

ESE 531: Digital Signal Processing

Lec 7: February 1st, 2018
Sampling and Reconstruction



Lecture Outline

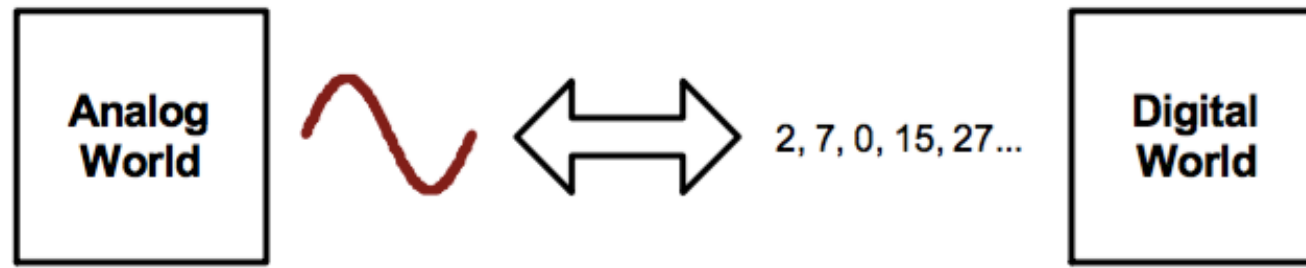
- ❑ Sampling
- ❑ Frequency Response of Sampled Signal
- ❑ Reconstruction

Video Example



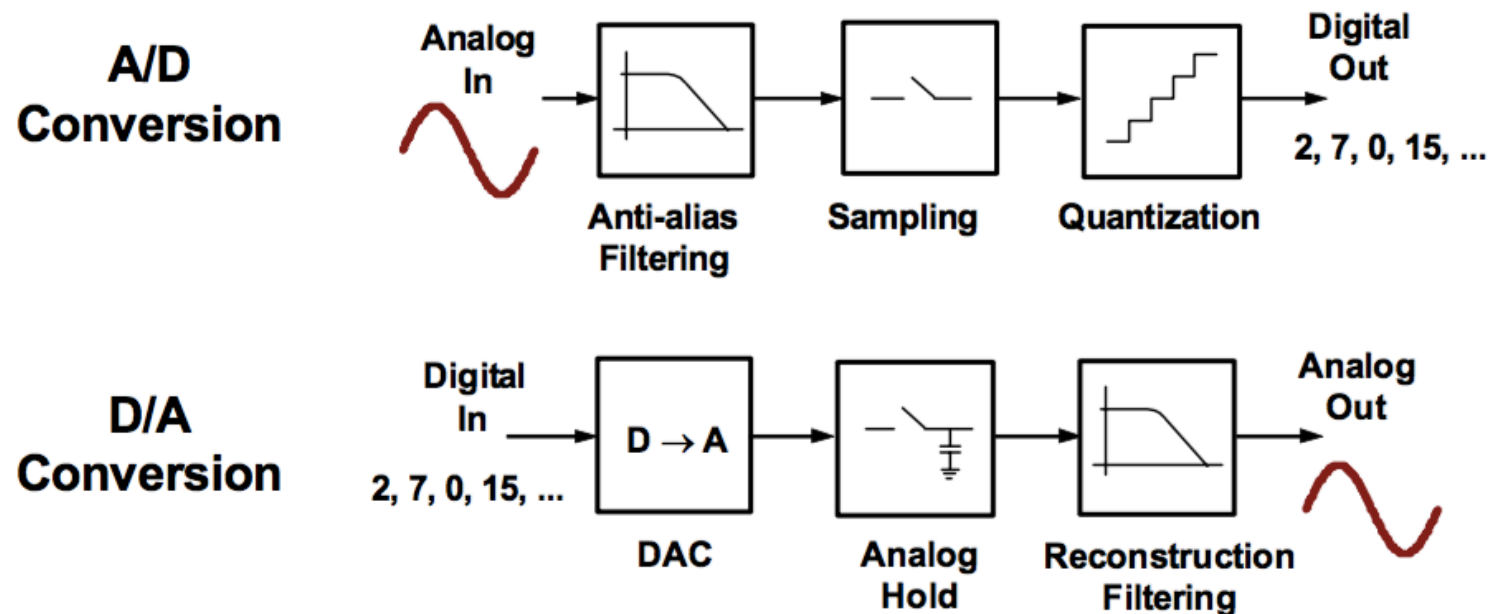
❑ <https://www.youtube.com/watch?v=ByTsISFXUoY>

The Data Conversion Problem



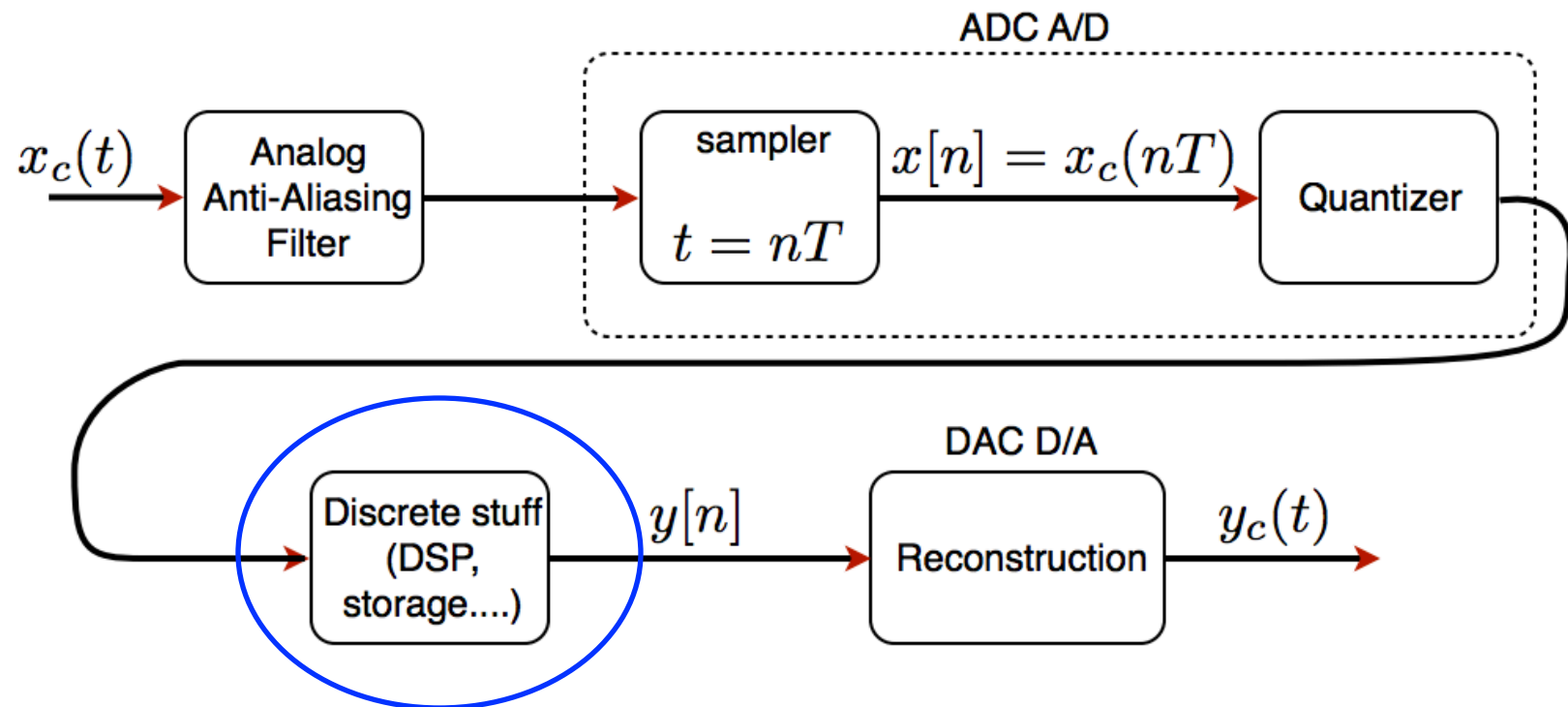
- ❑ Real world signals
 - Continuous time, continuous amplitude
- ❑ Digital abstraction
 - Discrete time, discrete amplitude
- ❑ Two problems
 - How to go discretize in time and amplitude
 - A/D conversion
 - How to "undescretize" in time and amplitude
 - D/A conversion

Overview

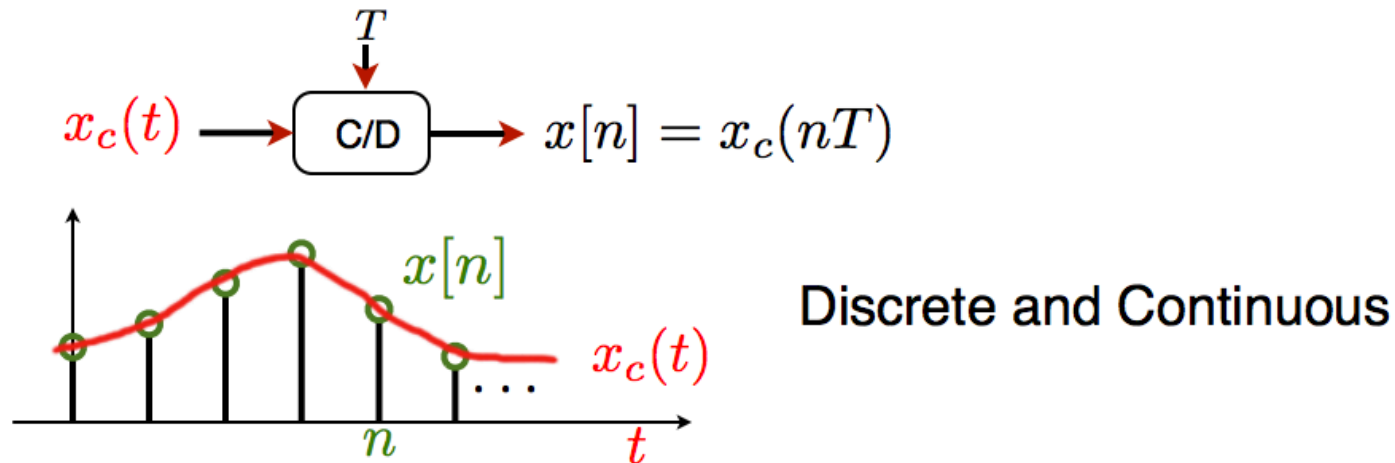


- ❑ We'll first look at these building blocks from a functional, "black box" perspective
 - Later refine and look at implementation

DSP System

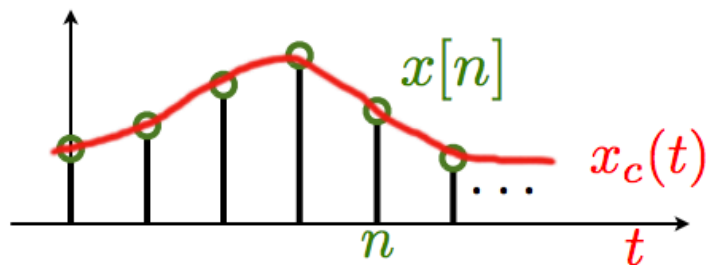
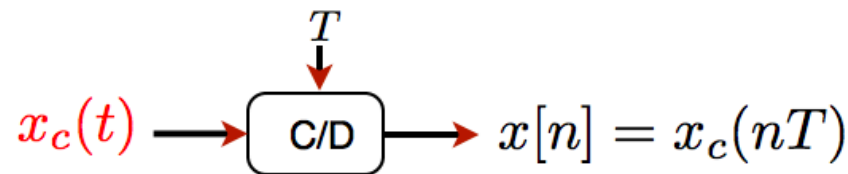


Ideal Sampling Model



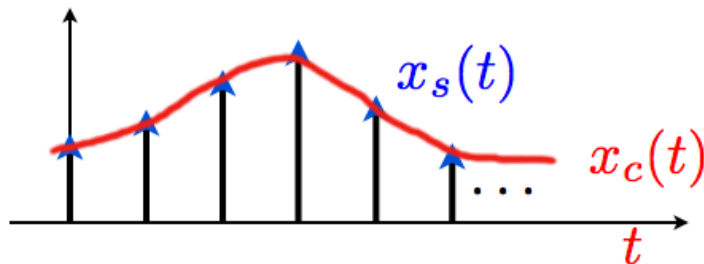
- ❑ Ideal continuous-to-discrete time (C/D) converter
 - T is the sampling period
 - $f_s = 1/T$ is the sampling frequency
 - $\Omega_s = 2\pi/T$

Ideal Sampling Model



Discrete and Continuous

define impulsive sampling:



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t - T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- ❑ Dirac delta function, $\delta(t)$
 - Infinitely high and thin, area of 1
 - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- ❑ Three signals. How are they related? In time? In frequency?



Frequency Domain Analysis

- How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Frequency Domain Analysis

- How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x[n] \quad \text{D.T} \qquad X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}$$

Frequency Domain Analysis

- How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) \quad : \text{C.T.}$$

$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

$$x[n] \quad : \text{D.T.}$$

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \qquad \omega = \Omega T$$


Frequency Domain Analysis

- How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) \quad : \text{C.T.}$$

$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

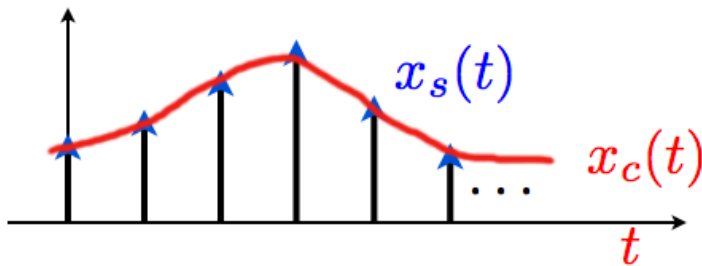
$$x[n] \quad : \text{D.T.}$$

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \qquad \omega = \Omega T$$

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$$

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T}$$

Frequency Domain Analysis



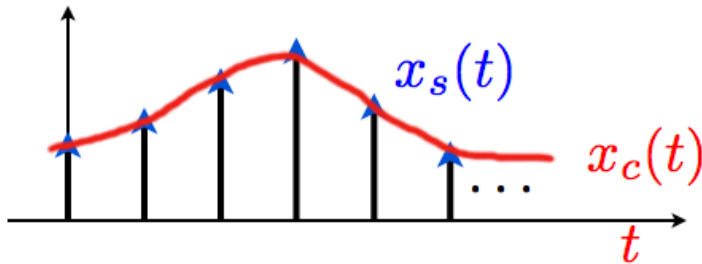
Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t - T) + \cdots$$

$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t - T) + \cdots$$

$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$s(t) \leftrightarrow S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$

Ω_s



Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$



Frequency Domain Analysis

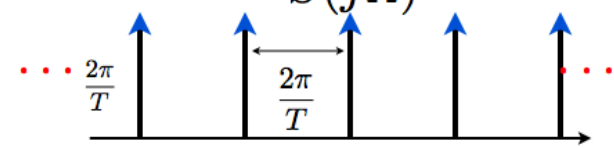
$$x_s(t) = s(t) \cdot x_c(t)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

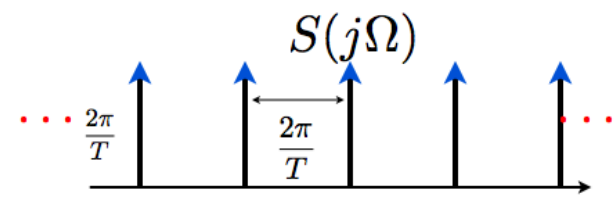
$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$


The diagram illustrates the frequency spectrum $S(j\Omega)$ as a periodic impulse train. The impulses are represented by vertical blue arrows on a horizontal axis. The period between the impulses is labeled $\frac{2\pi}{T}$. The label $S(j\Omega)$ is positioned above the central impulse. Red dots and ellipses indicate the sequence continues in both directions.

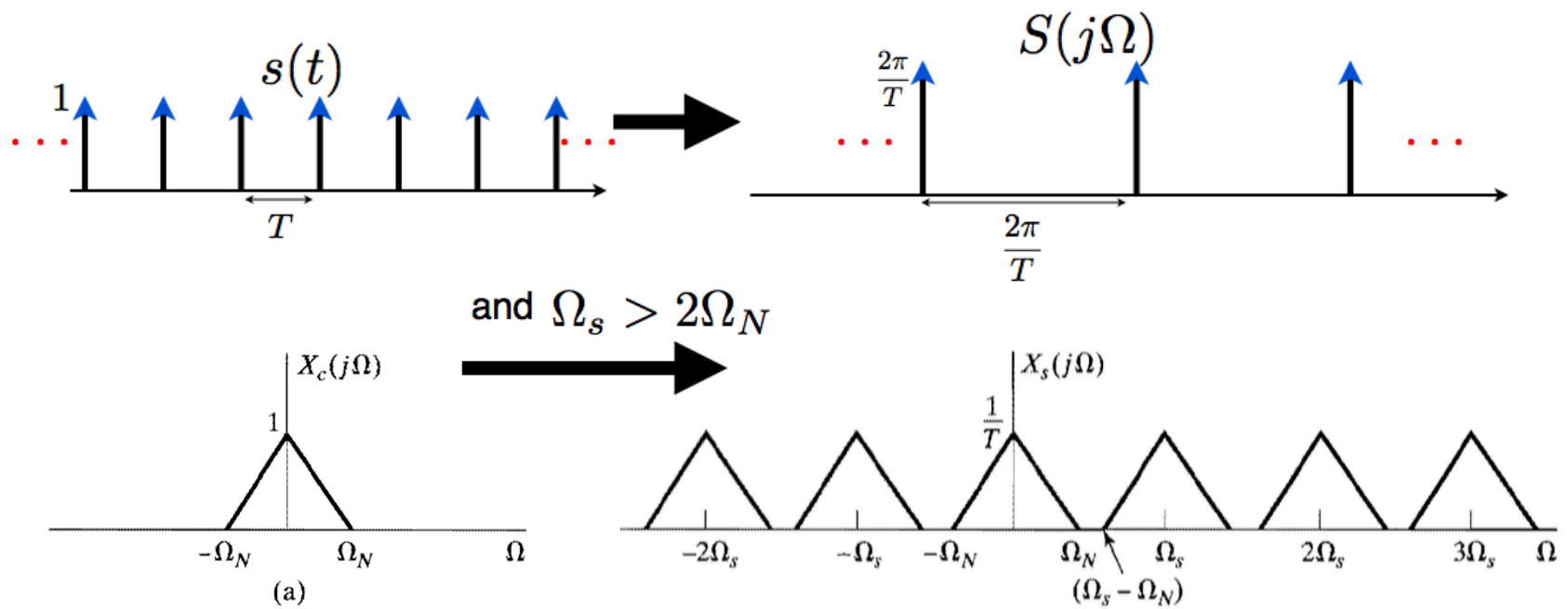
Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

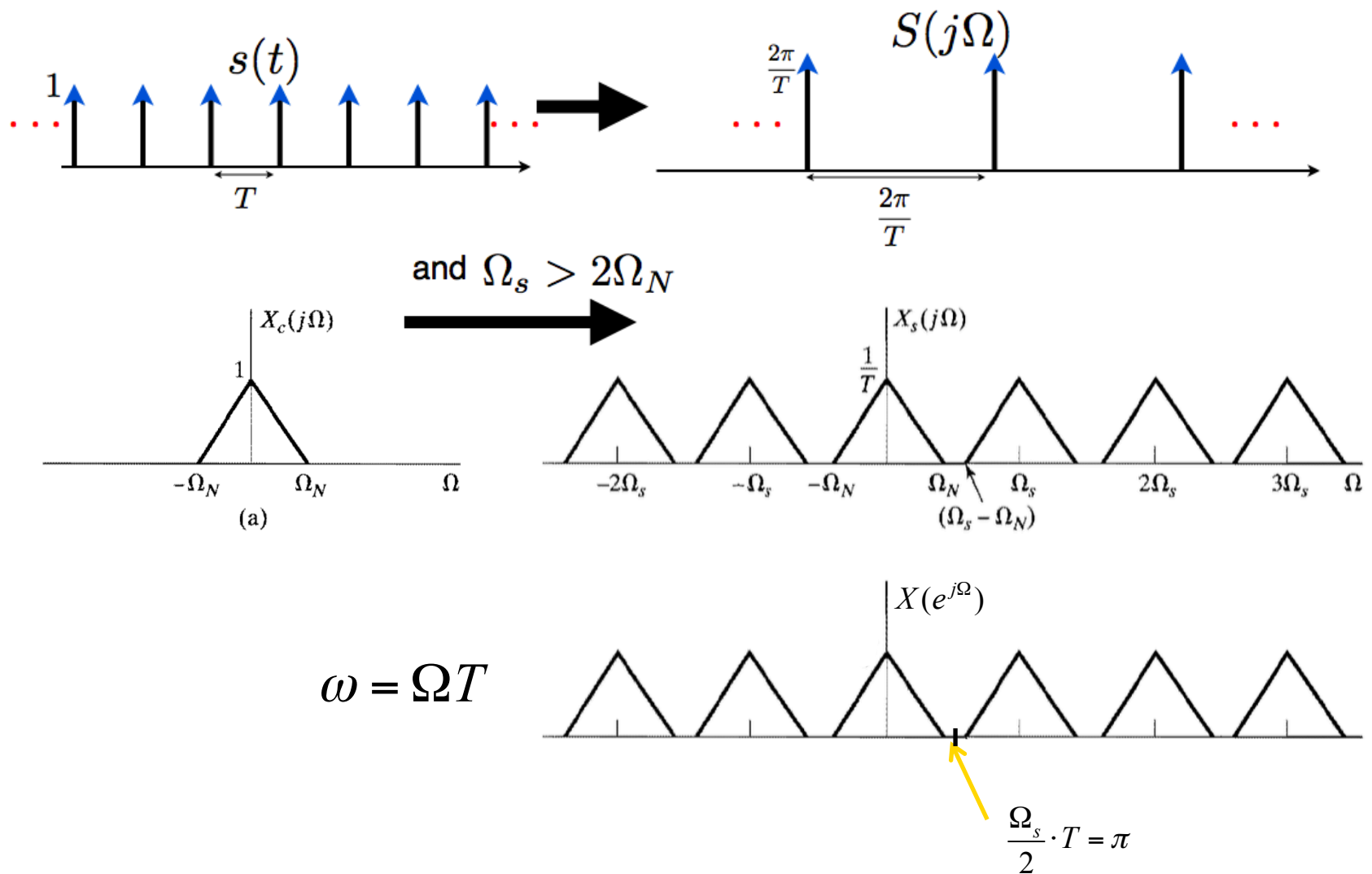
$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T} \end{aligned}$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$


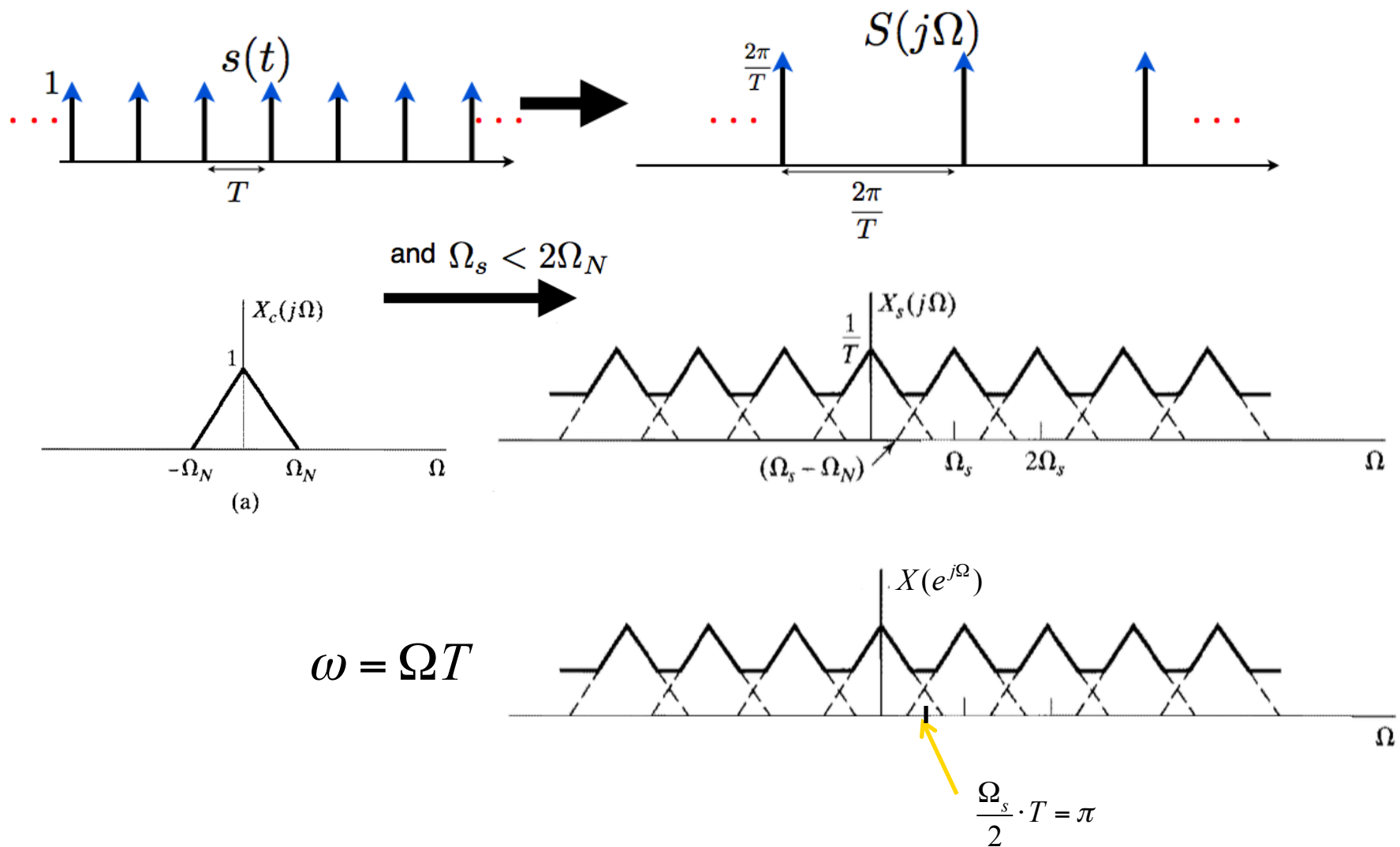
Frequency Domain Analysis



Frequency Domain Analysis



Frequency Domain Analysis

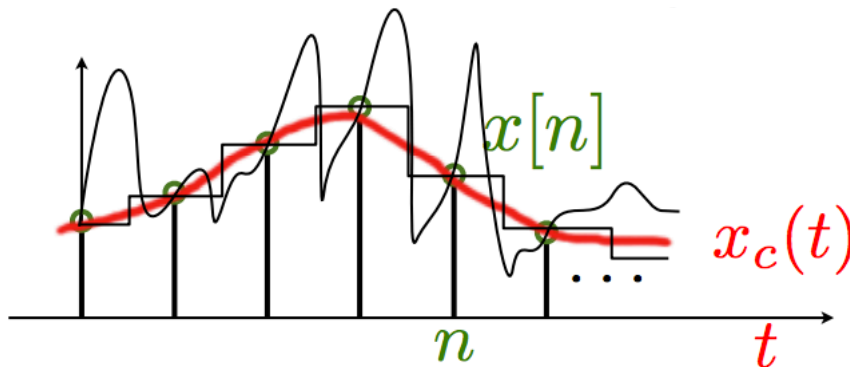


Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

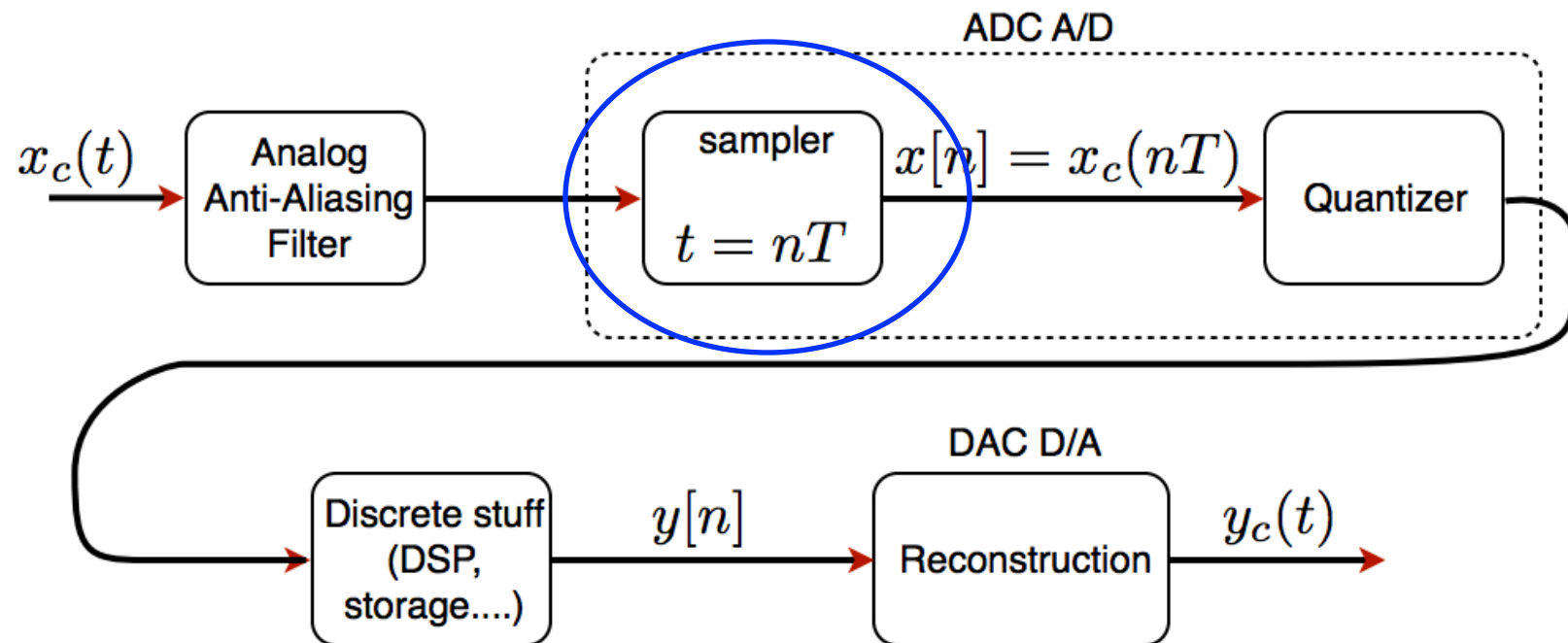
$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$
- Bandlimitedness is the key to uniqueness

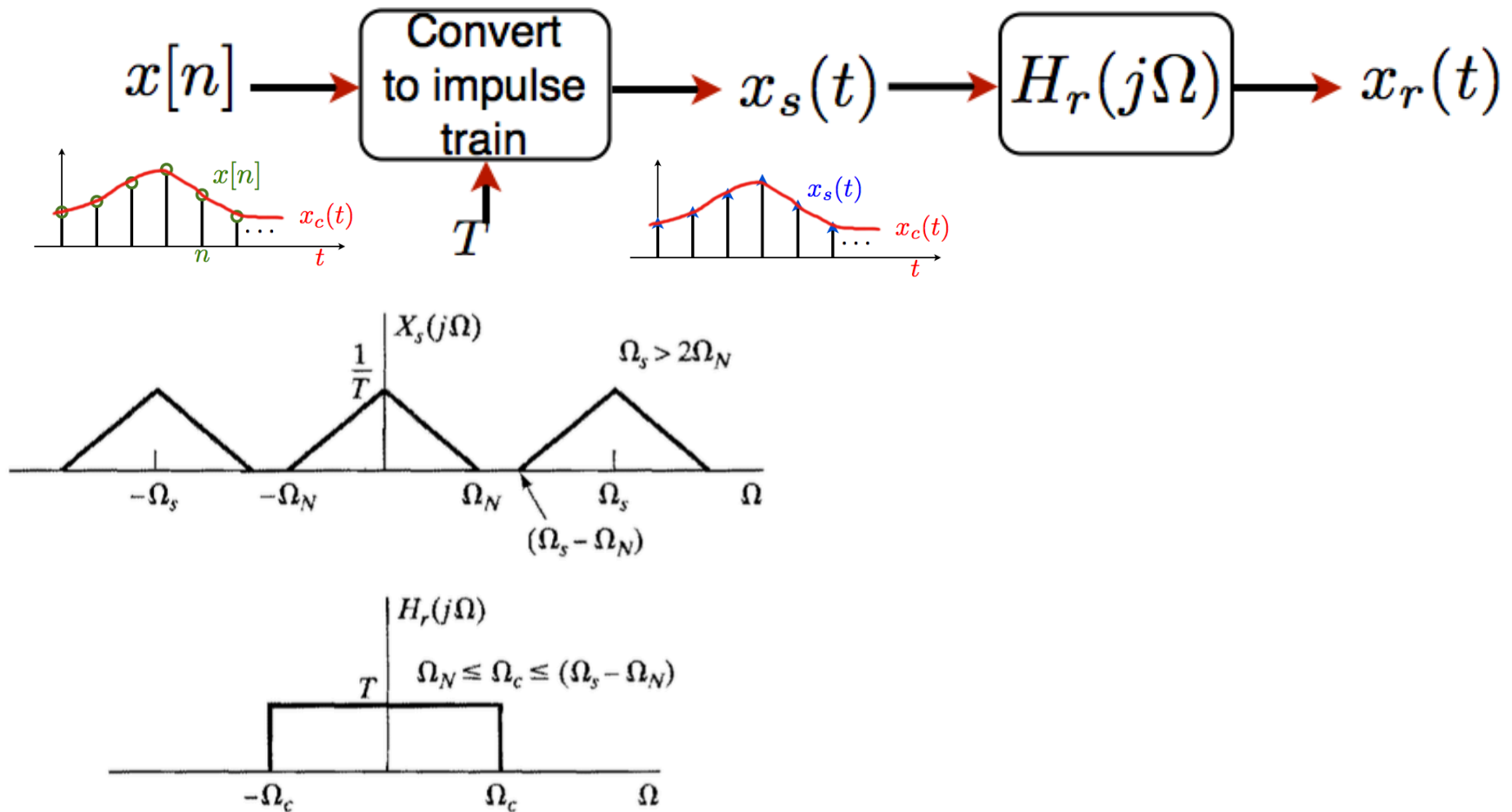


Multiple signals go through the samples, but only one is bandlimited within our sampling band

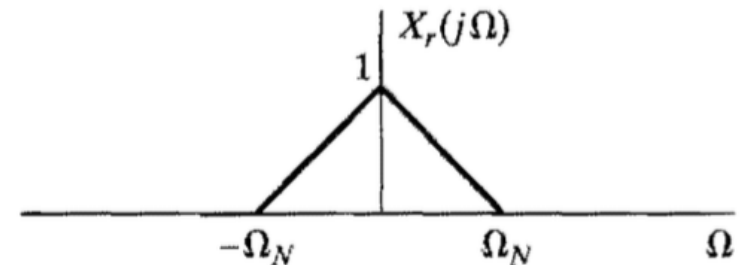
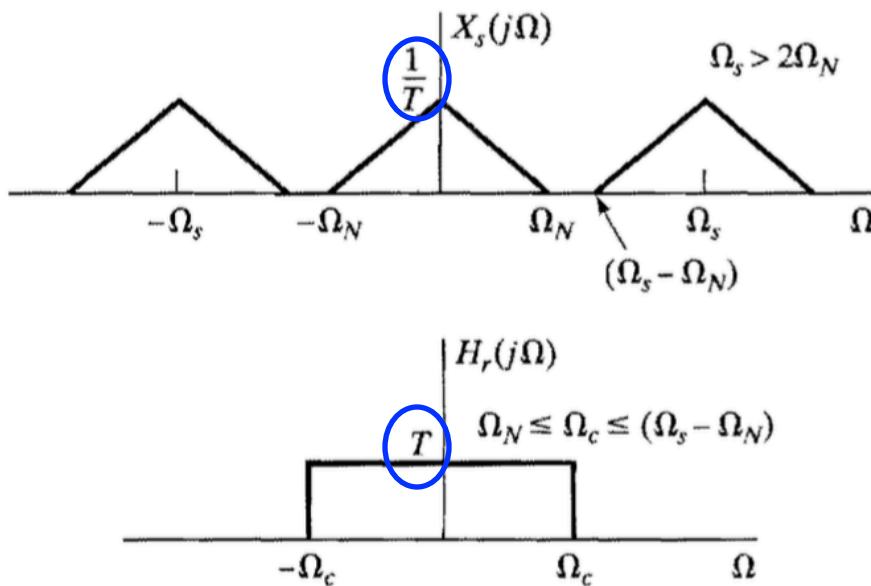
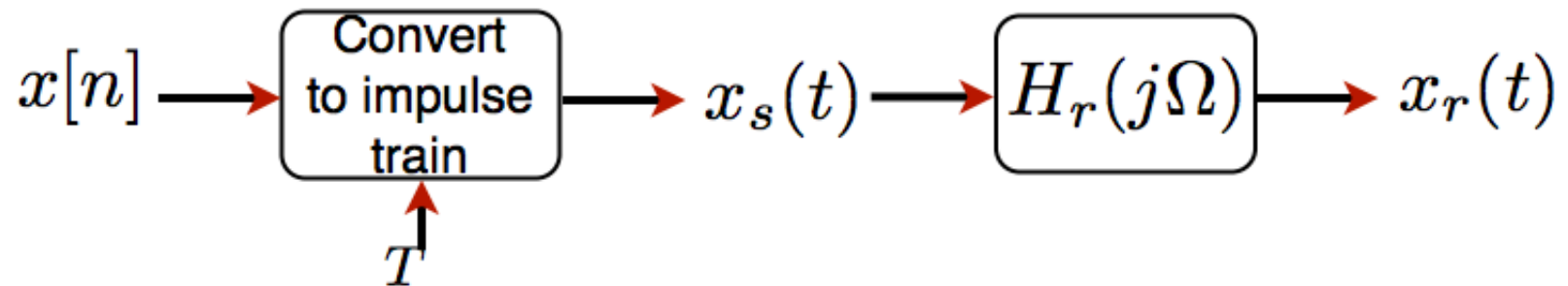
DSP System



Reconstruction in Frequency Domain



Reconstruction in Frequency Domain



Example: Cosine Input

- Sample the continuous-time signal $x_c(t) = \cos(4000 \pi t)$ with sampling period $T = 1/6000$ ($f_s = 6\text{kHz}$)

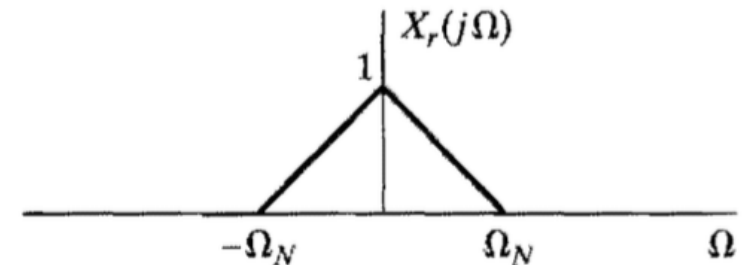
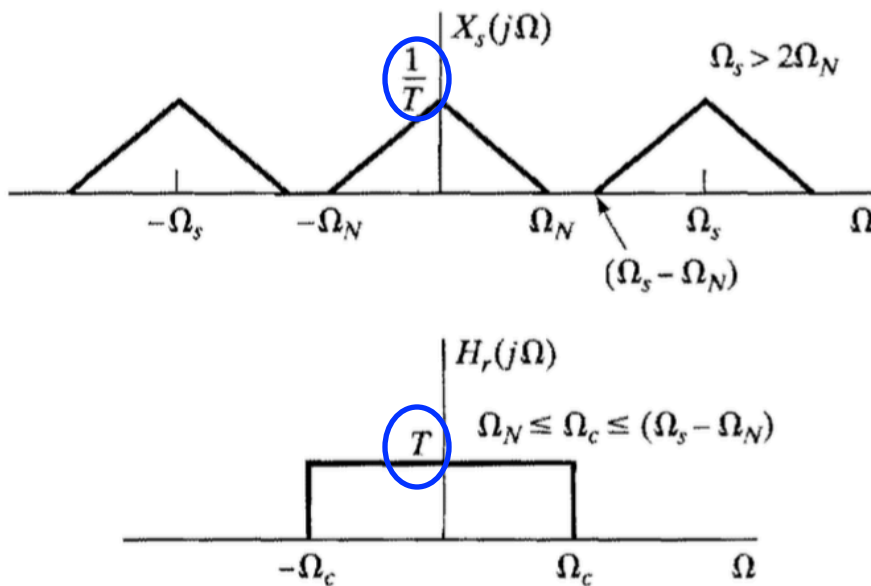
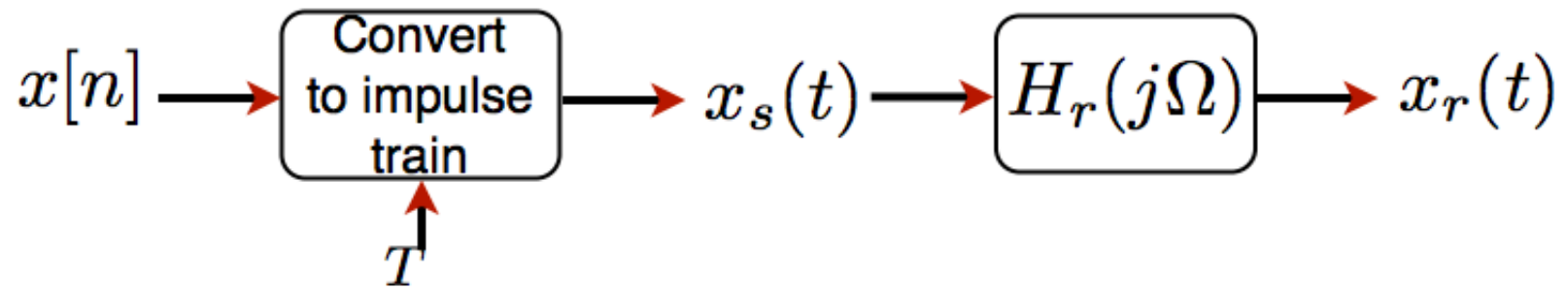
$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T} \end{aligned}$$



Example: Cosine Input

- ❑ Sample the continuous-time signal $x_c(t) = \cos(16000 \pi t)$ with sampling period $T = 1/6000$ ($f_s = 6\text{kHz}$)

Reconstruction in Frequency Domain



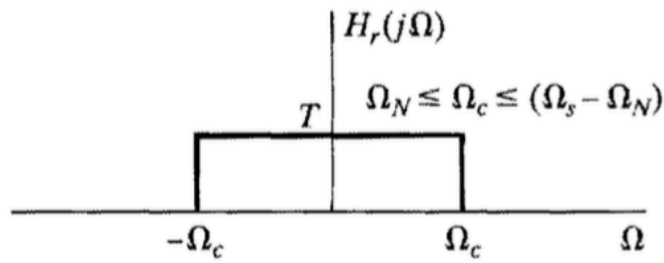
$$2\Omega_N = \Omega_s$$

$$\Omega_N = \Omega_c = \Omega_s/2$$

Sample at Nyquist
and filter at signal
bandwidth

Reconstruction in Time Domain

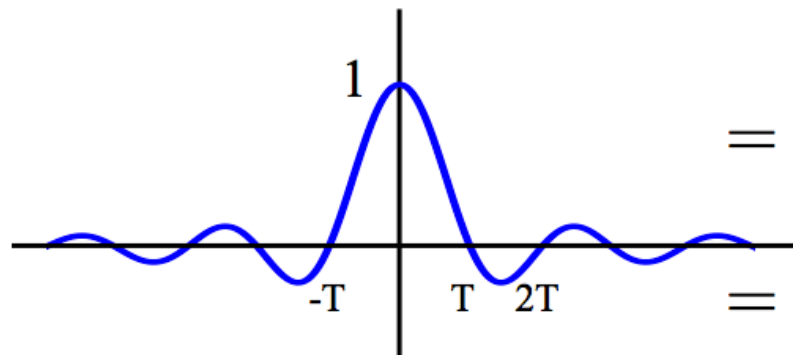
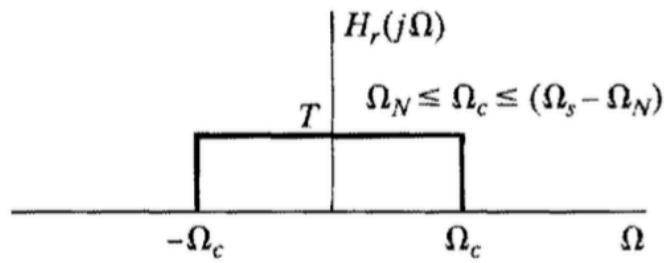
$$\Omega_N = \Omega_C = \Omega_s/2$$



$$h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Reconstruction in Time Domain

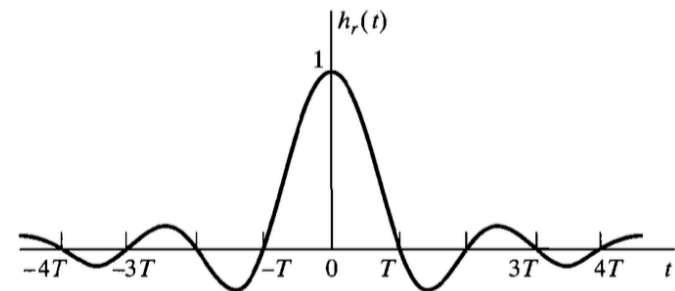
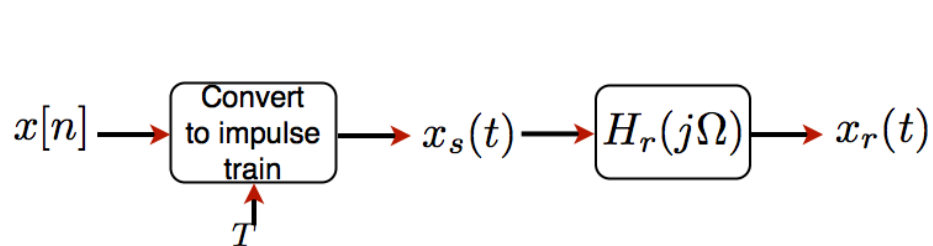
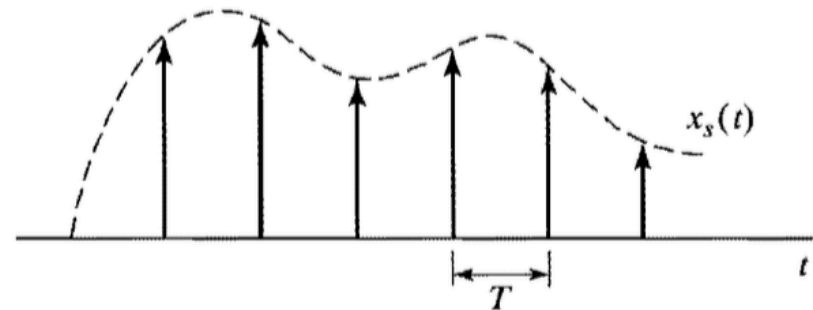
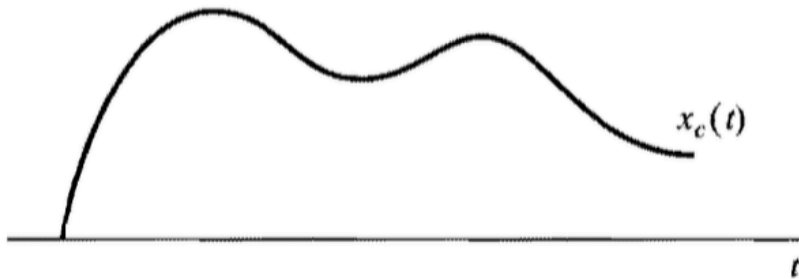
$$\Omega_N = \Omega_C = \Omega_s/2$$



$$\begin{aligned} h_r(t) &= \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \\ &= \frac{T}{2\pi} \frac{1}{jt} s^{j\Omega t} \Big|_{-\Omega_s/2}^{\Omega_s/2} \\ &= \frac{T}{\pi t} \frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\ &= \frac{T}{\pi t} \sin\left(\frac{\Omega_s}{2}t\right) = \frac{T}{\pi t} \sin\left(\frac{\pi}{T}t\right) \\ &= \text{sinc}\left(\frac{t}{T}\right) \end{aligned}$$

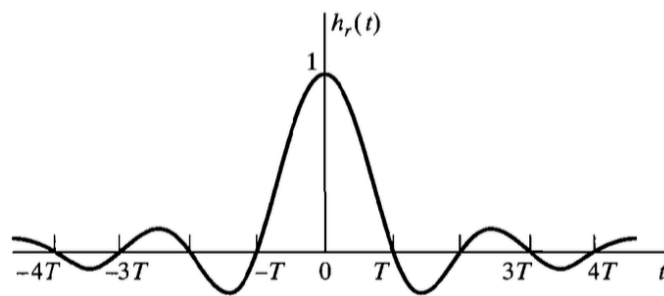
Reconstruction in Time Domain

$$\begin{aligned}
 x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n] \delta(t - nT) \right) * h_r(t) \\
 &= \sum_n x[n] h_r(t - nT)
 \end{aligned}$$

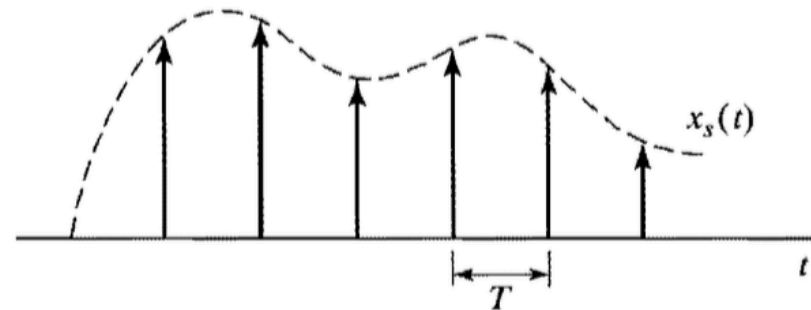


Reconstruction in Time Domain

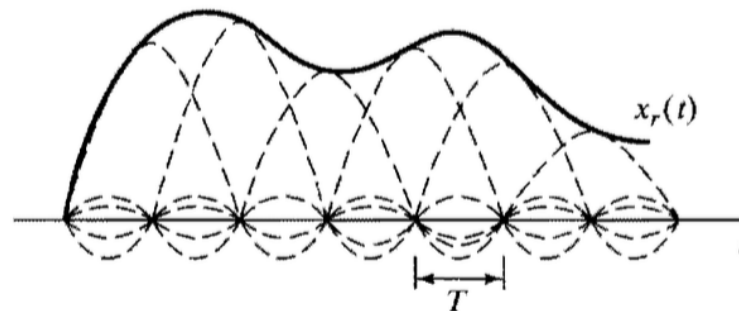
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 \end{aligned}$$



*



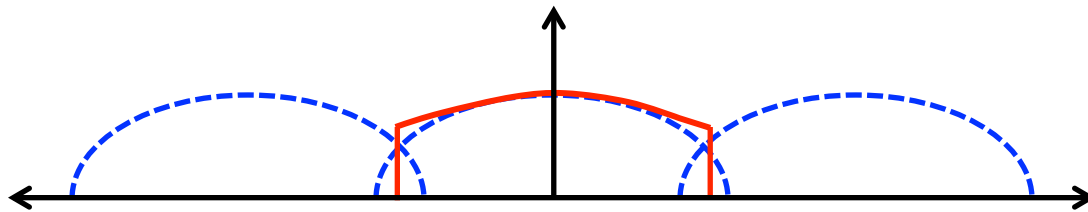
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The sum of “sincs”
gives $x_r(t) \rightarrow$ unique
signal that is
bandlimited by
sampling bandwidth

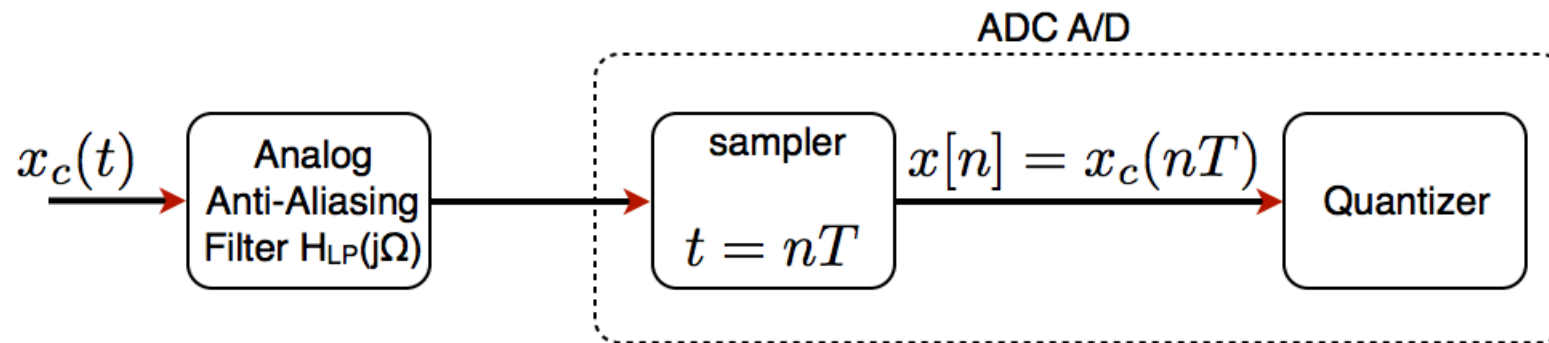
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

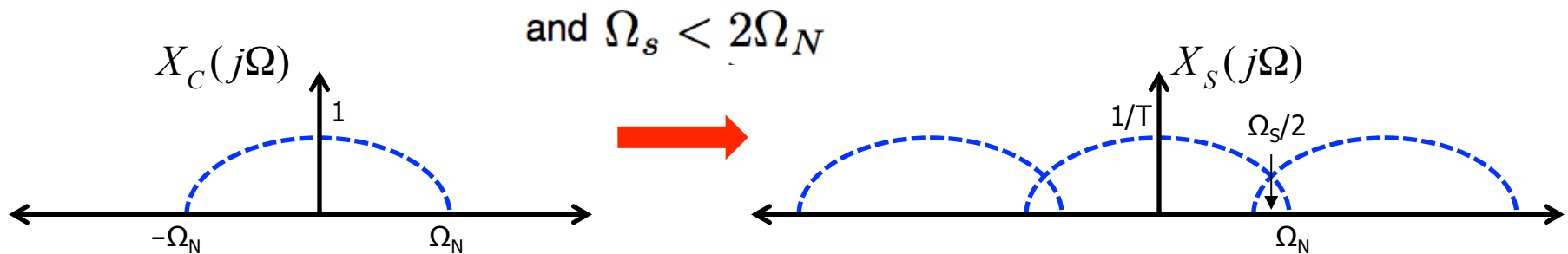
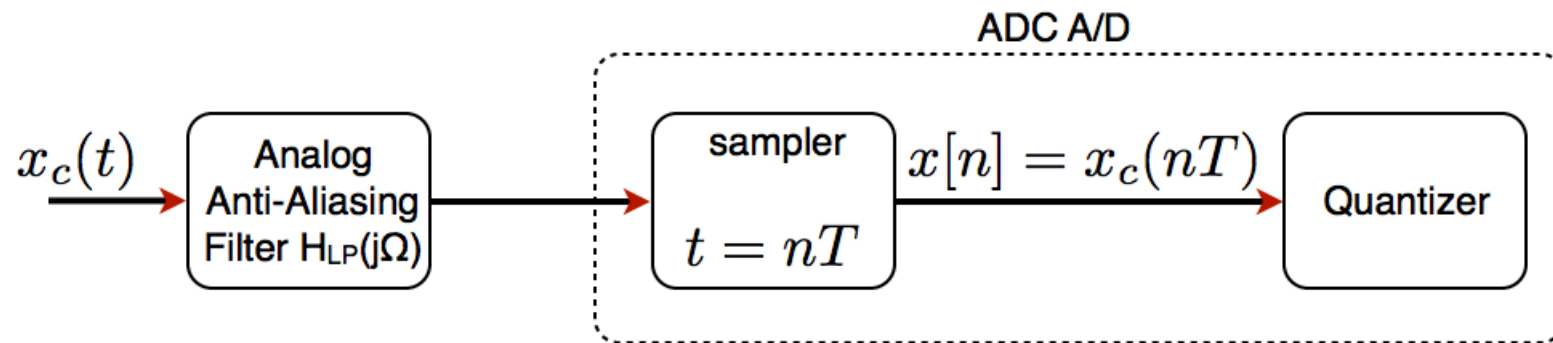


$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

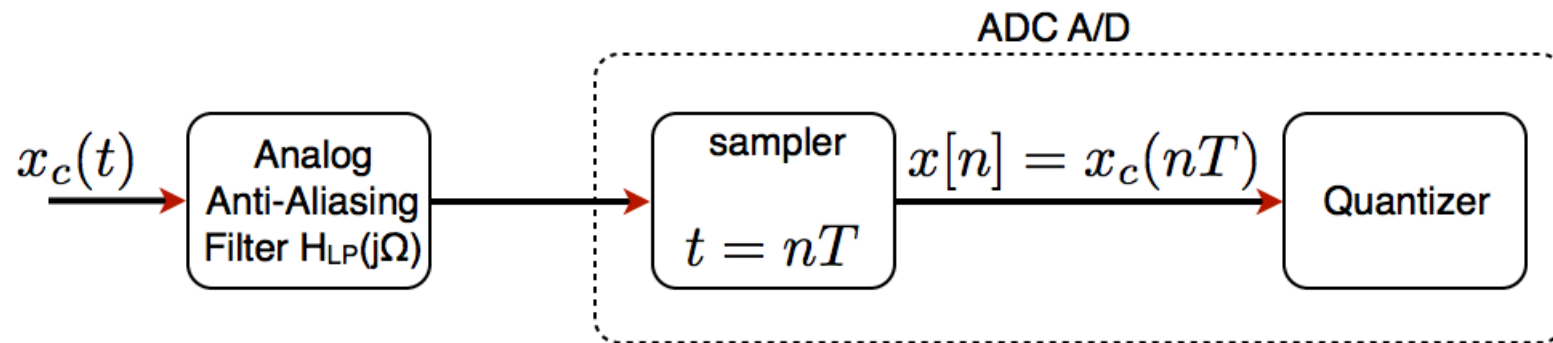
Anti-Aliasing Filter



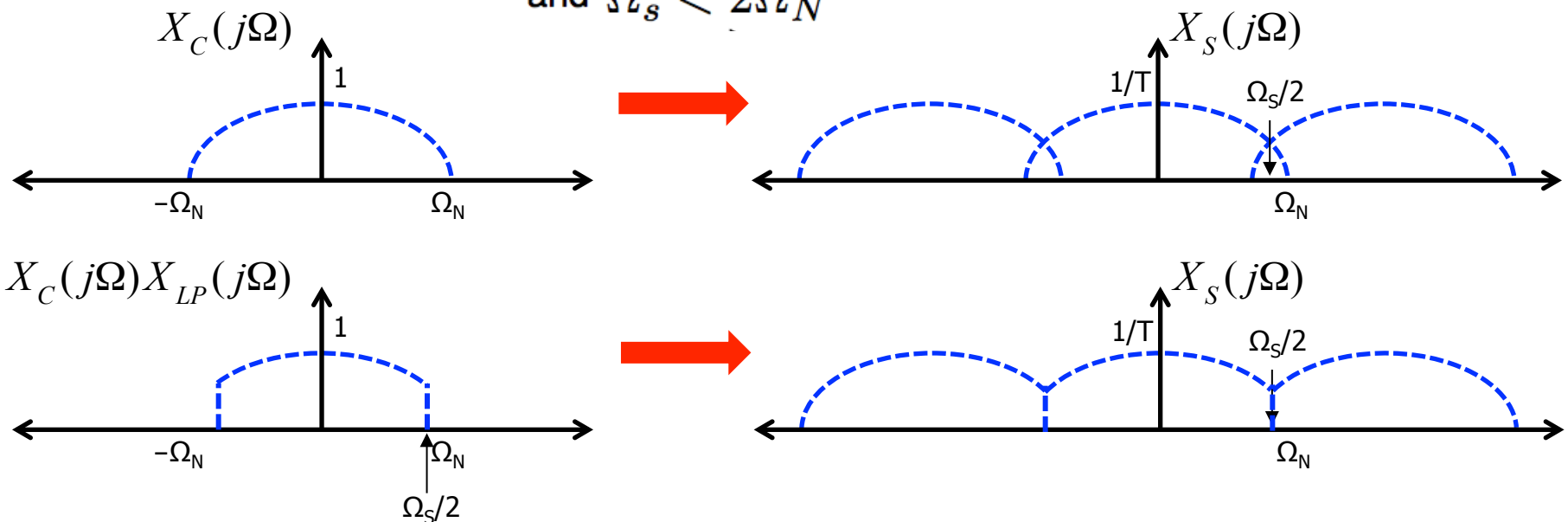
Anti-Aliasing Filter



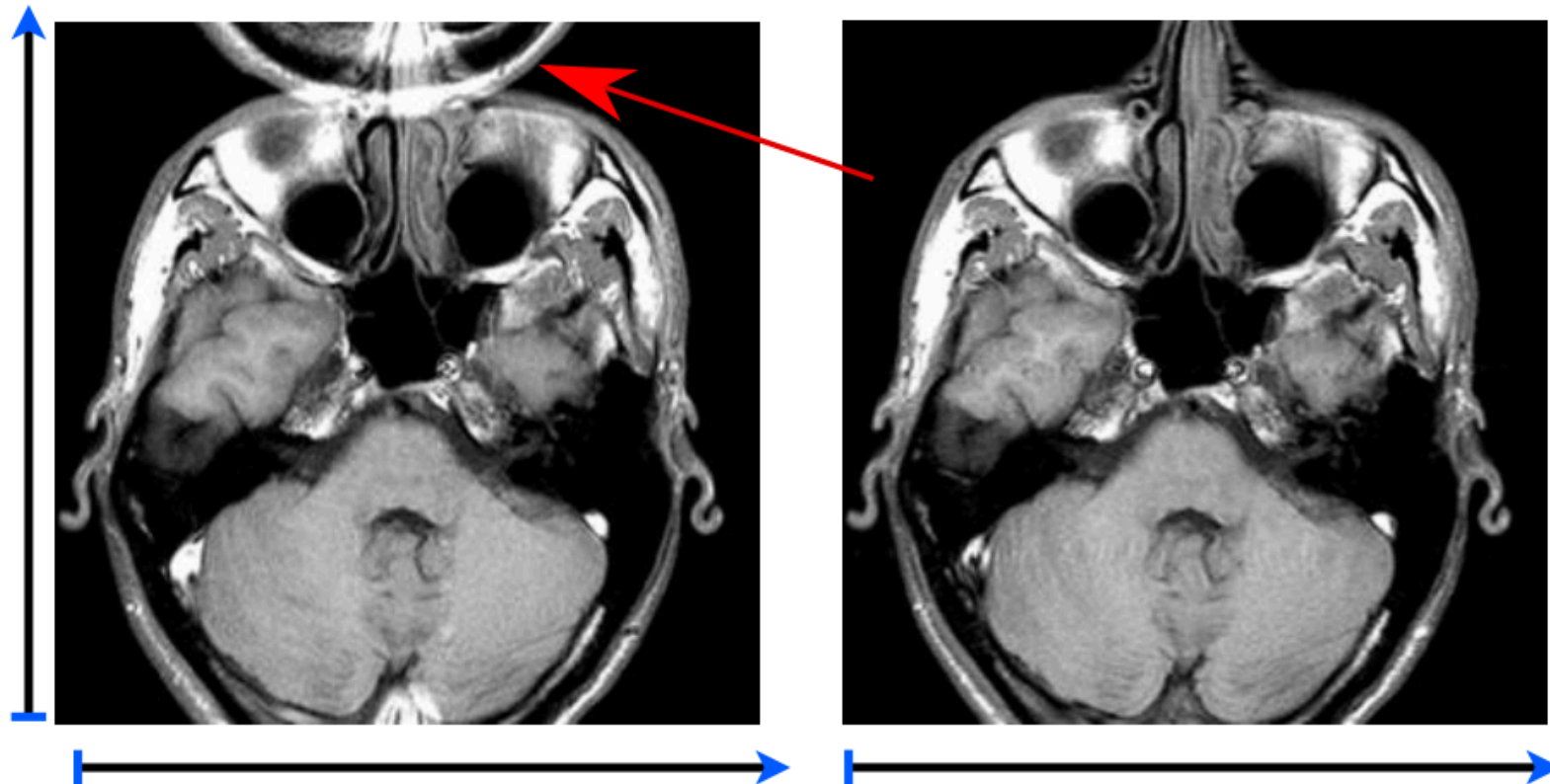
Anti-Aliasing Filter



and $\Omega_s < 2\Omega_N$



MRI aliasing example



MRI anti-aliasing example



MRI anti-aliasing example





Big Ideas

- ❑ Sampling
 - Ideal sampling modeled as impulsive sampling
 - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- ❑ Frequency Response of Sampled Signal
 - Sampled signal is period replicated input CT signal
- ❑ Reconstruction
 - Low pass/reconstruction filter results in sum of sincs
- ❑ Anti-aliasing filtering
 - Force input signal to be bandlimited



Admin

- ❑ HW 2 due tomorrow at midnight
- ❑ HW 3 posted after class
 - Due 2/9 at midnight