

ESE 531: Digital Signal Processing

Lec 3: January 18, 2018

Discrete Time Signals and Systems



Lecture Outline

- ❑ Discrete Time Systems
- ❑ LTI Systems
- ❑ LTI System Properties
- ❑ Difference Equations

Discrete-Time Systems

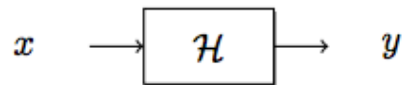


Discrete Time Systems

DEFINITION

A discrete-time **system** \mathcal{H} is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples:
 - A speech recognition system converts acoustic waves of speech into text
 - A radar system transforms the received radar pulse to estimate the position and velocity of targets
 - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
 - A 30 day moving average smooths out the day-to-day variability in a stock price



System Properties

❑ Causality

- $y[n]$ only depends on $x[m]$ for $m \leq n$

❑ Linearity

- Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
 - $Ax_1[n] + Bx_2[n] \rightarrow Ay_1[n] + By_2[n]$

❑ Memoryless

- $y[n]$ depends only on $x[n]$

❑ Time Invariance

- Shifted input results in shifted output
 - $x[n-q] \rightarrow y[n-q]$

❑ BIBO Stability

- A bounded input results in a bounded output (ie. max signal value exists for output if max)



Examples

❑ Causal? Linear? Time-invariant? Memoryless?
BIBO Stable?

❑ Time Shift:

- $y[n] = x[n - m]$

❑ Accumulator:

- $$y[n] = \sum_{k=-\infty}^n x[k]$$

❑ Compressor ($M > 1$):

$$y[n] = x[Mn]$$



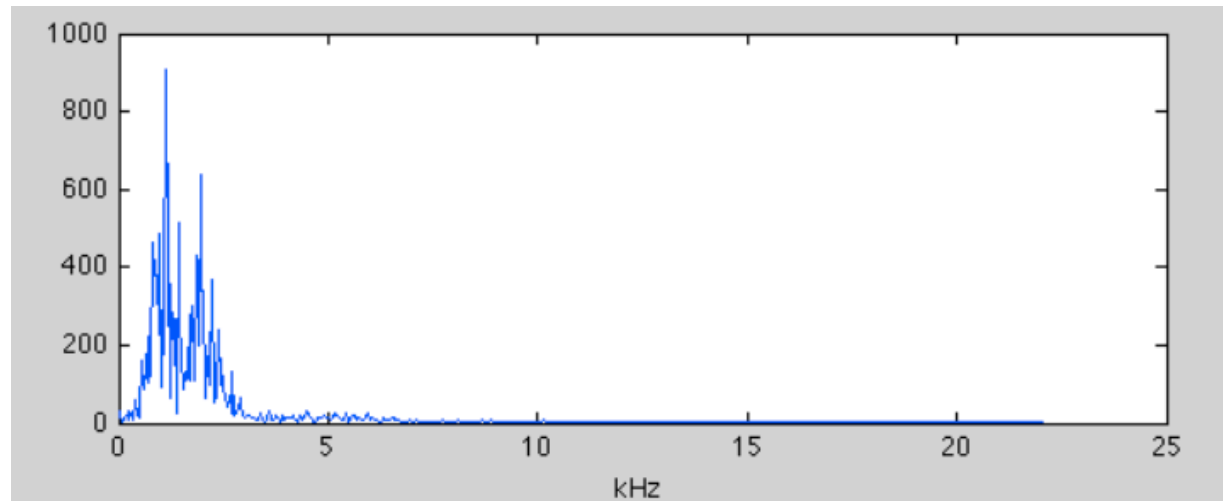
Non-Linear System Example

□ Median Filter

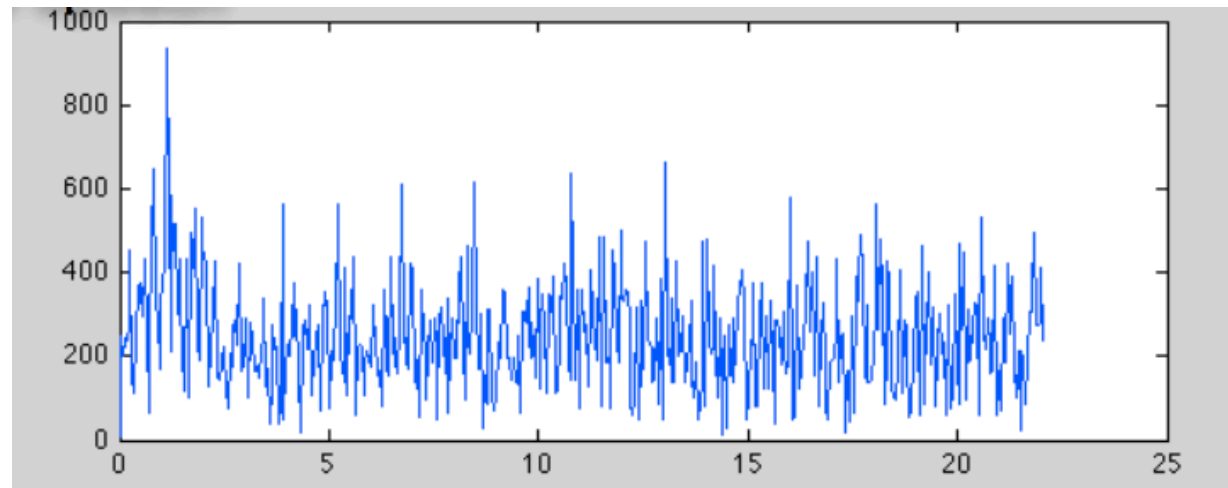
- $y[n] = \text{MED} \{x[n-k], \dots, x[n+k]\}$
- Let $k=1$
- $y[n] = \text{MED} \{x[n-1], x[n], x[n+1]\}$

Spectrum of Speech

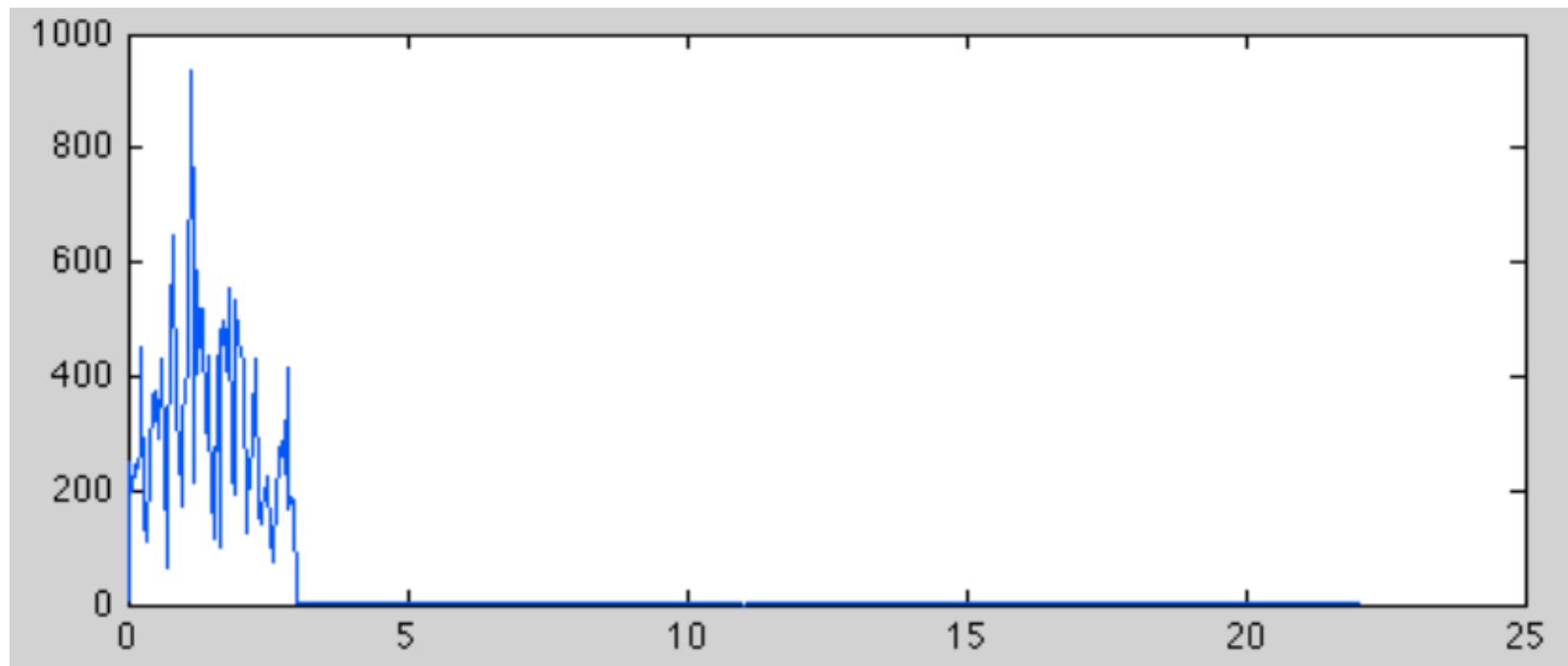
Speech



Corrupted
Speech

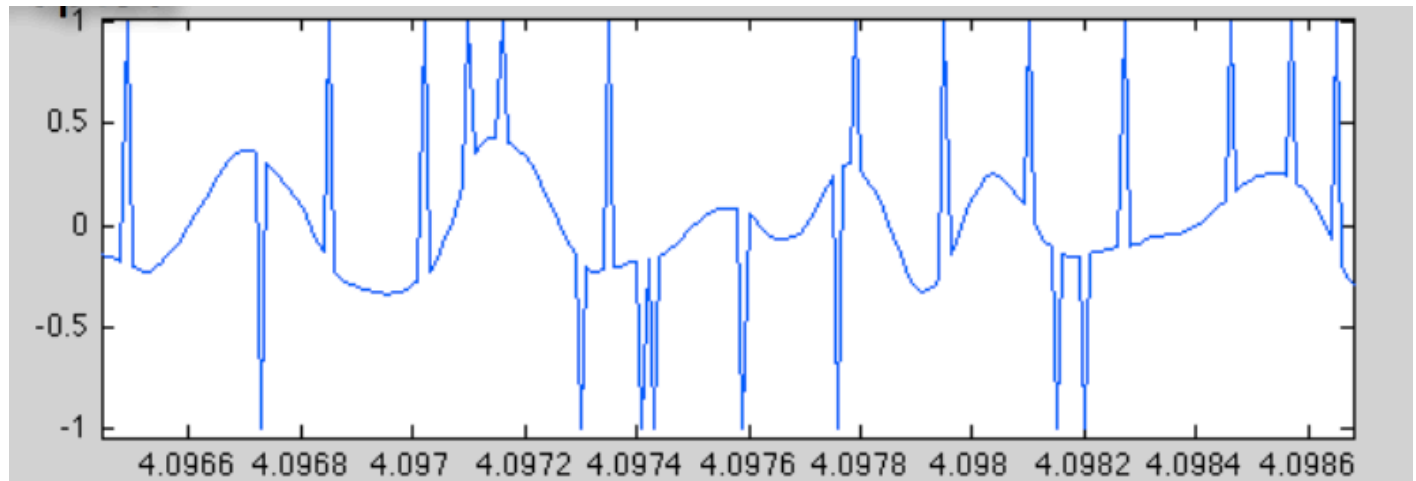


Low Pass Filtering

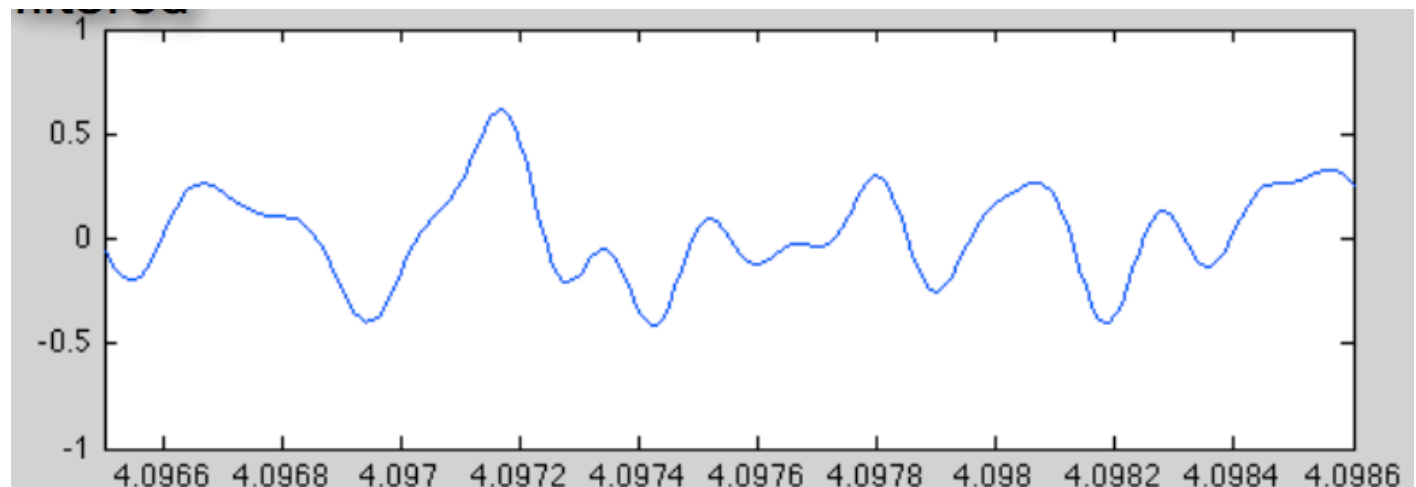


Low Pass Filtering

Corrupted
Speech

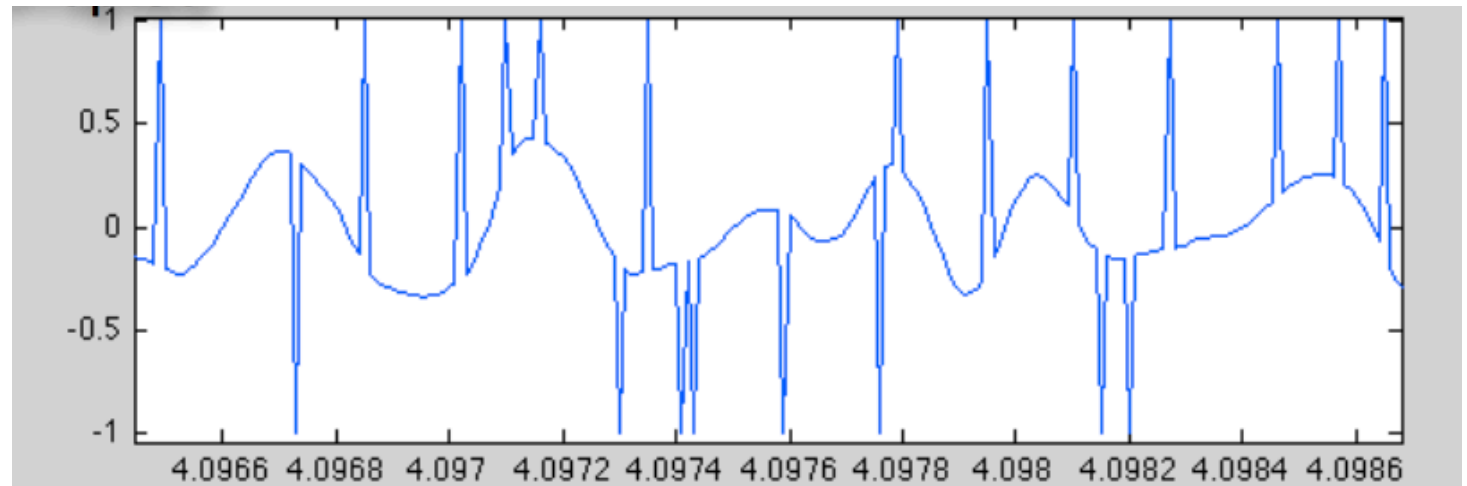


LP-Filtered
Speech

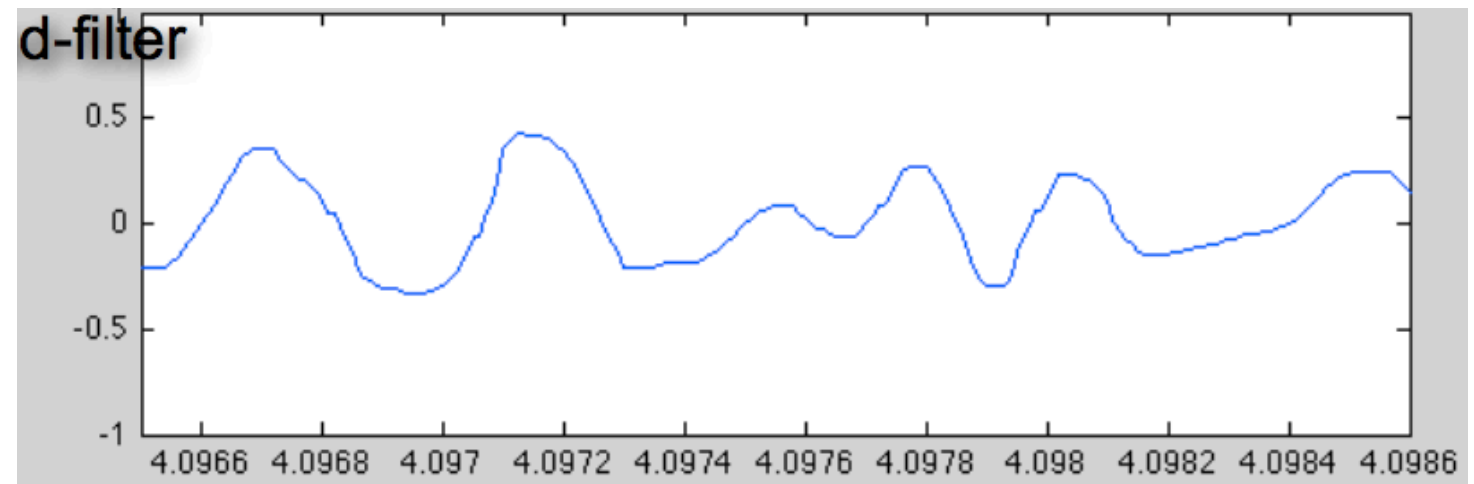


Median Filtering

Corrupted
Speech



Med-Filter
Speech



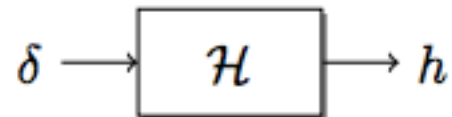
LTI Systems

LTI Systems

DEFINITION

A system \mathcal{H} is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



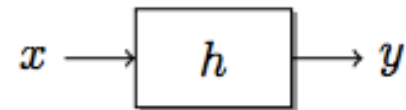
- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

A block diagram showing a general input-output relationship. An input signal x enters a rectangular block labeled h , and the output is y .

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

$$y[n] = x[n] * h[n]$$

Convolution



- Convolution formula

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- To compute the entry $y[n]$ in the output vector y :

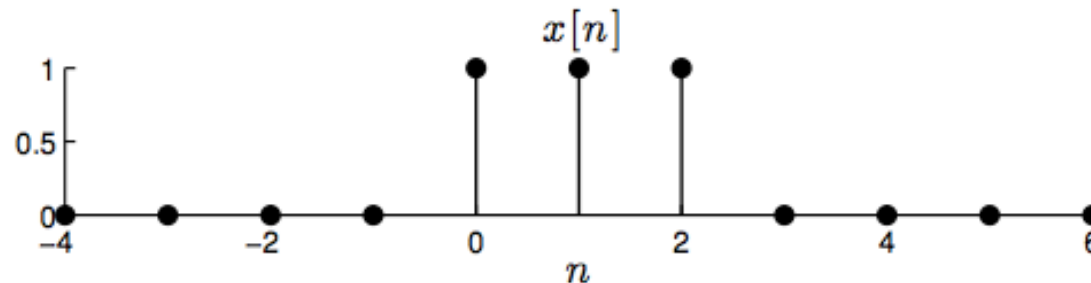
- 1 **Time reverse** the impulse response vector h and **shift** it n time steps to the right (delay)
- 2 Compute the **inner product** between the shifted impulse response and the input vector x

- Repeat for every n

Convolution Example

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- Convolve a unit pulse with itself



Convolution is Commutative

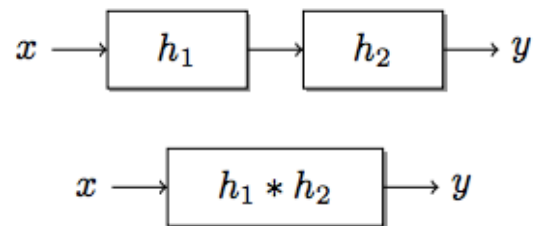
- **Fact:** Convolution is commutative: $x * h = h * x$

- These block diagrams are equivalent: $x \longrightarrow \boxed{h} \longrightarrow y$ $h \longrightarrow \boxed{x} \longrightarrow y$

- Enables us to pick either h or x to flip and shift (or stack into a matrix) when convolving

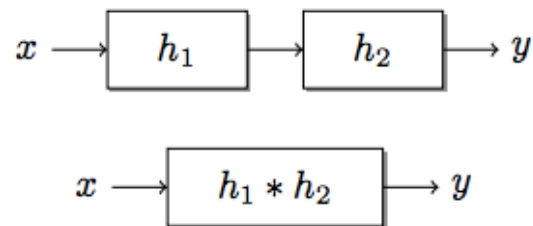
LTI Systems in Series

- Impulse response of the **cascade** (aka series connection) of two LTI systems:

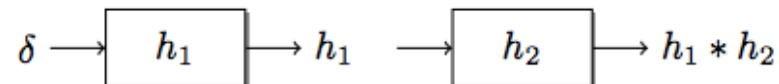


LTI Systems in Series

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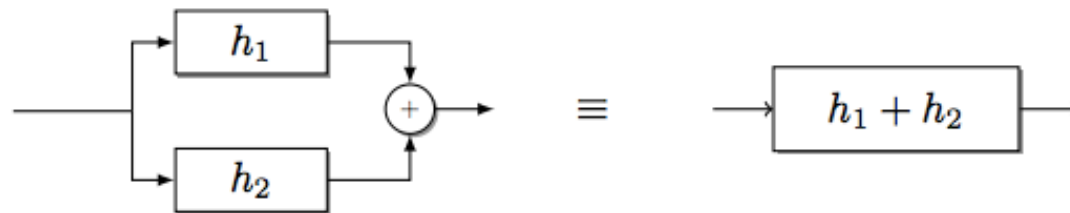


- Easy proof by picture; find impulse response the old school way



LTI Systems in Parallel

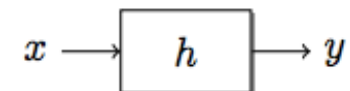
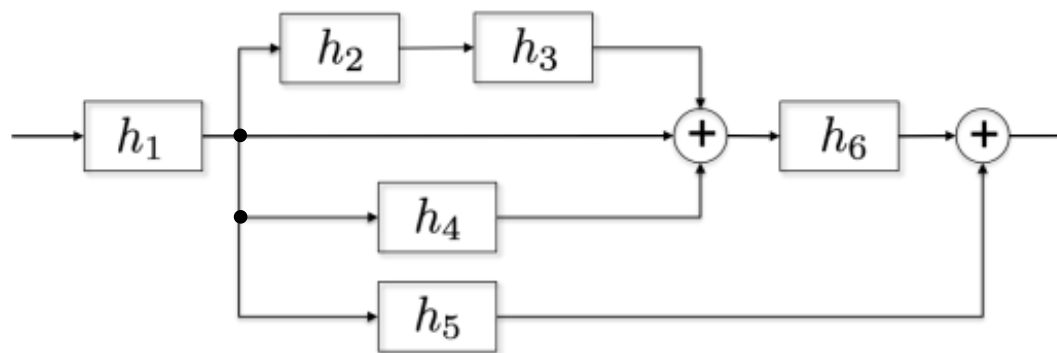
- Impulse response of the **parallel connection** of two LTI systems



- Proof is an easy application of the linearity of an LTI system

Example

- Compute the overall effective impulse response of the following system

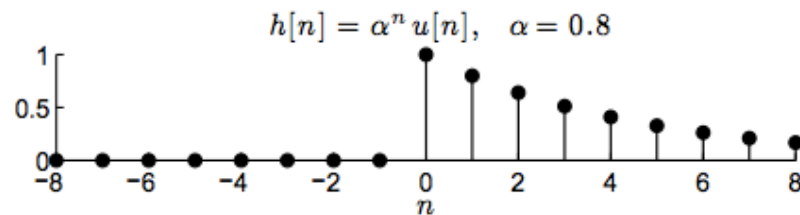


Causal System Revisited

DEFINITION

A system \mathcal{H} is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- **Fact:** An LTI system is causal if its impulse response is causal: $h[n] = 0$ for $n < 0$



- To prove, note that the convolution

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

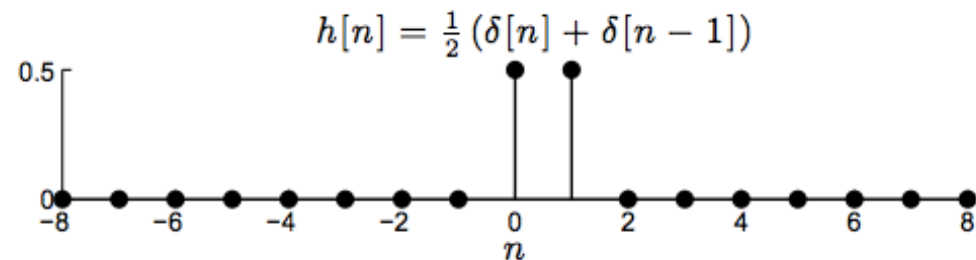
does not look into the future if $h[n-m] = 0$ when $m > n$; equivalently, $h[n'] = 0$ when $n' < 0$

Duration of Impulse

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

- Example: Moving average system $y[n] = \mathcal{H}\{x[n]\} = \frac{1}{2} (x[n] + x[n - 1])$

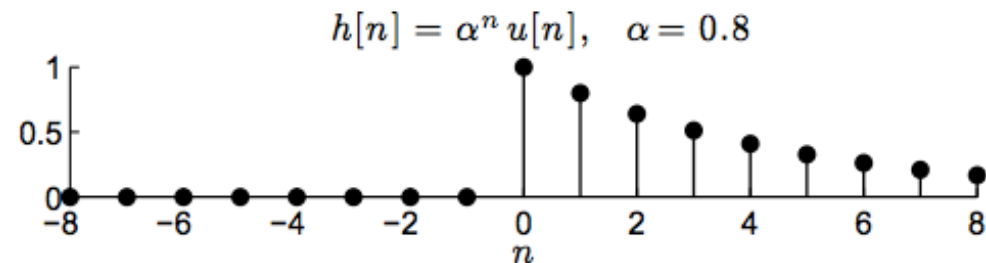


Duration of Impulse

DEFINITION

An LTI system has an **infinite impulse response (IIR)** if the duration of its impulse response h is infinite

- Example: Recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha y[n - 1]$

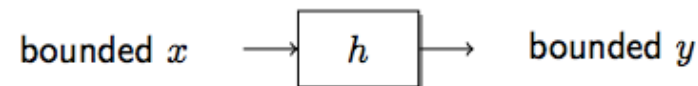


- Note: Obviously the FIR/IIR distinction applies only to infinite-length signals

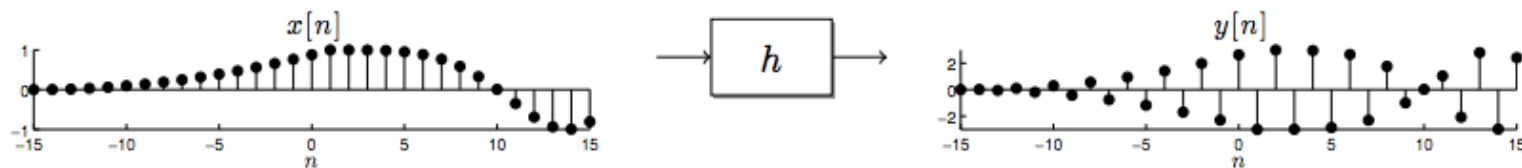
BIBO Stability Revisited

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



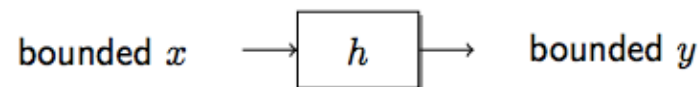
- Bounded input and output means $\|x\|_{\infty} < \infty$ and $\|y\|_{\infty} < \infty$, or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all n



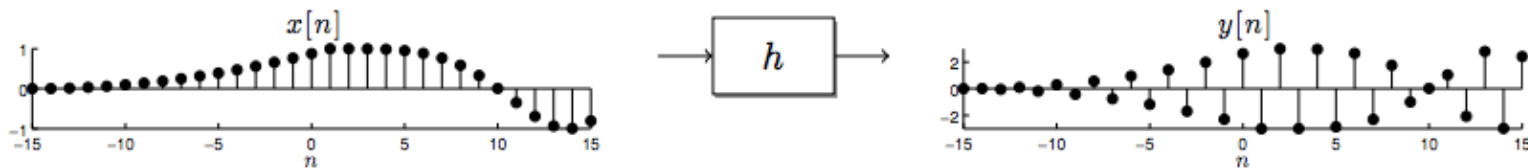
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- Bounded input and output means $\|x\|_{\infty} < \infty$ and $\|y\|_{\infty} < \infty$



- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$



BIBO Stability – Sufficient Condition

- Prove that if $\|h\|_1 < \infty$ then the system is BIBO stable – for any input $\|x\|_\infty < \infty$ the output $\|y\|_\infty < \infty$



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- Recall that $\|x\|_\infty < \infty$ means there exist a constant A such that $|x[n]| < A < \infty$ for all n
- Let $\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| = B < \infty$

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$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} h[n-m] x[m] \right|$$

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$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} h[n-m] x[m] \right| \leq \sum_{m=-\infty}^{\infty} |h[n-m]| |x[m]|$$

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$$\begin{aligned} |y[n]| &= \left| \sum_{m=-\infty}^{\infty} h[n-m] x[m] \right| \leq \sum_{m=-\infty}^{\infty} |h[n-m]| |x[m]| \\ &< \sum_{m=-\infty}^{\infty} |h[n-m]| A = A \sum_{k=-\infty}^{\infty} |h[k]| = AB = C < \infty \end{aligned}$$

- Since $|y[n]| < C < \infty$ for all n , $\|y\|_\infty < \infty$ ✓

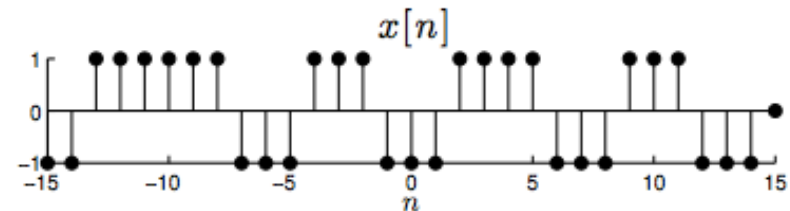
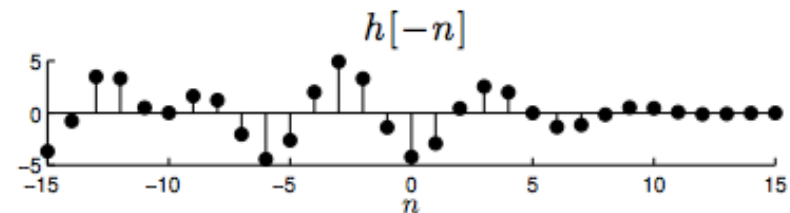
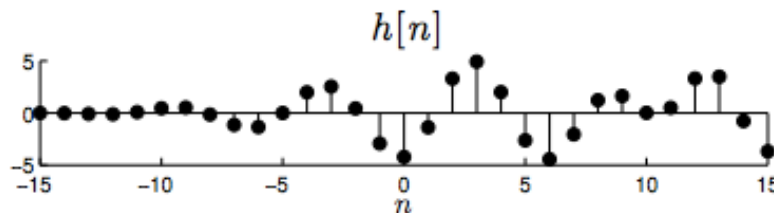


BIBO Stability – Necessary Condition

- Prove that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
 - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical

BIBO Stability – Necessary Condition

- Prove that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
 - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical
- Given an impulse response h with $\|h\|_1 = \infty$ (assume complex-valued), form the tricky special signal $x[n] = \text{sgn}(h[-n])$
 - $x[n]$ is the \pm sign of the time-reversed impulse response $h[-n]$
 - Note that x is bounded: $|x[n]| \leq 1$ for all n





BIBO Stability – Necessary Condition

- We are proving that that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
- Armed with the tricky special signal x , compute the output $y[n]$ at the time point $n = 0$

$$y[0] = \sum_{m=-\infty}^{\infty} h[0 - m] x[m]$$

BIBO Stability – Necessary Condition

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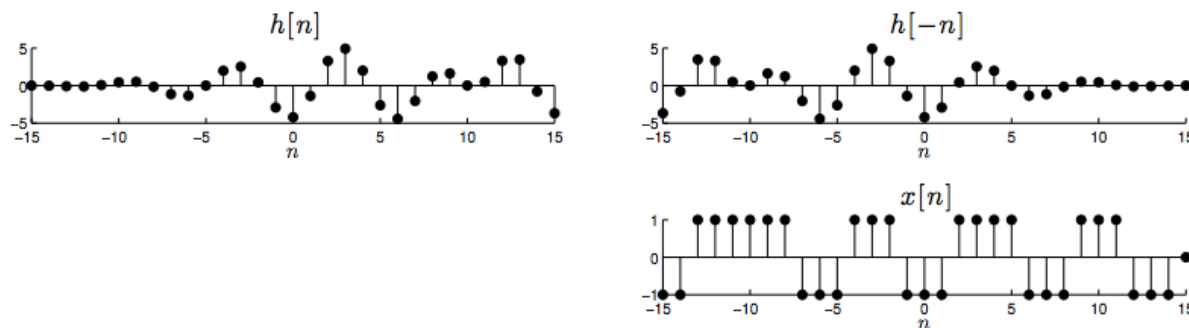
$$\begin{aligned} y[0] &= \sum_{m=-\infty}^{\infty} h[0-m] x[m] = \sum_{m=-\infty}^{\infty} h[-m] \operatorname{sgn}(h[-m]) \\ &= \sum_{m=-\infty}^{\infty} |h[-m]| = \sum_{k=-\infty}^{\infty} |h[k]| = \infty \end{aligned}$$

BIBO Stability – Necessary Condition

- We are proving that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
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$$\begin{aligned} y[0] &= \sum_{m=-\infty}^{\infty} h[0-m] x[m] = \sum_{m=-\infty}^{\infty} h[-m] \operatorname{sgn}(h[-m]) \\ &= \sum_{m=-\infty}^{\infty} |h[-m]| = \sum_{k=-\infty}^{\infty} |h[k]| = \infty \end{aligned}$$

- So, even though x was bounded, y is not bounded; so system is not BIBO stable



Examples

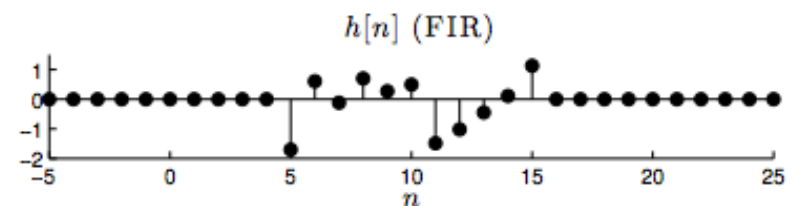
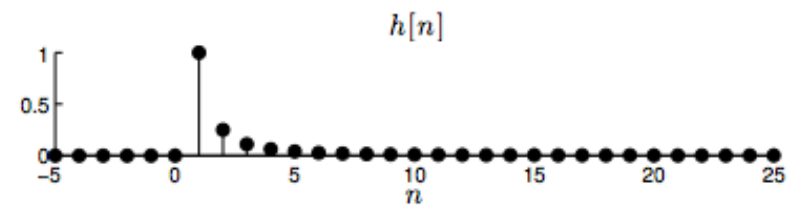
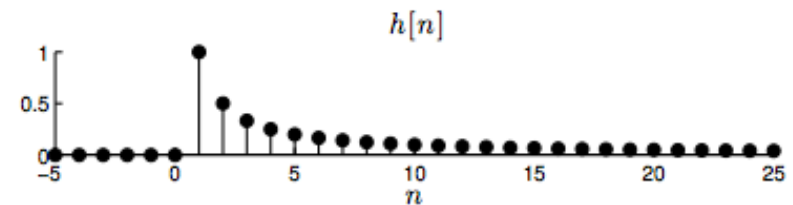
- Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n} \right| = \infty \Rightarrow \text{not BIBO}$$

- Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right| = \frac{\pi^2}{6} \Rightarrow \text{BIBO}$$

- Example: h FIR \Rightarrow BIBO





Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha y[n - 1]$
- Impulse response: $h[n] = \alpha^n u[n]$

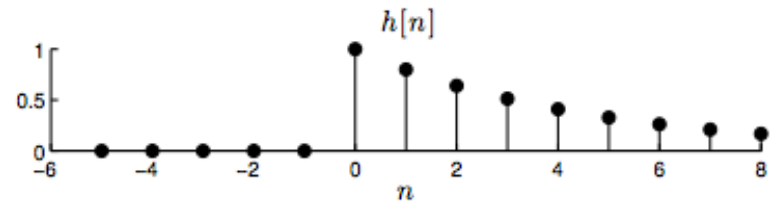
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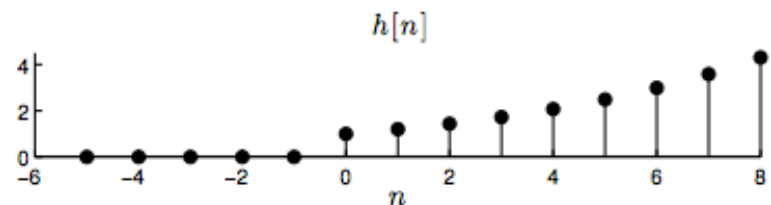
■ For $|\alpha| < 1$

$$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty \Rightarrow \text{BIBO}$$



■ For $|\alpha| > 1$

$$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \infty \Rightarrow \text{not BIBO}$$





Difference Equations

□ Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



Difference Equations

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$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

Difference Equations

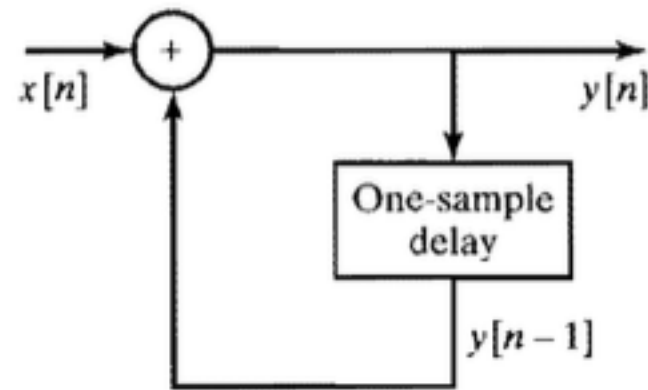
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$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$



Example: Difference Equation

- ❑ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- ❑ Let $M_1=0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$



Big Ideas

- ❑ LTI Systems are a special class of systems with significant signal processing applications
 - Can be characterized by the impulse response
- ❑ LTI System Properties
 - Causality and stability can be determined from impulse response
- ❑ Difference equations suggest implementation of systems
 - Give insight into complexity of system



Admin

- ❑ HW 1 out now
 - Due 1/26 at midnight
 - Submit in Canvas