ESE 531: Digital Signal Processing

Lec 25: April 24, 2018

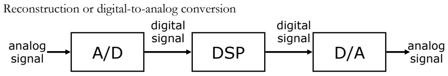
Review

Course Content

- Introduction
- Discrete Time Signals & Systems
- Discrete Time Fourier Transform
- Z-Transform
- Inverse Z-Transform
- Sampling of Continuous Time Signals
- Frequency Domain of Discrete Time Series
- Downsampling/Upsampling
- Data Converters, Sigma Delta Modulation
- Oversampling, Noise Shaping
- Frequency Response of LTI Systems
- Basic Structures for IIR and FIR Systems
- Design of IIR and FIR Filters
- Filter Banks
- Adaptive Filters
- Computation of the Discrete Fourier Transform
- Fast Fourier Transform
- Spectral Analysis
- Wavelet Transform
- Compressive Sampling

Digital Signal Processing

- Represent signals by a sequence of numbers
  - Sampling and quantization (or analog-to-digital conversion)
- Perform processing on these numbers with a digital processor
  - Digital signal processing
- Reconstruct analog signal from processed numbers
  - Reconstruction or digital-to-analog conversion



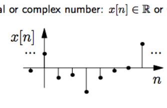
- Analog input → analog output
  - Eg. Digital recording music
- Analog input → digital output
  - Eg. Touch tone phone dialing, speech to text
- Digital input → analog output
  - Eg. Text to speech
- Digital input → digital output
  - Eg. Compression of a file on computer

Discrete Time Signals and Systems

Signals are Functions

DEFINITION: A **signal** is a function that maps an independent variable to a dependent variable.

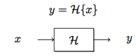
- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
  - Independent variable is an **integer**: $n \in \mathbb{Z}$ (will refer to as time)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$



Discrete Time Systems

DEFINITION: A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

System Properties

- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max)

LTI Systems

DEFINITION: A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response

$$\delta \longrightarrow \boxed{\mathcal{H}} \longrightarrow h$$

- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

$$x \longrightarrow \boxed{h} \longrightarrow y \qquad y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

Discrete Time Fourier Transform

Penn

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}$$

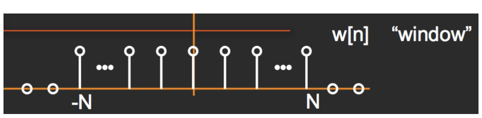
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi f k}$$

$$x[n] = \int_{-0.5}^{0.5} X(f) e^{j2\pi f n} df$$

Example: Window DTFT



$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k] e^{-j\omega k}$$

$$= \sum_{k=-N}^{N} e^{-j\omega k}$$

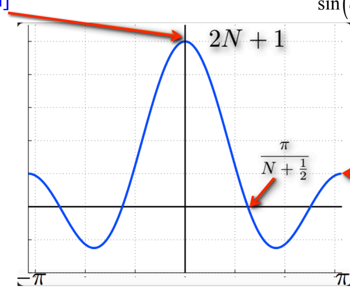
Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Also, Σ x[n]

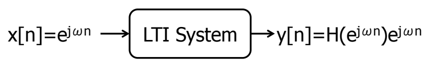
=1 why?

Plot for N=2



LTI System Frequency Response

- Fourier Transform of impulse response

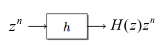


$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

z-Transform

- The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- Very useful for designing and analyzing signal processing systems
- Properties are very similar to the DTFT with a few caveats

Complex Exponentials as Eigenfunctions



- Prove that complex exponentials are the eigenvectors of LTI systems simply by computing the convolution with input z^n

$$z^n * h[n] = \sum_{m=-\infty}^{\infty} z^{n-m}\, h[m] = \sum_{m=-\infty}^{\infty} z^n\, z^{-m}\, h[m]$$

$$= \left(\sum_{m=-\infty}^{\infty} h[m]\, z^{-m}\right) z^n$$

$$= H(z)z^n$$

$$\sum_{n=-\infty}^{\infty} h[n]\, z^{-n} = H(z)$$

Region of Convergence (ROC)

DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its z-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| < \infty$$

Inverse z-Transform

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion
  - Power series expansion

Partial Fraction Expansion

- Let
$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$
- M zeros and N poles at nonzero locations

Partial Fraction Expansion

- If M<N and the poles are 1[st] order

$$X(z)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}=\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

- where

$$A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Example: 2[nd]-Order z-Transform

- 2[nd]-order = two poles

Right sided

$$X(z)=\frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)},\qquad ROC=\left\{z:\ \frac{1}{2}<|z|\right\}$$

5. $a^n u[n]$ $\quad \frac{1}{1-az^{-1}}$ $\quad |z|>|a|$

$$x[n]=-\left(\frac{1}{4}\right)^n u[n]+2\left(\frac{1}{2}\right)^n u[n]$$

Partial Fraction Expansion

- If M≥N and the poles are 1[st] order

$$X(z)=\sum_{r=0}^{M-N}B_r z^{-r}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

- Where B_k is found by long division

$$A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}},\qquad ROC=\left\{z:\ 1<|z|\right\}$$

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & \underline{z^{-2}-3z^{-1}+2} \\ & \quad 5z^{-1}-1 \end{array}$$

$$X(z)=2+\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=2-\frac{9}{1-\frac{1}{2}z^{-1}}+\frac{8}{1-z^{-1}},\qquad ROC=\left\{z:\ 1<|z|\right\}$$

$$x[n]=2\delta[n]-9\left(\frac{1}{2}\right)^n u[n]+8u[n]$$

Power Series Expansion

- Expansion of the z-transform definition

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$
$$=\cdots+x[-2]z^{2}+x[-1]z+x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z)=z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$=z^2-\frac{1}{2}z-1+\frac{1}{2}z^{-1}$$

$$x[n]=\begin{cases}1, & n=-2\\ -1/2, & n=-1\\ -1, & n=0\\ 1/2, & n=1\\ 0, & else\end{cases}$$

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

$$=\cdots+x[-2]z^2+x[-1]z+x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots$$

Difference Equation to z-Transform

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{m=0}^{M}b_m x[n-m]$$

$$y[n]=-\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k]+\sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-m]$$

- Difference equations of this form behave as causal LTI systems
  - when the input is zero prior to n=0
  - Initial rest equations are imposed prior to the time when input becomes nonzero
    - i.e y[-N]=y[-N+1]=…=y[-1]=0

Difference Equation to z-Transform

$$y[n]=-\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k]+\sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)x[n-m]$$

$$Y(z)=-\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)z^{-k}Y(z)+\sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)z^{-k}X(z)$$

$$\sum_{k=0}^{N}\left(\frac{a_k}{a_0}\right)z^{-k}Y(z)=\sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)z^{-k}X(z)\Rightarrow Y(z)=\frac{\sum_{m=0}^{M}(b_k)z^{-k}}{\sum_{k=0}^{N}(a_k)z^{-k}}X(z)$$

$$H(z)=\frac{\sum_{m=0}^{M}(b_k)z^{-k}}{\sum_{k=0}^{N}(a_k)z^{-k}}$$

Example: 1st-Order System

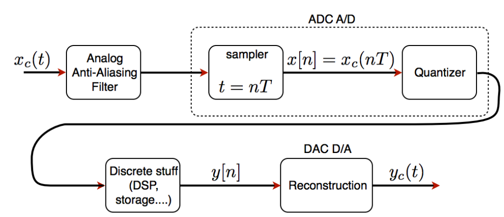
$$H(z)=\frac{\sum_{m=0}^{M}(b_k)z^{-k}}{\sum_{k=0}^{N}(a_k)z^{-k}}$$

$$y[n]=ay[n-1]+x[n]$$

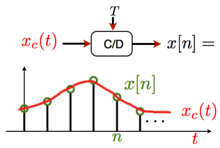
$$H(z)=\frac{1}{1-az^{-1}} \qquad h[n]=a^n u[n]$$

Sampling and Reconstruction

DSP System



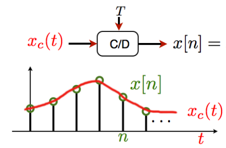
Ideal Sampling Model



Discrete and Continuous

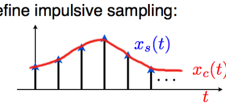
- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - f_s=1/T is the sampling frequency
  - $\Omega_s = 2\pi/T$

Ideal Sampling Model



Discrete and Continuous

define impulsive sampling:

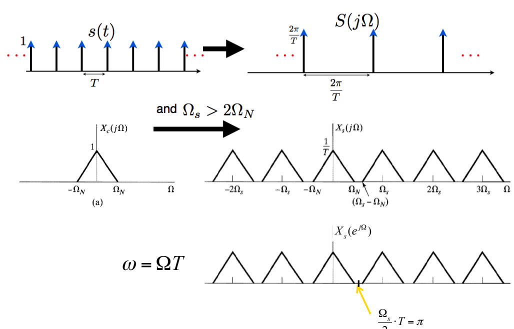


Continuous

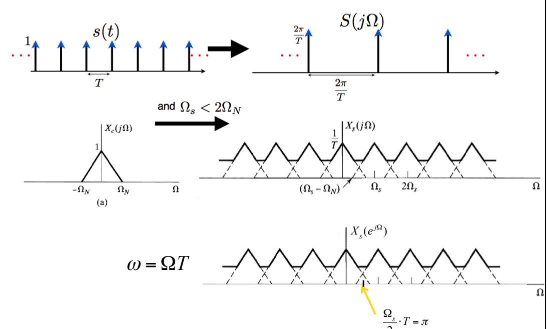
$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis

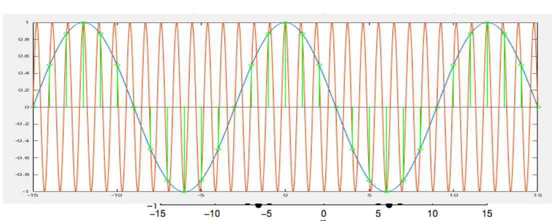


Frequency Domain Analysis



Aliasing Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$

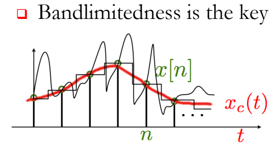


Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

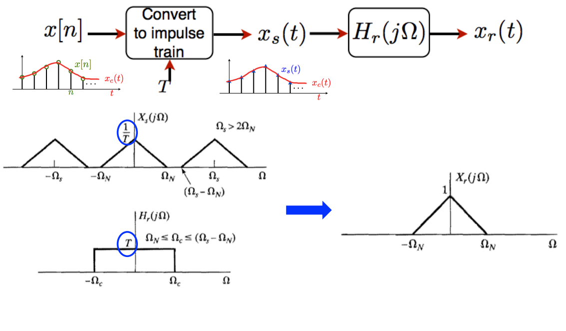
$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness



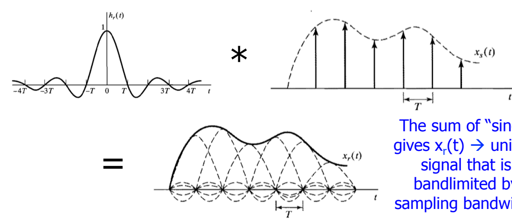
Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

Reconstruction in Frequency Domain



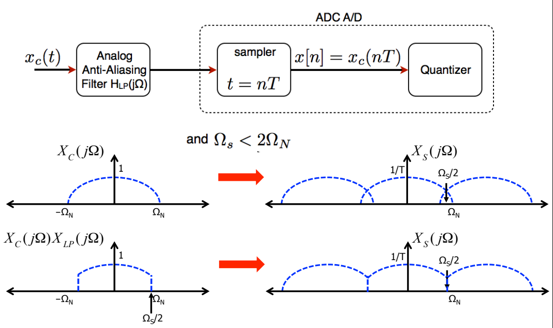
Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t)$$
$$= \sum_n x[n] h_r(t-nT)$$



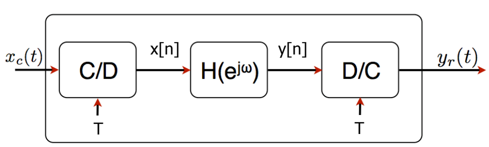
The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

Anti-Aliasing Filter



DT and CT processing

Discrete-Time Processing of Continuous Time



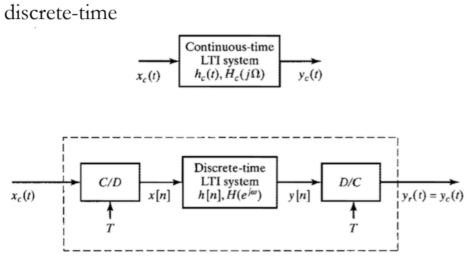
$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right] \qquad y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

- If $x_c(t)$ is bandlimited by $\Omega_s/T = \pi/T$, then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{eff}(j\Omega) = \begin{cases} H(e^{j\omega})\big|_{\omega=\Omega T} & |\Omega| < \Omega_s/T \\ 0 & else \end{cases}$$

Impulse Invariance

- Want to implement continuous-time system in discrete-time



Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\omega / T), \qquad |\omega| < \pi$$

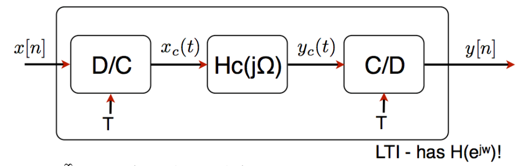
- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi / T$$

$$\boxed{h[n] = T h_c(nT)}$$

Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time



$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

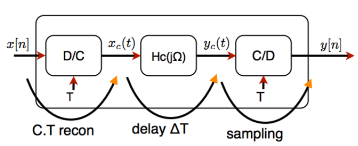
$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

Example: Non-integer Delay

- What is the time domain operation when Δ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

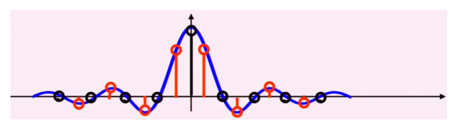
Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ delay of ΔT in time



Example: Non-integer Delay

- My delay system has an impulse response of a sinc with a continuous time delay

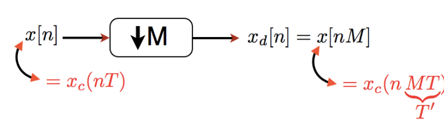
$$h[n] = \text{sinc}(n - \Delta)$$



Rate Re-Sampling

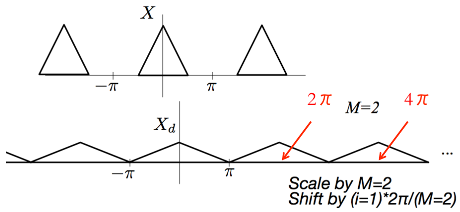
Downsampling

- Definition: Reducing the sampling rate by an integer number



Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M}-\frac{2\pi}{M}i\right)}\right)$$



Example

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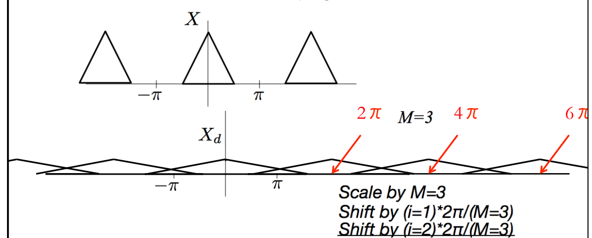
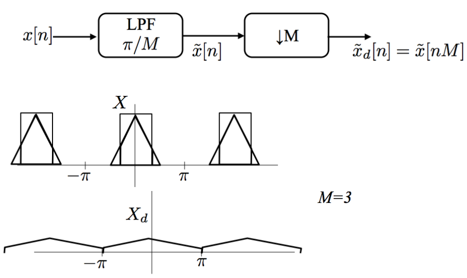


Example



Upsampling

- Definition: Increasing the sampling rate by an integer number

$x[n] = x_c(nT)$

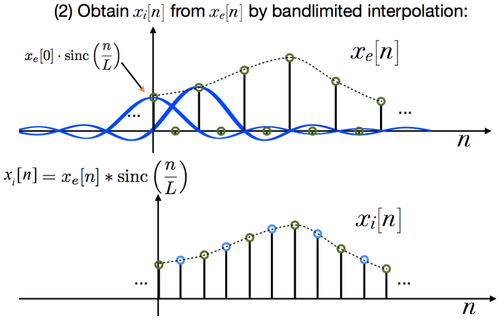
$x_i[n] = x_c(nT')$ where $T' = \frac{T}{L}$ L integer

Obtain $x_i[n]$ from $x[n]$ in two steps:

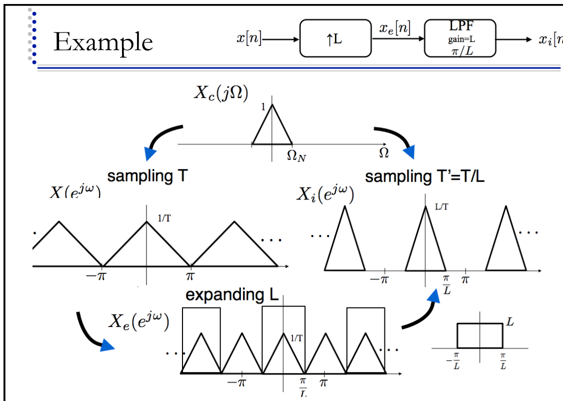
(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:

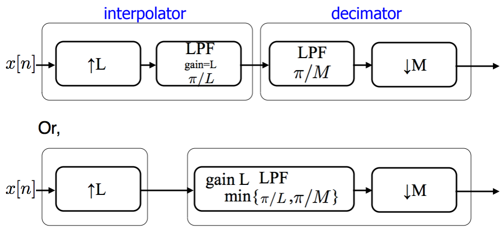


Example

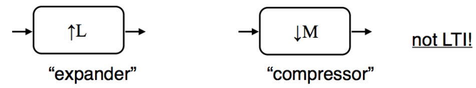


Non-integer Sampling

- T'=TM/L
  - Upsample by L, then downsample by M



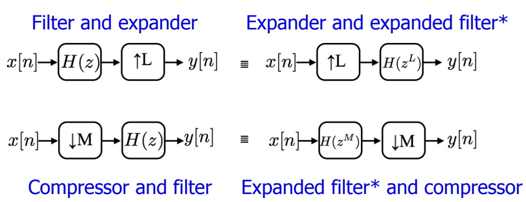
Interchanging Operations



Upsampling
-expanding in time
-compressing in frequency

Downsampling
-compressing in time
-expanding in frequency

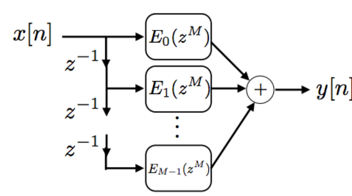
Interchanging Operations - Summary



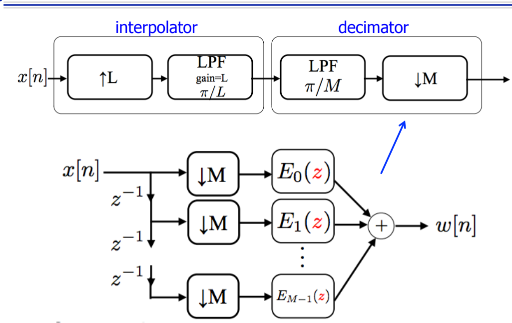
*Expanded filter = expanded impulse response, compressed freq response

Polyphase Decomposition

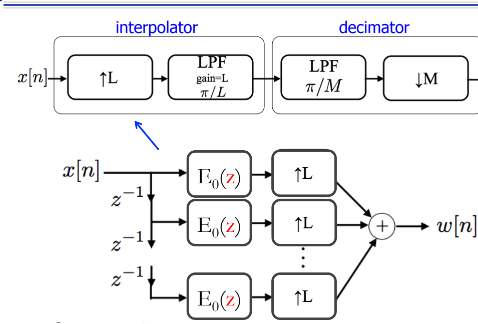
$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$$



Polyphase Implementation of Decimator

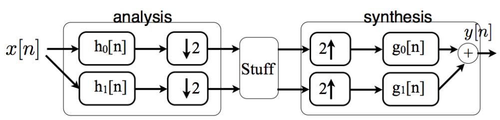


Polyphase Implementation of Interpolation

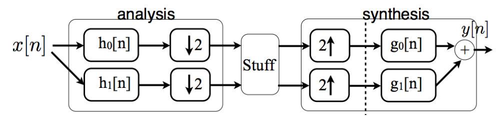


Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering
- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π



Perfect Reconstruction non-Ideal Filters

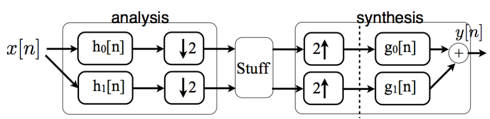


$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]X(e^{j(\omega-\pi)})$$

need to cancel!

aliasing

Quadrature Mirror Filters



Quadrature mirror filters

$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$
$$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$$
$$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$$

Frequency Response of Systems

Frequency Response of LTI System

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- We can define a magnitude response

$$\left|Y(e^{j\omega})\right| = \left|H(e^{j\omega})\right|\left|X(e^{j\omega})\right|$$

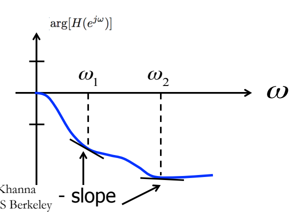
- And a phase response

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

Stable and causal if all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0 \prod_{k=1}^{M}(1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

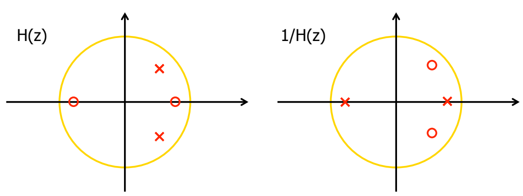
General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

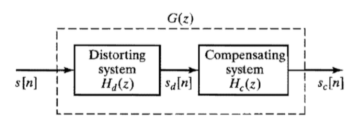
Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle



Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\, H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z)$ → $H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

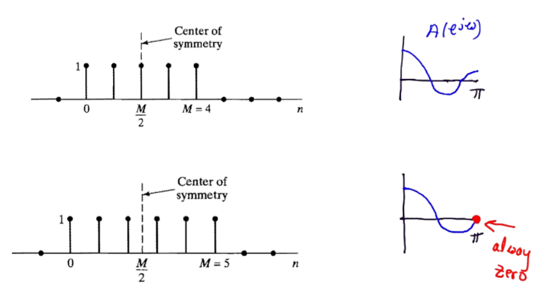
Generalized Linear Phase

- An LTI system has generalized linear phase if frequency response $H(e^{j\omega})$ can be expressed as:

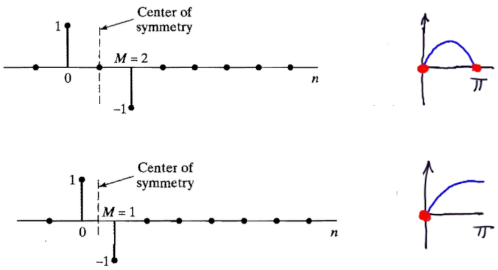
$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha + j\beta}, \quad |\omega| < \pi$$

- Where $A(\omega)$ is a real function.
- What is the group delay?

FIR GLP: Type I and II



FIR GLP: Type III and IV



Zeros of GLP System

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

- If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

- If zero is real and not on unit circle (θ =0)

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

FIR Filter Design

Penn

FIR Design by Windowing

- Given desired frequency response, $H_d(e^{j\omega})$, find an impulse response

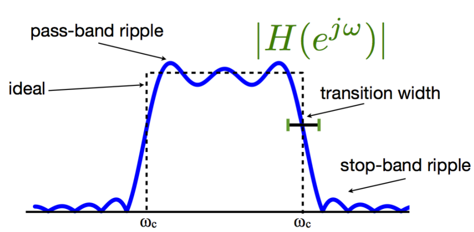
$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

ideal

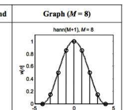
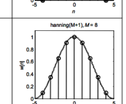
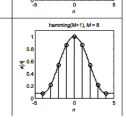
- Obtain the M^{th} order causal FIR filter by truncating/windowing it

$$h[n] = \left\{ \begin{array}{ll} h_d[n]w[n] & 0 \le n \le M \\ 0 & \text{otherwise} \end{array} \right\}$$

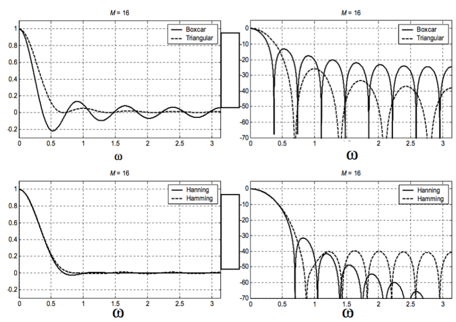
FIR Design by Windowing



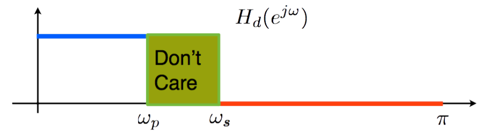
Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1 + \cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1 + \cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54 + 0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`hamming(M+1)`	

Tradeoff – Ripple vs. Transition Width



Optimality



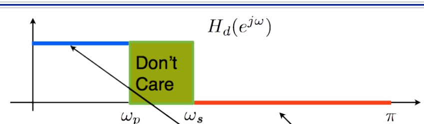
- Least Squares:

$$\text{minimize} \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- Variation: Weighted Least Squares:

$$\text{minimize} \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Least-Squares Linear Phase Filter



Given M, ω_p, ω_s find the best LS filter:

$$A = \begin{bmatrix} 1 & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_p) \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix} \quad b = [1, 1, \cdots, 1, 0, 0, \cdots, 0]^T$$

Least-Squares

$$\text{argmin}_{\tilde{h}} \quad ||A\tilde{h} - b||^2$$

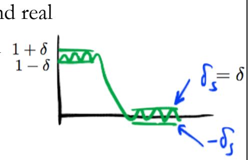
Solution:

$$\tilde{h} = (A^*A)^{-1}A^*b$$

- Result will generally be non-symmetric and complex valued.
- However, if $\tilde{H}(e^{j\omega})$ is real, $\tilde{h}[n]$ should have symmetry!

Min-Max Ripple Design

- Recall, $\tilde{H}(e^{j\omega})$ is symmetric and real
- Given ω_p, ω_s, M, find δ, $\tilde{h}_+$



$$\begin{aligned} &\text{minimize} \quad \delta \\ &\text{Subject to :} \\ &1-\delta \le \tilde{H}(e^{j\omega_k}) \le 1+\delta \qquad 0 \le \omega_k \le \omega_p \\ &-\delta \le \tilde{H}(e^{j\omega_k}) \le \delta \qquad \omega_s \le \omega_k \le \pi \\ &\delta > 0 \end{aligned}$$

- Formulation is a linear program with solution δ, $\tilde{h}_+$
- A well studied class of problems

IIR Filter Design

Penn

IIR Filter Design

- Transform continuous-time filter into a discrete-time filter meeting specs
  - Pick suitable transformation from *s* (Laplace variable) to *z* (or *t* to *n*)
  - Pick suitable analog $H_c(s)$ allowing specs to be met, transform to $H(z)$
- We've seen this before… impulse invariance

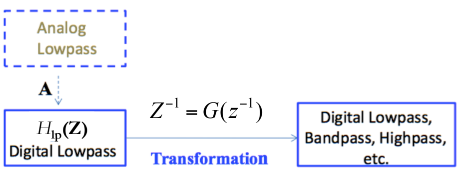
Bilinear Transformation

- The technique uses an algebraic transformation between the variables s and *z* that maps the entire j Ω -axis in the s-plane to one revolution of the unit circle in the z-plane.

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right);$$

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right).$$

Transformation of DT Filters



- Map Z-plane→z-plane with transformation G

$$H(z) = H_{lp}(Z)\big|_{Z^{-1}=G(z^{-1})}$$

General Transformations

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - az^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k}z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2} - \frac{2\alpha}{1+k}z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

Discrete Fourier Transform

Discrete Fourier Transform

- The DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \quad \text{DFT, analysis}$$

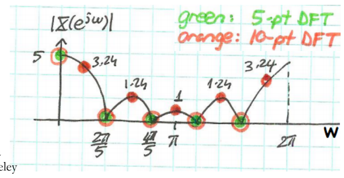
- It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \le n \le N-1$$
$$X[k] = 0 \quad \text{outside } 0 \le k \le N-1$$

DFT vs DTFT

- Back to example

$$X[k] = \sum_{n=0}^{4} W_{10}^{nk}$$
$$= e^{-j\frac{4\pi}{10}k} \frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}$$



Circular Convolution

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \circledast_N x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

  - Very useful!! (for linear convolutions with DFT)

Linear Convolution via Circular Convolution

- Zero-pad x[n] by P-1 zeros

$$x_{zp}[n] = \begin{cases} x[n] & 0 \le n \le L-1 \\ 0 & L \le n \le L+P-2 \end{cases}$$

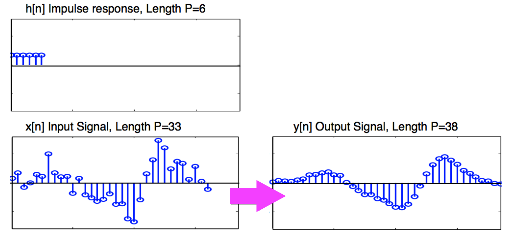
- Zero-pad h[n] by L-1 zeros

$$h_{zp}[n] = \begin{cases} h[n] & 0 \le n \le P-1 \\ 0 & P \le n \le L+P-2 \end{cases}$$

- Now, both sequences are length M=L+P-1

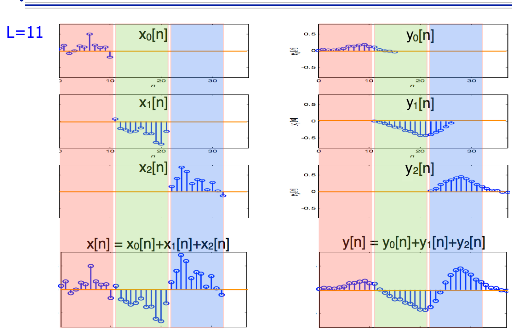
Block Convolution

Example:

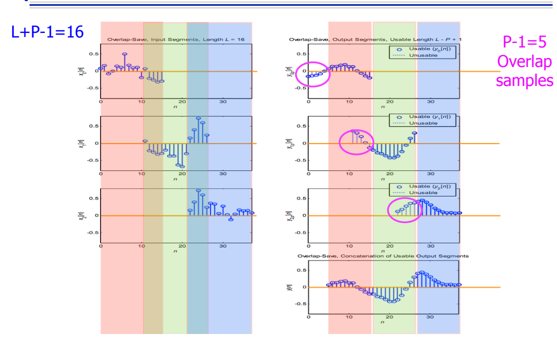


Example of Overlap-Add

L+P-1=16



Example of Overlap-Save



Circular Conv. as Linear Conv. w/ Aliasing

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise}, \end{cases}$$

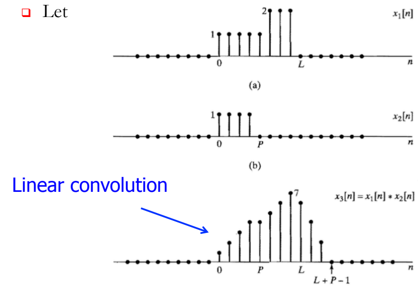
- Therefore

$$x_{3_p}[n] = x_1[n] \circledN x_2[n]$$

- The N-point circular convolution is the sum of linear convolutions shifted in time by N

Example 2:

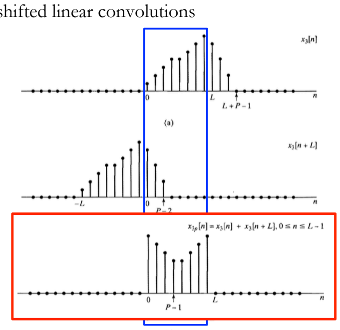
- Let



- What does the L-point circular convolution look like?

Example 2:

- The L-shifted linear convolutions



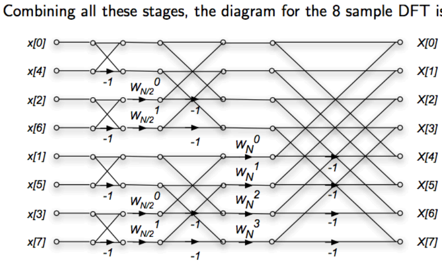
FFT

Fast Fourier Transform

- Enable computation of an N-point DFT (or DFT^{-1}) with the order of just N · $\log_2$ N complex multiplications.
- Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
  - Decimation-in-time algorithms
  - Decimation-in-frequency
- Historically, power-of-2 DFTs had highest efficiency
- Modern computing has led to non-power-of-2 FFTs with high efficiency
- Sparsity leads to reduce computation on order K · logN

Decimation-in-Time FFT

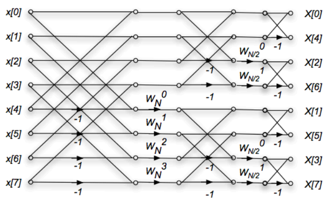
Combining all these stages, the diagram for the 8 sample DFT is:



- 3=$\log_2$(N)=$\log_2$(8) stages
- 4=N/2=8/2 multiplications in each stage
  - 1st stage has trivial multiplication

Decimation-in-Frequency FFT

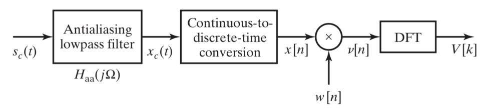
The diagram for and 8-point decimation-in-frequency DFT is as follows



This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

Spectral Analysis Using the DFT

- Steps for processing continuous time (CT) signals



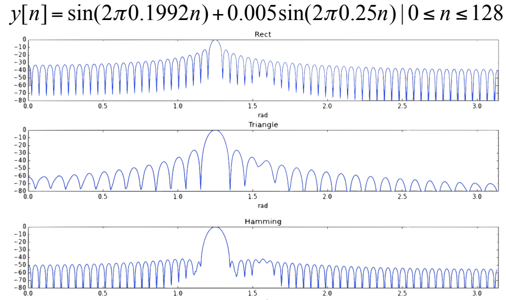
Spectral Analysis Using the DFT

- Two important tools:
  - Applying a window → reduced artifacts
  - Zero-padding → increases spectral sampling

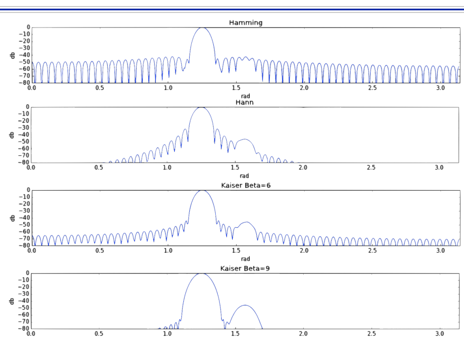
Parameter	Symbol	Units
Sampling interval	T	s
Sampling frequency	$\Omega_s = \frac{2\pi}{T}$	rad/s
Window length	L	unitless
Window duration	$L \cdot T$	s
DFT length	$N \geq L$	unitless
DFT duration	$N \cdot T$	s
Spectral resolution	$\frac{\Omega_s}{L} = \frac{2\pi}{L \cdot T}$	rad/s
Spectral sampling interval	$\frac{\Omega_s}{N} = \frac{2\pi}{N \cdot T}$	rad/s

Window Comparison Example

$$y[n] = \sin(2\pi 0.1992n) + 0.005\sin(2\pi 0.25n) \mid 0 \leq n \leq 128$$



Window Comparison Example



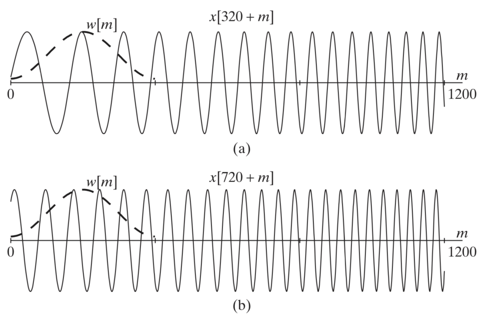
Time Dependent Fourier Transform

- Also called short-time Fourier transform
- To get temporal information, use part of the signal around every time point

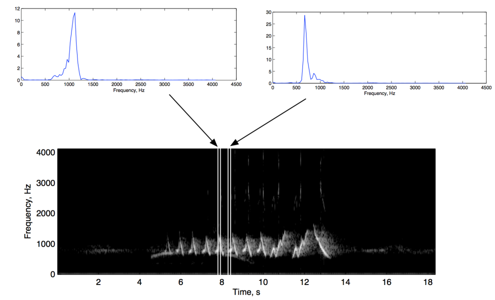
$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

- Mapping from 1D → 2D, n discrete, λ cont.
- Simply slide a window and compute DTFT

Time Dependent Fourier Transform



Spectrogram Example



Final Project

- Due today @midnight – Project must be submitted into Canvas
  - No late projects accepted

Final Exam

- Final – 4/30
  - Location Towne 100
  - Starts at exactly 3:00pm, ends at exactly 5:00pm (120 minutes)
  - Cumulative – covers entire course
    - Except data converters, noise shaping (lec 11), adaptive filters (lec 22), wavelet transform compressive sampling (lec 24)
  - Closed book
    - Data/Equation sheet provided by me
    - 2 8.5x11 two-sided cheat sheet allowed
    - Calculators allowed, no smart phones
  - Old exams posted
  - Review session by Yexuan and Linyan on 4/26, 3-4pm in Towne 309
  - Wednesday 4/25 – last day for office hours
    - Piazza still available