

ESE 531: Digital Signal Processing

Lec 2: January 16, 2018
Discrete Time Signals and Systems



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Lecture Outline

- Discrete Time Signals
- Signal Properties
- Discrete Time Systems

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Discrete Time Signals



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Signals

Signal (n): A detectable physical quantity ... by which messages or information can be transmitted (Merriam-Webster)

- Signals carry information
- Examples:
 - Speech signals transmit language via acoustic waves
 - Radar signals transmit the position and velocity of targets via electromagnetic waves
 - Electrophysiology signals transmit information about processes inside the body
 - Financial signals transmit information about events in the economy
- Signal processing systems manipulate the information carried by signals

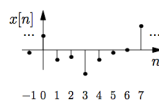
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Signals are Functions

DEFINITION A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
 - Independent variable is an **integer**: $n \in \mathbb{Z}$ (will refer to as **time**)
 - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or \mathbb{C}

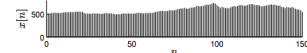


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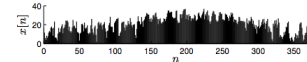
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A Menagerie of Signals

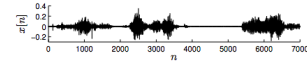
- Google Share daily share price for 5 months



- Temperature at Houston Intercontinental Airport in 2013 (Celcius)



- Excerpt from Shakespeare's *Hamlet*

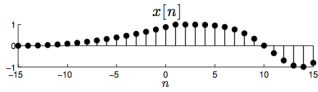


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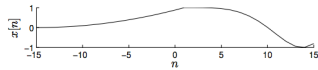
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Plotting Signals Correctly

- In a discrete-time signal $x[n]$, the independent variable n is discrete (integer)
- To plot a discrete-time signal in a program like Matlab, you should use the `stem` or similar command and not the `plot` command
- Correct:



- Incorrect:

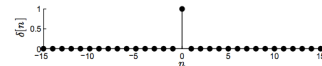


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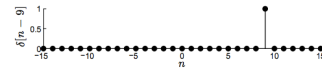
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Unit Sample

DEFINITION The delta function (aka unit impulse) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



- The shifted delta function $\delta[n - m]$ peaks up at $n = m$; here $m = 9$

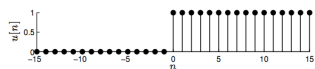


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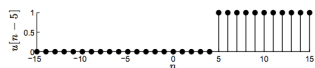
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Unit Step

DEFINITION The unit step $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



- The shifted unit step $u[n - m]$ jumps from 0 to 1 at $n = m$; here $m = 5$



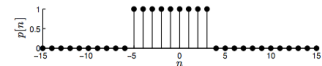
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Unit Pulse

DEFINITION The unit pulse (aka boxcar) $p[n] = \begin{cases} 0 & n < N_1 \\ 1 & N_1 \leq n \leq N_2 \\ 0 & n > N_2 \end{cases}$

- Ex: $p[n]$ for $N_1 = -5$ and $N_2 = 3$



- One of many different formulas for the unit pulse

$$p[n] = u[n - N_1] - u[n - (N_2 + 1)]$$

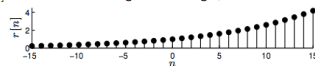
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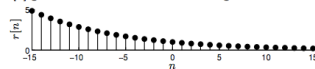
Real Exponential

DEFINITION The real exponential $r[n] = a^n$, $a \in \mathbb{R}$, $a \geq 0$

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$



- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$

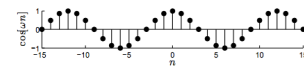


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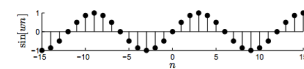
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Sinusoids

- There are two natural real-valued sinusoids: $\cos(\omega n + \phi)$ and $\sin(\omega n + \phi)$
- Frequency:** ω (units: radians/sample)
- Phase:** ϕ (units: radians)
- $\cos(\omega n)$



- $\sin(\omega n)$

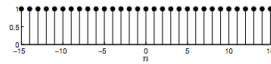


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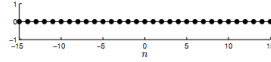
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Sinusoid Examples

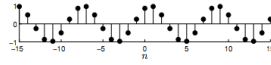
■ $\cos(0n)$



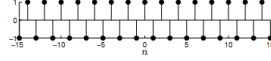
■ $\sin(0n)$



■ $\sin(\frac{\pi}{4}n + \frac{2\pi}{6})$



■ $\cos(\pi n)$



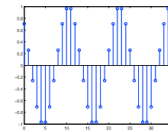
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Sinusoid in Matlab

- It's easy to play around in Matlab to get comfortable with the properties of sinusoids

```
N=36;
n=0:N-1;
omega=pi/6;
phi=pi/4;
x=cos(omega*n+phi);
stem(n,x)
```



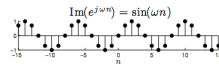
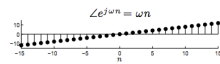
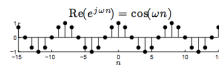
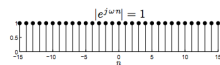
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Complex Sinusoid

- The complex-valued sinusoid combines both the cos and sin terms (via Euler's identity)

$$e^{j(\omega n + \phi)} = \cos(\omega n + \phi) + j \sin(\omega n + \phi)$$

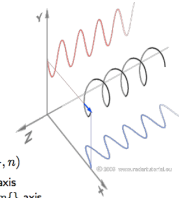


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Complex Sinusoid as Helix

$$e^{j(\omega n + \phi)} = \cos(\omega n + \phi) + j \sin(\omega n + \phi)$$



- A complex sinusoid is a **helix** in 3D space ($\text{Re}\{\cdot\}, \text{Im}\{\cdot\}, n$)
 - Real part** (cos term) is the projection onto the $\text{Re}\{\cdot\}$ axis
 - Imaginary part** (sin term) is the projection onto the $\text{Im}\{\cdot\}$ axis
- Frequency ω determines rotation **speed** and **direction** of helix
 - $\omega > 0 \Rightarrow$ anticlockwise rotation
 - $\omega < 0 \Rightarrow$ clockwise rotation

Animation: https://upload.wikimedia.org/wikipedia/commons/4/41/Rising_circular.gif

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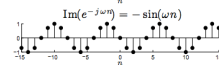
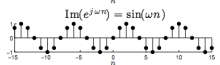
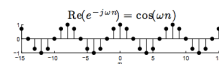
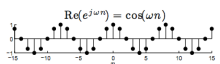
Negative Frequency

- Negative frequency is nothing to be afraid of! Consider a sinusoid with a negative frequency $-\omega$

$$e^{j(-\omega)n} = e^{-j\omega n} = \cos(-\omega n) + j \sin(-\omega n) = \cos(\omega n) - j \sin(\omega n)$$

- Also note: $e^{j(-\omega)n} = e^{-j\omega n} = (e^{j\omega n})^*$

- Bottom line: negating the frequency is equivalent to complex conjugating a complex sinusoid, which flips the sign of the imaginary, sin term



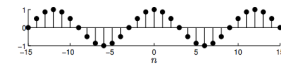
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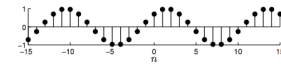
Phase of a Sinusoid

- ϕ is a (frequency independent) shift that is referenced to one period of oscillation

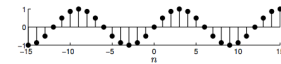
■ $\cos(\frac{\pi}{8}n - 0)$



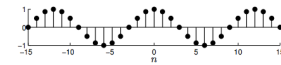
■ $\cos(\frac{\pi}{8}n - \frac{\pi}{4})$



■ $\cos(\frac{\pi}{8}n - \frac{\pi}{2}) = \sin(\frac{\pi}{8}n)$



■ $\cos(\frac{\pi}{8}n - 2\pi) = \cos(\frac{\pi}{8}n)$



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Complex Exponentials

- Complex sinusoid $e^{j(\omega n + \phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
- Generalize to $e^{\text{General Complex Numbers}}$
- Consider the general complex number $z = |z| e^{j\omega}$, $z \in \mathbb{C}$
 - $|z|$ = magnitude of z
 - $\omega = \angle(z)$, phase angle of z
 - Can visualize $z \in \mathbb{C}$ as a point in the complex plane

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Complex Exponentials

- Complex sinusoid $e^{j(\omega n + \phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$
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 - $|z|$ = magnitude of z
 - $\omega = \angle(z)$, phase angle of z
 - Can visualize $z \in \mathbb{C}$ as a point in the complex plane

- Now we have

$$z^n = (|z| e^{j\omega})^n = |z|^n (e^{j\omega})^n = |z|^n e^{j\omega n}$$

- $|z|^n$ is a real exponential (a^n with $a = |z|$)
- $e^{j\omega n}$ is a complex sinusoid

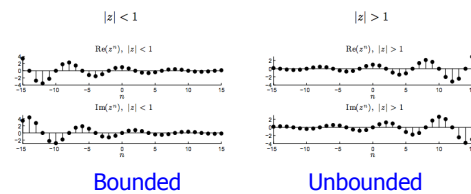
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Complex Exponentials

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

- $|z|^n$ is a real exponential envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a complex sinusoid

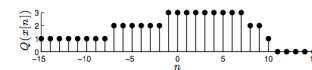


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Digital Signals

- Digital signals are a special sub-class of discrete-time signals
 - Independent variable is still an integer: $n \in \mathbb{Z}$
 - Dependent variable is from a finite set of integers: $x[n] \in \{0, 1, \dots, D-1\}$
 - Typically, choose $D = 2^q$ and represent each possible level of $x[n]$ as a digital code with q bits
 - Ex: Digital signal with $q = 2$ bits $\Rightarrow D = 2^2 = 4$ levels



- Ex: Compact discs use $q = 16$ bits $\Rightarrow D = 2^{16} = 65536$ levels

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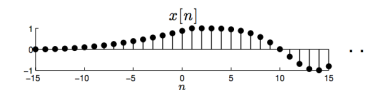
Signal Properties



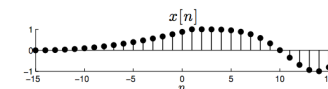
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Finite/Infinite Length Sequences

- An infinite-length discrete-time signal $x[n]$ is defined for all $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$



- A finite-length discrete-time signal $x[n]$ is defined only for a finite range of $N_1 \leq n \leq N_2$



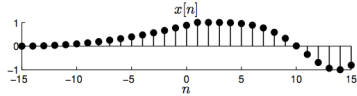
- Important: a finite-length signal is undefined for $n < N_1$ and $n > N_2$

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Windowing

- Converts a longer signal into a shorter one $y[n] = \begin{cases} x[n] & N_1 \leq n \leq N_2 \\ 0 & \text{otherwise} \end{cases}$



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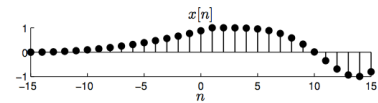
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Zero Padding

- Converts a shorter signal into a longer one

- Say $x[n]$ is defined for $N_1 \leq n \leq N_2$

- Given $N_0 \leq N_1 \leq N_2 \leq N_3$ $y[n] = \begin{cases} 0 & N_0 \leq n < N_1 \\ x[n] & N_1 \leq n \leq N_2 \\ 0 & N_2 < n \leq N_3 \end{cases}$



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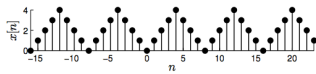
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Periodic Signals

DEFINITION

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall m \in \mathbb{Z}$$



Notes:

- The period N must be an integer
- A periodic signal is infinite in length

DEFINITION

A discrete-time signal is **aperiodic** if it is not periodic

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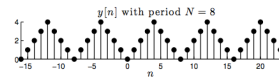
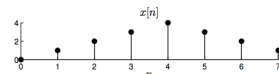
Periodization

- Converts a finite-length signal into an infinite-length, periodic signal

- Given finite-length $x[n]$, replicate $x[n]$ periodically with period N

$$y[n] = \sum_{m=-\infty}^{\infty} x[n - mN], \quad n \in \mathbb{Z}$$

$$= \dots + x[n + 2N] + x[n + N] + x[n] + x[n - N] + x[n - 2N] + \dots$$



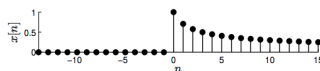
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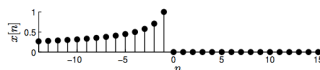
Causal Signals

DEFINITION

A signal $x[n]$ is **causal** if $x[n] = 0$ for all $n < 0$.



- A signal $x[n]$ is **anti-causal** if $x[n] = 0$ for all $n \geq 0$



- A signal $x[n]$ is **acausal** if it is not causal

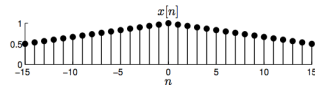
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Even Signals

DEFINITION

A real signal $x[n]$ is **even** if $x[-n] = x[n]$



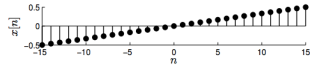
- Even signals are symmetrical around the point $n = 0$

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Odd Signals

DEFINITION A real signal $x[n]$ is **odd** if $x[-n] = -x[n]$



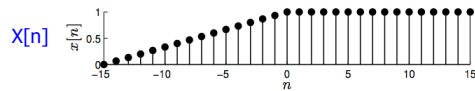
- Odd signals are anti-symmetrical around the point $n = 0$

Signal Decomposition

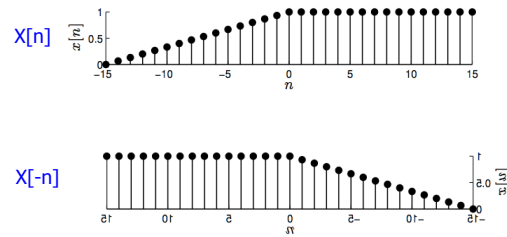
- Useful fact:** Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part
- Even part:** $e[n] = \frac{1}{2}(x[n] + x[-n])$ (easy to verify that $e[n]$ is even)
- Odd part:** $o[n] = \frac{1}{2}(x[n] - x[-n])$ (easy to verify that $o[n]$ is odd)
- Decomposition** $x[n] = e[n] + o[n]$
- Verify the decomposition:**

$$\begin{aligned} e[n] + o[n] &= \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n]) \\ &= \frac{1}{2}(x[n] + x[-n] + x[n] - x[-n]) \\ &= \frac{1}{2}(2x[n]) = x[n] \quad \checkmark \end{aligned}$$

Decomposition Example



Decomposition Example



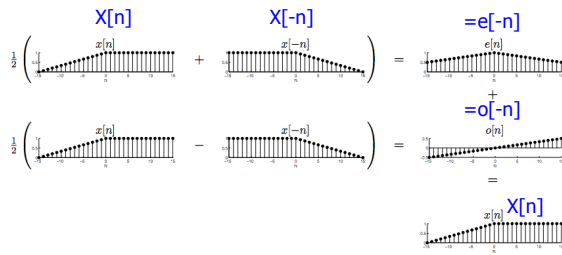
Decomposition Example

$$\begin{aligned} \frac{1}{2} \left(\begin{array}{c} x[n] \\ x[-n] \end{array} \right) &= e[-n] \\ \frac{1}{2} \left(\begin{array}{c} x[n] \\ -x[-n] \end{array} \right) &= o[-n] \end{aligned}$$

Decomposition Example

$$\begin{aligned} \frac{1}{2} \left(\begin{array}{c} x[n] \\ x[-n] \end{array} \right) &= e[n] \\ \frac{1}{2} \left(\begin{array}{c} x[n] \\ -x[-n] \end{array} \right) &= o[n] \end{aligned}$$

Decomposition Example



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Discrete-Time Sinusoids

- Discrete-time sinusoids $e^{j(\omega n + \phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property #1: Aliasing
- Weird property #2: Aperiodicity

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Property #1: Aliasing of Sinusoids

- Consider two sinusoids with two different frequencies
 - $\omega \Rightarrow x_1[n] = e^{j(\omega n + \phi)}$
 - $\omega + 2\pi \Rightarrow x_2[n] = e^{j((\omega + 2\pi)n + \phi)}$

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Property #1: Aliasing of Sinusoids

- Consider two sinusoids with two different frequencies
 - $\omega \Rightarrow x_1[n] = e^{j(\omega n + \phi)}$
 - $\omega + 2\pi \Rightarrow x_2[n] = e^{j((\omega + 2\pi)n + \phi)}$
- But note that

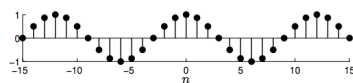
$$x_2[n] = e^{j((\omega + 2\pi)n + \phi)} = e^{j(\omega n + \phi)} e^{j2\pi n} = e^{j(\omega n + \phi)} = x_1[n]$$
- The signals x_1 and x_2 have different frequencies but are **identical**!
- We say that x_1 and x_2 are **aliases**; this phenomenon is called **aliasing**
- Note: Any integer multiple of 2π will do; try with $x_3[n] = e^{j((\omega + 2\pi m)n + \phi)}$, $m \in \mathbb{Z}$

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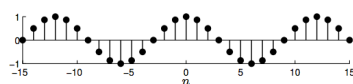
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Aliasing Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$



- $x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$

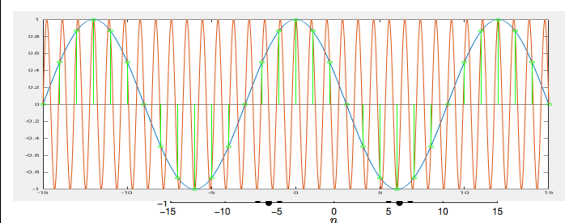


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Aliasing Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$



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Alias-Free Frequencies

- Since

$$x_3[n] = e^{j(\omega+2\pi m)n+\phi} = e^{j(\omega n+\phi)} = x_1[n] \quad \forall m \in \mathbb{Z}$$

the only frequencies that lead to unique (distinct) sinusoids lie in an interval of length 2π

- Two intervals are typically used in the signal processing literature (and in this course)

- $0 \leq \omega < 2\pi$
- $-\pi < \omega \leq \pi$

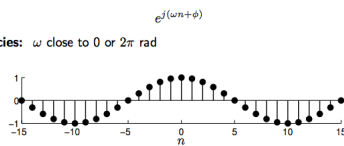
Which is higher in frequency?

- $\cos(\pi n)$ or $\cos(3\pi/2n)$?

Low and High Frequencies

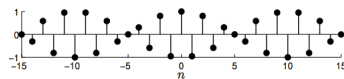
- Low frequencies:** ω close to 0 or 2π rad

Ex: $\cos(\frac{\pi}{10}n)$

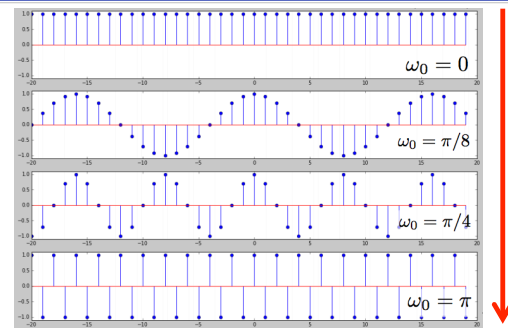


- High frequencies:** ω close to π or $-\pi$ rad

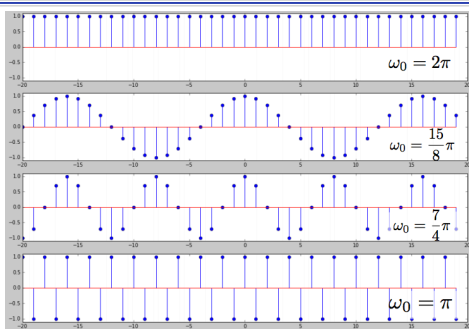
Ex: $\cos(\frac{9\pi}{10}n)$



Increasing Frequency



Decreasing Frequency



Property #2: Periodicity of Sinusoids

- Consider $x_1[n] = e^{j(\omega n + \phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (harmonic frequency)

- It is easy to show that x_1 is periodic with period N , since

$$x_1[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} = e^{j(\omega n + \phi)} e^{j(\frac{2\pi k}{N}N)} = x_1[n] \quad \checkmark$$

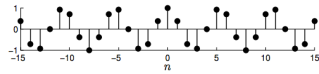
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- Ex: $x_1[n] = \cos(\frac{2\pi}{16}n)$, $N = 16$



- Note: x_1 is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer

Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)

Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is x_2 periodic?

$$x_2[n+N] = e^{j(\omega(n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} \neq x_2[n] \quad \text{NO!}$$

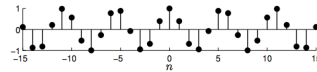
Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is x_2 periodic?

$$x_2[n+N] = e^{j(\omega(n+N) + \phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)} e^{j(\omega N)} \neq x_2[n] \quad \text{NO!}$$

- Ex: $x_2[n] = \cos(1.16n)$



- If its frequency ω is not harmonic, then a sinusoid oscillates but is not periodic!

Harmonic Sinusoids

$$e^{j(\omega n + \phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with **harmonic frequencies**

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

- Which means that

- Most discrete-time sinusoids are **not** periodic!
- The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)

Periodic or not?

□ $\cos(5/7 \pi n)$

□ $\cos(\pi/5n)$

- What are N and k ? (I.e. How many samples is one period?)

Periodic or not?

- $\cos(5/7 \pi n)$
 - $N=14, k=5$
 - $\cos(5/14 * 2 \pi n)$
 - Repeats every $N=14$ samples
- $\cos(\pi / 5n)$
 - $N=10, k=1$
 - $\cos(1/10 * 2 \pi n)$
 - Repeats every $N=10$ samples

Periodic or not?

- $\cos(5/7 \pi n)$
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- $\cos(\pi / 5n)$
 - $N=10, k=1$
 - $\cos(1/10 * 2 \pi n)$
 - Repeats every $N=10$ samples
- $\cos(5/7 \pi n) + \cos(\pi / 5n) ?$

Periodic or not?

- $\cos(5/7 \pi n) + \cos(\pi / 5n) ?$
 - $N = \text{SCM}\{10, 14\} = 70$
 - $\cos(5/7 * \pi n) + \cos(\pi / 5n)$
 - $n=N=70 \rightarrow \cos(5/7 * 70 \pi) + \cos(\pi / 5 * 70) = \cos(25 * 2 \pi) + \cos(7 * 2 \pi)$

Discrete-Time Systems



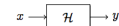
Discrete Time Systems

DEFINITION A discrete-time **system** \mathcal{H} is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$

- Systems manipulate the information in signals
- Examples:
 - A speech recognition system converts acoustic waves of speech into text
 - A radar system transforms the received radar pulse to estimate the position and velocity of targets
 - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
 - A 30 day moving average smooths out the day-to-day variability in a stock price

Signal Length and Systems



- Recall that there are two kinds of signals: infinite-length and finite-length
- Accordingly, we will consider two kinds of systems:
 - Systems that transform an infinite-length-signal x into an infinite-length signal y
 - Systems that transform a length- N signal x into a length- N signal y
(Such systems can also be used to process periodic signals with period N)
- For generality, we will assume that the input and output signals are complex valued

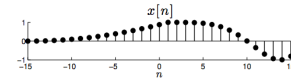
System Examples

- Identity $y[n] = x[n] \quad \forall n$
- Scaling $y[n] = 2x[n] \quad \forall n$
- Offset $y[n] = x[n] + 2 \quad \forall n$
- Square signal $y[n] = (x[n])^2 \quad \forall n$
- Shift $y[n] = x[n + 2] \quad \forall n$
- Decimate $y[n] = x[2n] \quad \forall n$
- Square time $y[n] = x[n^2] \quad \forall n$

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System Examples



- Shift system ($m \in \mathbb{Z}$ fixed) $y[n] = x[n - m] \quad \forall n$
- Moving average (combines shift, sum, scale) $y[n] = \frac{1}{2}(x[n] + x[n - 1]) \quad \forall n$
- Recursive average $y[n] = x[n] + \alpha y[n - 1] \quad \forall n$

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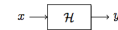
System Properties

- Memoryless
- Linearity
- Time Invariance
- Causality
- BIBO Stability

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Memoryless



- $y[n]$ depends only on $x[n]$
- Examples:
 - Ideal delay system (or shift system): $y[n] = x[n - m]$ **memoryless?**
 - Square system: $y[n] = (x[n])^2$ **memoryless?**

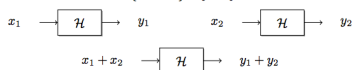
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Linear Systems

A system \mathcal{H} is (zero-state) **linear** if it satisfies the following two properties:

- Scaling $\mathcal{H}\{\alpha x\} = \alpha \mathcal{H}\{x\} \quad \forall \alpha \in \mathbb{C}$
- Additivity If $y_1 = \mathcal{H}\{x_1\}$ and $y_2 = \mathcal{H}\{x_2\}$ then $\mathcal{H}\{x_1 + x_2\} = y_1 + y_2$



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Proving Linearity

- A system that is not linear is called **nonlinear**
- To prove that a system is linear, you must prove rigorously that it has **both** the scaling and additivity properties for **arbitrary** input signals
- To prove that a system is nonlinear, it is sufficient to exhibit a **counterexample**

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Linearity Example: Moving Average

$$x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- **Scaling:** (Strategy to prove – Scale input x by $\alpha \in \mathbb{C}$, compute output y via the formula at top, and verify that it is scaled as well)

- Let

$$x'[n] = \alpha x[n], \quad \alpha \in \mathbb{C}$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1])$$

Linearity Example: Moving Average

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- Let

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- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}(\alpha x[n] + \alpha x[n-1]) = \alpha \left(\frac{1}{2}(x[n] + x[n-1]) \right) = \alpha y[n] \quad \checkmark$$

Linearity Example: Moving Average

$$x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)

- Let

$$x'[n] = x_1[n] + x_2[n]$$

- Let $y'/y_1/y_2$ denote the output when $x'/x_1/x_2$ is input

Linearity Example: Moving Average

$$x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)

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- Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}((x_1[n] + x_2[n]) + (x_1[n-1] + x_2[n-1]))$$

Linearity Example: Moving Average

$$x[n] \xrightarrow{\mathcal{H}} y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)

- Let

$$x'[n] = x_1[n] + x_2[n]$$

- Let $y'/y_1/y_2$ denote the output when $x'/x_1/x_2$ is input

- Then

$$\begin{aligned} y'[n] &= \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}((x_1[n] + x_2[n]) + (x_1[n-1] + x_2[n-1])) \\ &= \frac{1}{2}(x_1[n] + x_1[n-1]) + \frac{1}{2}(x_2[n] + x_2[n-1]) = y_1[n] + y_2[n] \quad \checkmark \end{aligned}$$

Example: Squaring is Nonlinear

$$x[n] \xrightarrow{\mathcal{H}} y[n] = (x[n])^2$$

- **Additivity:** Input two signals into the system and see what happens

- Let

$$y_1[n] = (x_1[n])^2, \quad y_2[n] = (x_2[n])^2$$

- Set

$$x'[n] = x_1[n] + x_2[n]$$

- Then

$$y'[n] = (x'[n])^2 = (x_1[n] + x_2[n])^2 = (x_1[n])^2 + 2x_1[n]x_2[n] + (x_2[n])^2 \neq y_1[n] + y_2[n]$$

- **Nonlinear!**

Time-Invariant Systems

A system \mathcal{H} processing infinite-length signals is **time-invariant** (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal



- Intuition: A time-invariant system behaves the same no matter when the input is applied
- A system that is not time-invariant is called **time-varying**

Example: Moving Average

$$x[n] \rightarrow \mathcal{H} \rightarrow y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- Let

$$x'[n] = x[n-q], \quad q \in \mathbb{Z}$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}(x[n-q] + x[n-q-1]) = y[n-q] \quad \checkmark$$

Example: Decimation

$$x[n] \rightarrow \mathcal{H} \rightarrow y[n] = x[2n]$$

- This system is time-varying; demonstrate with a counter-example

- Let

$$x'[n] = x[n-1]$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] = x'[2n] = x[2n-1] \neq x[2(n-1)] = y[n-1]$$

Causal Systems

A system \mathcal{H} is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- Forward difference system:

$$y[n] = x[n+1] - x[n] \quad \text{causal?}$$

- Backward difference system:

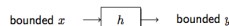
$$y[n] = x[n] - x[n-1] \quad \text{causal?}$$

Stability

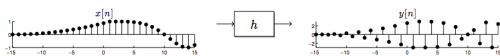
- BIBO Stability

- Bounded-input bounded-output Stability

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$, or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all n



System Properties - Summary

- Causality

- $y[n]$ only depends on $x[m]$ for $m \leq n$

- Linearity

- Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs

$$Ax_1[n] + Bx_2[n] \rightarrow Ay_1[n] + By_2[n]$$

- Memoryless

- $y[n]$ depends only on $x[n]$

- Time Invariance

- Shifted input results in shifted output

$$x[n-q] \rightarrow y[n-q]$$

- BIBO Stability

- A bounded input results in a bounded output (ie. max signal value exists for output if max)

Examples

- Causal? Linear? Time-invariant? Memoryless? BIBO Stable?

- Time Shift:

- $y[n] = x[n - m]$

- Accumulator:

- $$y[n] = \sum_{k=-\infty}^n x[k]$$

- Compressor ($M > 1$):

- $$y[n] = x[Mn]$$

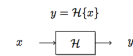
Big Ideas

- Discrete Time Signals

- Unit impulse, unit step, exponential, sinusoids, complex sinusoids
 - Can be finite length, infinite length
 - Properties
 - Even, odd, causal
 - Periodicity and aliasing
 - Discrete frequency bounded!

- Discrete Time Systems

- Transform one signal to another
 - Properties
 - Linear, Time-invariance, memoryless, causality, BIBO stability



Admin

- Enroll in Piazza site:
 - piazza.com/upenn/spring2018/ese531
- HW 1 out after class