ESE 531: Digital Signal Processing

Lec 3: January 18, 2018

Discrete Time Signals and Systems

Lecture Outline

- Discrete Time Systems
- LTI Systems
- LTI System Properties
- Difference Equations

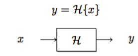
Discrete-Time Systems

Discrete Time Systems

DEFINITION

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

System Properties

- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - x[n-q] → y[n-q]
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max)

Examples

- Causal? Linear? Time-invariant? Memoryless? BIBO Stable?
- Time Shift:
  - $y[n] = x[n-m]$
- Accumulator:
  - $$y[n] = \sum_{k=-\infty}^{n} x[k]$$
- Compressor (M>1):

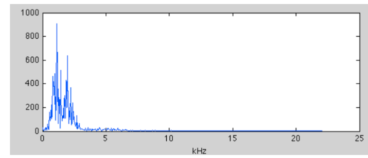
$$y[n] = x[Mn]$$

Non-Linear System Example

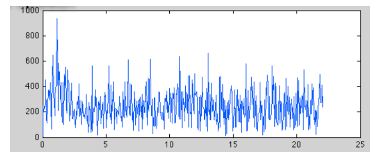
- Median Filter
  - y[n]=MED{x[n-k], …x[n+k]}
  - Let k=1
  - y[n]=MED{x[n-1], x[n], x[n+1]}

Spectrum of Speech

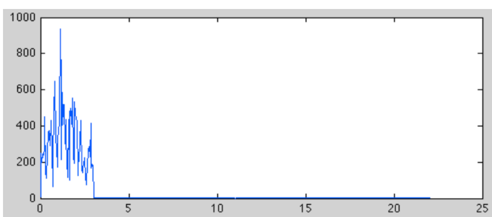
Speech



Corrupted Speech

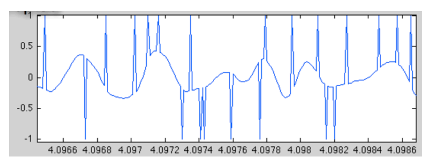


Low Pass Filtering

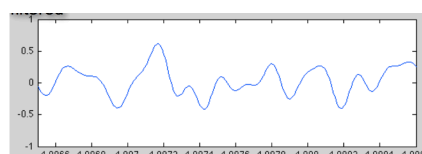


Low Pass Filtering

Corrupted Speech

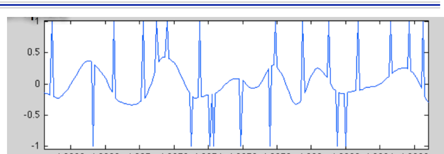


LP-Filtered Speech

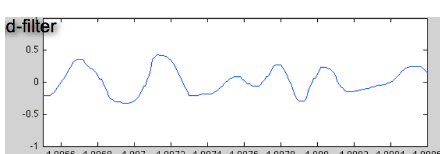


Median Filtering

Corrupted Speech



Med-Filter Speech



LTI Systems

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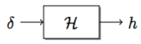
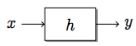
LTI Systems

> **DEFINITION** A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



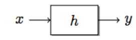
- Then the output for an arbitrary input is a sum of weighted, delay impulse responses



$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m]$$

$$y[n] = x[n] * h[n]$$

Convolution



- Convolution formula

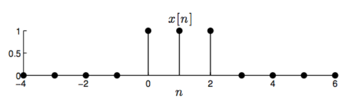
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m]$$

- To compute the entry $y[n]$ in the output vector y:
  1. **Time reverse** the impulse response vector h and **shift** it n time steps to the right (delay)
  2. Compute the **inner product** between the shifted impulse response and the input vector x
- Repeat for every n

Convolution Example

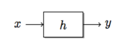
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m]$$

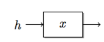
- Convolve a unit pulse with itself



Convolution is Commutative

- **Fact:** Convolution is commutative: $x * h = h * x$
- These block diagrams are equivalent:

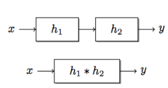




- Enables us to pick either h or x to flip and shift (or stack into a matrix) when convolving

LTI Systems in Series

- Impulse response of the **cascade** (aka series connection) of two LTI systems:



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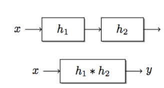
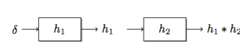
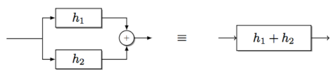


- Easy proof by picture; find impulse response the old school way



LTI Systems in Parallel

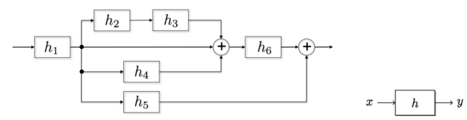
- Impulse response of the **parallel connection** of two LTI systems



- Proof is an easy application of the linearity of an LTI system

Example

- Compute the overall effective impulse response of the following system

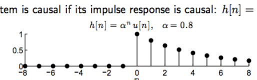


Causal System Revisited

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- **Fact:** An LTI system is causal if its impulse response is causal: $h[n] = 0$ for $n < 0$



- To prove, note that the convolution

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

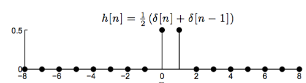
does not look into the future if $h[n-m] = 0$ when $m > n$; equivalently, $h[n'] = 0$ when $n' < 0$

Duration of Impulse

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

- Example: Moving average system $\quad y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; \frac{1}{2}\left(x[n] + x[n-1]\right)$

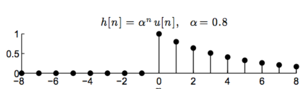


Duration of Impulse

DEFINITION

An LTI system has an **infinite impulse response** (IIR) if the duration of its impulse response h is infinite

- Example: Recursive average system $\quad y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; x[n] + \alpha\, y[n-1]$

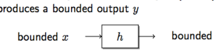


- Note: Obviously the FIR/IIR distinction applies only to infinite-length signals

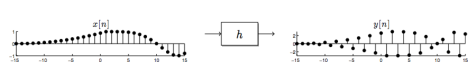
BIBO Stability Revisited

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$, or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all n



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bounded x → h → bounded y

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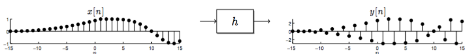

- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

BIBO Stability – Sufficient Condition

- Prove that if $\|h\|_1 < \infty$ then the system is BIBO stable – for any input $\|x\|_\infty < \infty$ the output $\|y\|_\infty < \infty$

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- Let $\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| = B < \infty$

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- Compute a bound on $|y[n]|$ using the convolution of x and h and the bounds A and B

$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} h[n-m]\, x[m] \right|$$

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$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} h[n-m]\, x[m] \right| \leq \sum_{m=-\infty}^{\infty} |h[n-m]|\, |x[m]|$$

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$$|y[n]| = \left| \sum_{m=-\infty}^{\infty} h[n-m]\, x[m] \right| \leq \sum_{m=-\infty}^{\infty} |h[n-m]|\, |x[m]|$$

$$< \sum_{m=-\infty}^{\infty} |h[n-m]|\, A = A \sum_{k=-\infty}^{\infty} |h[k]| = AB = C < \infty$$

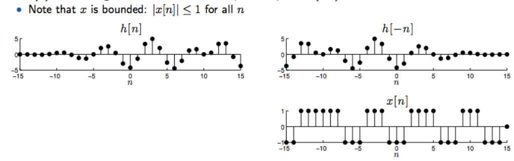
- Since $|y[n]| < C < \infty$ for all n, $\|y\|_\infty < \infty$ ✓

BIBO Stability – Necessary Condition

- Prove that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
  - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical

BIBO Stability – Necessary Condition

- Prove that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
  - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical
- Given an impulse response h with $\|h\|_1 = \infty$ (assume complex-valued), form the tricky special signal $x[n] = \text{sgn}(h[-n])$
  - $x[n]$ is the $\pm$ sign of the time-reversed impulse response $h[-n]$
  - Note that x is bounded: $|x[n]| \leq 1$ for all n



BIBO Stability – Necessary Condition

- We are proving that that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
- Armed with the tricky special signal x, compute the output $y[n]$ at the time point $n = 0$

$$y[0] = \sum_{m=-\infty}^{\infty} h[0-m]\,x[m]$$

BIBO Stability – Necessary Condition

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$$y[0] = \sum_{m=-\infty}^{\infty} h[0-m]\,x[m] = \sum_{m=-\infty}^{\infty} h[-m]\,\text{sgn}(h[-m]) = \sum_{m=-\infty}^{\infty} |h[-m]| = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

BIBO Stability – Necessary Condition

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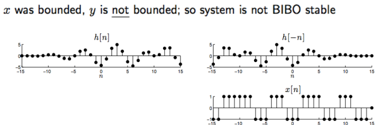
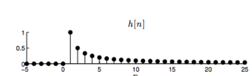
- So, even though x was bounded, y is not bounded; so system is not BIBO stable



Examples

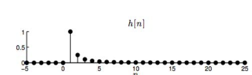
- Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

  $\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right| = \infty \Rightarrow$ not BIBO

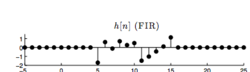


- Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

  $\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right| = \frac{\pi^2}{6} \Rightarrow$ BIBO



- Example: h FIR $\Rightarrow$ BIBO



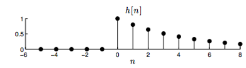
Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$
- Impulse response: $h[n] = \alpha^n u[n]$

Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$
- Impulse response: $h[n] = \alpha^n u[n]$
- For $|\alpha| < 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty \Rightarrow$ BIBO



- For $|\alpha| > 1$

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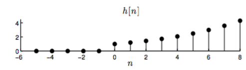

Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$

Difference Equations

- Accumulator example

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$$y[n] - y[n-1] = x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Difference Equations

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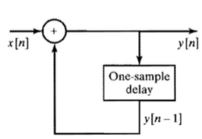

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Example: Difference Equation

- Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- Let M_1=0 (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Big Ideas

- LTI Systems are a special class of systems with significant signal processing applications
  - Can be characterized by the impulse response
- LTI System Properties
  - Causality and stability can be determined from impulse response
- Difference equations suggest implementation of systems
  - Give insight into complexity of system

Admin

- HW 1 out now
  - Due 1/26 at midnight
  - Submit in Canvas