ESE 531: Digital Signal Processing

Lec 4: January 23, 2018

Discrete Time Fourier Transform

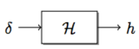
Lecture Outline

- LTI Systems
- Difference Equations
- Eigenfunctions
- Discrete Time Fourier Transform
  - Definition
  - Properties
- Frequency Response of LTI Systems

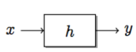
LTI Systems

DEFINITION: A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



- Then the output for an arbitrary input is a sum of weighted, delay impulse responses



$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

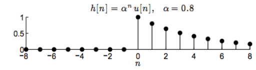
$$y[n] = x[n] * h[n]$$

Causal System Revisited

DEFINITION: A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- **Fact:** An LTI system is causal if its impulse response is causal: $h[n] = 0$ for $n < 0$

$h[n] = \alpha^n u[n], \quad \alpha = 0.8$



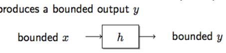
- To prove, note that the convolution

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

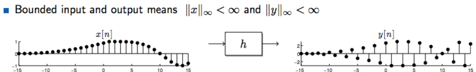
does not look into the future if $h[n-m] = 0$ when $m > n$; equivalently, $h[n'] = 0$ when $n' < 0$

BIBO Stability Revisited

DEFINITION: An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$

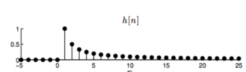


- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

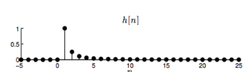
Examples

- Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$



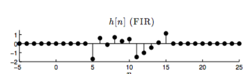
$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right| = \infty \;\Rightarrow$ not BIBO

- Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$



$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right| = \frac{\pi^2}{6} \;\Rightarrow$ BIBO

- Example: h FIR $\;\Rightarrow$ BIBO



Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$
- Impulse response: $h[n] = \alpha^n u[n]$

Example

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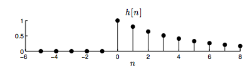
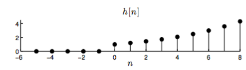
- Impulse response: $h[n] = \alpha^n u[n]$
- For $|\alpha| < 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty \Rightarrow$ BIBO



- For $|\alpha| > 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \infty \Rightarrow$ not BIBO



Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$

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$$\boxed{y[n] - y[n-1] = x[n]}$$

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

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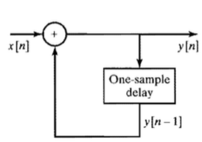

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

Example: Difference Equation

- Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- Let M_1=0 (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Eigenfunctions

Eigenfunction

- $x[n]=e^{j\omega n}$

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$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$
$$=\sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k]$$
$$=e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$
$$=H(e^{j\omega})e^{j\omega n}$$

Eigenvalue (frequency response)

- $x[n]=e^{j\omega n}$

$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$
$$=\sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k]$$
$$=e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$
$$=H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega})=\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase

CT vs DT Frequency Response

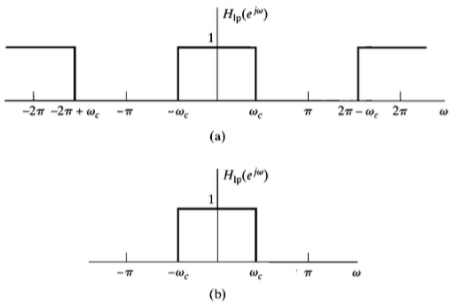
- $H(e^{j(\omega+2\pi)n})$?

CT vs DT Frequency Response

- $H(e^{j(\omega+2\pi)n})$?

$$H(e^{j(\omega+2\pi)n})=\sum_{k=-\infty}^{\infty} h[k]e^{-j(\omega+2\pi)k}$$
$$=\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{-j2\pi k}$$
$$=\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$
$$=H(e^{j\omega n})$$

Periodicity of Low Pass Freq Response

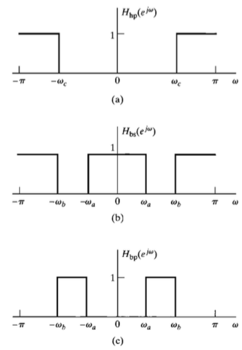


Other Filters

High-pass

Band-stop

Band-pass



Discrete-Time Fourier Transform (DTFT)

Penn

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

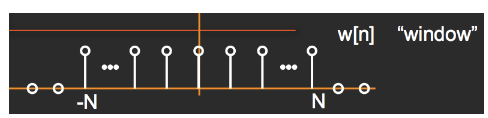
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$$

$$x[n] = \int_{-0.5}^{0.5} X(f)e^{j2\pi fn}df$$

Example: Window DTFT



$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k]e^{-j\omega k}$$

$$= \sum_{k=-N}^{N} e^{-j\omega k}$$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$
$$= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N}$$
$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$
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Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$
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Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

$$p = e^{j\omega} \qquad M = 2N$$

$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$
$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}}$$

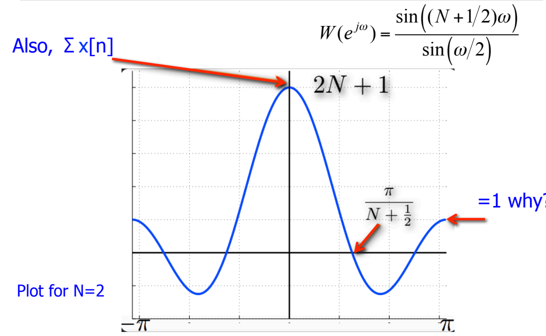
Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$
$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}}$$
$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}}$$
$$= \frac{e^{-j\omega(N+1/2)} - e^{j\omega(N+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}} = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Periodic sinc

Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$



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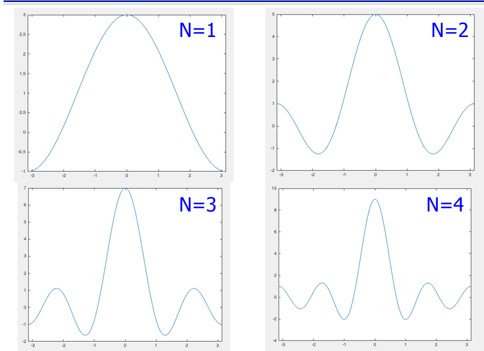


Properties of the DTFT

- Linearity:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

- Conjugate Symmetry:

$$X^*(e^{j\omega}) = X(e^{-j\omega}) \quad \text{If x[n] real}$$

$$\mathrm{Re}\{X(e^{-j\omega})\} = \mathrm{Re}\{X(e^{j\omega})\}$$

$$\mathrm{Im}\{X(e^{-j\omega})\} = -\mathrm{Im}\{X(e^{j\omega})\}$$

Properties of the DTFT

- Time Reversal:

$$x[n] \leftrightarrow X(e^{j\omega}) \qquad \text{If x[n] real}$$

$$x[-n] \leftrightarrow X(e^{-j\omega}) \qquad x[-n] \leftrightarrow X^*(e^{j\omega})$$

- Time/Freq Shifting:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$x[n-n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

Properties of the DTFT

- Differentiation in Frequency:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

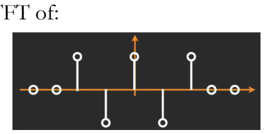
- Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Example: Windowed cos(π n)

- What is DTFT of:



$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

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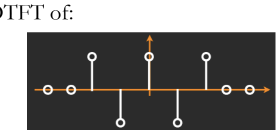

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$$x[-n] \leftrightarrow X^*(e^{-j\omega})$$

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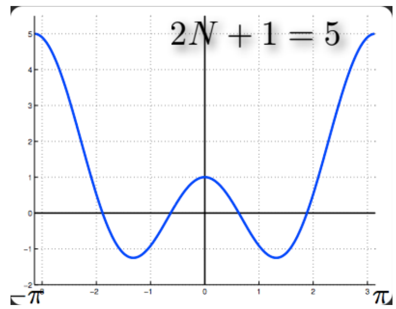
$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

Example: Windowed cos(π n)



Frequency Response of LTI Systems

LTI Systems

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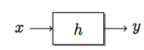
- LTI system can be completely characterized by its impulse response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

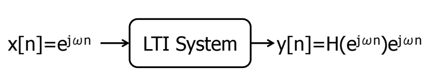


$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

LTI System Frequency Response

- (DT)Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \ldots + x[n]}{M+1}$$

Impulse response

Example: Moving Average

- Moving Average Filter
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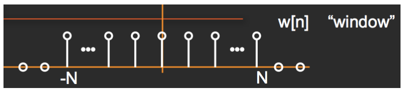
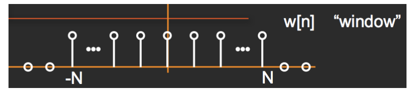
Impulse response

Scaled &Time Shifted window



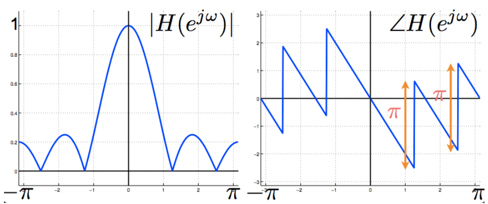
Example: Moving Average



$$w[n]\leftrightarrow W(e^{j\omega})=\frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$\frac{1}{M+1}w[n-M/2]\leftrightarrow W(e^{j\omega})=\frac{e^{-j\omega M/2}}{M+1}\frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average

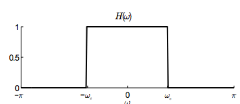


M=4

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega=0$) but blocks high frequencies (near $\omega=\pm\pi$)

$$H(\omega)=\begin{cases}1 & -\omega_c\le\omega\le\omega_c\\ 0 & \text{otherwise}\end{cases}$$



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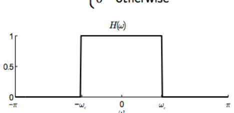

- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n]=\int_{-\pi}^{\pi}H(\omega)\,e^{j\omega n}\frac{d\omega}{2\pi}=\int_{-\omega_c}^{\omega_c}e^{j\omega n}\frac{d\omega}{2\pi}=\left.\frac{e^{j\omega n}}{2\pi jn}\right|_{-\omega_c}^{\omega_c}=\frac{e^{j\omega_c n}-e^{-j\omega_c n}}{2\pi jn}=\frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n}$$

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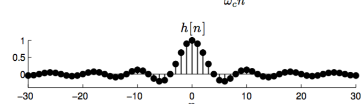
$$H(\omega)=\begin{cases}1 & -\omega_c\le|\omega|\le\omega_c\\ 0 & \text{otherwise}\end{cases}$$

$$h[n]\;=\;2\omega_c\frac{\sin(\omega_c n)}{\omega_c n}$$



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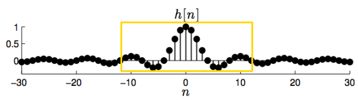
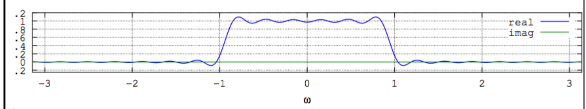
$$H(\omega) = \begin{cases} 1 & -\omega_c \le |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncation and shift $h_{LP}[n] = w_N[n-N] \cdot h[n-N]$

Example: Practical LP Filter



- Pass band smeared and rippled
  - Smearing determined by width of main lobe
  - Rippling determined by size of main lobes

Big Ideas

- Difference Equations
  - Help see implementation… more later with z-transform
- Discrete Time Fourier Transform
  - Represent signals as a sum of scaled and phase shifted complex sinusoids (eigenfunctions)
  - Continuous in frequency over 2π

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}$$

- Frequency Response of LTI Systems

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

  - Frequency response of impulse response
  - Describes scaling and phase shifting of a pure frequency

$x[n]=e^{j\omega n}$ → LTI System → $y[n]=H(e^{j\omega n})e^{j\omega n}$

Admin

- HW 1 out now
  - Due 1/26 at midnight
  - Submit in Canvas
- Office hour locations
  - In limbo
  - Keep an eye on Piazza