ESE 531: Digital Signal Processing

Lec 5: January 25, 2018

z-Transform

Lecture Outline

- z-Transform
  - Regions of convergence (ROC) & properties
  - z-Transform properties
- Inverse z-transform

z-Transform

z-Transform

- The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- Very useful for designing and analyzing signal processing systems
- Properties are very similar to the DTFT with a few caveats

Reminder: DTFT Definition

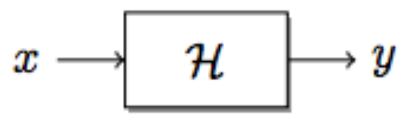
- The core "basis functions" (I.e eigenfunctions) of the DTFT are the complex sinusoids $e^{j\omega n}$ with arbitrary frequencies ω
- The sinusoids $e^{j\omega n}$ are eigenvectors of LTI systems for infinite-length signals

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad -\infty \le n < \infty$$

Reminder: Frequency Response of LTI System

- We can use the DTFT to characterize an LTI system



$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- and relate the DTFTs of the input and output

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad H(\omega) = \sum_{m=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$Y(\omega) \;=\; X(\omega) H(\omega)$$

Reminder: Complex Exponentials

- Complex sinusoid $e^{j(\omega n+\phi)}$ is of the form $e^{\text{Purely Imaginary Numbers}}$

- Generalize to $e^{\text{General Complex Numbers}}$

- Consider the general complex number $z = |z|\, e^{j\omega}$, $z \in \mathbb{C}$
  - $|z| =$ magnitude of z
  - $\omega = \angle(z)$, phase angle of z
  - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**

- Now we have

$$z^n = (|z|e^{j\omega})^n = |z|^n(e^{j\omega})^n = |z|^n e^{j\omega n}$$

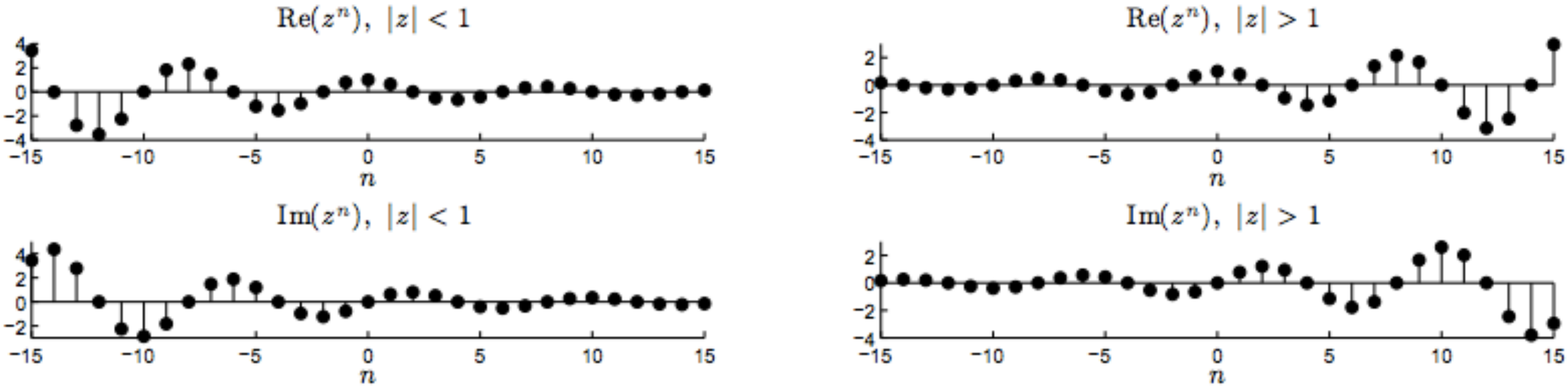
  - $|z|^n$ is a **real exponential** (a^n with $a = |z|$)
  - $e^{j\omega n}$ is a **complex sinusoid**

Reminder: Complex Exponentials

$$z^n \;=\; \left(|z|\, e^{j\omega n}\right)^n \;=\; |z|^n\, e^{j\omega n}$$

- $|z|^n$ is a **real exponential** envelope $(a^n$ with $a = |z|)$
- $e^{j\omega n}$ is a **complex sinusoid**

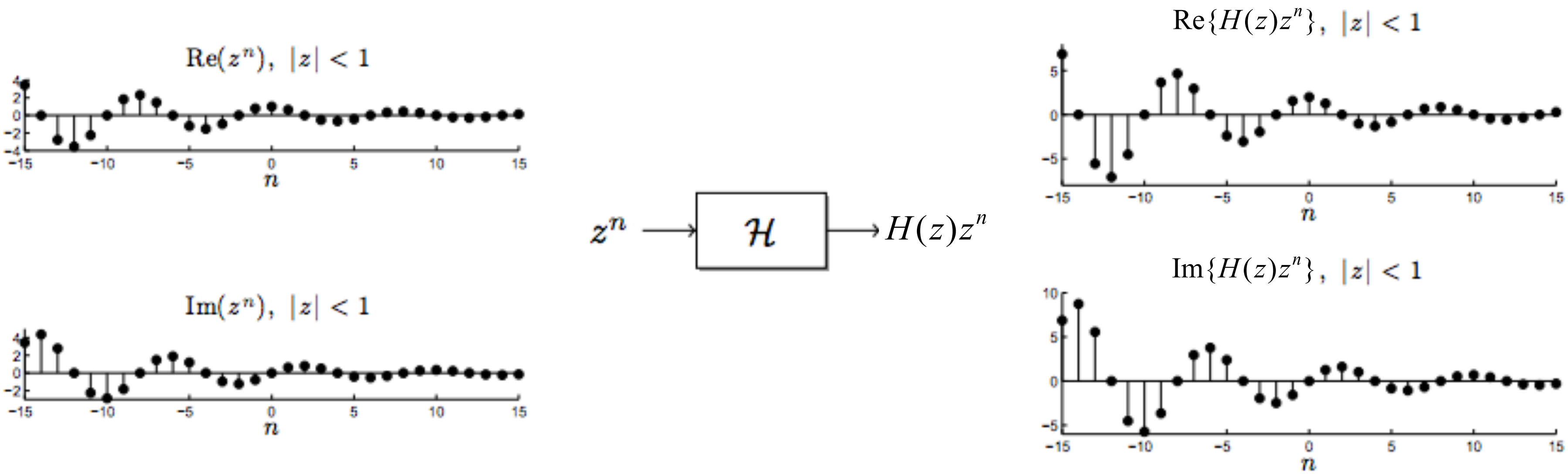
$|z| < 1$ $\qquad\qquad$ $|z| > 1$



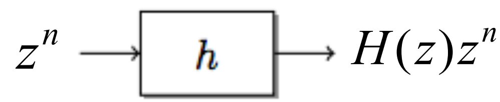
Bounded $\qquad\qquad$ Unbounded

Complex Exponentials as Eigenfunctions

- Fact: A more general set of eigenfunctions of an LTI system are the complex exponentials z^n, $z \in C$



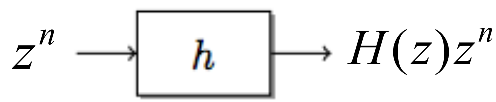
Complex Exponentials as Eigenfunctions



- Prove that complex exponentials are the eigenvectors of LTI systems simply by computing the convolution with input z^n

$$z^n * h[n] \quad =$$

Complex Exponentials as Eigenfunctions



- Prove that complex exponentials are the eigenvectors of LTI systems simply by computing the convolution with input z^n

$$
\begin{aligned}
z^n * h[n] &= \sum_{m=-\infty}^{\infty} z^{n-m}\, h[m] = \sum_{m=-\infty}^{\infty} z^n\, z^{-m}\, h[m] \\
&= \left(\sum_{m=-\infty}^{\infty} h[m]\, z^{-m} \right) z^n \\
&= H(z) z^n
\end{aligned}
$$

$$
\boxed{\sum_{n=-\infty}^{\infty} h[n]\, z^{-n} = H(z)}
$$

z-Transform

- Define the **forward z-transform** of x[n] as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

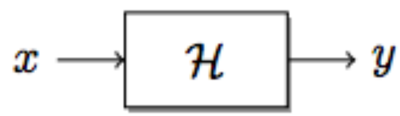
- The core "basis functions" of the z-transform are the complex exponentials z^n with arbitrary $z \in C$; these are the eigenfunctions of LTI systems for infinite-length signals

- **Notation abuse alert:** We use X(•) to represent both the DTFT X(ω) and the z-transform X(z); they are, in fact, intimately related

$$X_{\mathrm{DTFT}}(\omega) = X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

Transfer Function of LTI System

- We can use the z-Transform to characterize an LTI system



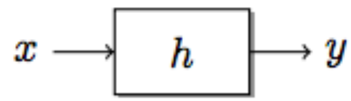
$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- and relate the z-transforms of the input and output

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \qquad H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

$$Y(z) \;=\; X(z)\, H(z)$$

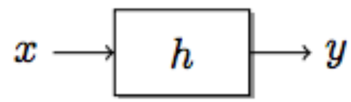
Proof



- Compute the z-transform of the result of the convolution of x and h
  (Note: we are a little cavalier about exchanging the infinite sums below)

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m]\, h[n-m] \right) z^{-n}$$

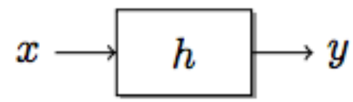
Proof



- Compute the z-transform of the result of the convolution of x and h (Note: we are a little cavalier about exchanging the infinite sums below)

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m]\, h[n-m] \right) z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{n=-\infty}^{\infty} h[n-m]\, z^{-n} \right) \end{aligned}$$

Proof



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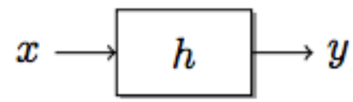
Y(z) &= \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m]\, h[n-m] \right) z^{-n} \\
&= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{n=-\infty}^{\infty} h[n-m]\, z^{-n} \right) \quad (\text{let } r = n - m) \\
&= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{r=-\infty}^{\infty} h[r]\, z^{-r-m} \right)
\end{aligned}
$$

Proof



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&= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{r=-\infty}^{\infty} h[r]\, z^{-r-m} \right) = \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m} \right) \left(\sum_{r=-\infty}^{\infty} h[r]\, z^{-r} \right) \\
&= X(z)\, H(z) \ \checkmark
\end{aligned}
$$

Reminder: Complex Exponentials

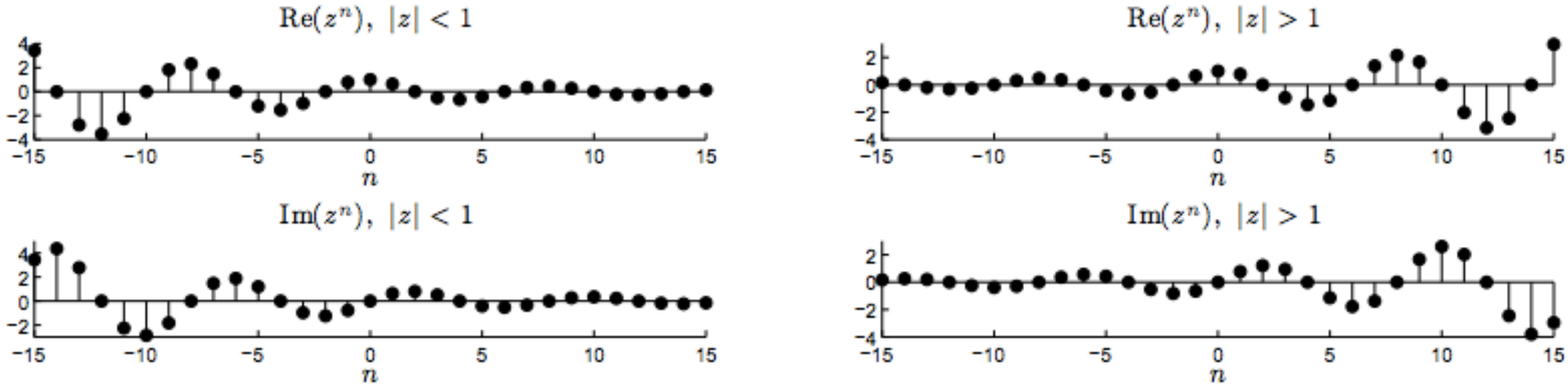
$$z^n = \left(|z|\, e^{j\omega n}\right)^n = |z|^n\, e^{j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- $|z|^n$ is a **real exponential** envelope $(a^n$ with $a = |z|)$
- $e^{j\omega n}$ is a **complex sinusoid**

What are we missing?

$|z| < 1$ $\qquad$ $|z| > 1$



Bounded $\qquad$ Unbounded

Region of Convergence (ROC)

Region of Convergence (ROC)

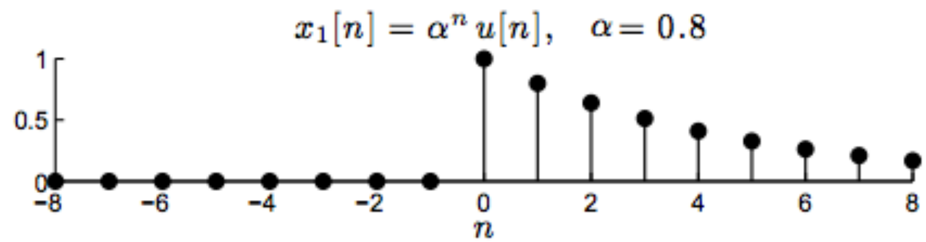
DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its z-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| \; < \; \infty$$

ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal) Right-sided sequence
- Example for $\alpha = 0.8$



- The **forward** z**-transform** of $x_1[n]$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n}$$

ROC Example 1

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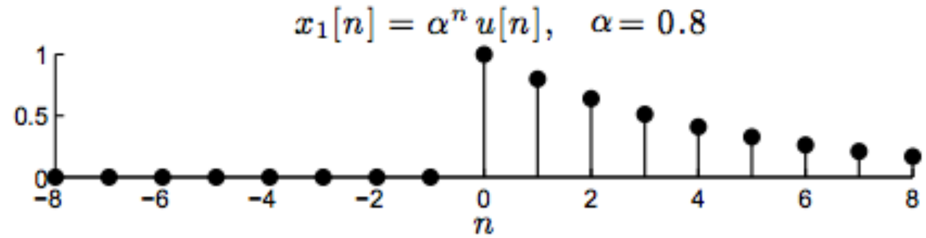


- The **forward** z**-transform** of $x_1[n]$

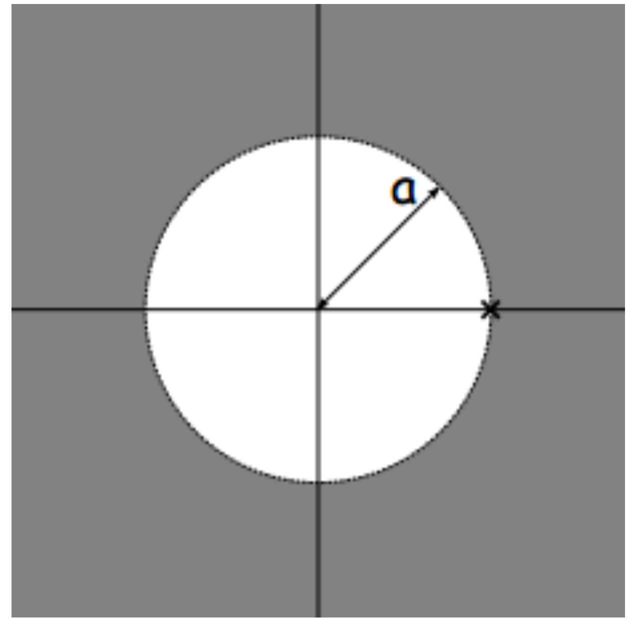
$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$

ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal)

$$ROC = \{z : |z| > |a|\}$$



- The **forward z-transform** of $x_1[n]$

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- **Important:** We can apply the geometric sum formula only when $|\alpha\, z^{-1}| < 1$ or $|z| > |\alpha|$

ROC Example 1

- What is the DTFT of $x_1[n]=a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad -\infty \le n < \infty$$

ROC Example 1

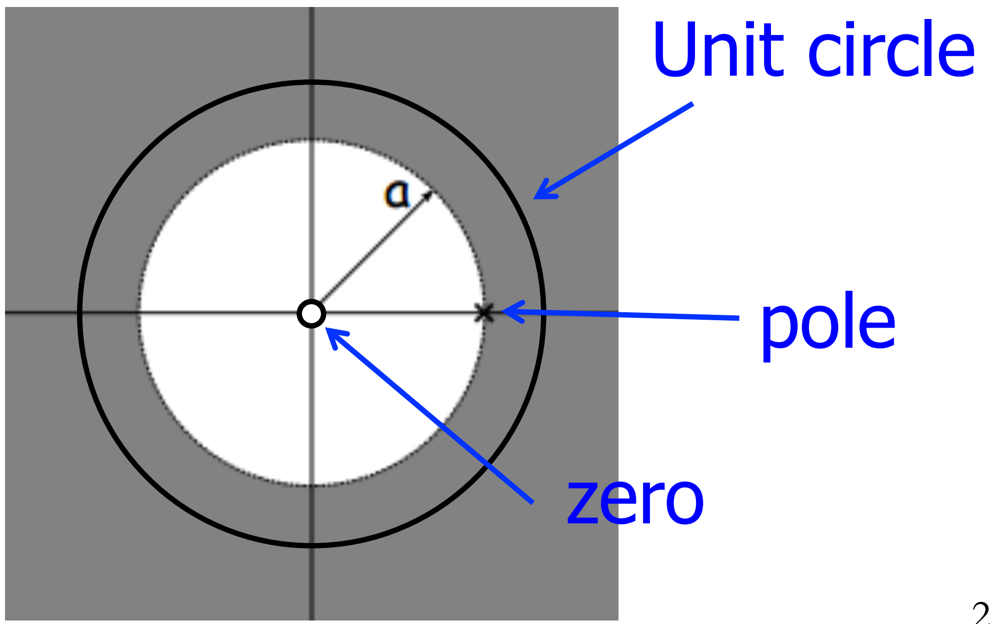
- What is the DTFT of $x_1[n]=a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad -\infty \le n < \infty$$

$$X_1(z) = \frac{z}{z-a}$$

$$ROC = \{z : \ |z| > |a|\}$$



ROC Example 2

- What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

ROC Example 2

- What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

- Hint:

$$x_1[n] = a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$$

ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Left-sided sequence

ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- The z-transform without ROC does not uniquely define a sequence!

ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

- Hint:

$$x_1[n] = a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \quad |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \quad |z| < |a|\}$$

ROC Example 5

- What is the z-transform of $x_5[n]$? ROC?

$$x_5[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

two-sided sequence

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

finite length sequence

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n+M-1]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}}$$

Zero cancels pole

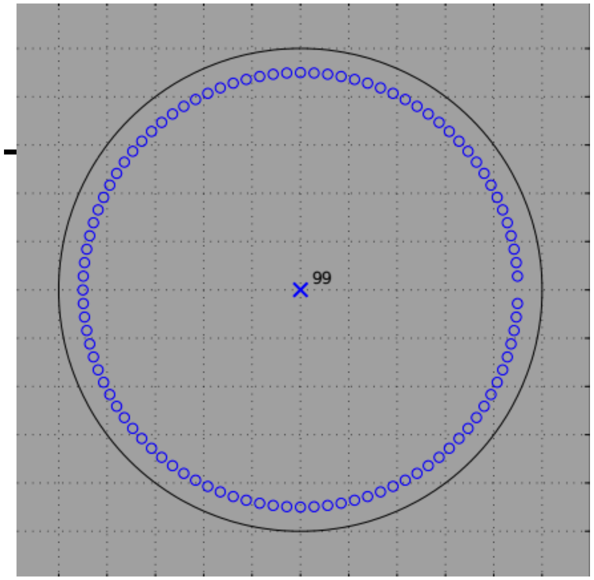
$$= \prod_{k=1}^{M-1} (1 - a e^{j2\pi k/M} z^{-1})$$

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M -$$



$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - a z^{-1}}$$

Zero cancels pole

$$= \prod_{k=1}^{M-1} \left(1 - a e^{j2\pi k/M} z^{-1}\right)$$

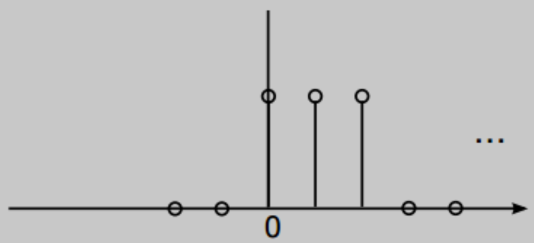
M=100

Properties of ROC

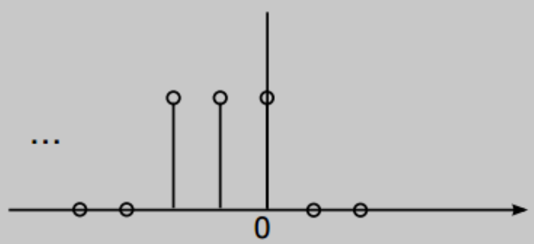
- A ring or a disk in Z-plane, centered at the origin
- DTFT converges *if and only if* ROC includes the unit circle
- ROC can't contain poles

Properties of ROC

- For finite duration sequences, ROC is the entire z-plane, except possibly z=0, z=∞



$X(z) = 1 + z^{-1} + z^{-2}$ ROC excludes $z = 0$



$X(z) = 1 + z^{1} + z^{2}$ ROC excludes $z = \infty$

Properties of ROC

- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  - Examples 1,2
- For left-sided: inwards from inner most pole to zero
  - Example 3
- For two-sided, ROC is a ring - or do not exist
  - Examples 4,5

Properties of z-Transform

- Linearity:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z)$$

- Time shifting:

$$x[n] \leftrightarrow X(z)$$

$$x[n - n_d] \leftrightarrow z^{-n_d} X(z)$$

- Multiplication by exponential sequence

$$x[n] \leftrightarrow X(z)$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$$

Properties of z-Transform

- Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

- Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$

- Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

ROC_Y at least $ROC_X \wedge ROC_H$

Inverse z-Transform

- Recall the **inverse DTFT**

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

- There exists a similar formula for the **inverse z-transform** via a contour integral in the complex z-plane

$$x[n] = \oint_C X(z)\, z^n\, \frac{dz}{j2\pi z}$$

- Evaluation of such integrals is fun, but beyond the scope of this course

Inverse z-Transform

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Power series expansion
  - Partial fraction expansion
  - Residue theorem

Big Ideas

- z-Transform
  - Uses complex exponential eigenfunctions to represent discrete time sequence
    - DTFT is z-Transform where $z=e^{j\omega}$, $|z|=1$
  - Draw pole-zero plots
  - Must specify region of convergence (ROC)

- z-Transform properties
  - Similar to DTFT
- Inverse z-transform
  - Avoid it!

Admin

- HW 1 due tomorrow at midnight
  - Submit **single pdf** in Canvas
  - Don't need to submit .m file, just the code in your pdf
- HW 2 posted tonight
- Advice
  - Want to try and develop intuition
  - Practice problems at end of chapter with answers in the text
  - Matlab resources
    - Mathworks website
    - See piazza for other references