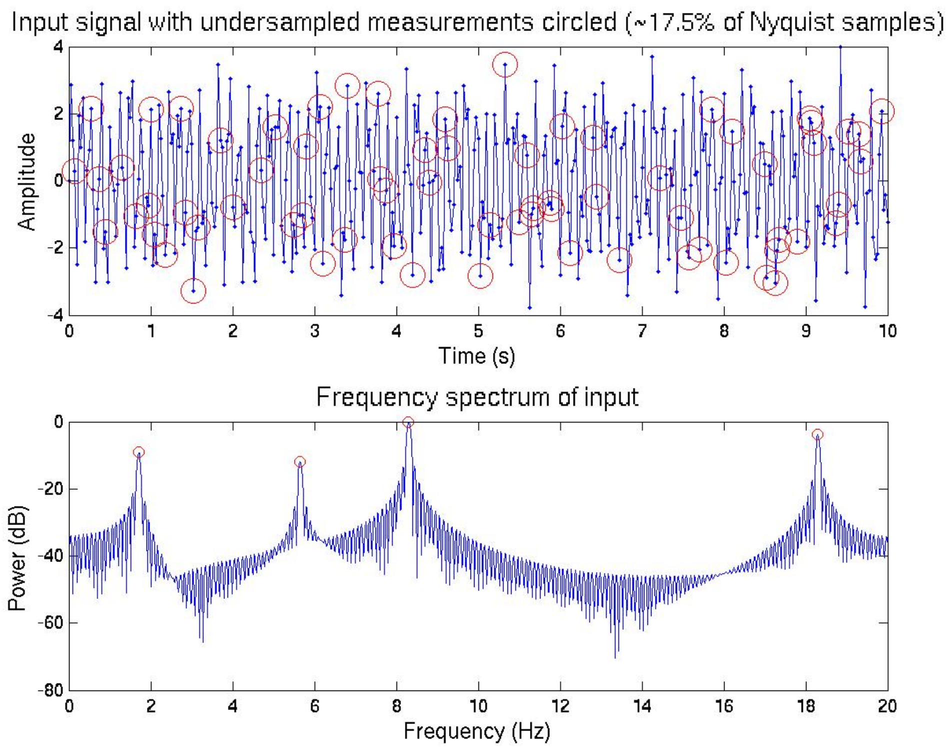


- Sense signal randomly M times
  - $M > C \cdot \mu^2(\Phi, \Psi) \cdot S \cdot \log N$
- Recover with linear program

$$\min \sum_{\omega} |\hat{g}(\omega)| \quad \text{subject to} \quad g(t_m) = f(t_m), \quad m = 1, \ldots, M$$

Compressive Sampling

$$\hat{f}(\omega) = \sum_{i=1}^{K} \alpha_i \delta(\omega_i - \omega) \overset{\mathcal{F}}{\Leftrightarrow} f(t) = \sum_{i=1}^{K} \alpha_i e^{i\omega_i t}$$



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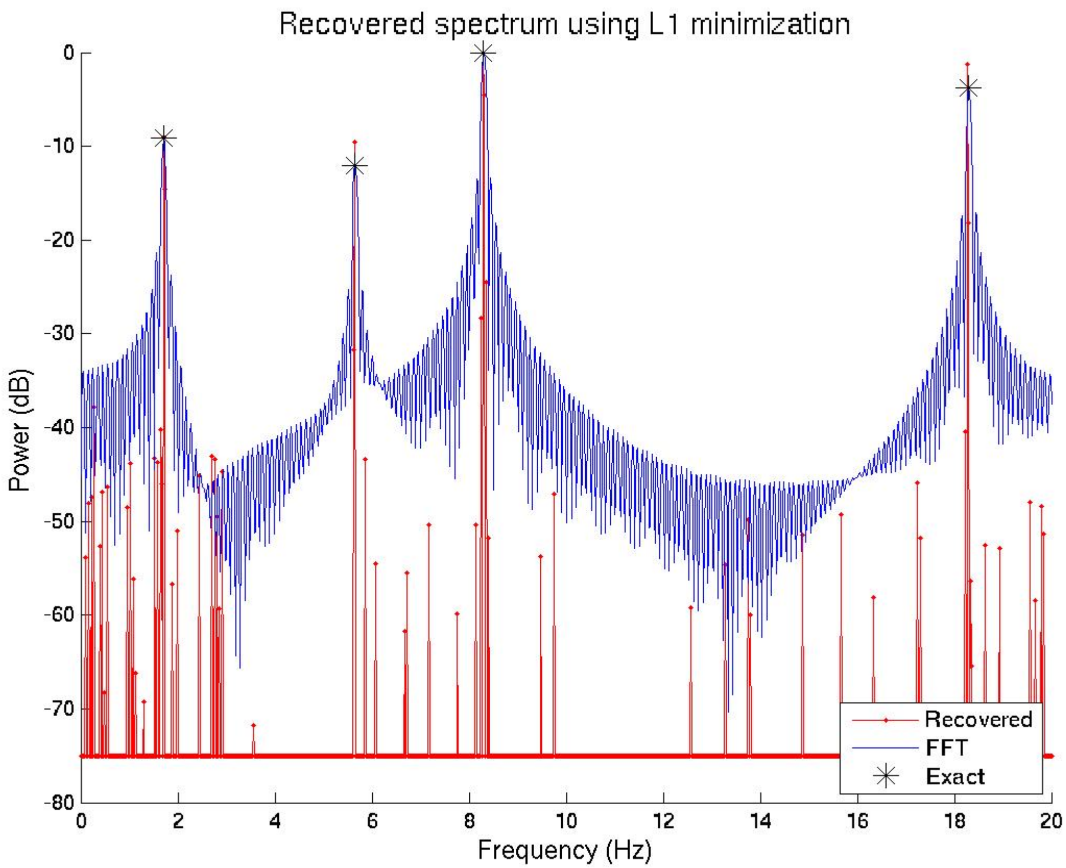
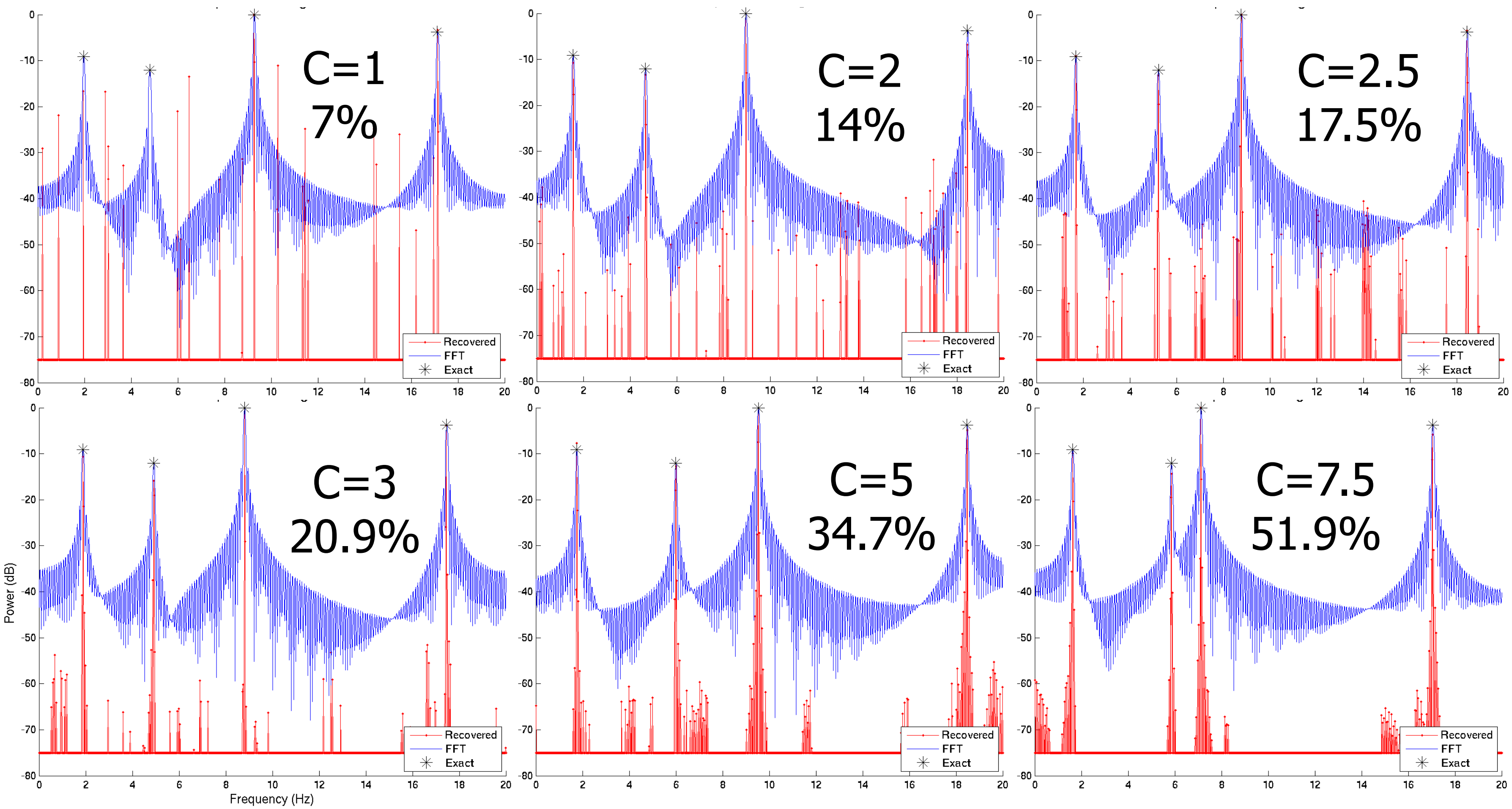
Example: Sum of Sinusoids

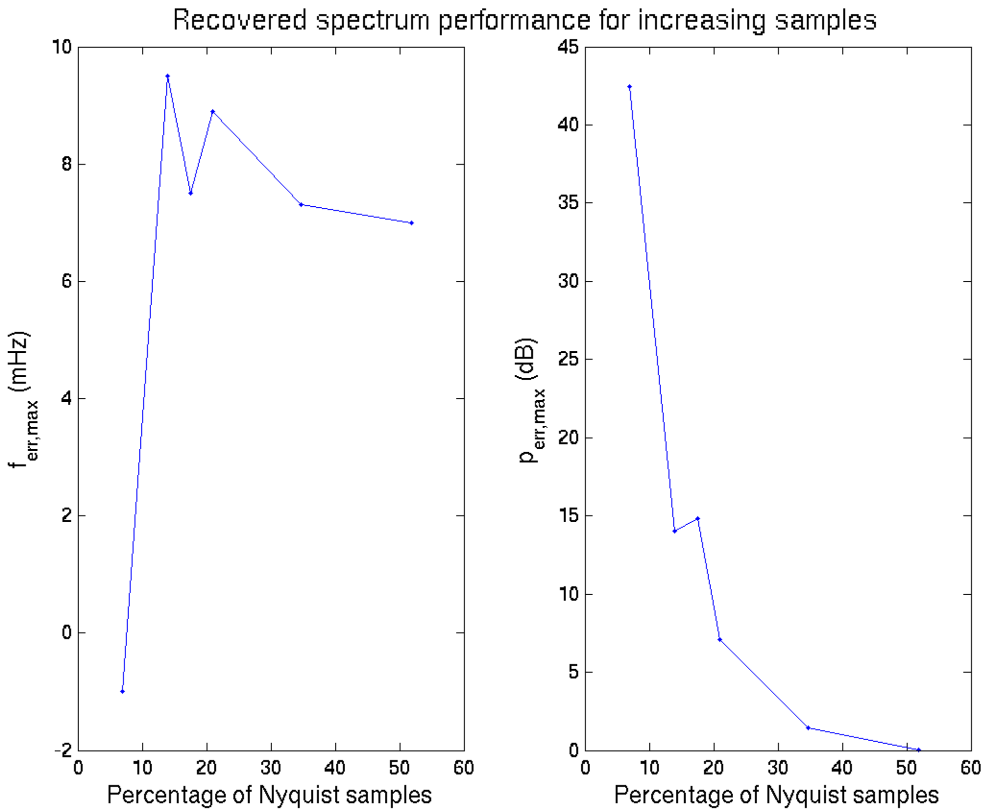


- Two relevant "knobs"
  - percentage of Nyquist samples as altered by adjusting optimization factor, *C*
  - input signal duration, *T*
    - Data block size

Example: Increasing C

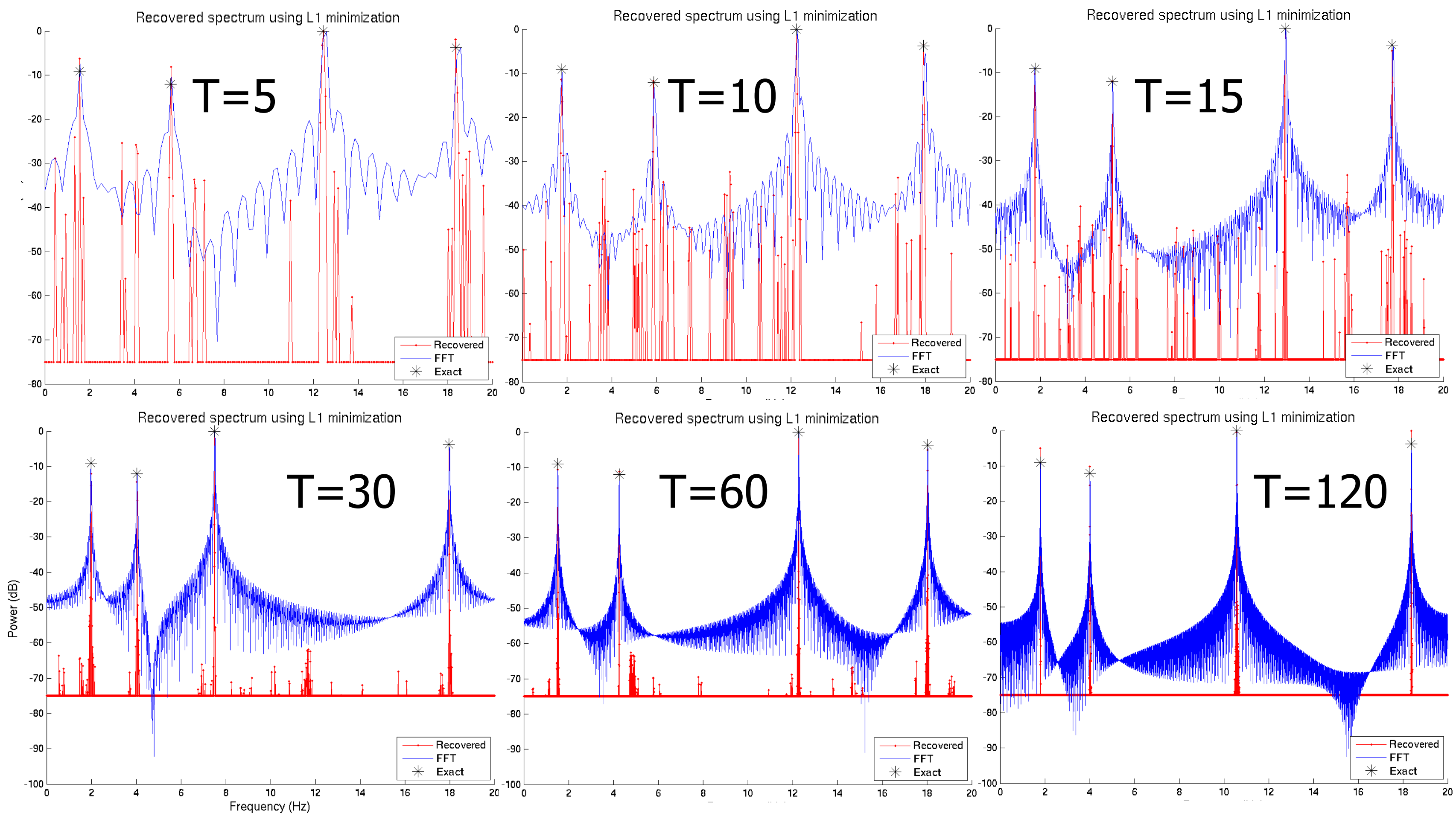


Example: Increasing C

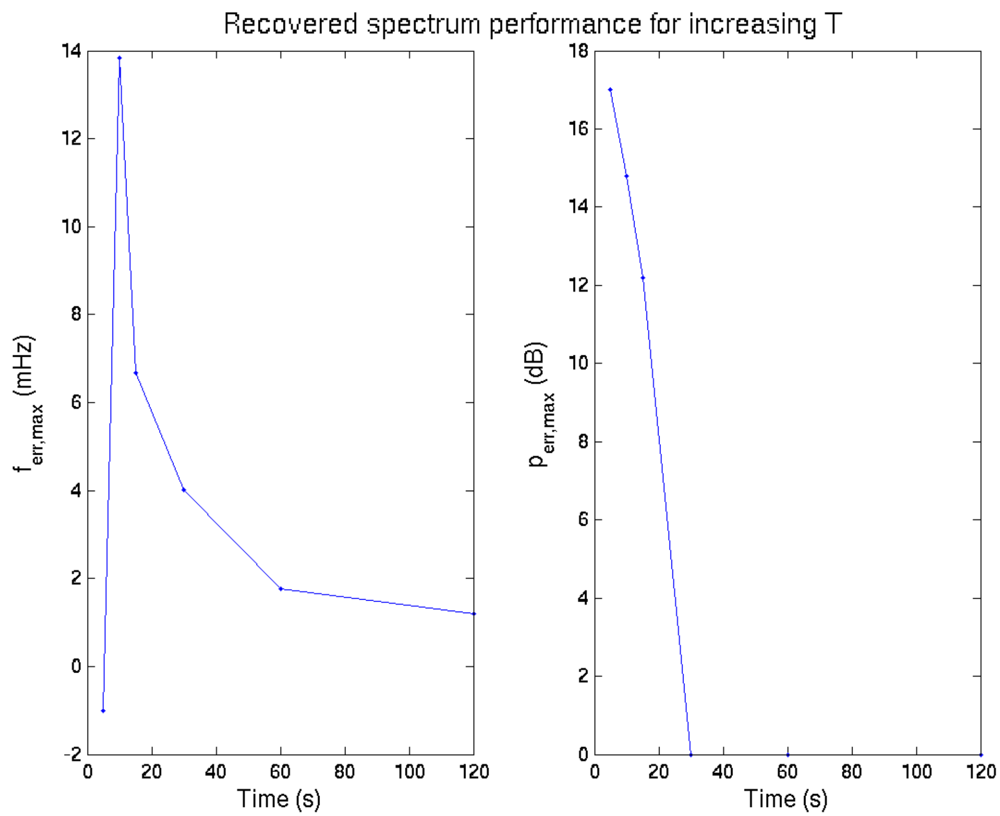


- **$f_{err,max}$ within 10 mHz**
- **$p_{err,max}$ decreasing**

Example: Increasing T

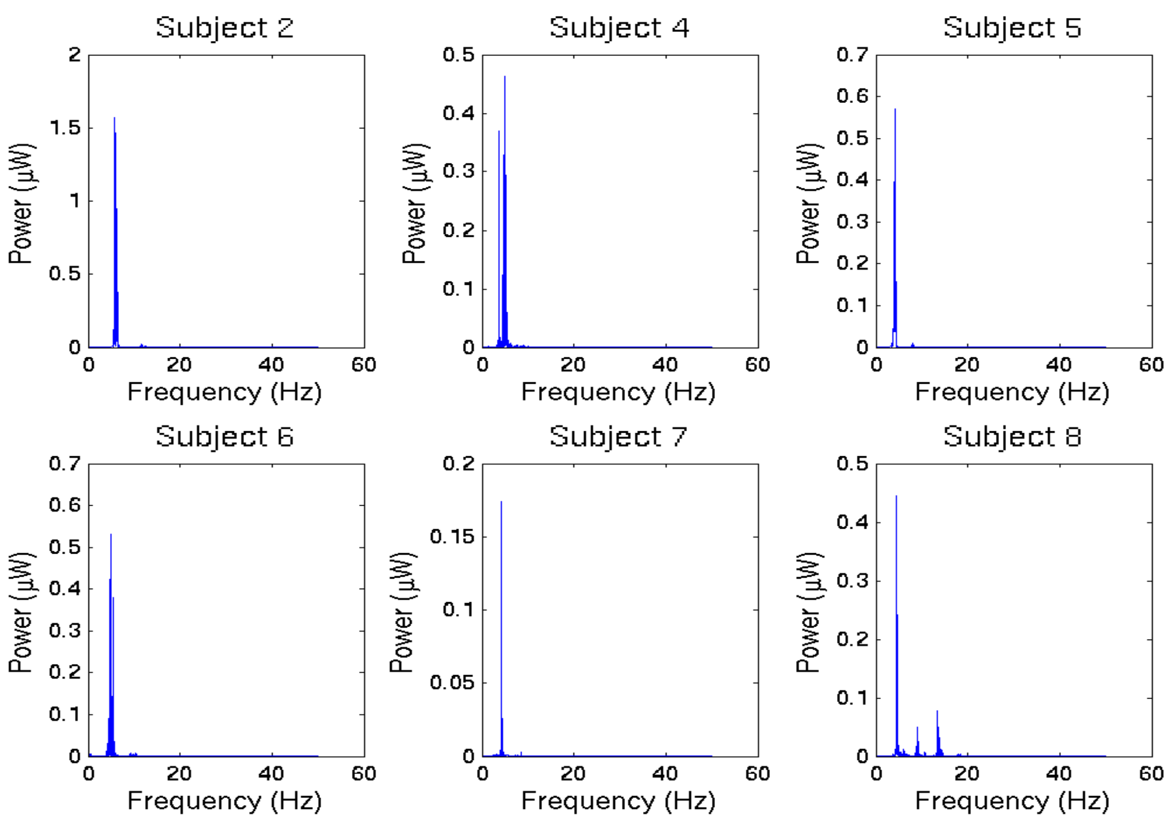


Example: Increasing T



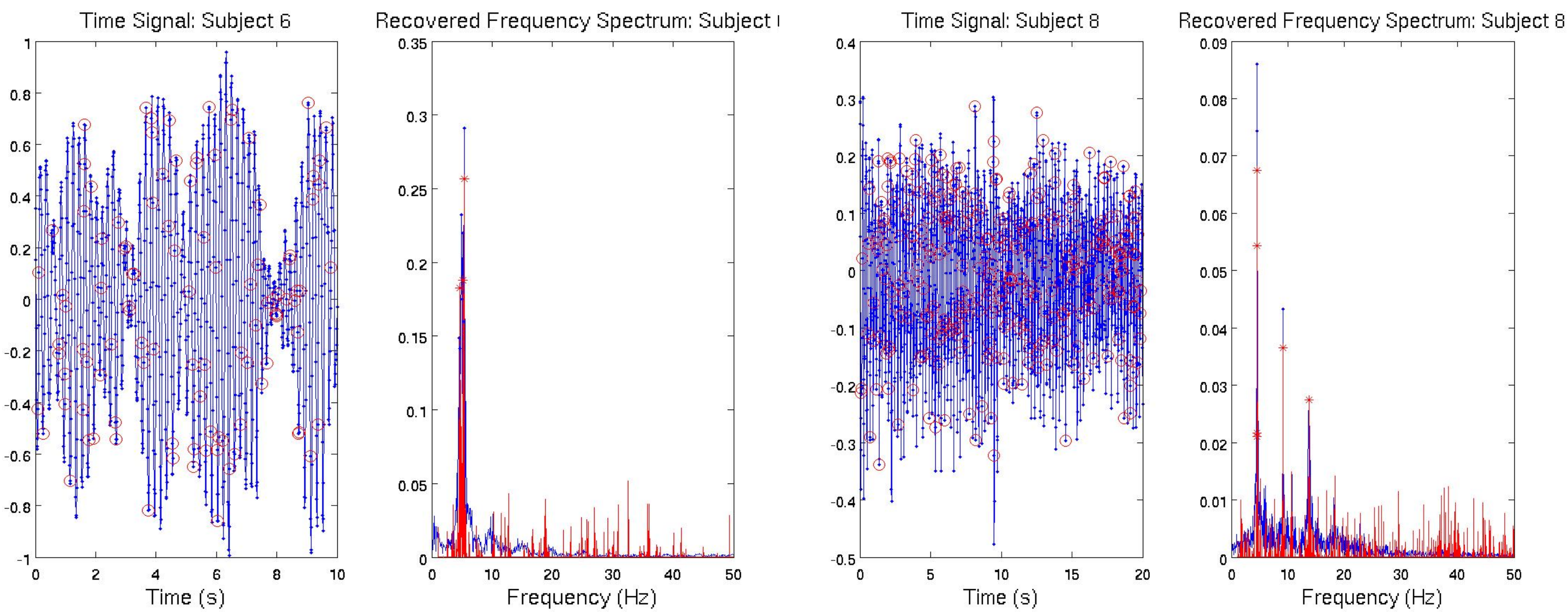
- $f_{err,max}$ **decreasing**
- $p_{err,max}$ **decreasing**

Biometric Example: Parkinson's Tremors

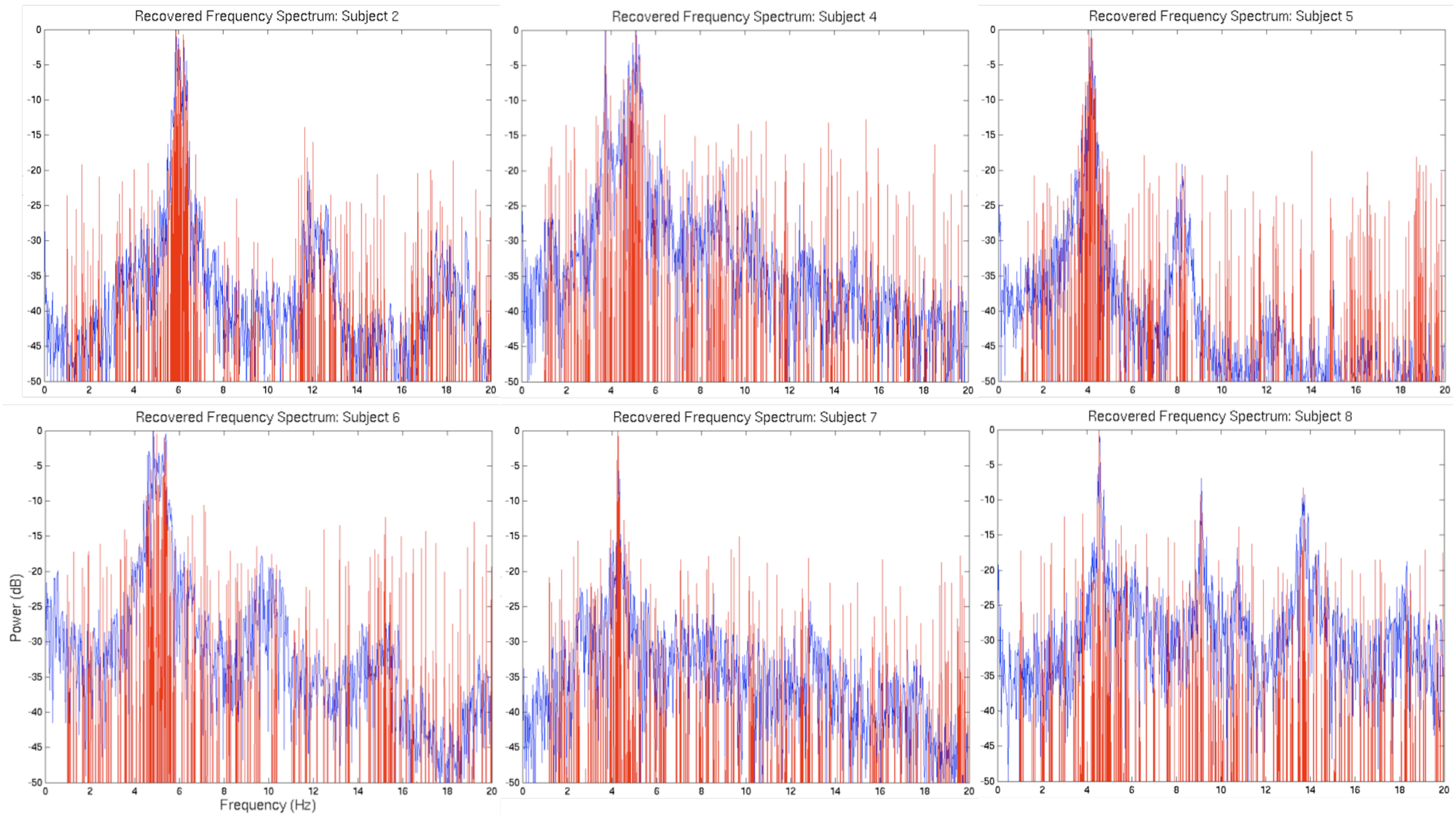


- 6 Subjects of real tremor data
  - collected using low intensity velocity-transducing laser recording aimed at reflective tape attached to the subjects' finger recording the finger velocity
  - All show Parkinson's tremor in the 4-6 Hz range.
  - Subject 8 shows activity at two higher frequencies
  - Subject 4 appears to have two tremors very close to each other in frequency

Compressive Sampling: Real Data

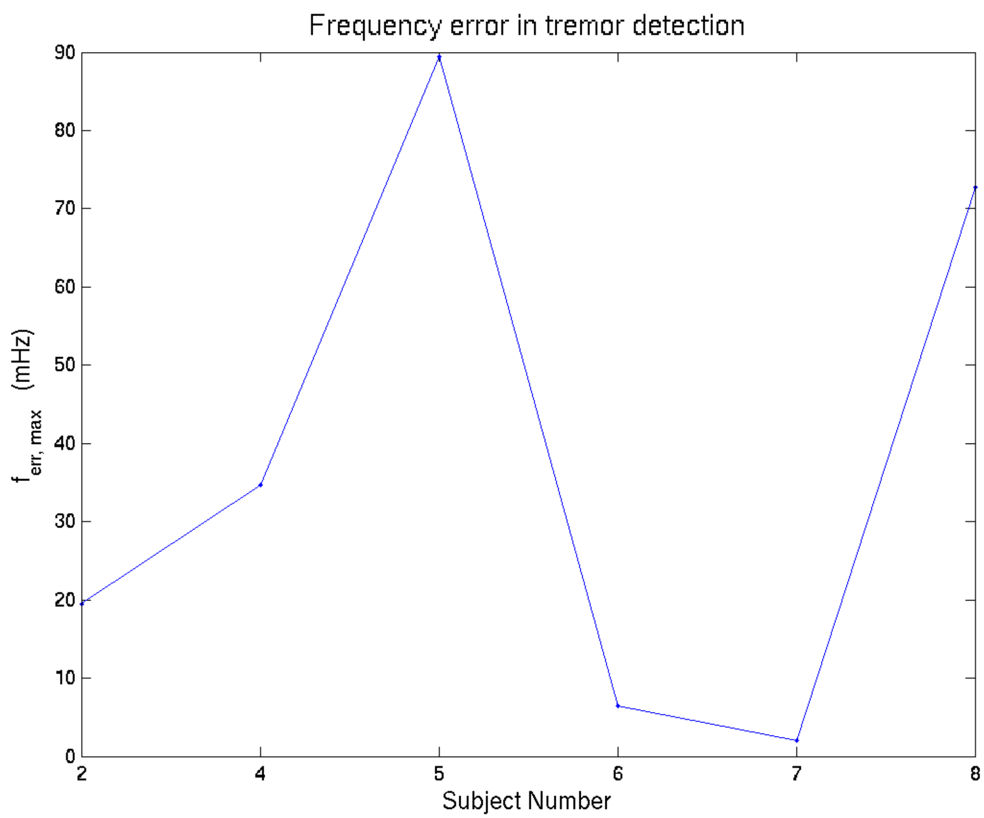


Biometric Example: Parkinson's Tremors



- **C=10.5, T=30**
  - 20% Nyquist required samples

Biometric Example: Parkinson's Tremors



- **Tremors detected within 100 mHz**
- **randomly sample 20% of the Nyquist required samples**

Requires post processing to randomly sample!

Implementing Compressive Sampling

- Devised a way to randomly sample 20% of the Nyquist required samples and still detect the tremor frequencies within 100mHz
  - Requires post processing to randomly sample!

- Implement hardware on chip to "choose" samples in real time
  - Only write to memory the "chosen" samples
    - Design random-like sequence generator
  - Only convert the "chosen" samples
    - Design low energy ADC

Big Ideas

- Wavelet transform
  - Capture temporal data with fewer coefficients than STFT
- Compressive Sampling
  - Integrated sensing/sampling, compression and processing
  - Based on sparsity and incoherency

Admin

- Tania extra office hours
  - Monday 1-3pm
- Tuesday lecture
  - Review
  - Final exam details

- Project
  - Due 4/24