# ESE 531: Digital Signal Processing

Lec 25: April 24, 2018

Review

# Course Content

- Introduction
- Discrete Time Signals & Systems
- Discrete Time Fourier Transform
- Z-Transform
- Inverse Z-Transform
- Sampling of Continuous Time Signals
- Frequency Domain of Discrete Time Series
- Downsampling/Upsampling
- Data Converters, Sigma Delta Modulation
- Oversampling, Noise Shaping
- Frequency Response of LTI Systems
- Basic Structures for IIR and FIR Systems
- Design of IIR and FIR Filters
- Filter Banks
- Adaptive Filters
- Computation of the Discrete Fourier Transform
- Fast Fourier Transform
- Spectral Analysis
- Wavelet Transform
- Compressive Sampling

# Digital Signal Processing

- Represent signals by a sequence of numbers
  - Sampling and quantization (or analog-to-digital conversion)
- Perform processing on these numbers with a digital processor
  - Digital signal processing
- Reconstruct analog signal from processed numbers
  - Reconstruction or digital-to-analog conversion


- Analog input → analog output
  - Eg. Digital recording music
- Analog input → digital output
  - Eg. Touch tone phone dialing, speech to text
- Digital input → analog output
  - Eg. Text to speech
- Digital input → digital output
  - Eg. Compression of a file on computer

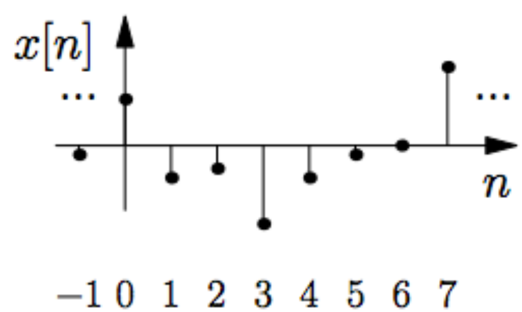
# Discrete Time Signals and Systems

# Signals are Functions

DEFINITION

A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of $n$ produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
  - Independent variable is an **integer**: $n \in \mathbb{Z}$ (will refer to as time)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$

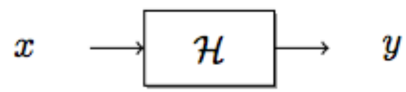


# Discrete Time Systems

**DEFINITION**

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal $x$ into a discrete-time output signal $y$

$$y = \mathcal{H}\{x\}$$




- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

# System Properties

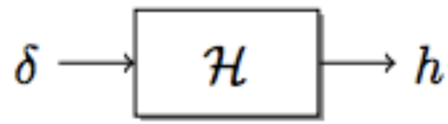
- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max )

# LTI Systems

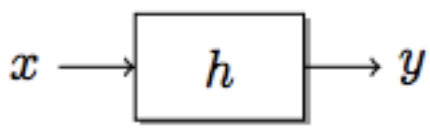
DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response




- Then the output for an arbitrary input is a sum of weighted, delay impulse responses




$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

# Discrete Time Fourier Transform

# DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

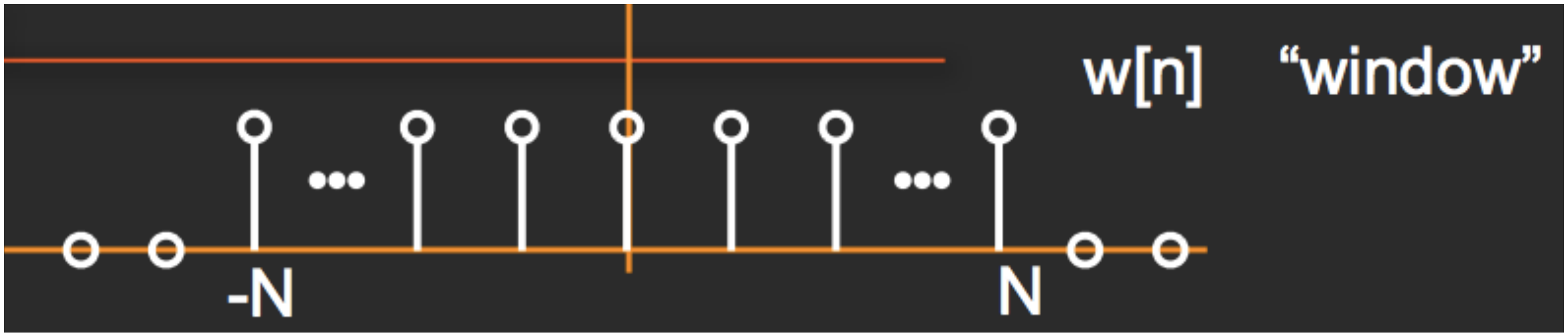
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$$

$$x[n] = \int_{-0.5}^{0.5} X(f)e^{j2\pi fn}df$$

# Example: Window DTFT



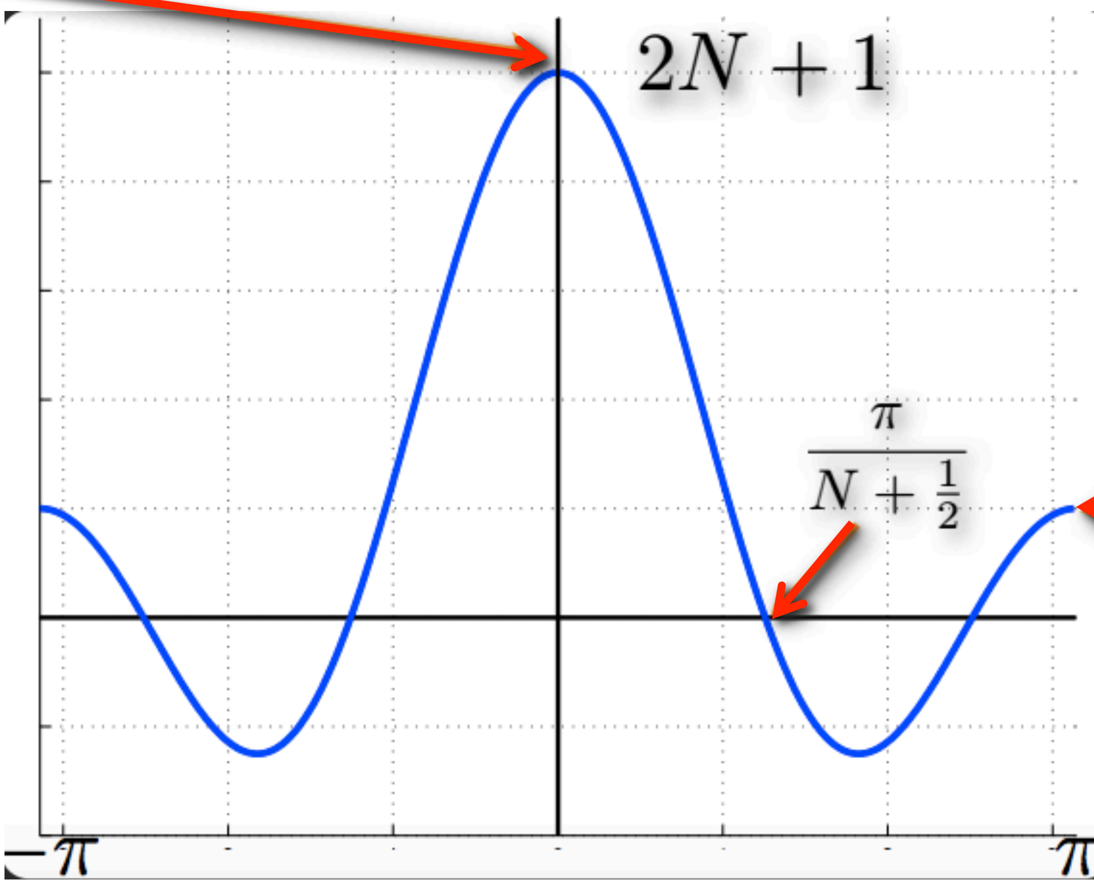

$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k]e^{-j\omega k}$$

$$= \sum_{k=-N}^{N} e^{-j\omega k}$$

# Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Also, Σ x[n]

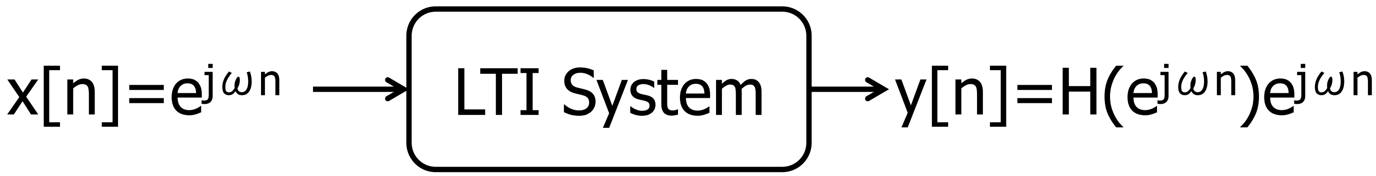



=1 why?

Plot for N=2

# LTI System Frequency Response

- Fourier Transform of impulse response

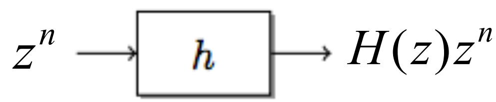



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

# z-Transform

- The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- Very useful for designing and analyzing signal processing systems
- Properties are very similar to the DTFT with a few caveats

# Complex Exponentials as Eigenfunctions




- Prove that complex exponentials are the eigenvectors of LTI systems simply by computing the convolution with input $z^n$

$$
\begin{aligned}
z^n * h[n] &= \sum_{m=-\infty}^{\infty} z^{n-m}\, h[m] = \sum_{m=-\infty}^{\infty} z^n\, z^{-m}\, h[m] \\
&= \left( \sum_{m=-\infty}^{\infty} h[m]\, z^{-m} \right) z^n \\
&= H(z) z^n
\end{aligned}
$$

$$
\boxed{\sum_{n=-\infty}^{\infty} h[n]\, z^{-n} = H(z)}
$$

# Region of Convergence (ROC)

DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its $z$-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| \;<\; \infty$$

# Inverse z-Transform

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion
  - Power series expansion

# Partial Fraction Expansion

- Let

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- M zeros and N poles at nonzero locations

# Partial Fraction Expansion

- If M<N and the poles are 1st order

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- where

$$A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k}$$

# Example: $2^{nd}$-Order z-Transform

- $2^{nd}$-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{ z : \frac{1}{2} < |z| \right\}$$

| | | |
|---|---|---|
| 5. $a^n u[n]$ | $\frac{1}{1 - az^{-1}}$ | $\lvert z \rvert > \lvert a \rvert$ |

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

# Partial Fraction Expansion

- If M≥N and the poles are 1st order

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- Where $B_k$ is found by long division

$$A_k = (1 - d_k z^{-1}) X(z) \Big|_{z=d_k}$$

# Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}, \qquad ROC = \{z:\ 1<|z|\}$$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\ \ 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & z^{-2} - 3z^{-1} + 2 \\ \hline & \quad\quad\quad 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

# Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}, \qquad ROC = \{z : 1 < |z|\}$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

# Power Series Expansion

- Expansion of the z-transform definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

# Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n = -2 \\ -1/2, & n = -1 \\ -1, & n = 0 \\ 1/2, & n = 1 \\ 0, & else \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

# Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = -\sum_{k=1}^{N} \left(\frac{a_k}{a_0}\right) y[n-k] + \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-m]$$

- Difference equations of this form behave as causal LTI systems
  - when the input is zero prior to n=0
  - Initial rest equations are imposed prior to the time when input becomes nonzero
    - i.e y[-N]=y[-N+1]=…=y[-1]=0

# Difference Equation to z-Transform

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-m]$$

$$Y(z) = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)z^{-k}Y(z) + \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)z^{-k}X(z)$$

$$\sum_{k=0}^{N}\left(\frac{a_k}{a_0}\right)z^{-k}Y(z) = \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)z^{-k}X(z) \Rightarrow Y(z) = \frac{\sum_{m=0}^{M}\left(b_k\right)z^{-k}}{\sum_{k=0}^{N}\left(a_k\right)z^{-k}}X(z)$$

$$H(z) = \frac{\sum_{m=0}^{M}\left(b_k\right)z^{-k}}{\sum_{k=0}^{N}\left(a_k\right)z^{-k}}$$

# Example: 1st-Order System

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

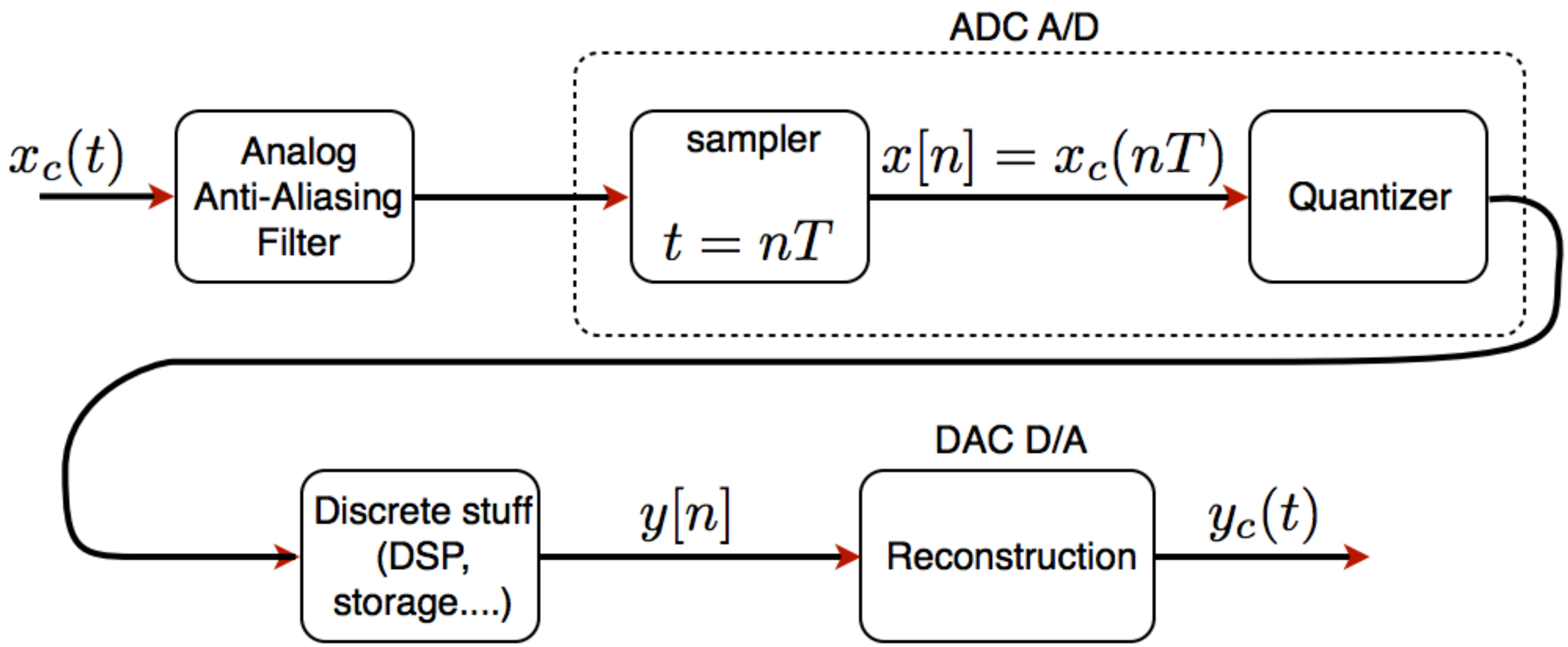
$$y[n] = ay[n-1] + x[n]$$

$$H(z) = \frac{1}{1 - az^{-1}}$$

$$h[n] = a^n u[n]$$

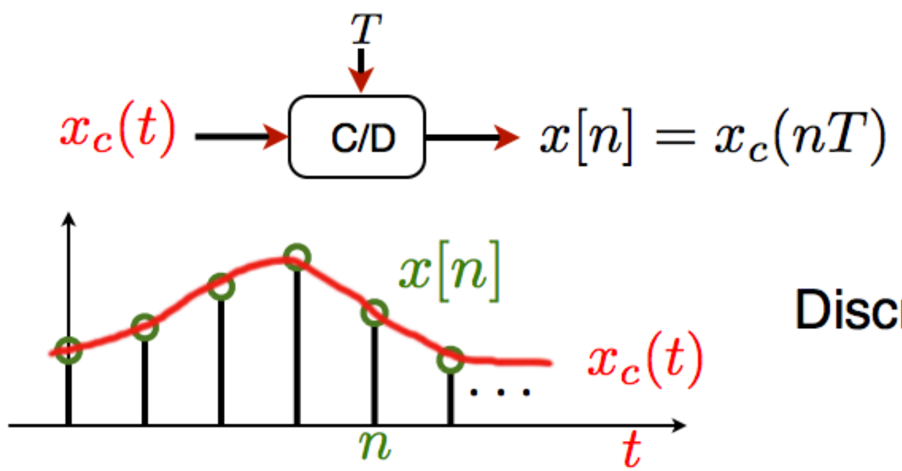
# Sampling and Reconstruction

# DSP System



ADC A/D
$x_c(t)$
Analog
Anti-Aliasing
Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff
(DSP,
storage....)
$y[n]$
Reconstruction
$y_c(t)$

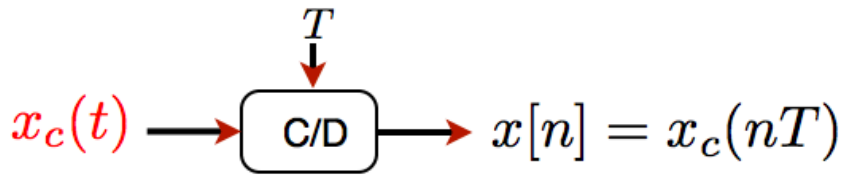
# Ideal Sampling Model

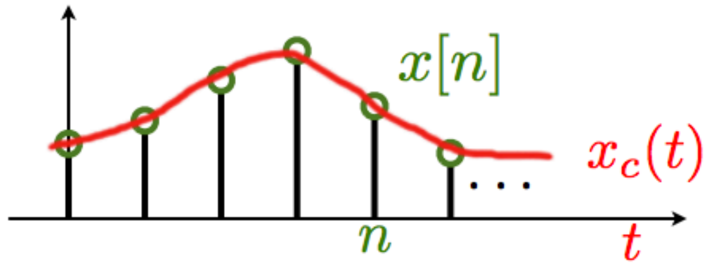



Discrete and Continuous

- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - $f_s$=1/T is the sampling frequency
  - $\Omega_s = 2\pi/T$

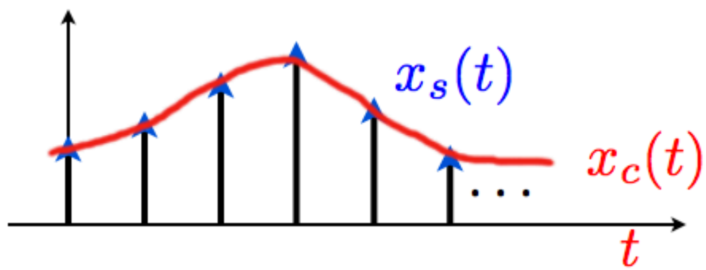
# Ideal Sampling Model




Discrete and Continuous

define impulsive sampling:

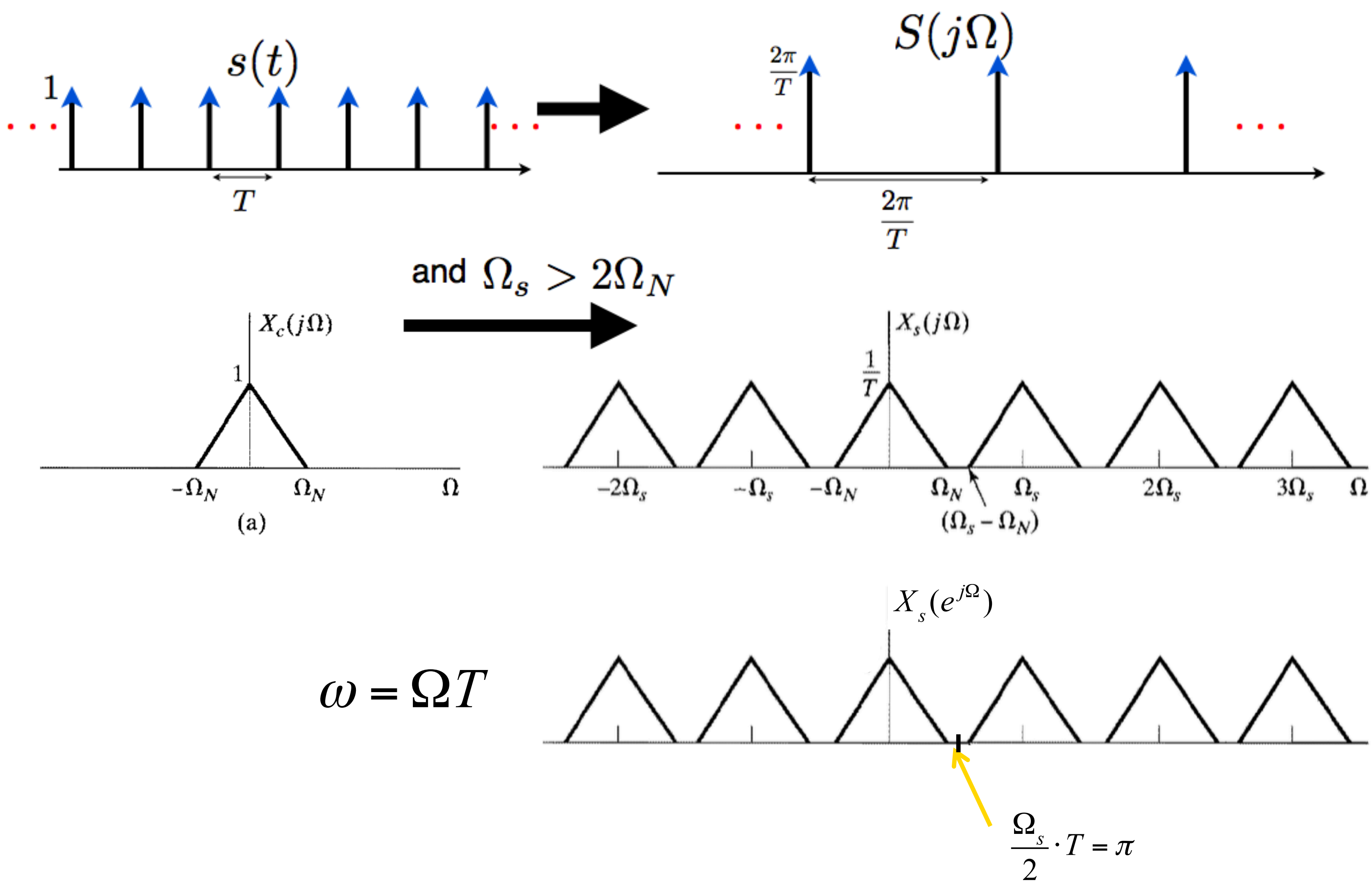



Continuous

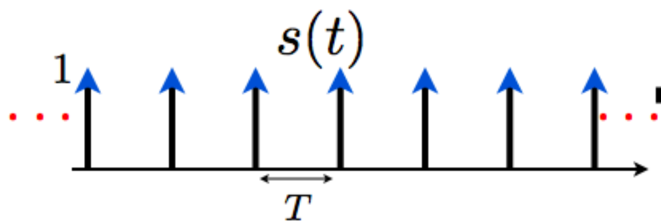
$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

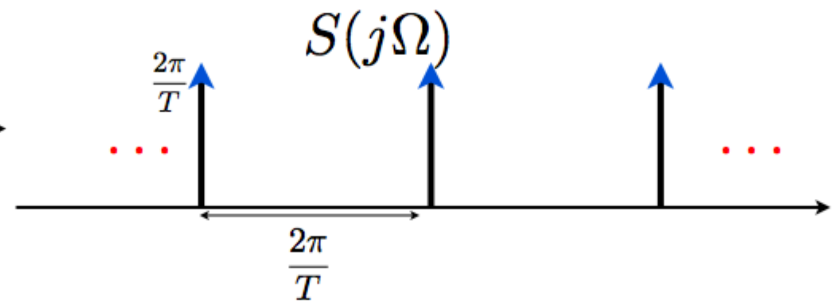
$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

# Frequency Domain Analysis

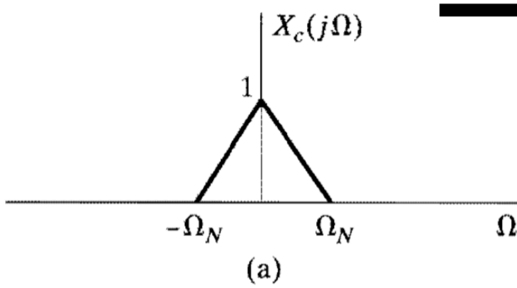


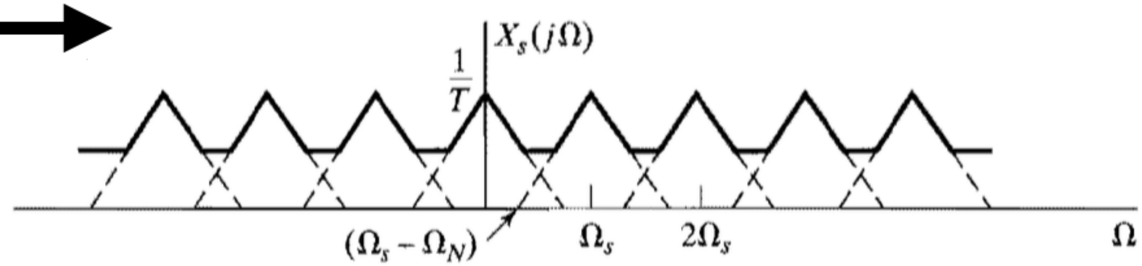
# Frequency Domain Analysis



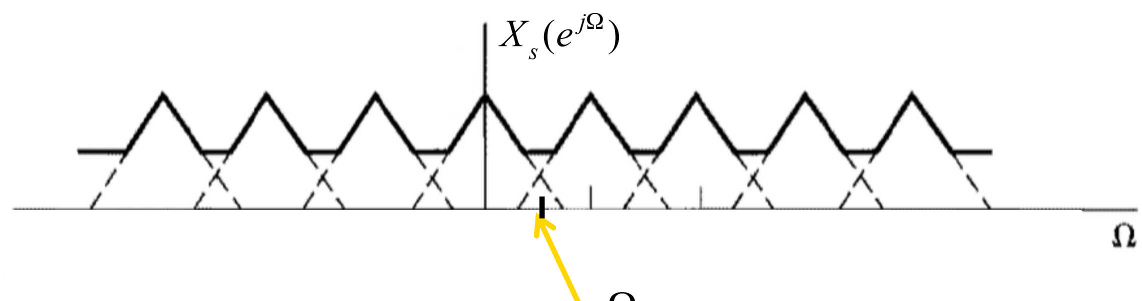




and $\Omega_s < 2\Omega_N$



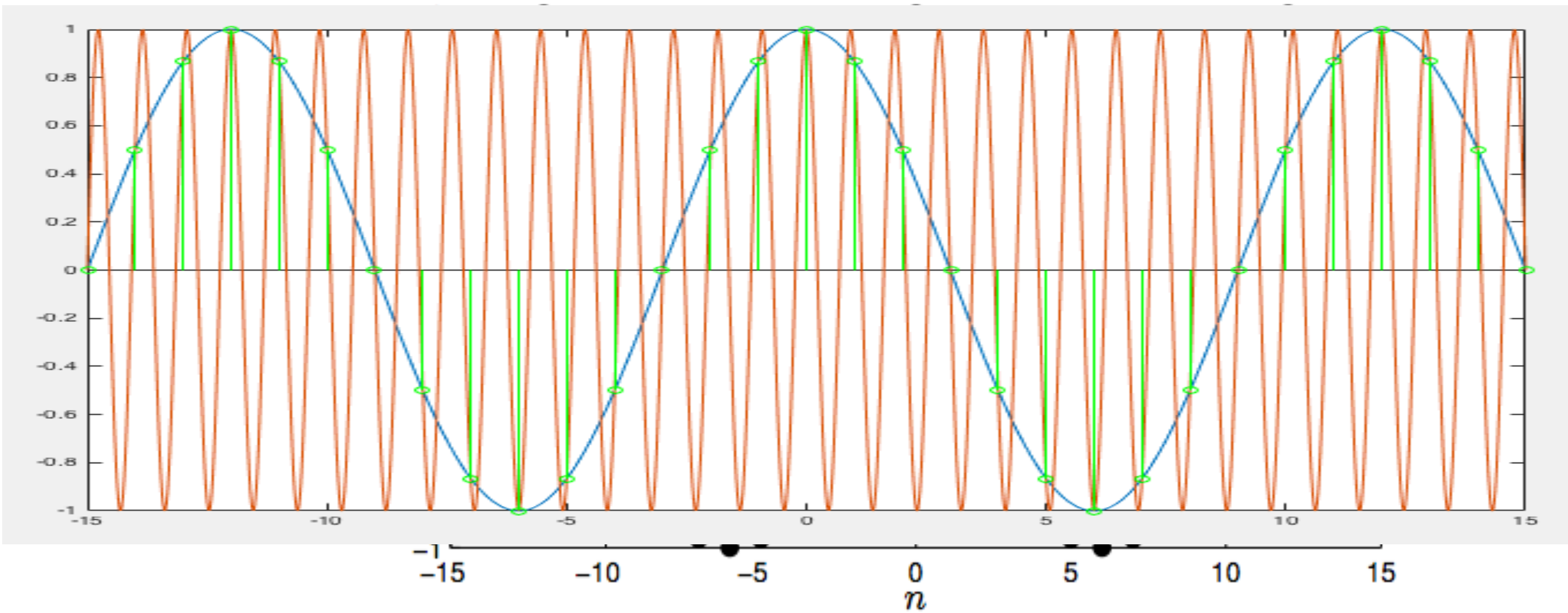

$\omega = \Omega T$




$\frac{\Omega_s}{2} \cdot T = \pi$

# Aliasing Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$

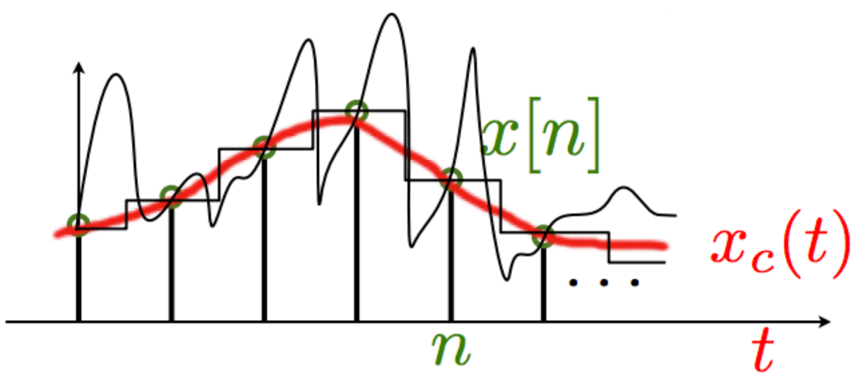


# Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

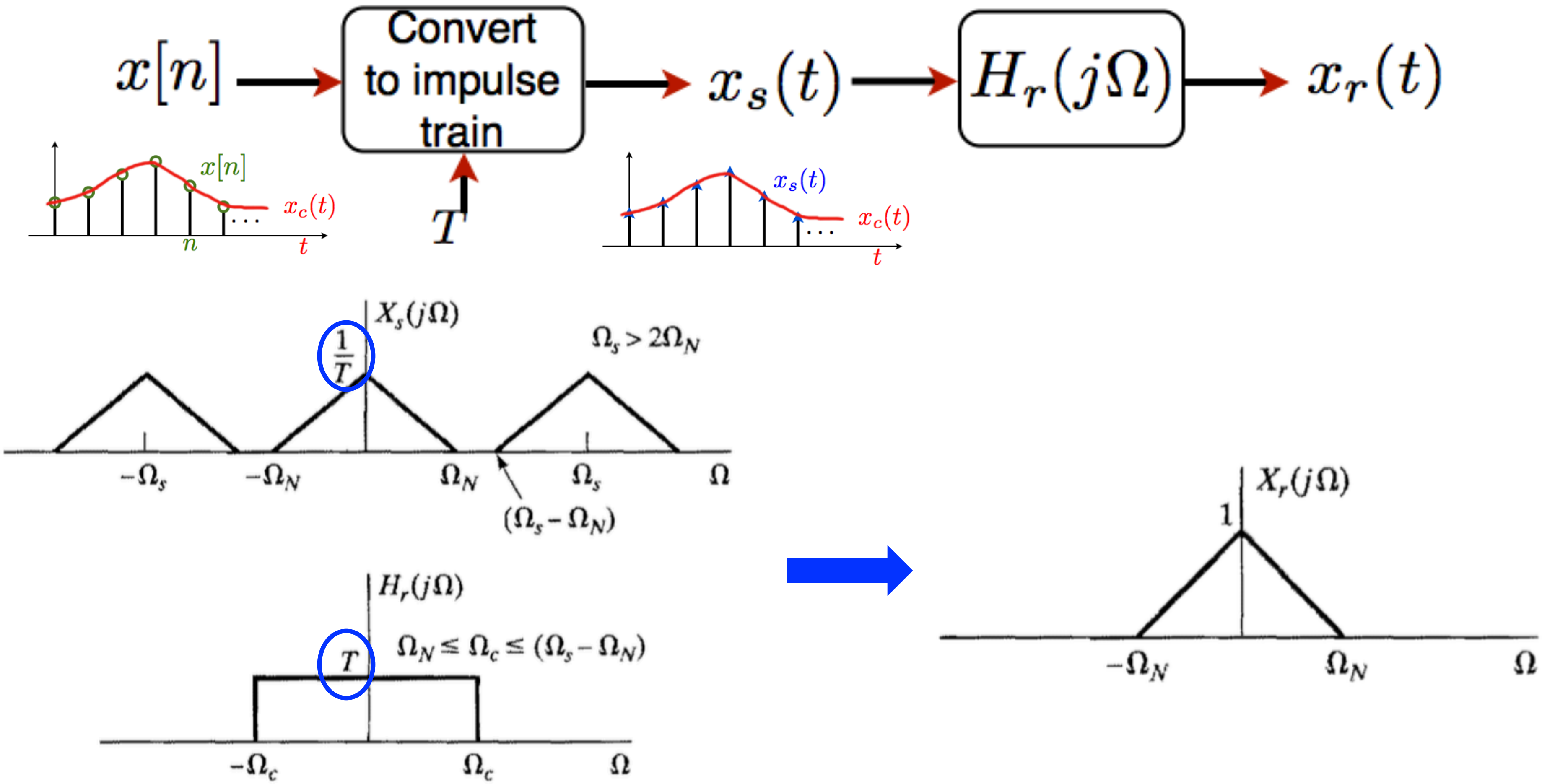
$$X_c(j\Omega) = 0 \;\; \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness




Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

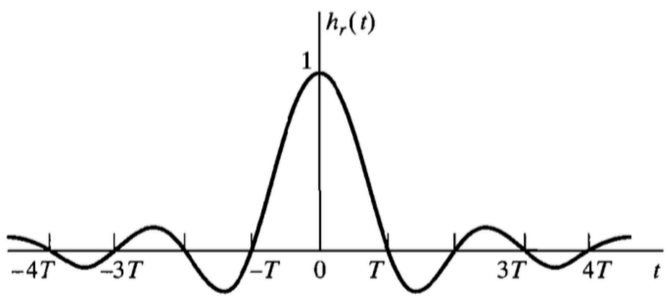
# Reconstruction in Frequency Domain



# Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT)\right) * h_r(t)$$

$$= \sum_n x[n]h_r(t - nT)$$

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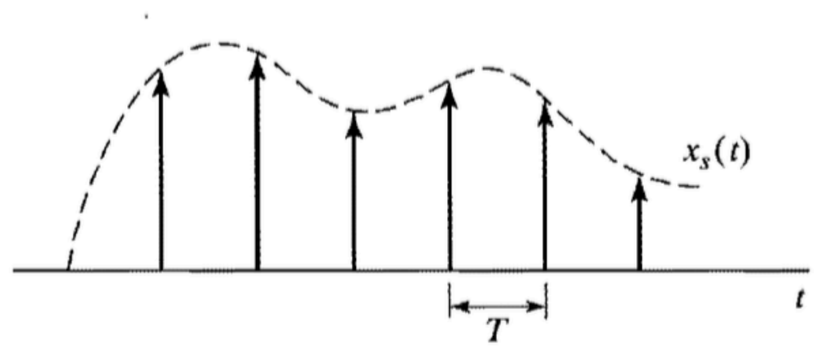


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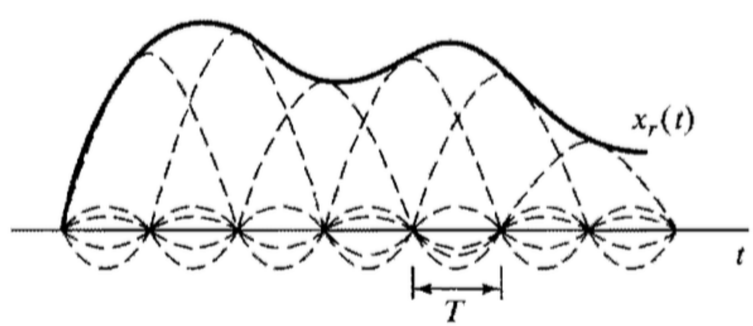
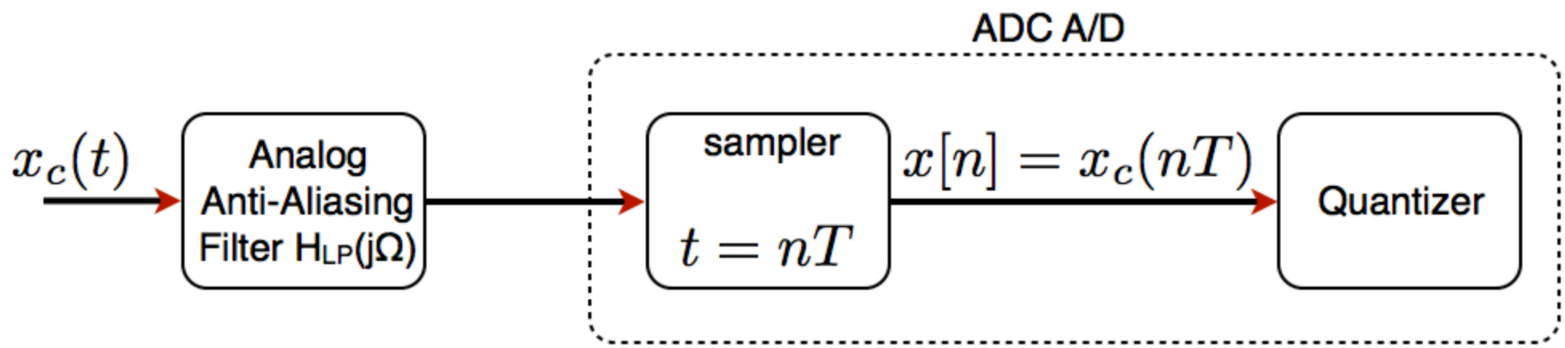
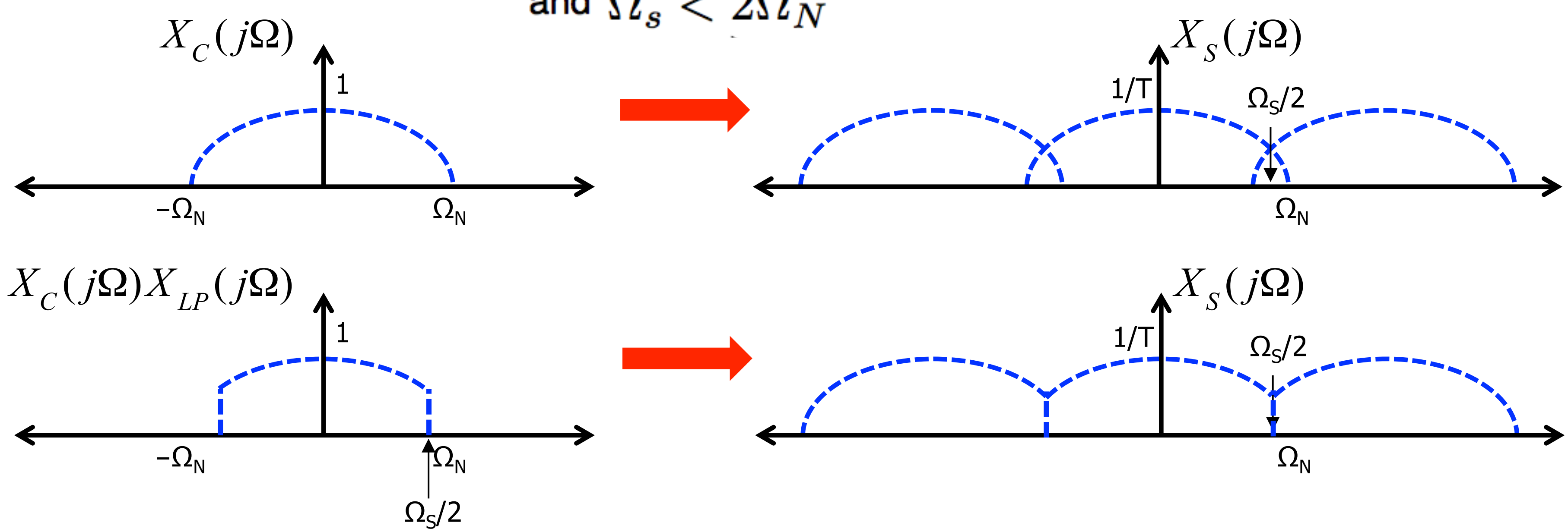



The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

# Anti-Aliasing Filter

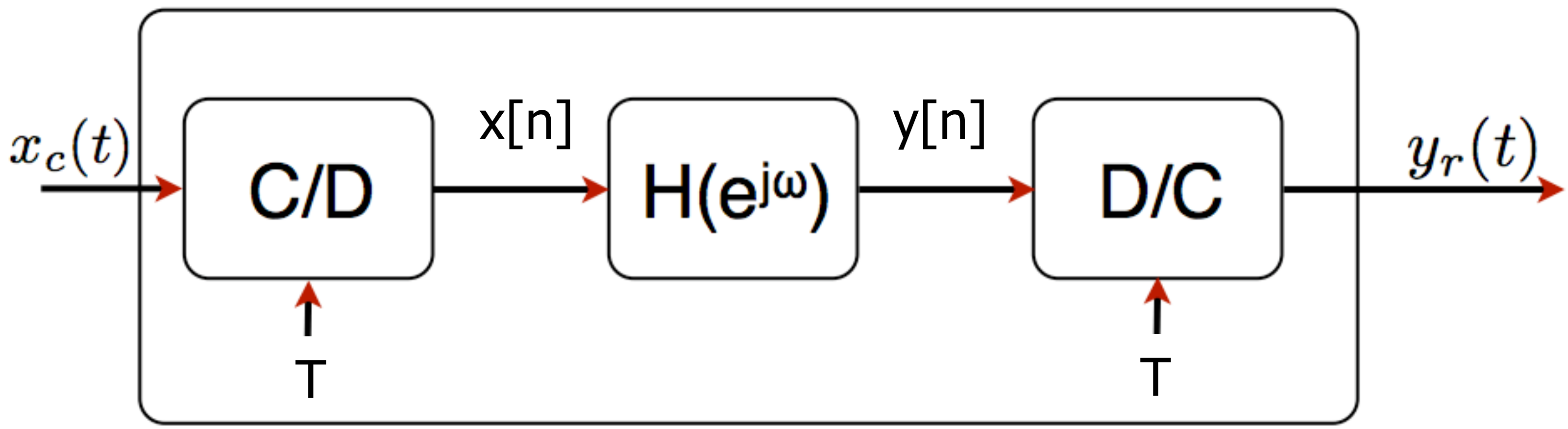



and $\Omega_s < 2\Omega_N$



# DT and CT processing

# Discrete-Time Processing of Continuous Time



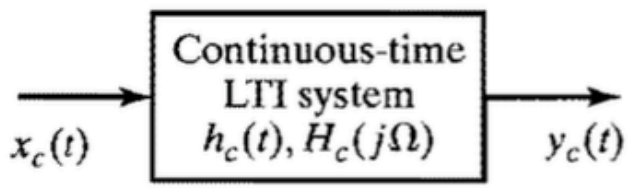

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right] \qquad y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

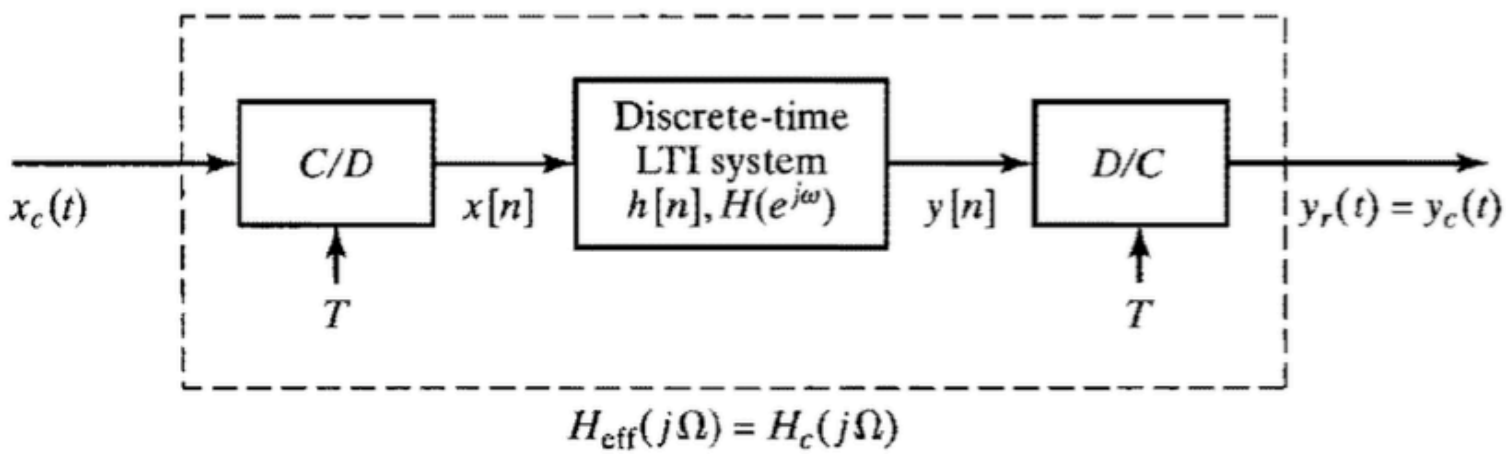
- If $x_c(t)$ is bandlimited by $\Omega_s/T = \pi/T$, then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{eff}(j\Omega) = \begin{cases} H(e^{j\omega})\Big|_{\omega=\Omega T} & |\Omega| < \Omega_s/T \\ 0 & else \end{cases}$$

# Impulse Invariance

- Want to implement continuous-time system in discrete-time



# Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\omega / T), \qquad |\omega| < \pi$$

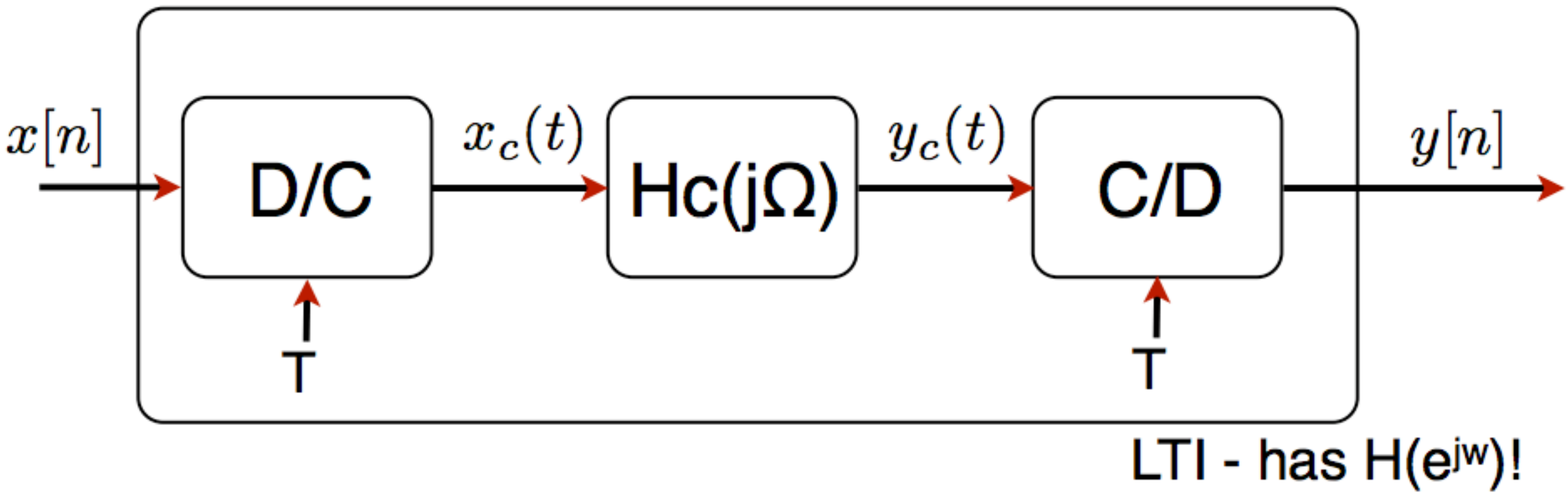
- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi / T$$

$$\boxed{h[n] = Th_c(nT)}$$

# Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time




$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

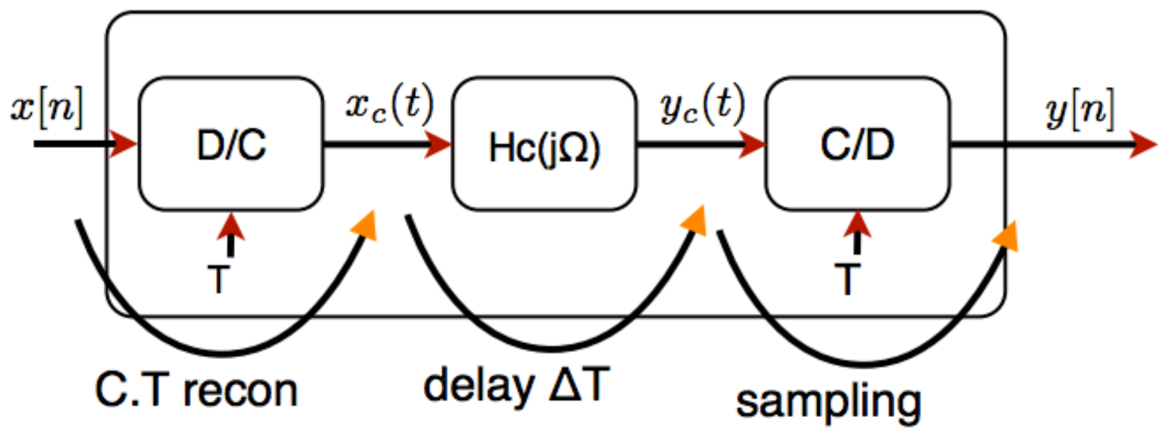
$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

# Example: Non-integer Delay

- What is the time domain operation when $\Delta$ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

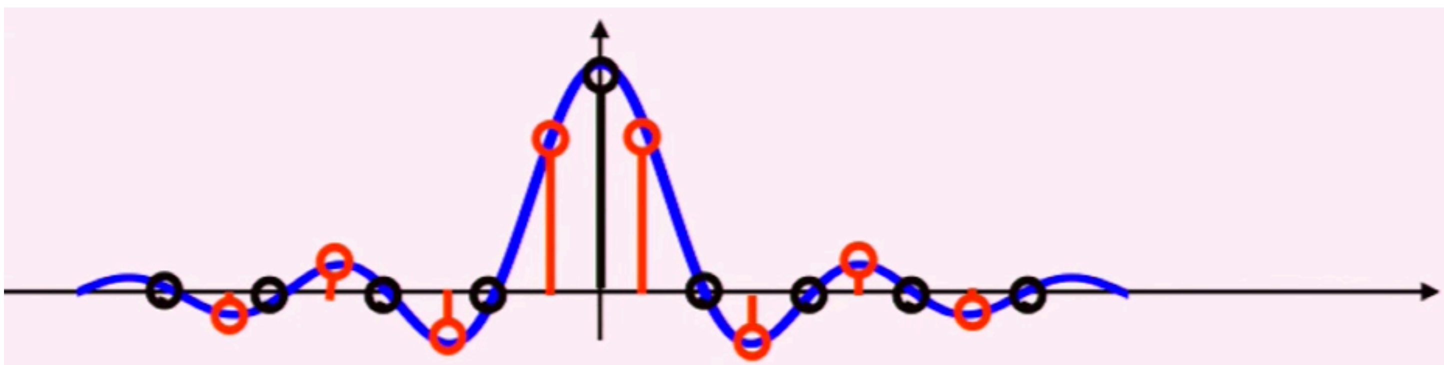
Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ <u>delay of ΔT in time</u>



# Example: Non-integer Delay

- My delay system has an impulse response of a sinc with a continuous time delay

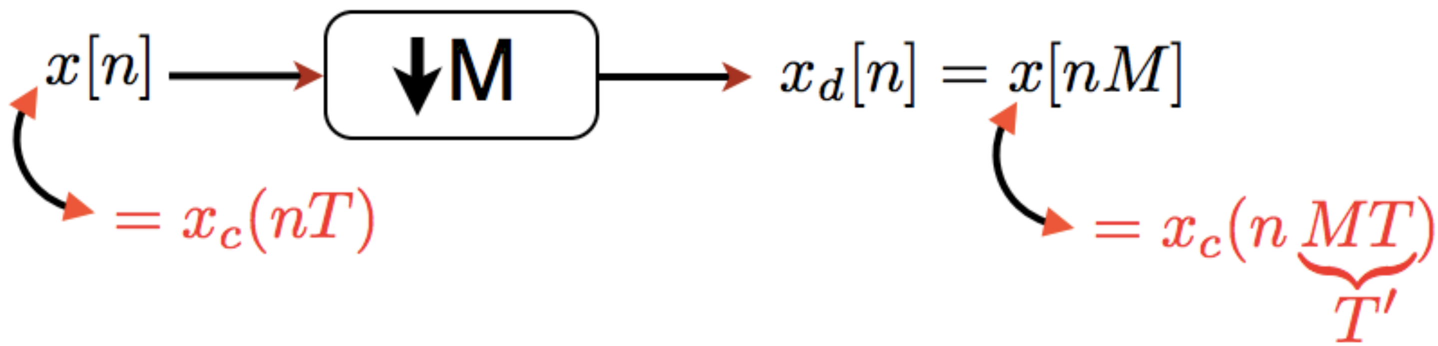
$$h[n] = \text{sinc}(n - \Delta)$$



# Rate Re-Sampling

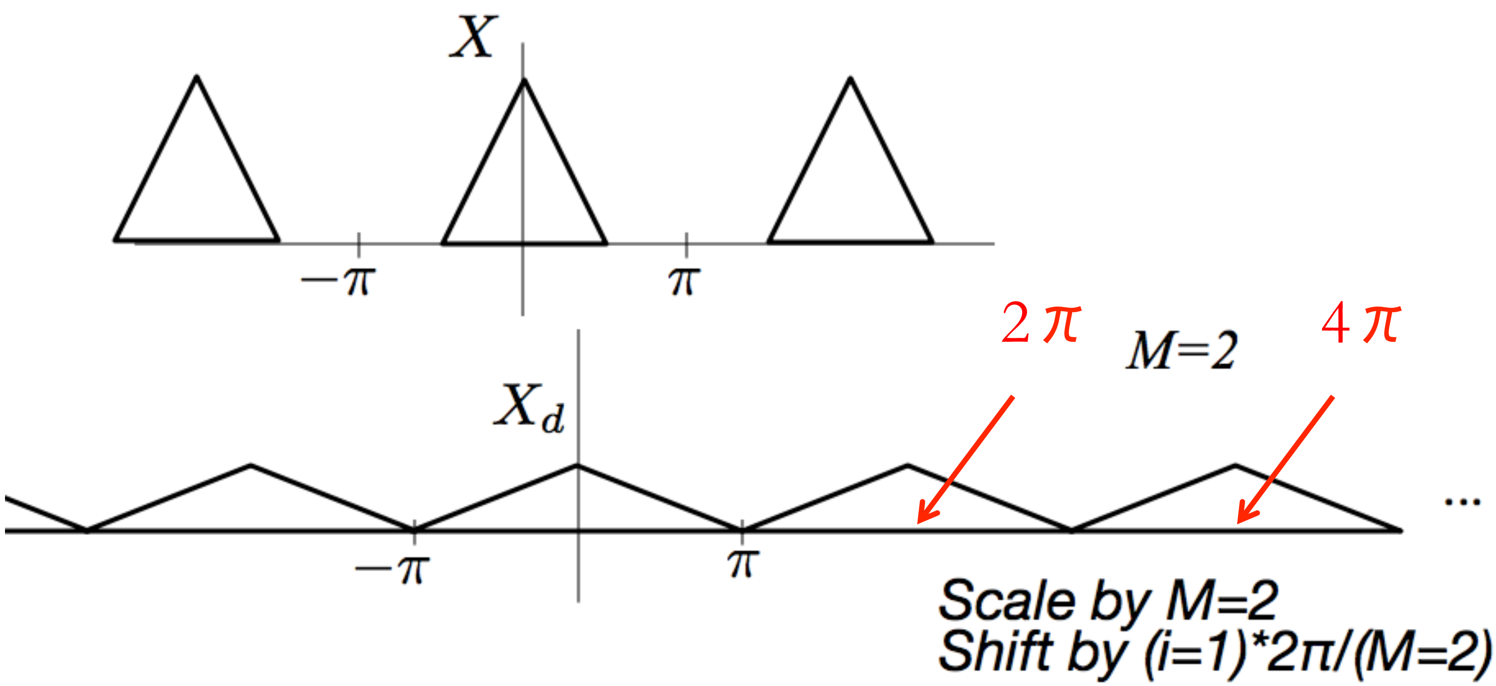
# Downsampling

- Definition: Reducing the sampling rate by an integer number



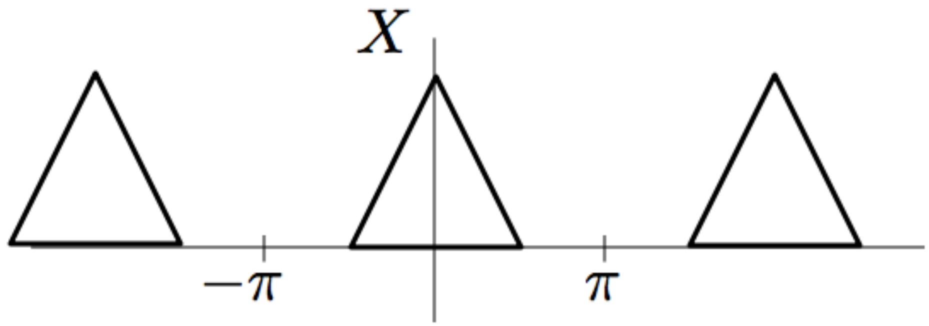
# Example

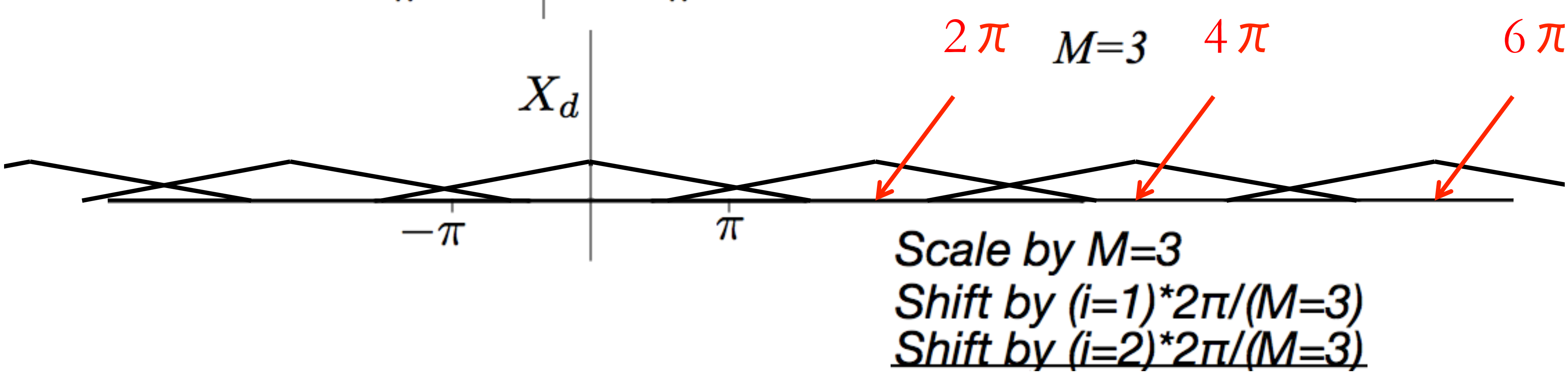
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



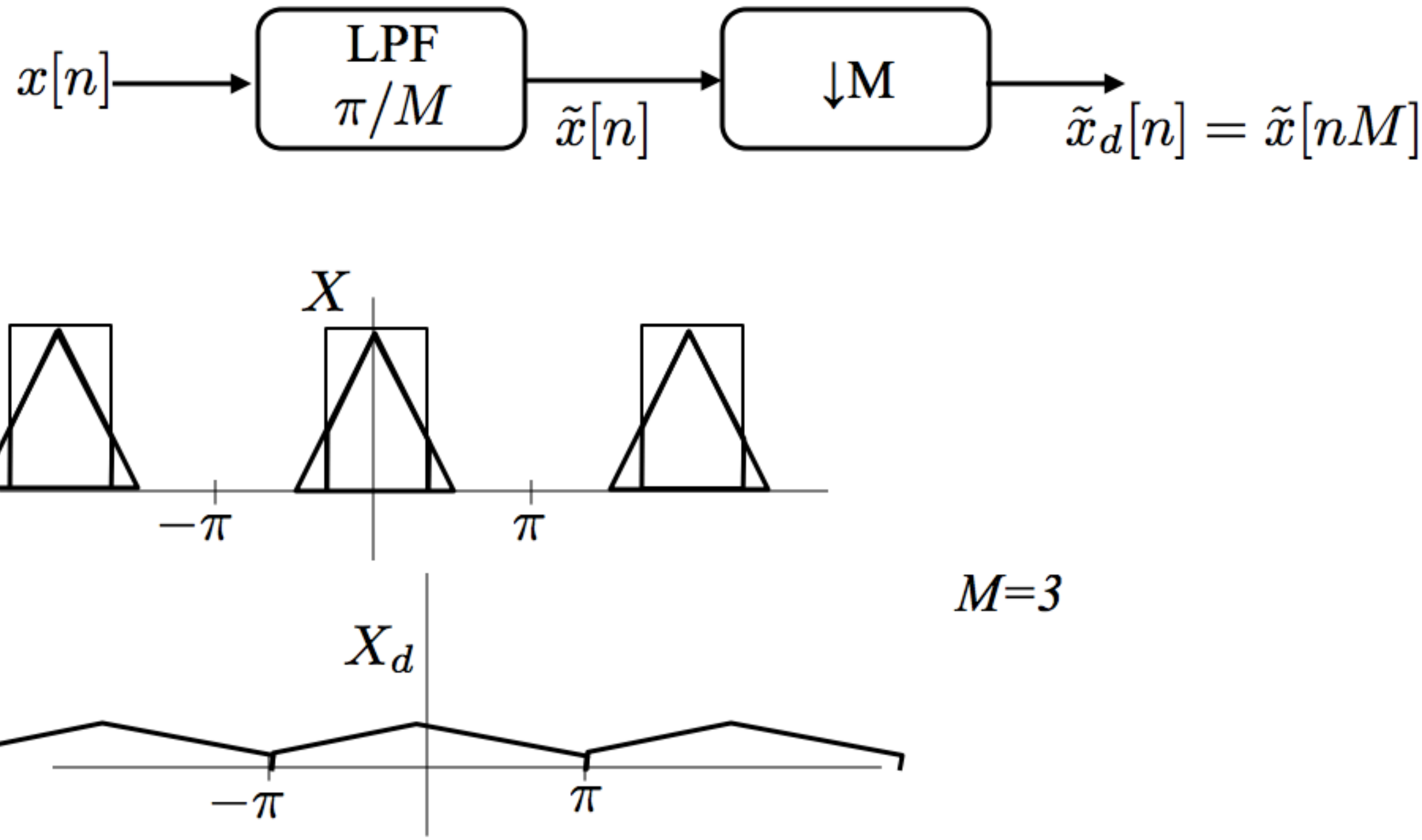
# Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M} i\right)}\right)$$



# Example




*M=3*

# Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

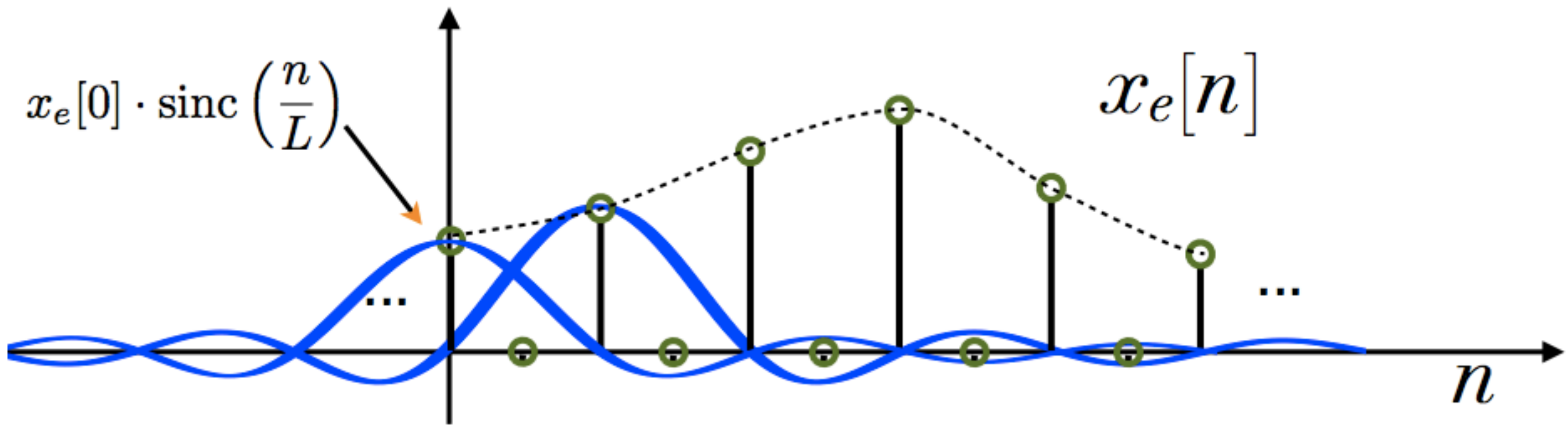
$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

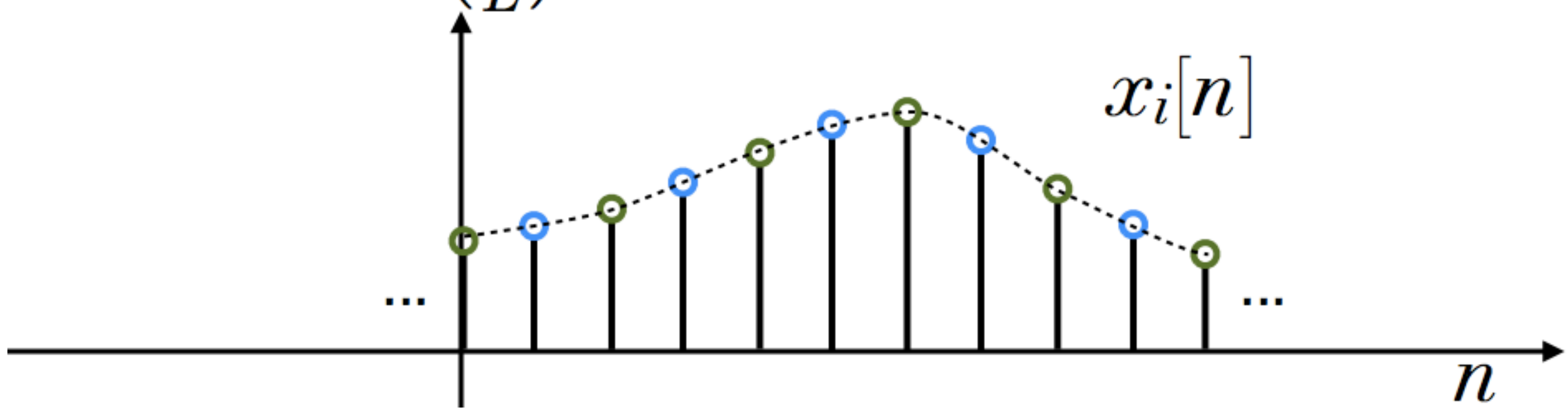
(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

# Upsampling

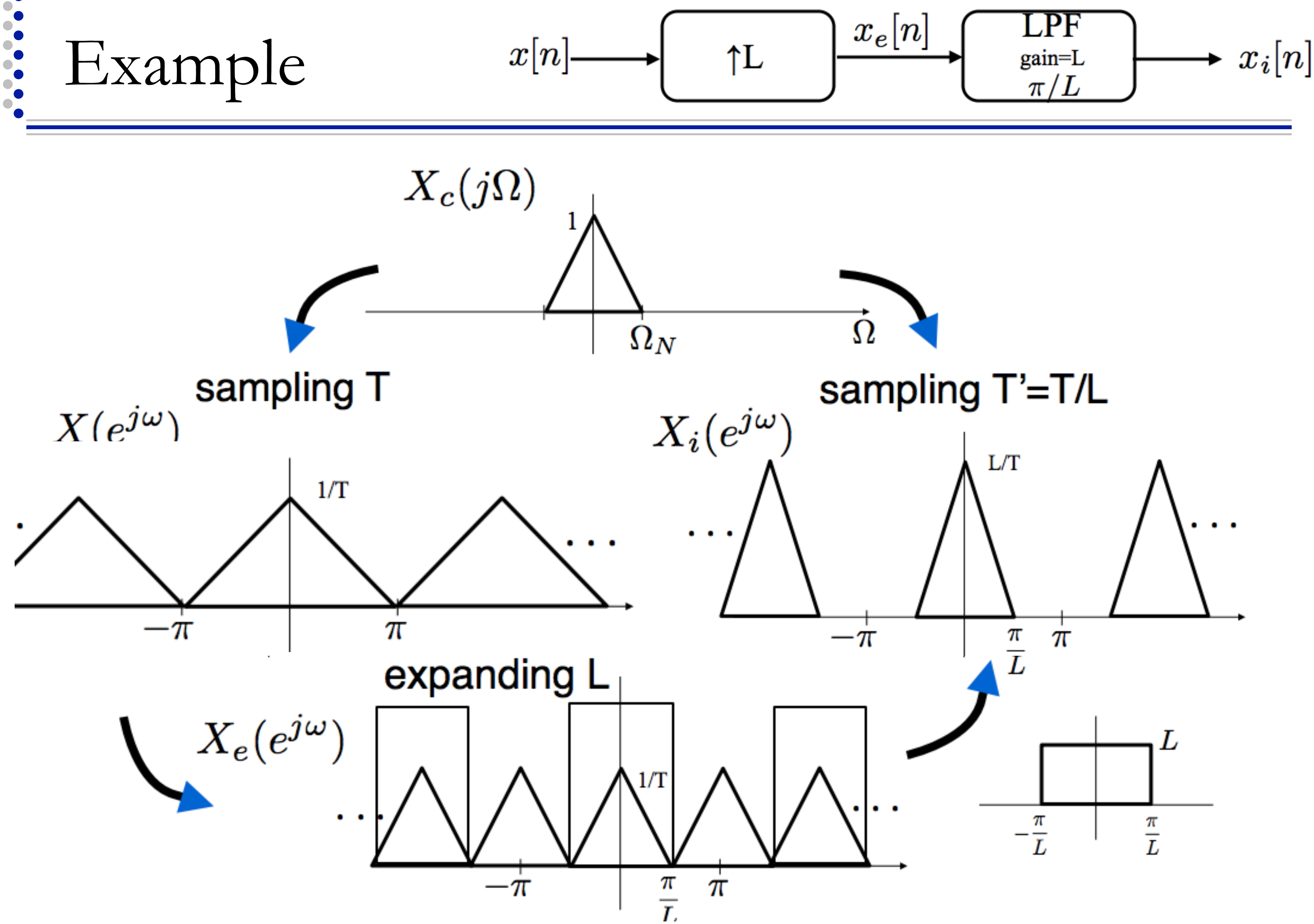
(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:




$$x_i[n] = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$$



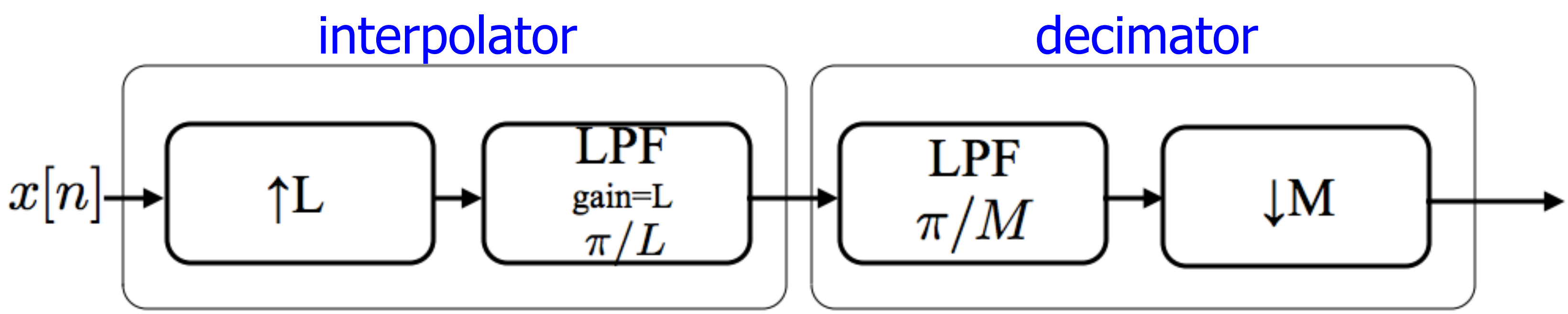
# Example



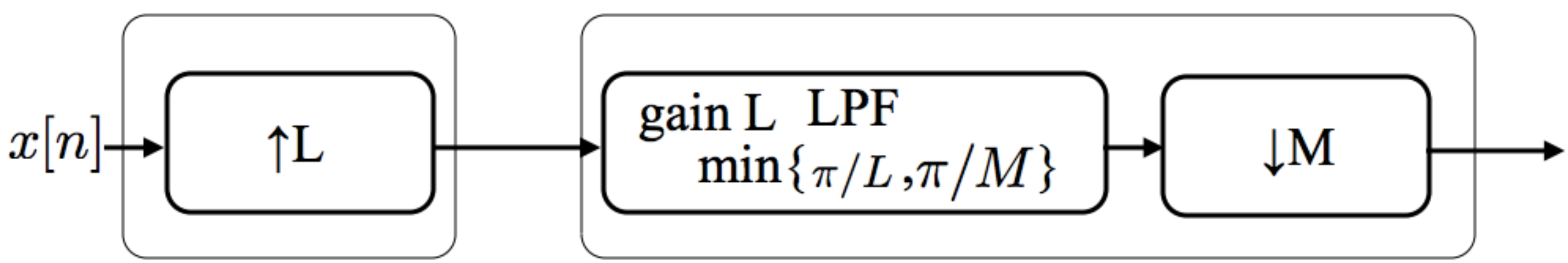
x[n]
↑L
x_e[n]
LPF
gain=L
π/L
x_i[n]
X_c(jΩ)
1
Ω_N
Ω
sampling T
sampling T'=T/L
X(e^{jω})
1/T
−π
π
X_i(e^{jω})
L/T
π/L
expanding L
X_e(e^{jω})
L
−π/L

# Non-integer Sampling

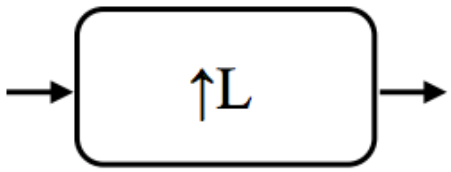
- T'=TM/L
  - Upsample by L, then downsample by M



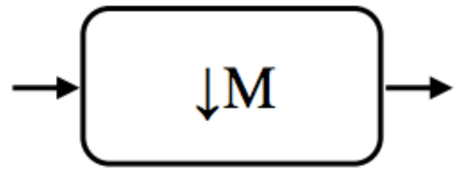

Or,



# Interchanging Operations




“expander”




“compressor”

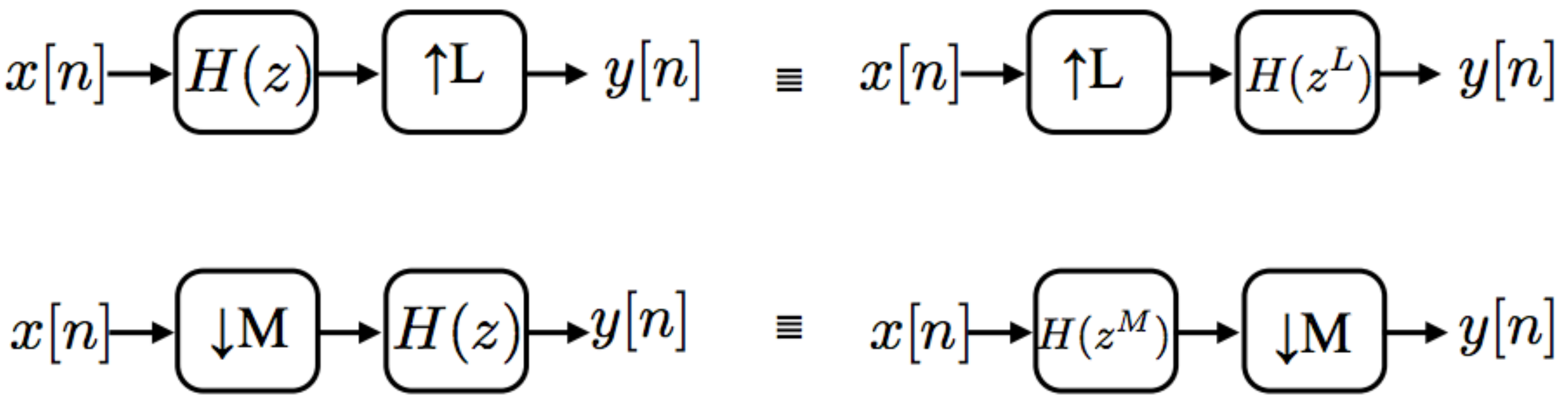
not LTI!

Upsampling
-expanding in time
-compressing in frequency

Downsampling
-compressing in time
-expanding in frequency

# Interchanging Operations - Summary

Filter and expander ≡ Expander and expanded filter*

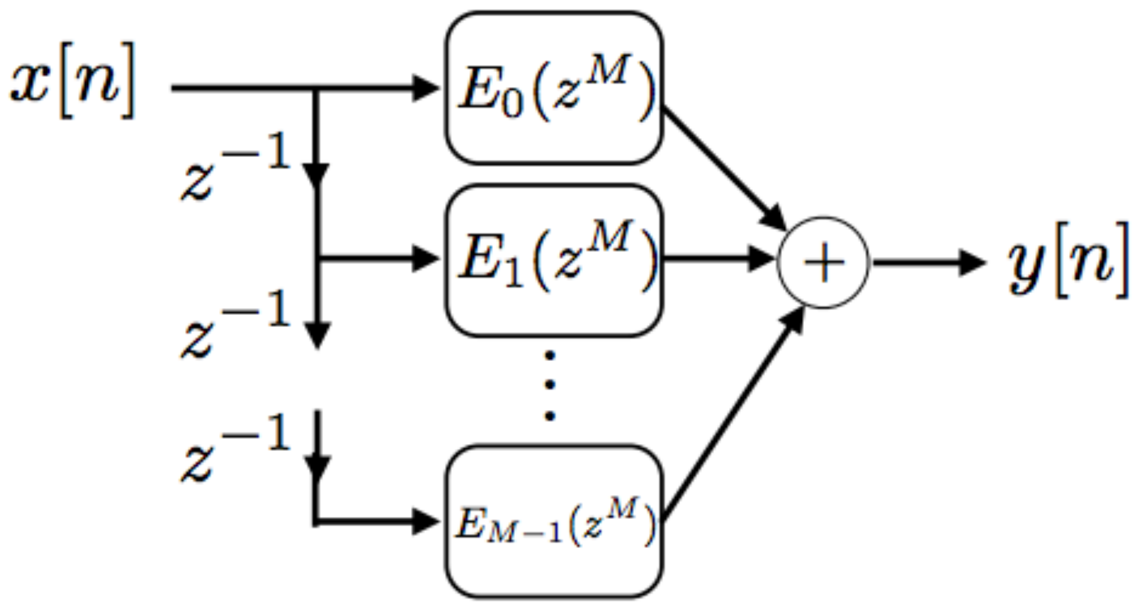



Compressor and filter ≡ Expanded filter* and compressor

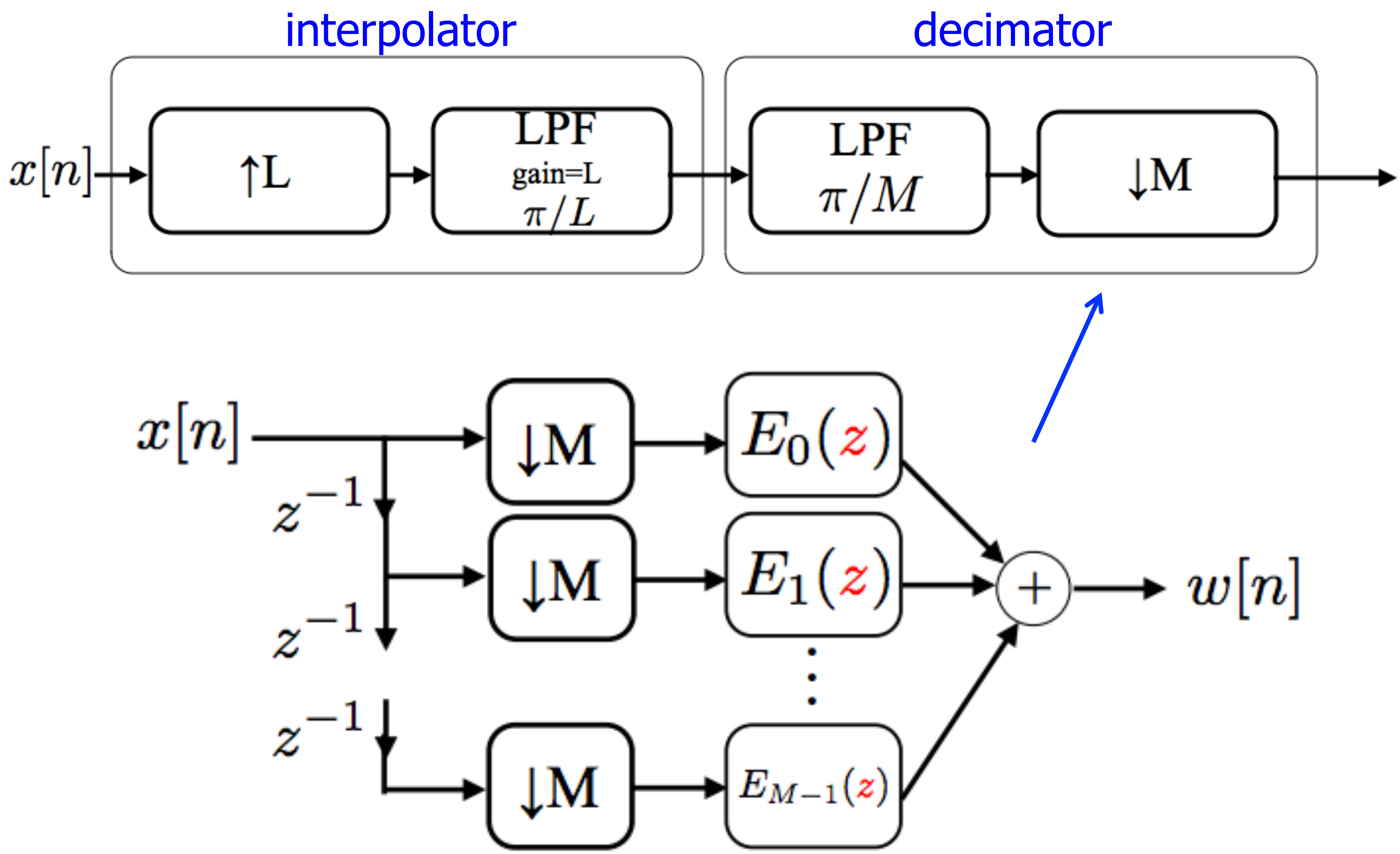
*Expanded filter = expanded impulse response, compressed freq response

# Polyphase Decomposition

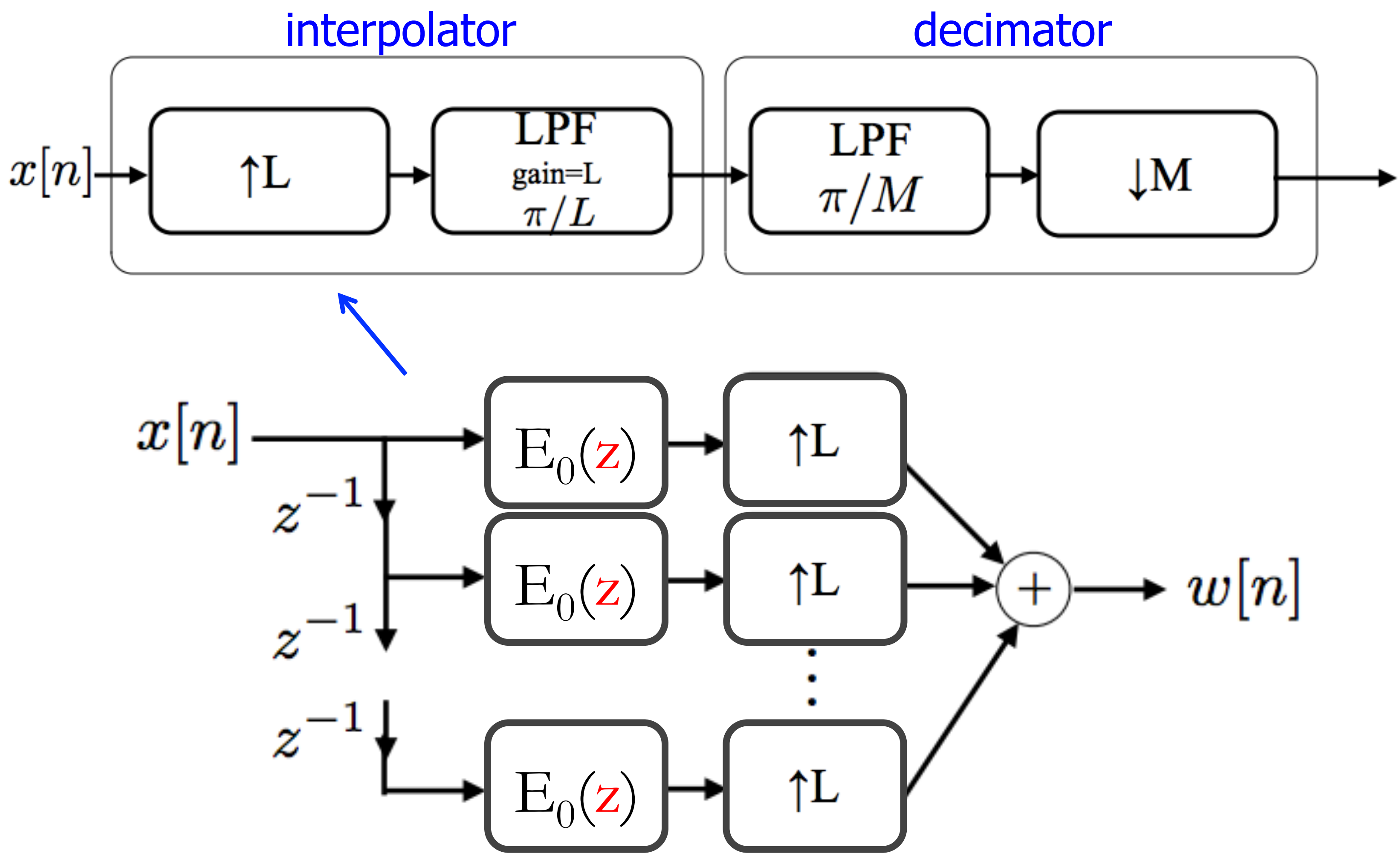
$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$$



# Polyphase Implementation of Decimator



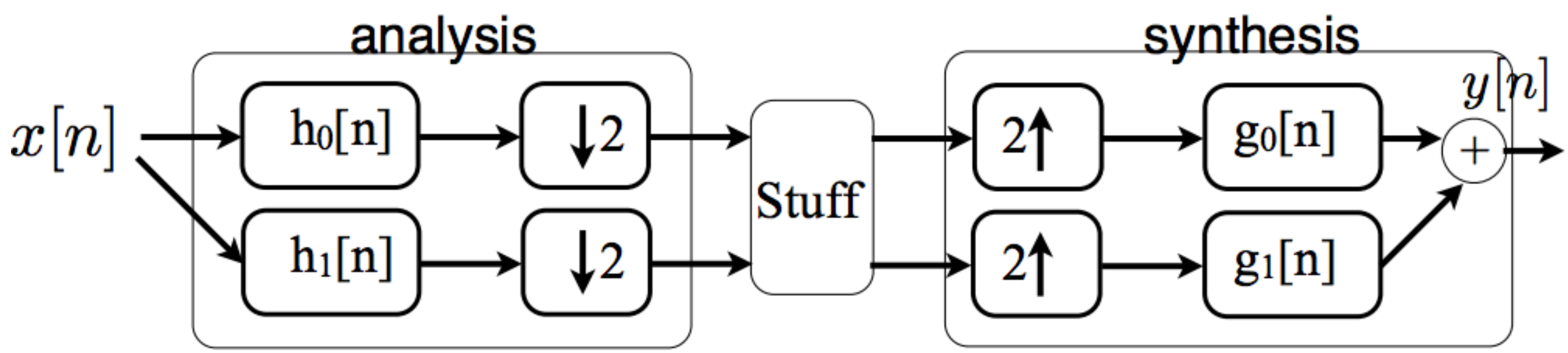
# Polyphase Implementation of Interpolation



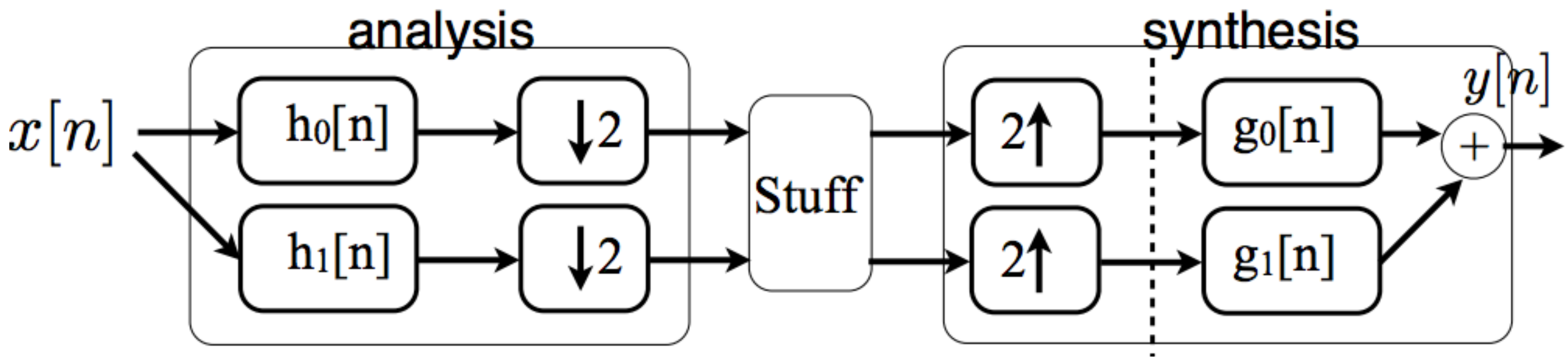
# Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by $\pi$



# Perfect Reconstruction non-Ideal Filters

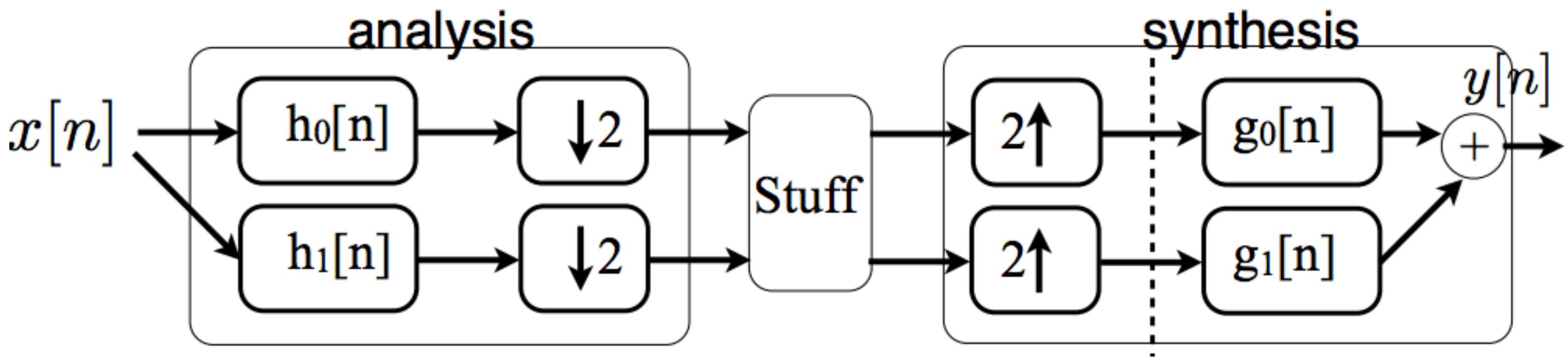



$$
\begin{aligned}
Y(e^{j\omega}) &= \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right] X(e^{j\omega}) \\
&+ \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right] X(e^{j(\omega-\pi)})
\end{aligned}
$$

need to cancel!

aliasing

# Quadrature Mirror Filters




Quadrature mirror filters

$$
\begin{aligned}
H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\
G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\
G_1(e^{j\omega}) &= -2H_1(e^{j\omega})
\end{aligned}
$$

# Frequency Response of Systems

# Frequency Response of LTI System

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- We can define a magnitude response

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right|\left|X\left(e^{j\omega}\right)\right|$$

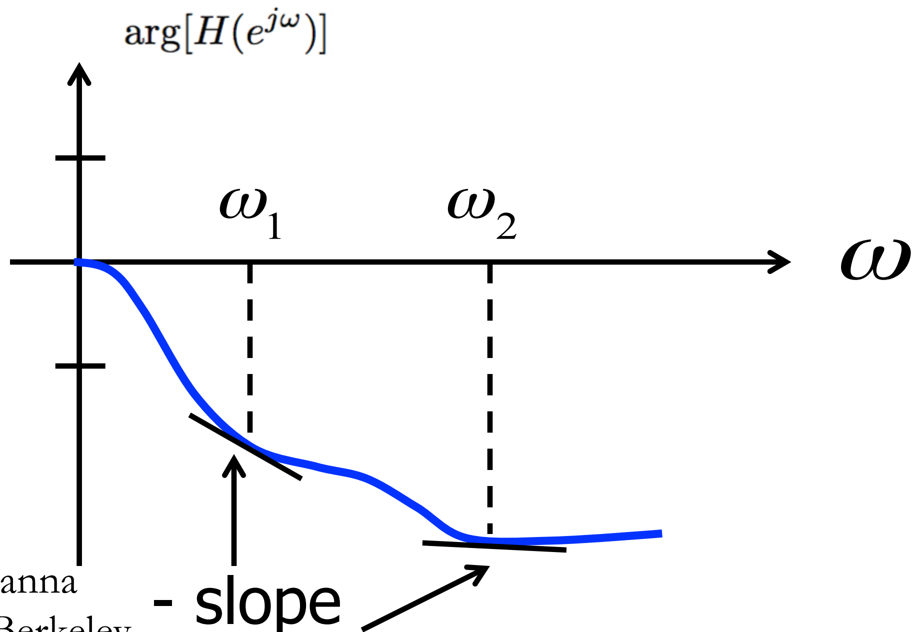
- And a phase response

$$\angle Y\left(e^{j\omega}\right) = \angle H\left(e^{j\omega}\right) + \angle X\left(e^{j\omega}\right)$$

# Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



# LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

Stable and causal if all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

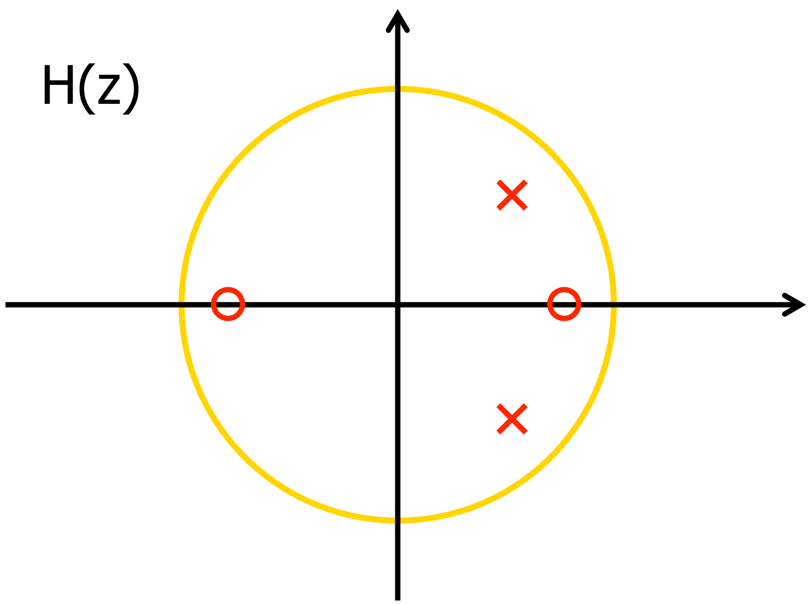
# General All-Pass Filter

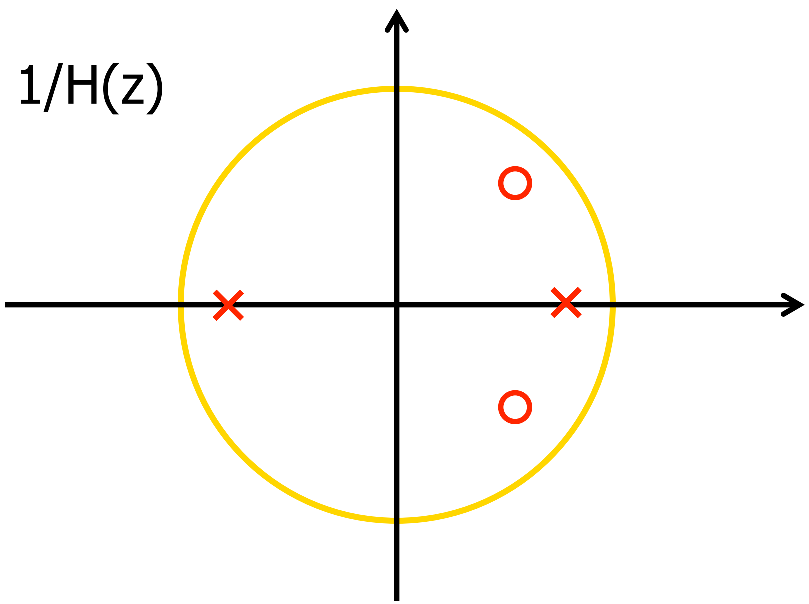
- $d_k$=real pole, $e_k$=complex poles paired w/ conjugate, $e_k^*$

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

# Minimum-Phase Systems

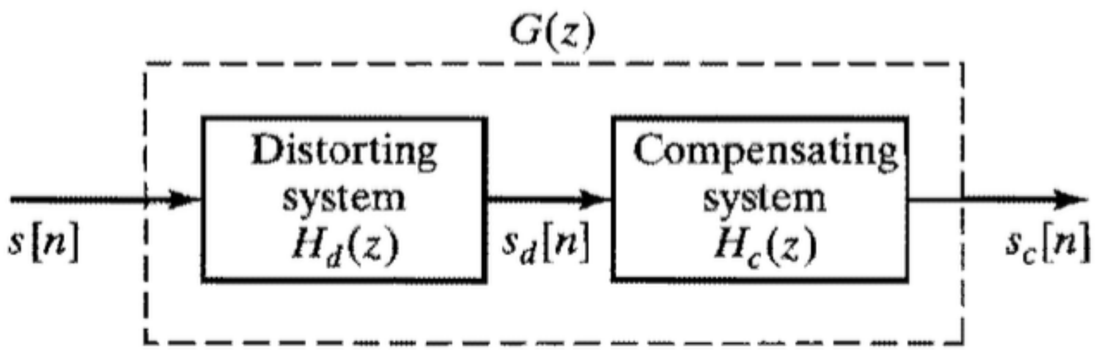
- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle



# Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:




- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z) \rightarrow H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

# Generalized Linear Phase

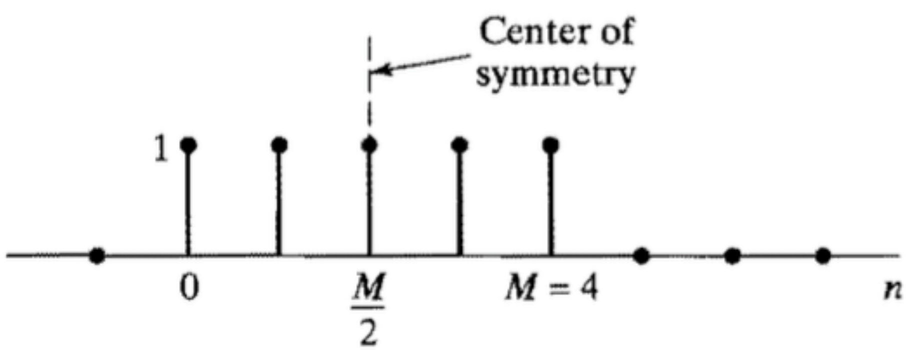
- An LTI system has generalized linear phase if frequency response $H(e^{j\omega})$ can be expressed as:

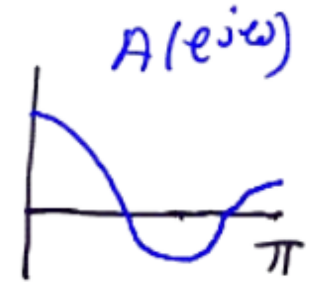
$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha + j\beta}, \quad |\omega| < \pi$$

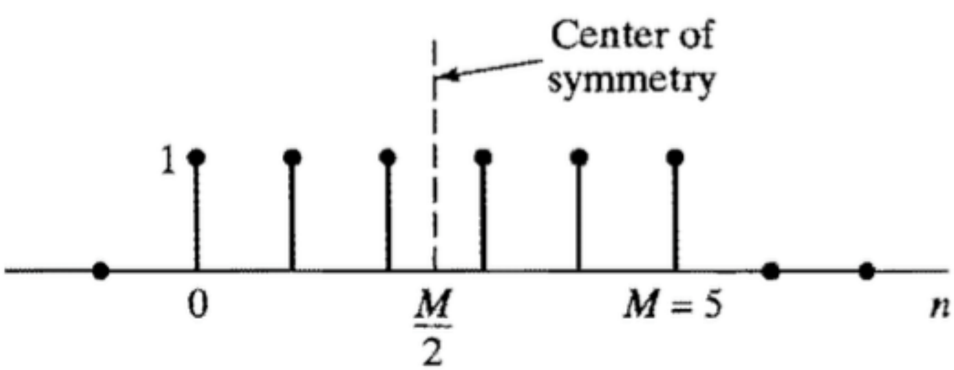
- Where A($\omega$) is a real function.

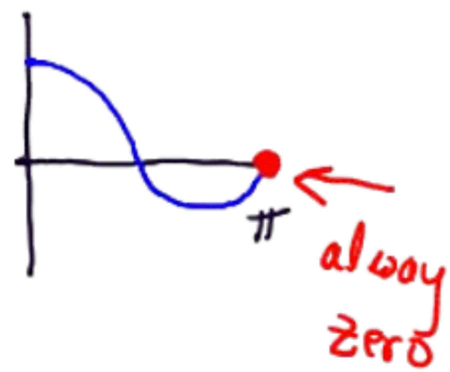
- What is the group delay?

# FIR GLP: Type I and II

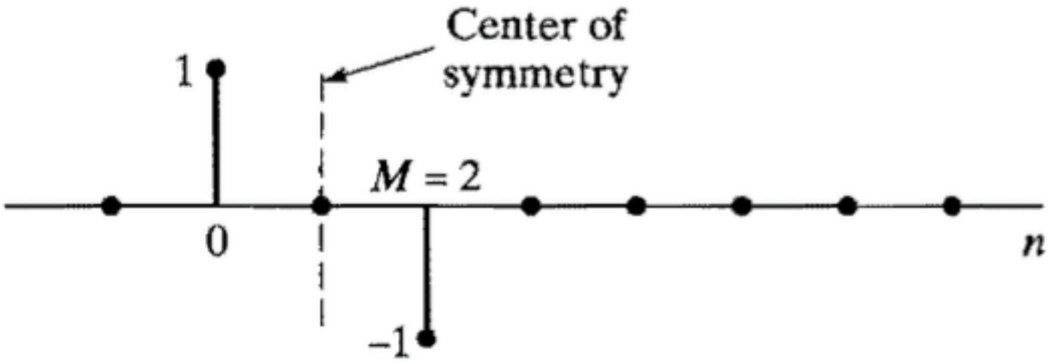


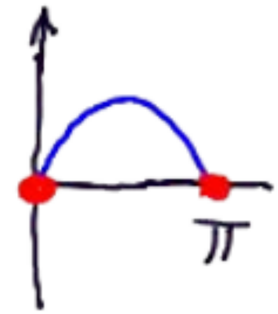



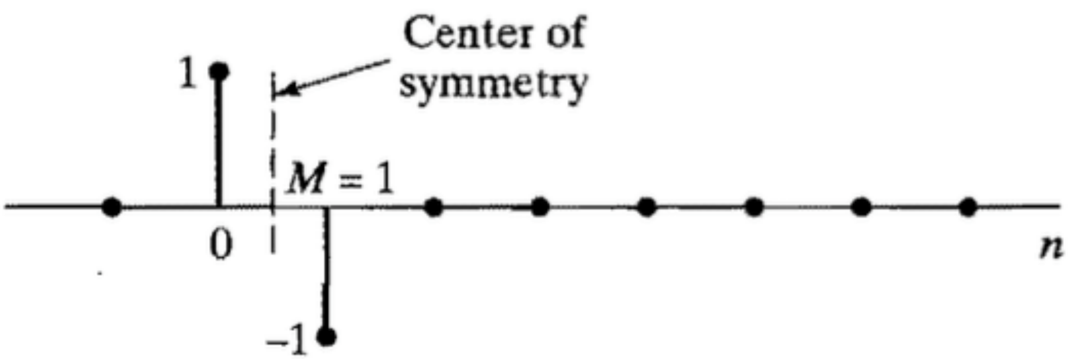



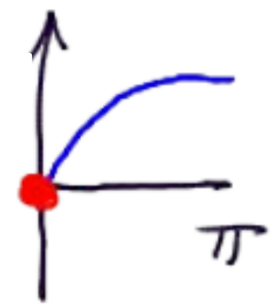



# FIR GLP: Type III and IV



# Zeros of GLP System

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

**Real system → zeros occur in conjugate-reciprocal groups of 4**

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

- If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

- If zero is real and not on unit circle ( $\theta$ =0)

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

# FIR Filter Design

# FIR Design by Windowing

- Given desired frequency response, $H_d(e^{j\omega})$ , find an impulse response

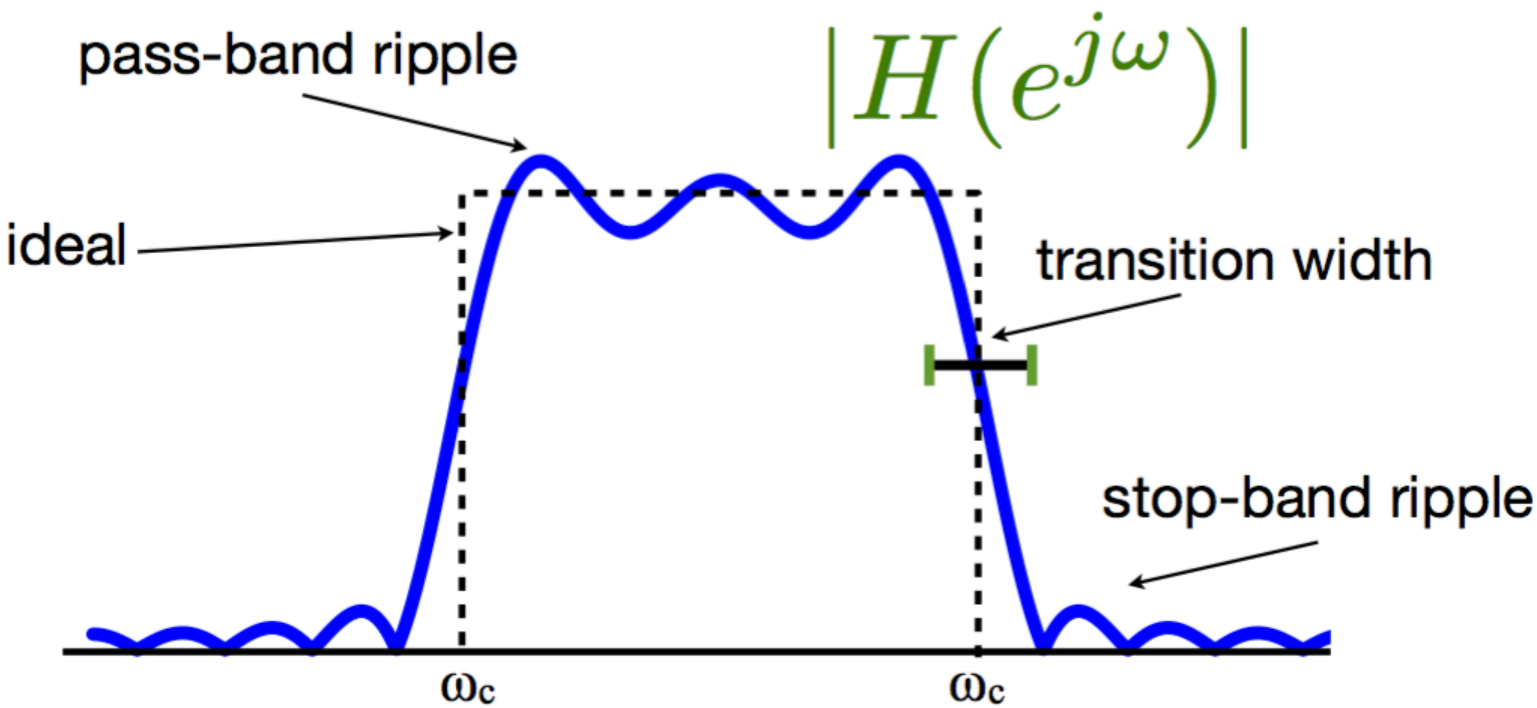
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

ideal

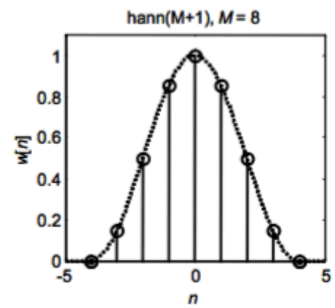
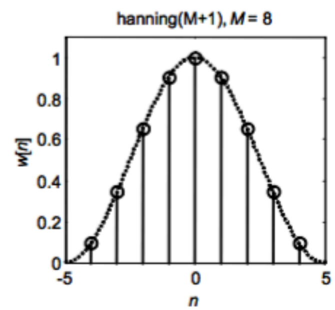
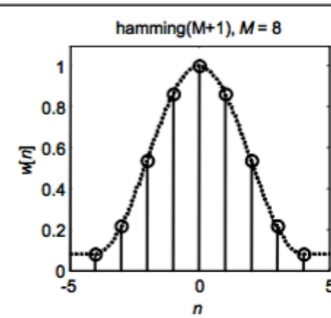
- Obtain the $M^{th}$ order causal FIR filter by truncating/windowing it

$$h[n] = \left\{ \begin{array}{ll} h_d[n]w[n] & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{array} \right\}$$

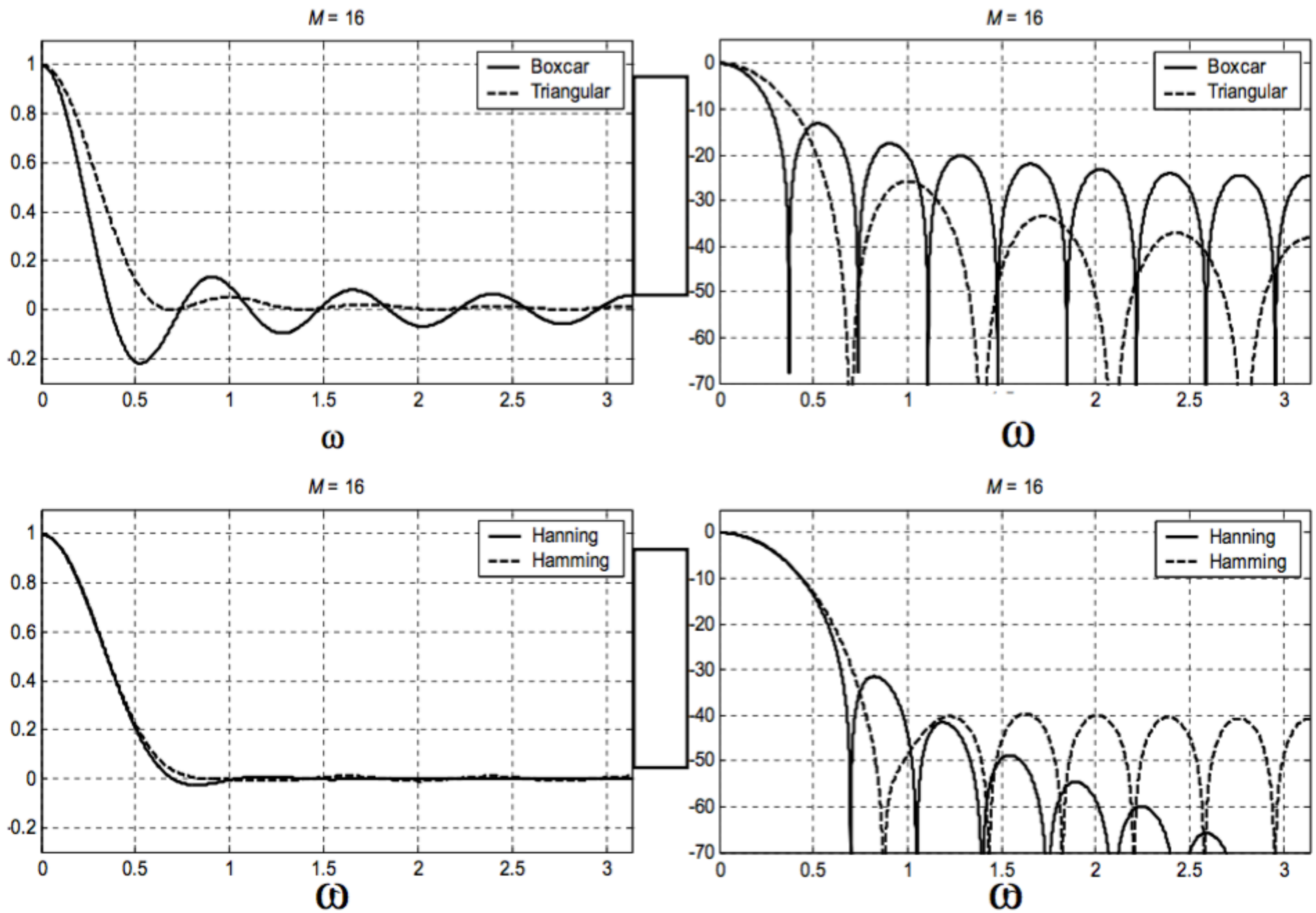
# FIR Design by Windowing



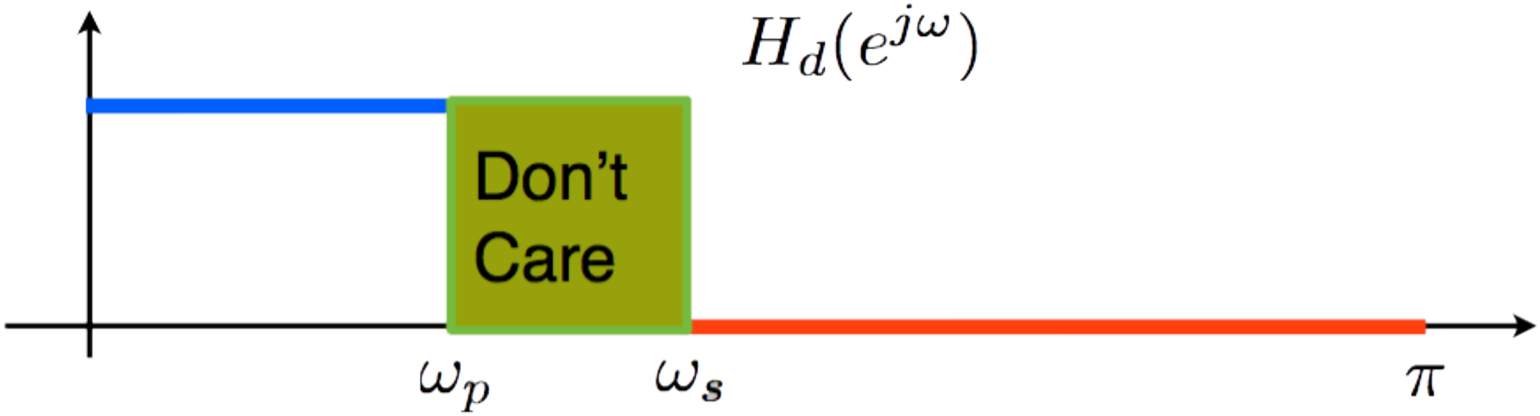
# Tapered Windows

| Name(s) | Definition | MATLAB Command | Graph ($M = 8$) |
|---|---|---|---|
| Hann | $w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$ | `hann(M+1)` |   |
| Hanning | $w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$ | `hanning(M+1)` |   |
| Hamming | $w[n] = \begin{cases} 0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$ | `hamming(M+1)` |   |

# Tradeoff – Ripple vs. Transition Width



# Optimality



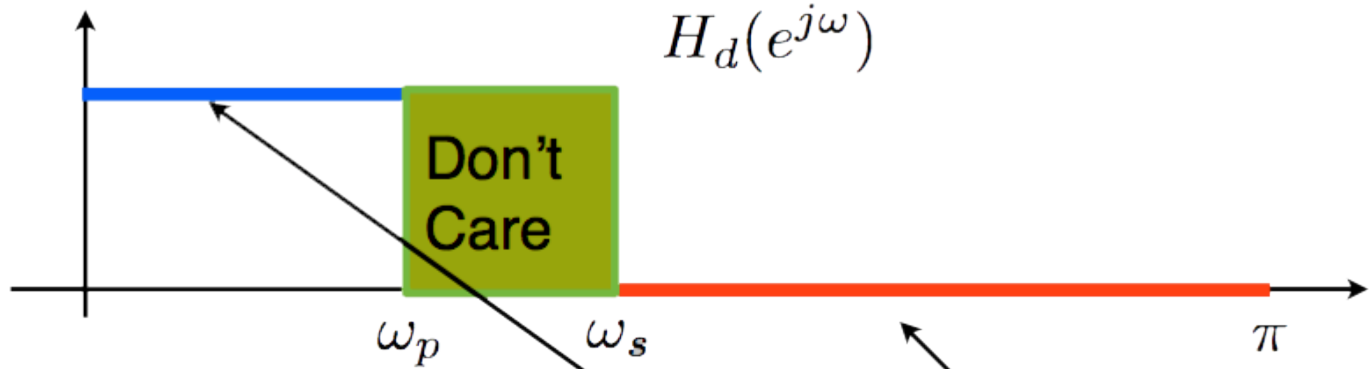

- Least Squares:

$$\text{minimize} \quad \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- Variation: Weighted Least Squares:

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

# Least-Squares Linear Phase Filter




Given M, $\omega_P$, $\omega_S$ find the best LS filter:

$$A = \begin{bmatrix} 1 & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_p) \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix}$$

$$b = [1, 1, \cdots, 1, 0, 0, \cdots, 0]^T$$

# Least-Squares

$$\text{argmin}_{\tilde{h}} \quad ||A\tilde{h} - b||^2$$

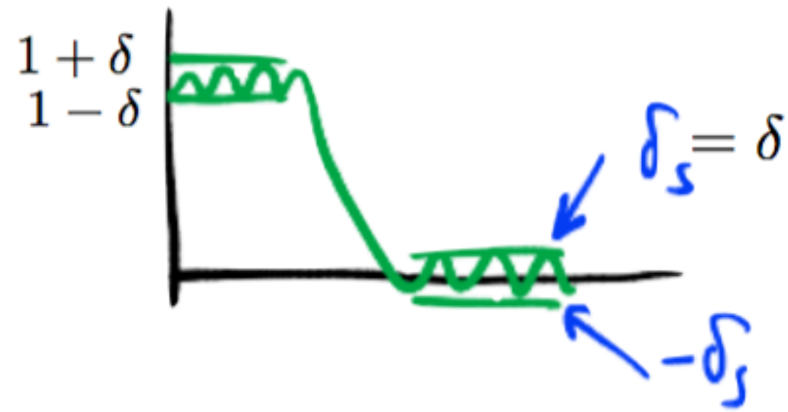
Solution:

$$\tilde{h} = (A^*A)^{-1}A^*b$$

- Result will generally be non-symmetric and complex valued.
- However, if $\tilde{H}(e^{j\omega})$ is real, $\tilde{h}[n]$ should have symmetry!

# Min-Max Ripple Design

- Recall, $\tilde{H}(e^{j\omega})$ is symmetric and real
- Given $\omega_p$, $\omega_s$, M, find $\delta$, $\tilde{h}_+$




$$
\begin{aligned}
&\text{minimize} \quad \delta \\
&\text{Subject to :} \\
&\quad 1-\delta \le \tilde{H}(e^{j\omega_k}) \le 1+\delta \qquad 0 \le \omega_k \le \omega_p \\
&\quad -\delta \le \tilde{H}(e^{j\omega_k}) \le \delta \qquad \omega_s \le \omega_k \le \pi \\
&\quad \delta > 0
\end{aligned}
$$

- Formulation is a linear program with solution $\delta$, $\tilde{h}_+$
- A well studied class of problems

# IIR Filter Design

# IIR Filter Design

- Transform continuous-time filter into a discrete-time filter meeting specs
  - Pick suitable transformation from $s$ (Laplace variable) to $z$ (or $t$ to $n$)
  - Pick suitable analog $H_c(s)$ allowing specs to be met, transform to $H(z)$

- We've seen this before… impulse invariance

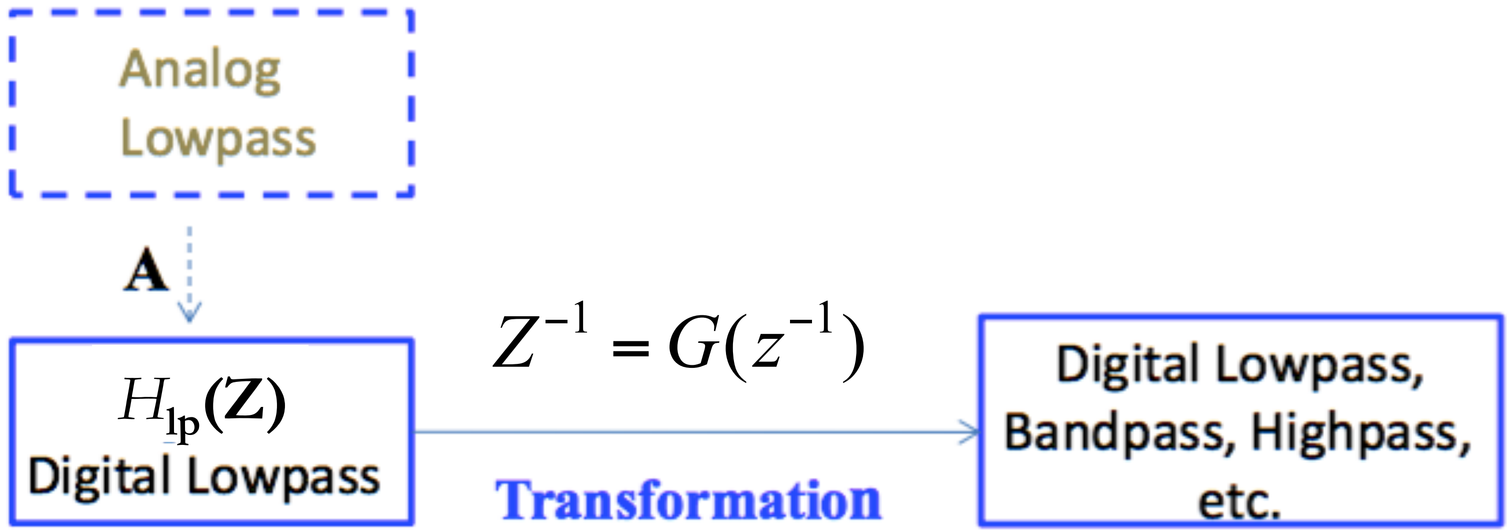
# Bilinear Transformation

- The technique uses an algebraic transformation between the variables s and $z$ that maps the entire $j\Omega$-axis in the s-plane to one revolution of the unit circle in the z-plane.

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right).$$

# Transformation of DT Filters




- Map Z-plane→z-plane with transformation G

$$H(z) = H_{lp}(Z)\Big|_{Z^{-1}=G(z^{-1})}$$

# General Transformations

**TABLE 7.1** TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY $\theta_p$ TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

| Filter Type | Transformations | Associated Design Formulas |
|---|---|---|
| Lowpass | $Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$ | $\alpha = \dfrac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$<br>$\omega_p$ = desired cutoff frequency |
| Highpass | $Z^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$ | $\alpha = -\dfrac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$<br>$\omega_p$ = desired cutoff frequency |
| Bandpass | $Z^{-1} = -\dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$ | $\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$<br>$k = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$<br>$\omega_{p1}$ = desired lower cutoff frequency<br>$\omega_{p2}$ = desired upper cutoff frequency |
| Bandstop | $Z^{-1} = \dfrac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$ | $\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$<br>$k = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$<br>$\omega_{p1}$ = desired lower cutoff frequency<br>$\omega_{p2}$ = desired upper cutoff frequency |

# Discrete Fourier Transform

# Discrete Fourier Transform

- The DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$ **Inverse DFT, synthesis**

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$ **DFT, analysis**

- It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \leq n \leq N-1$$

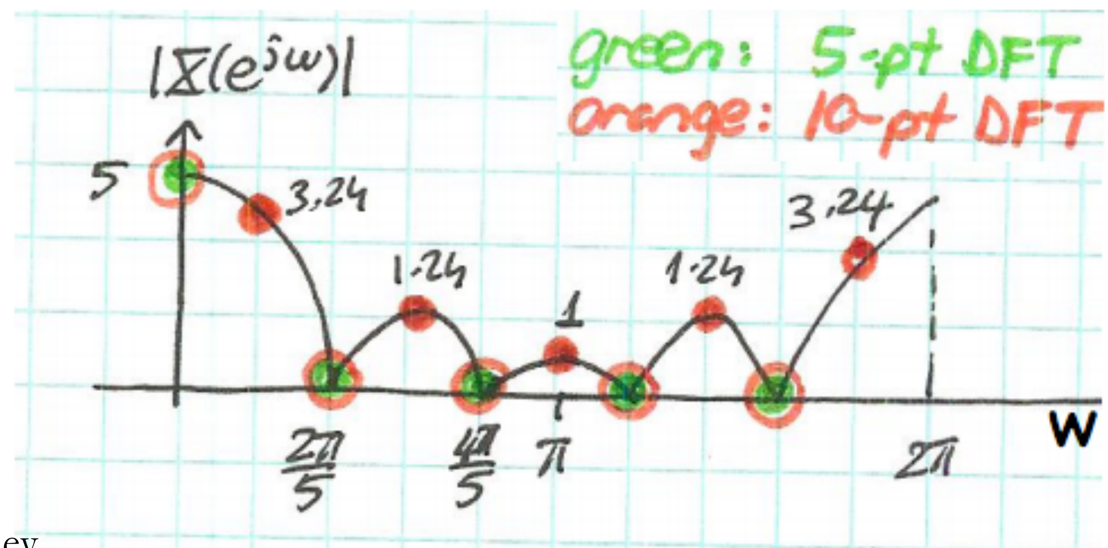
$$X[k] = 0 \quad \text{outside } 0 \leq k \leq N-1$$

# DFT vs DTFT

- Back to example

$$X[k] = \sum_{n=0}^{4} W_{10}^{nk}$$

$$= e^{-j\frac{4\pi}{10}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}$$



# Circular Convolution

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

  - Very useful!! (for linear convolutions with DFT)

# Linear Convolution via Circular Convolution

- Zero-pad x[n] by P-1 zeros

$$x_{\text{zp}}[n] = \begin{cases} x[n] & 0 \leq n \leq L-1 \\ 0 & L \leq n \leq L+P-2 \end{cases}$$

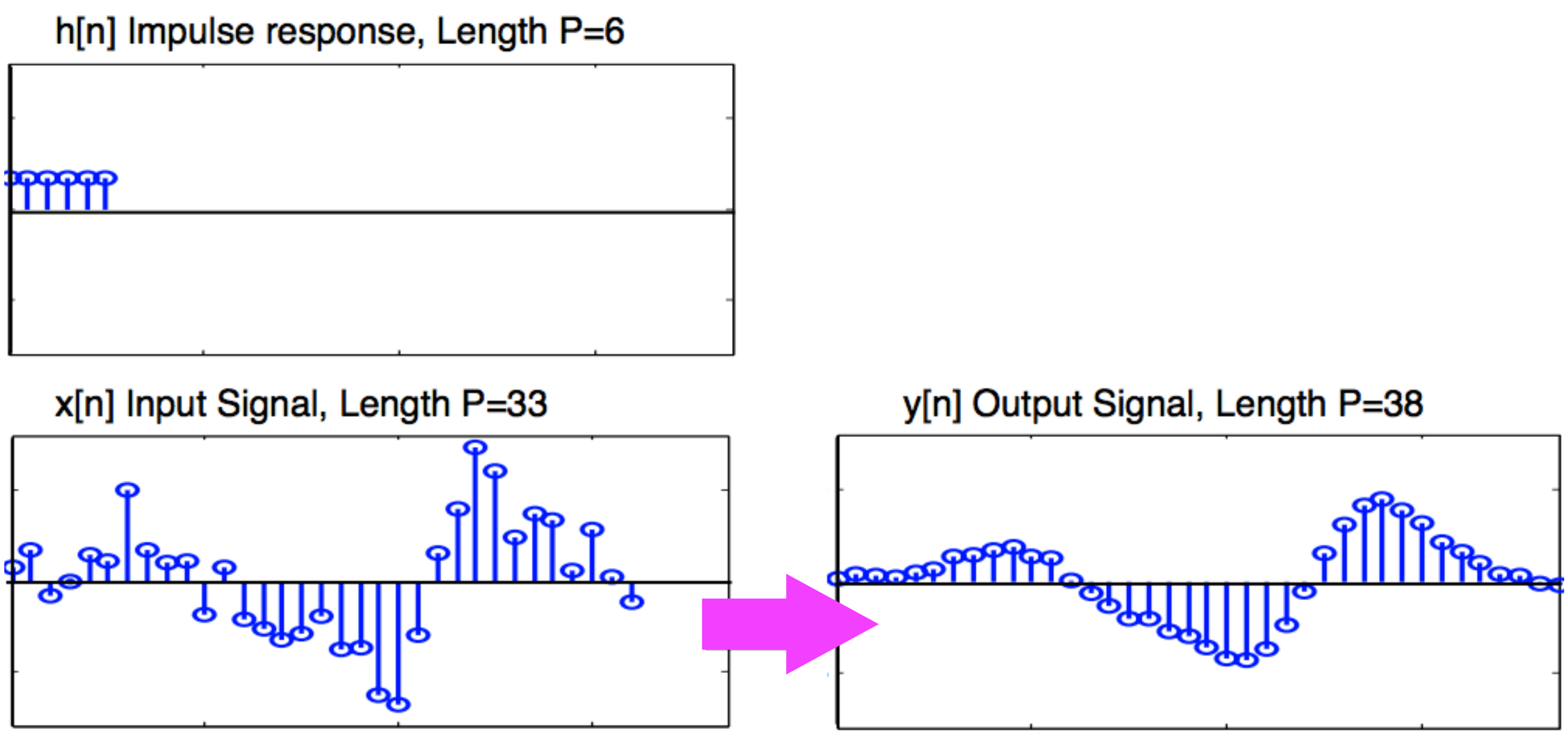
- Zero-pad h[n] by L-1 zeros

$$h_{\text{zp}}[n] = \begin{cases} h[n] & 0 \leq n \leq P-1 \\ 0 & P \leq n \leq L+P-2 \end{cases}$$

- Now, both sequences are length M=L+P-1

# Block Convolution

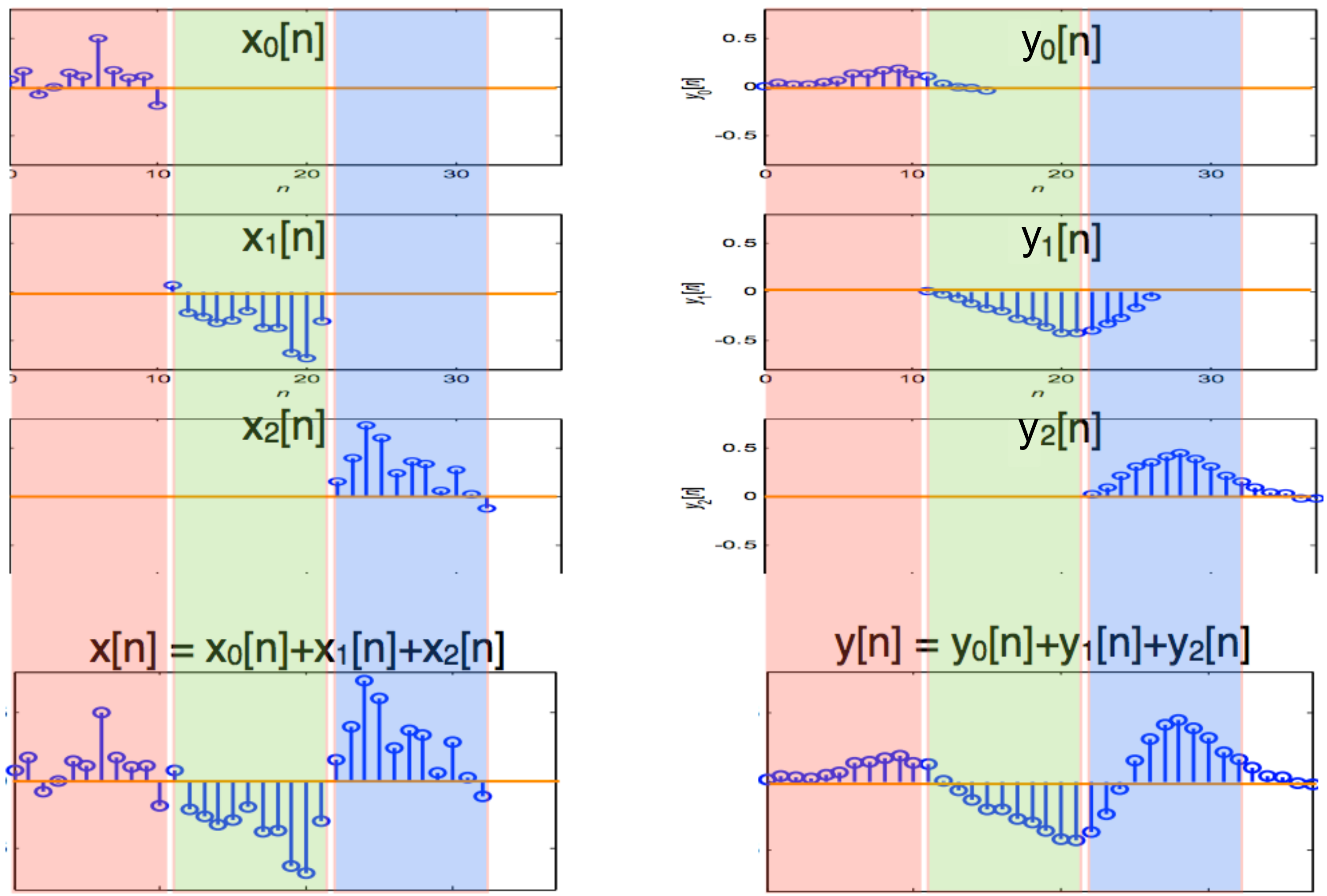
Example:



# Example of Overlap-Add

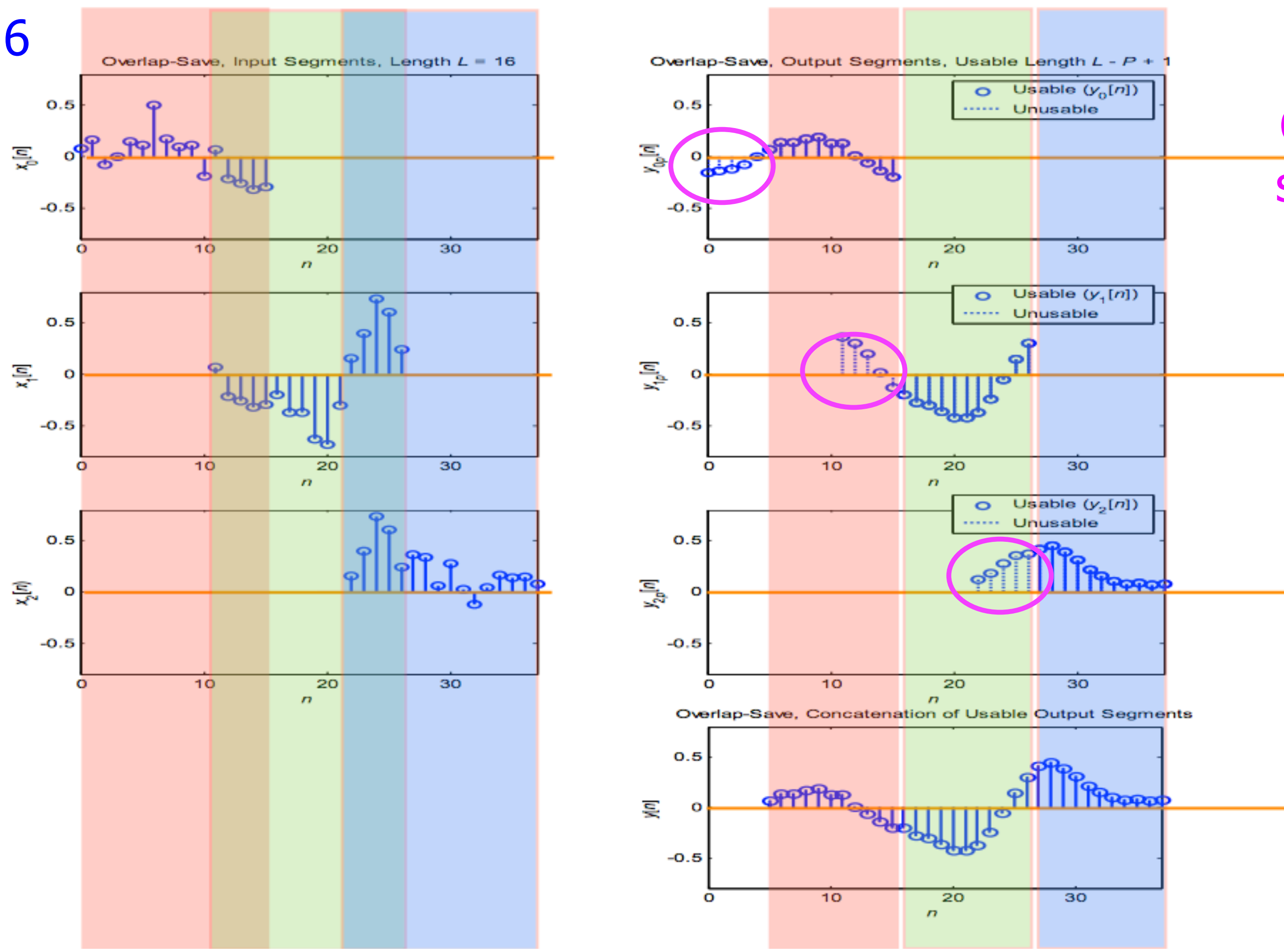
L+P-1=16

L=11



# Example of Overlap-Save

L+P-1=16




P-1=5
Overlap
samples

# Circular Conv. as Linear Conv. w/ Aliasing

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise}, \end{cases}$$

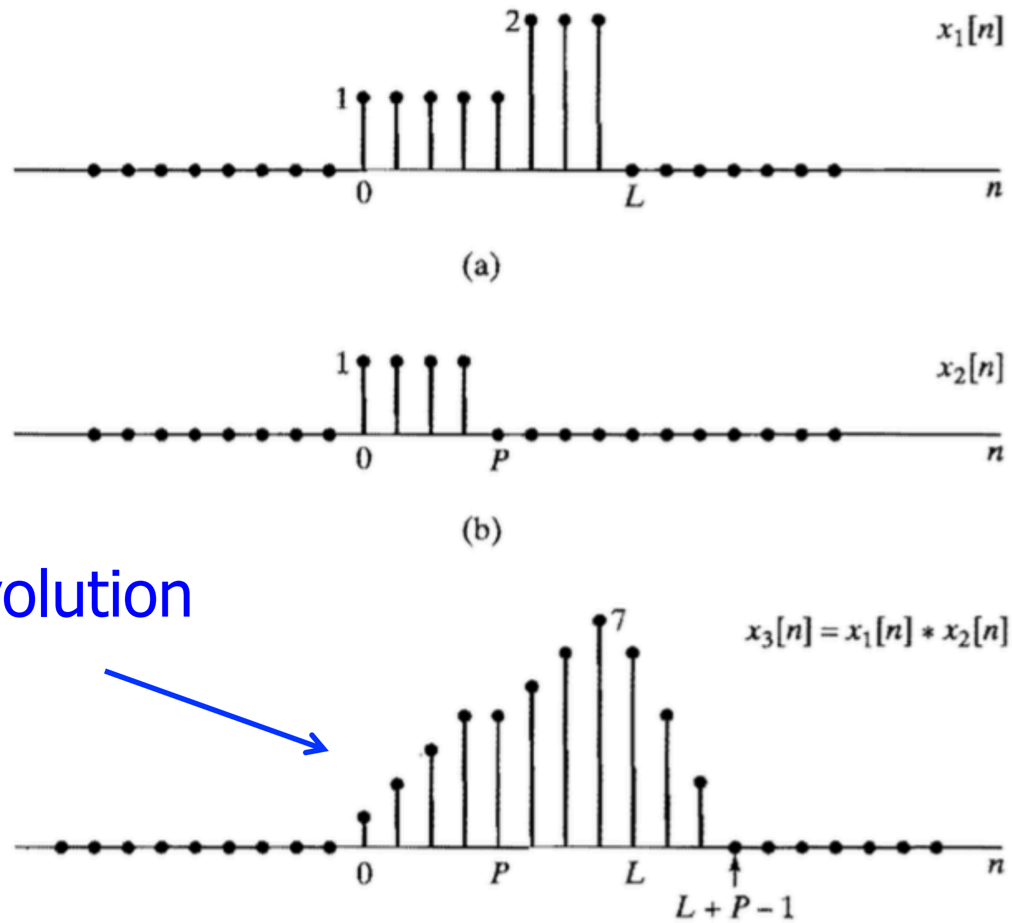
- Therefore

$$x_{3p}[n] = x_1[n] \, \textcircled{N} \, x_2[n]$$

- The N-point circular convolution is the sum of linear convolutions shifted in time by N

# Example 2:

- Let

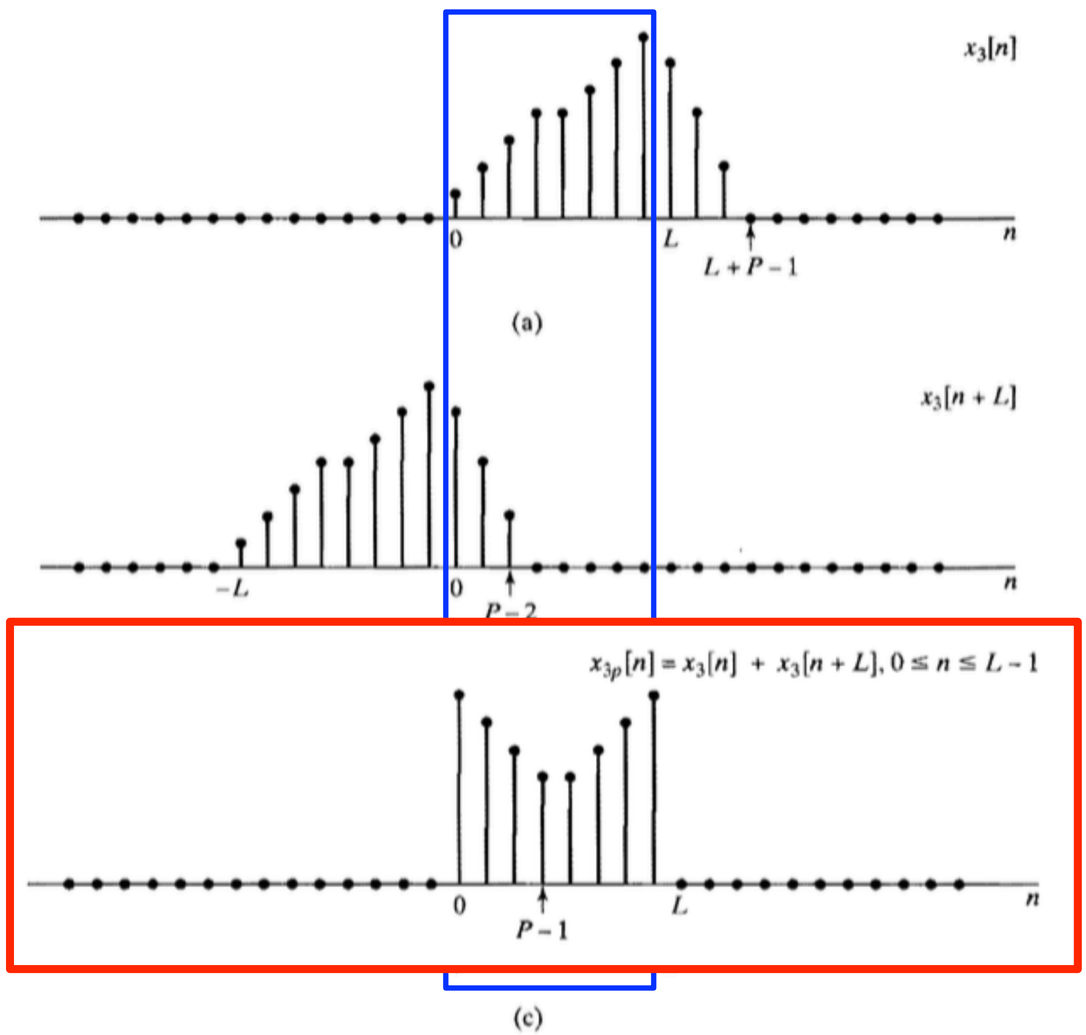



Linear convolution

- What does the L-point circular convolution look like?

# Example 2:

- The L-shifted linear convolutions



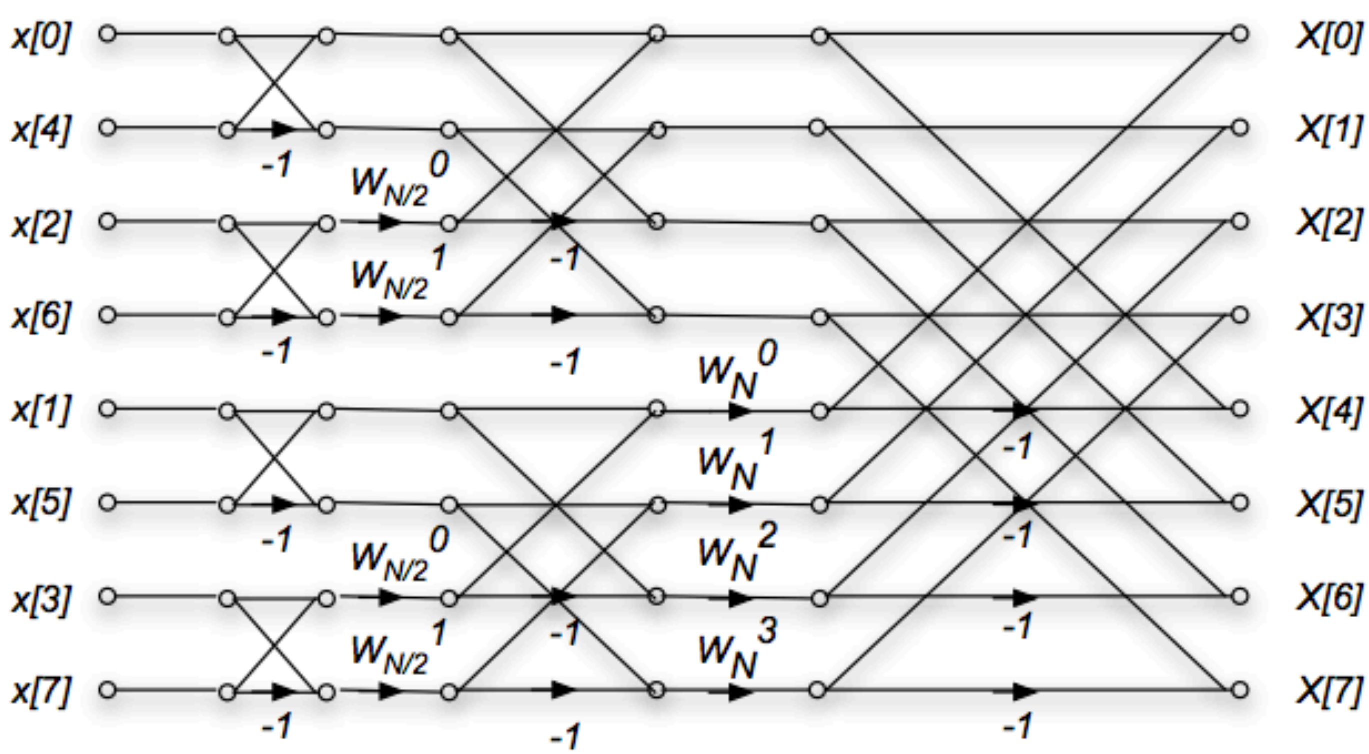
# FFT

# Fast Fourier Transform

- Enable computation of an N-point DFT (or $DFT^{-1}$) with the order of just $N \cdot \log_2 N$ complex multiplications.
- Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
  - Decimation-in-time algorithms
  - Decimation-in-frequency
- Historically, power-of-2 DFTs had highest efficiency
- Modern computing has led to non-power-of-2 FFTs with high efficiency
- Sparsity leads to reduce computation on order $K \cdot \log N$

# Decimation-in-Time FFT

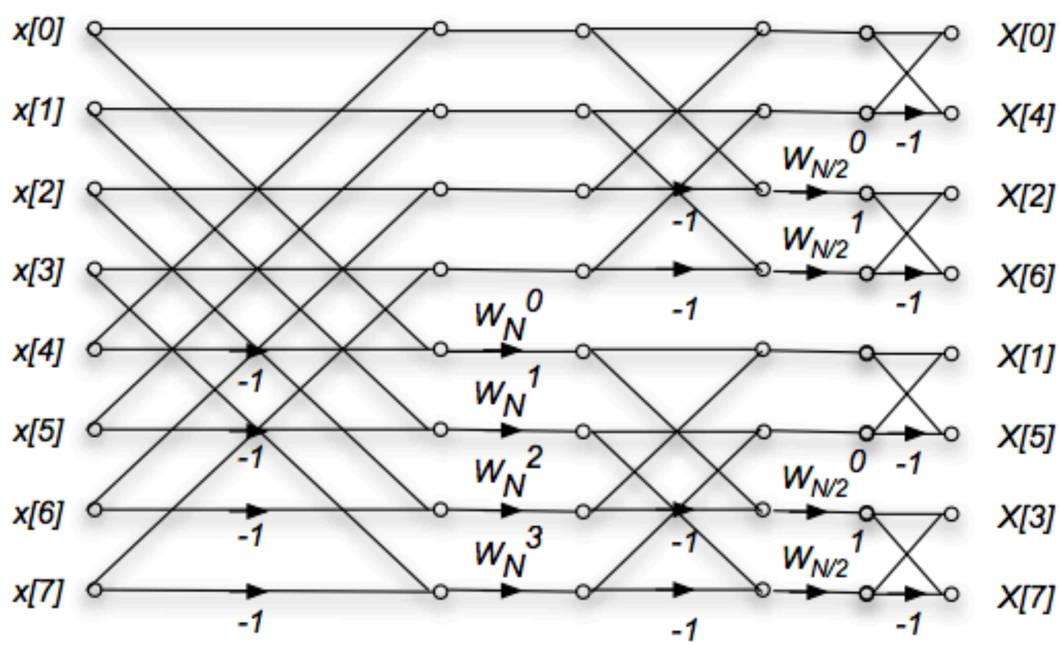
Combining all these stages, the diagram for the 8 sample DFT is:




- $3=\log_2(N)=\log_2(8)$ stages
- $4=N/2=8/2$ multiplications in each stage
  - 1st stage has trivial multiplication

# Decimation-in-Frequency FFT

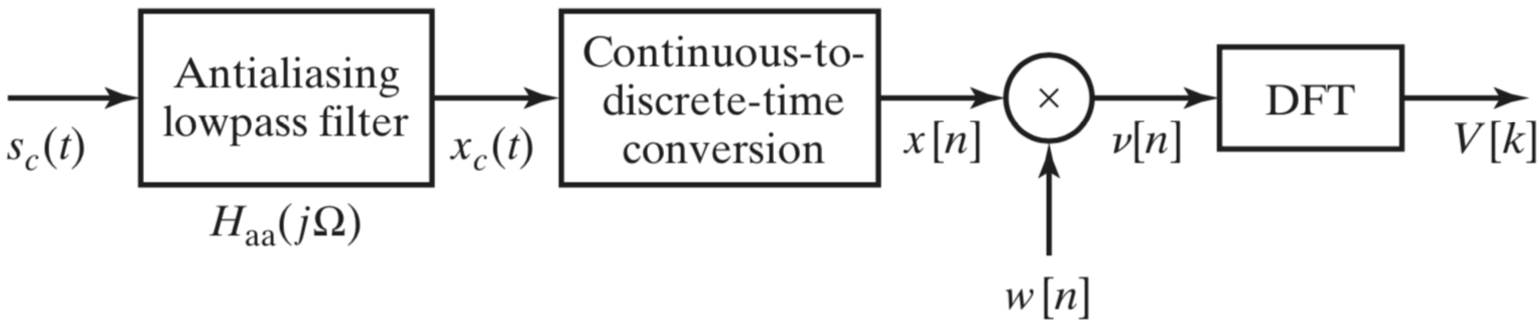
The diagram for and 8-point decimation-in-frequency DFT is as follows




This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

# Spectral Analysis Using the DFT

- Steps for processing continuous time (CT) signals



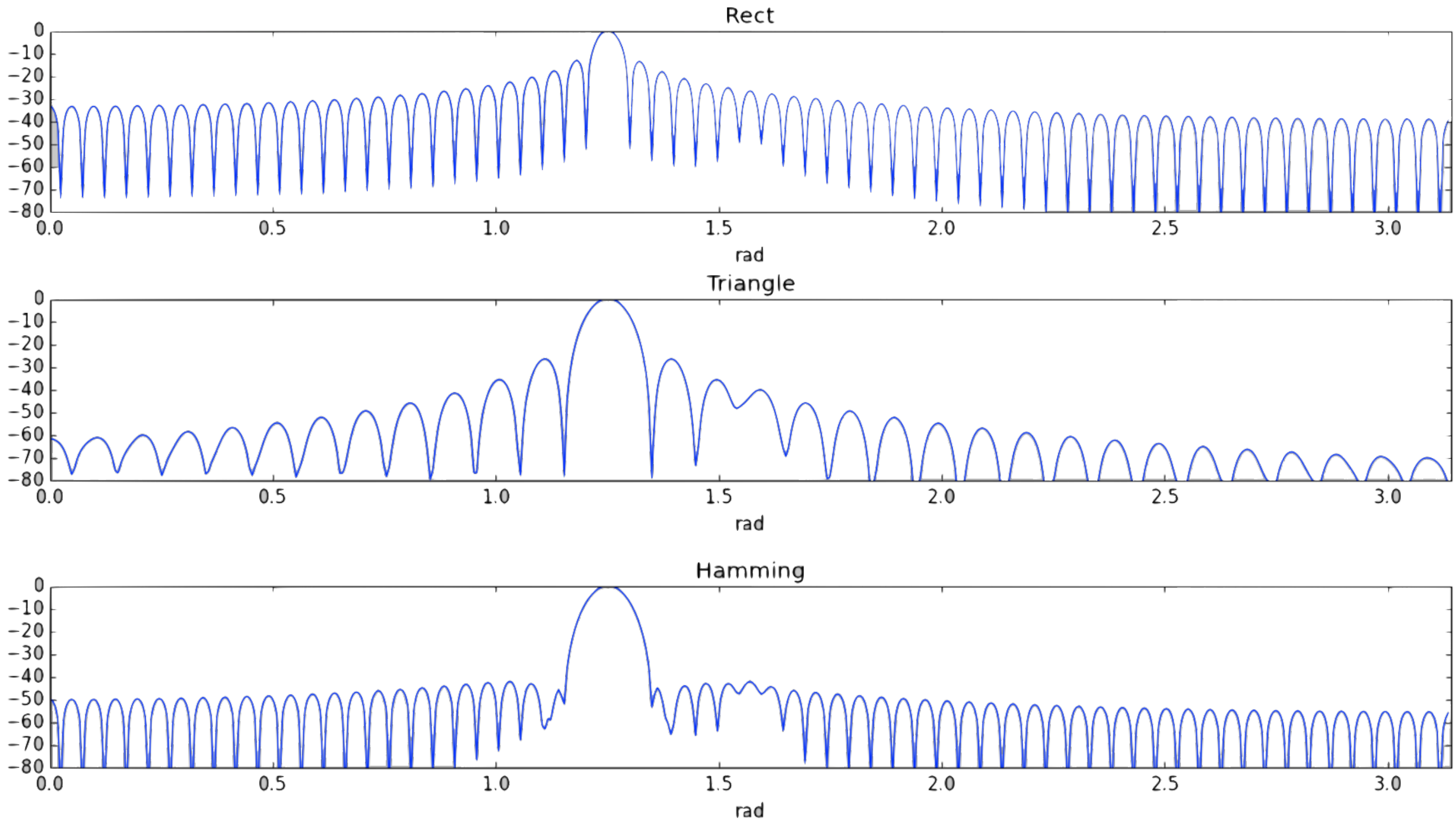
# Spectral Analysis Using the DFT

- Two important tools:
  - Applying a window → reduced artifacts
  - Zero-padding → increases spectral sampling

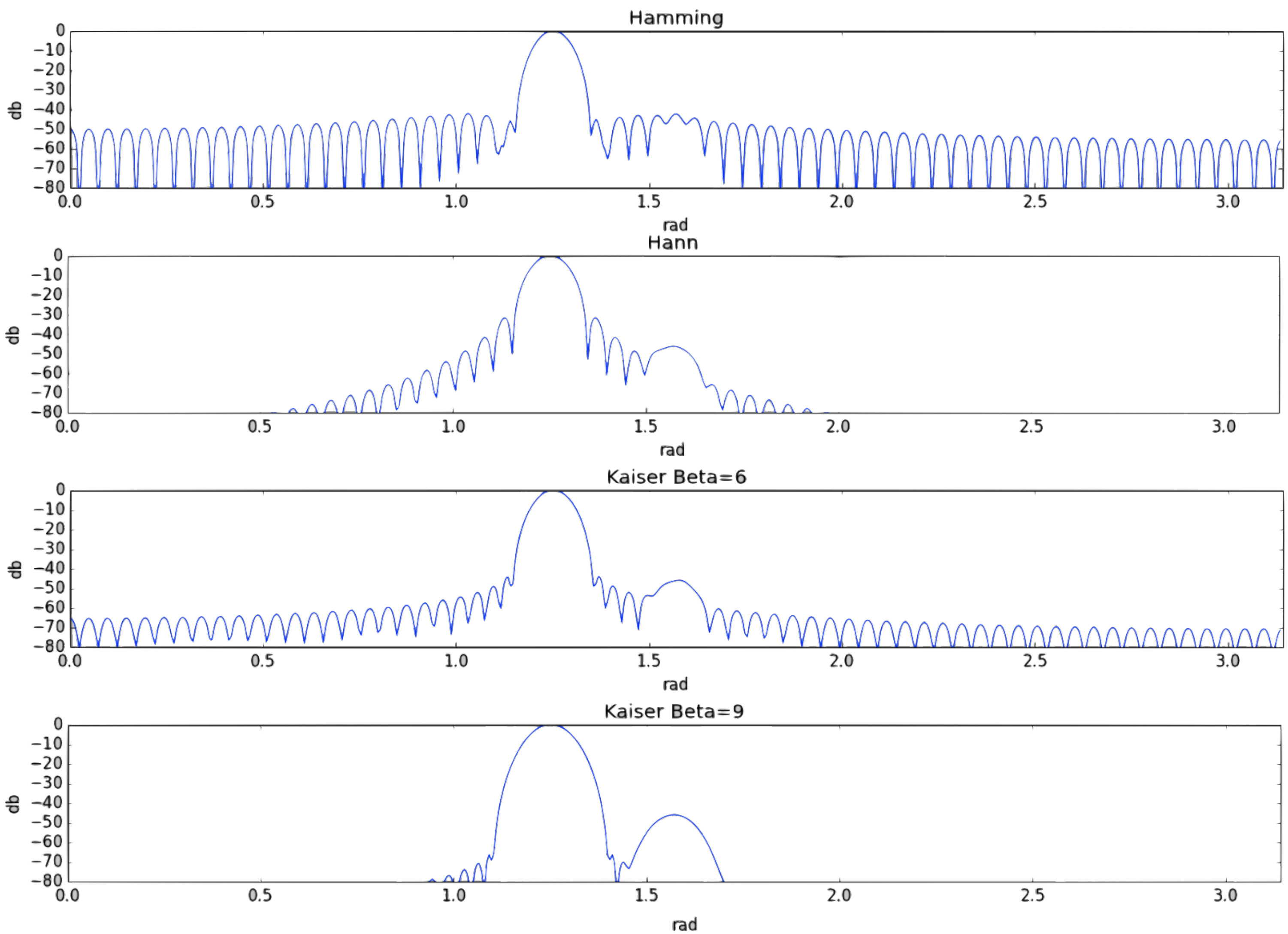
| **Parameter** | **Symbol** | **Units** |
|---|---|---|
| Sampling interval | $T$ | s |
| Sampling frequency | $\Omega_s = \frac{2\pi}{T}$ | rad/s |
| Window length | $L$ | unitless |
| Window duration | $L \cdot T$ | s |
| DFT length | $N \geq L$ | unitless |
| DFT duration | $N \cdot T$ | s |
| Spectral resolution | $\frac{\Omega_s}{L} = \frac{2\pi}{L \cdot T}$ | rad/s |
| Spectral sampling interval | $\frac{\Omega_s}{N} = \frac{2\pi}{N \cdot T}$ | rad/s |

# Window Comparison Example

$$y[n] = \sin(2\pi 0.1992n) + 0.005\sin(2\pi 0.25n) \mid 0 \leq n \leq 128$$



# Window Comparison Example



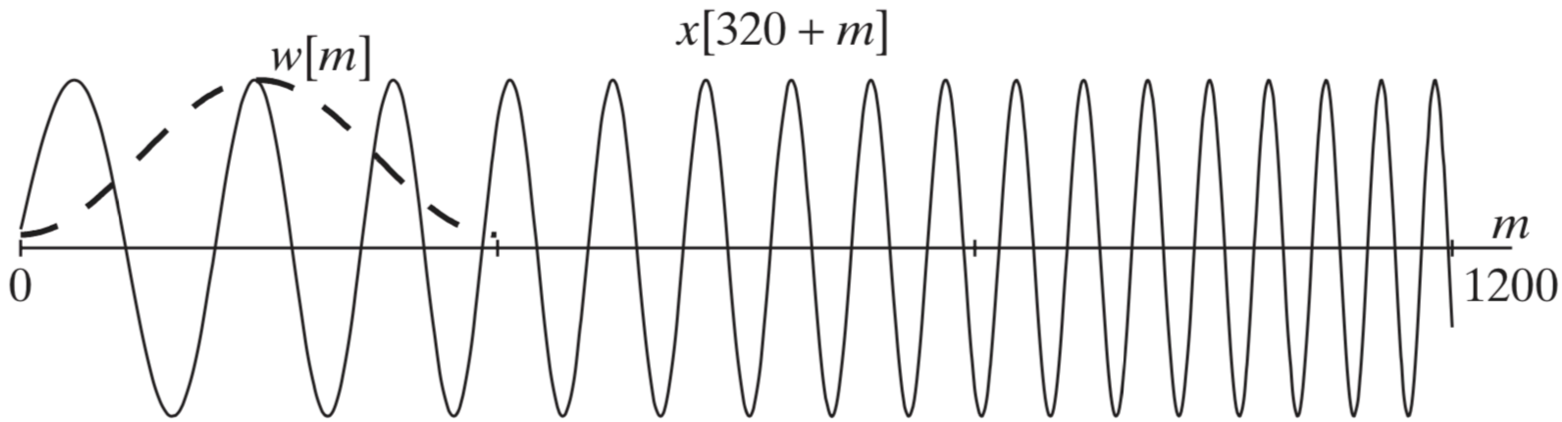
# Time Dependent Fourier Transform

- Also called short-time Fourier transform
- To get temporal information, use part of the signal around every time point

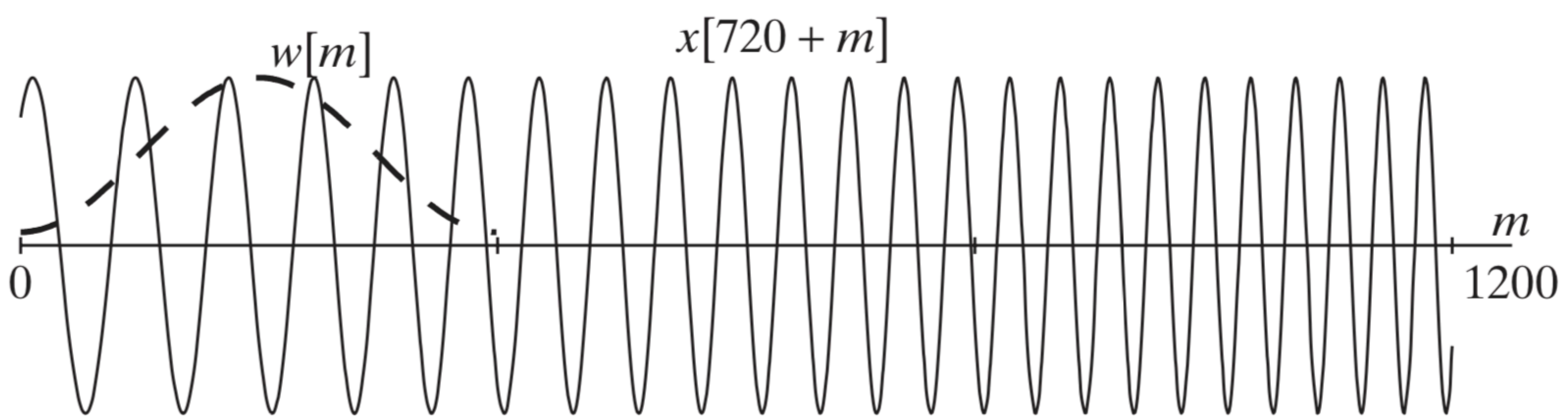
$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

- Mapping from 1D → 2D, n discrete, $\lambda$ cont.
- Simply slide a window and compute DTFT

# Time Dependent Fourier Transform

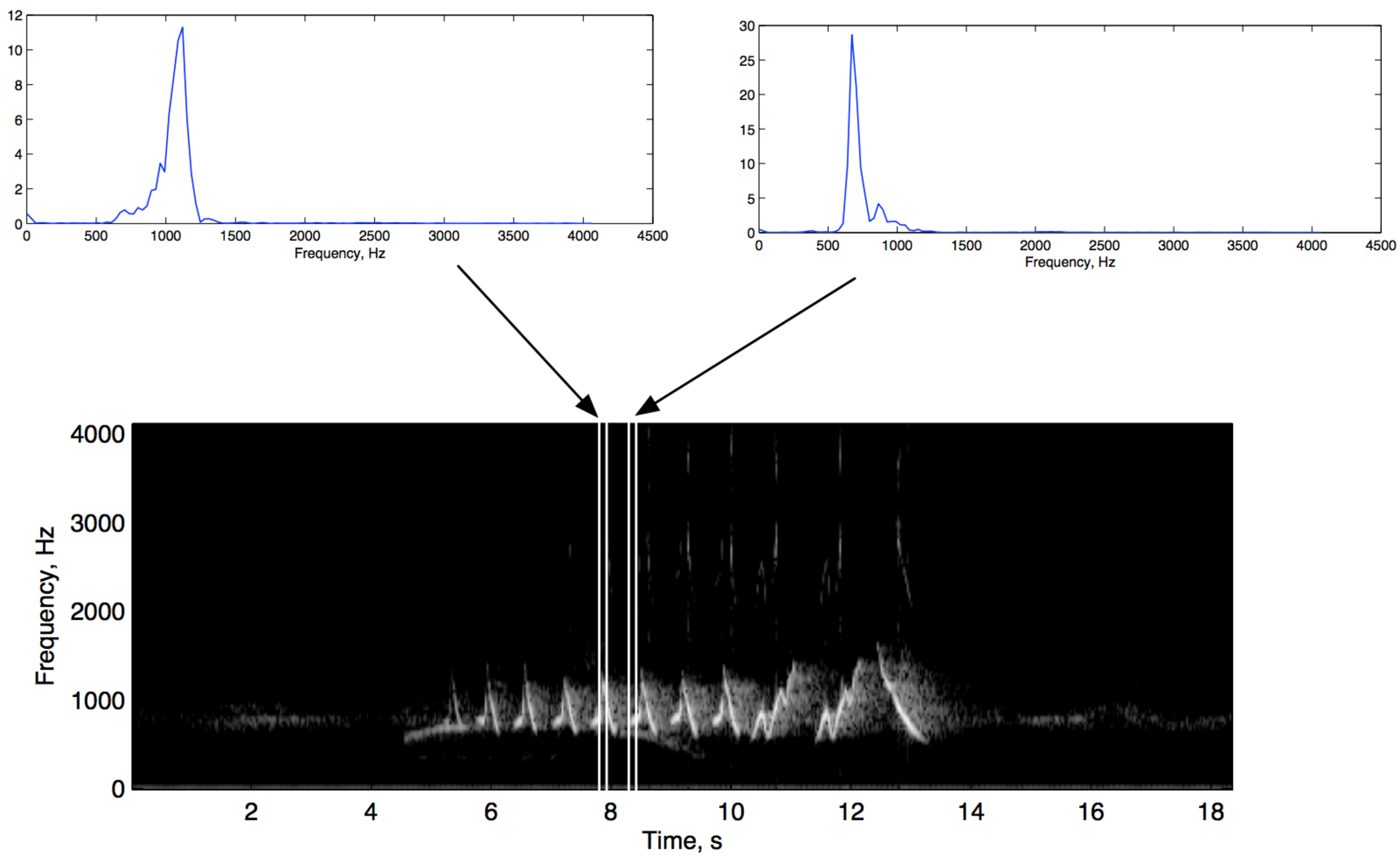


(a)



(b)

# Spectrogram Example



# Final Project

- Due today @midnight – Project must be submitted into Canvas
  - No late projects accepted

# Final Exam

- Final – 4/30
  - Location Towne 100
  - Starts at exactly 3:00pm, ends at exactly 5:00pm (120 minutes)
  - Cumulative – covers entire course
    - Except data converters, noise shaping (lec 11), adaptive filters (lec 22), wavelet transform compressive sampling (lec 24)
  - Closed book
    - Data/Equation sheet provided by me
    - 2 8.5x11 two-sided cheat sheet allowed
    - Calculators allowed, no smart phones
  - Old exams posted
  - Review session by Yexuan and Linyan on 4/26, 3-4pm in Towne 309
  - Wednesday 4/25 – last day for office hours
    - Piazza still available