ESE 531: Digital Signal Processing

Lec 7: February 1st, 2018

Sampling and Reconstruction

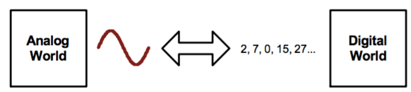
Lecture Outline

- Sampling
- Frequency Response of Sampled Signal
- Reconstruction

Video Example

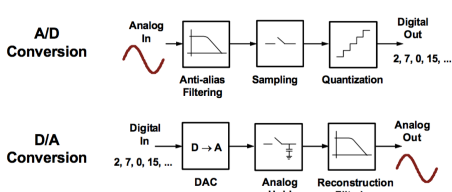
- https://www.youtube.com/watch?v=ByTsISFXUoY

The Data Conversion Problem



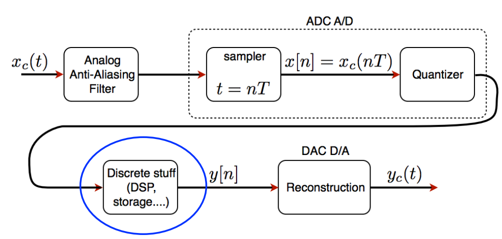
- Real world signals
  - Continuous time, continuous amplitude
- Digital abstraction
  - Discrete time, discrete amplitude
- Two problems
  - How to go discretize in time and amplitude
    - A/D conversion
  - How to "undescretize" in time and amplitude
    - D/A conversion

Overview

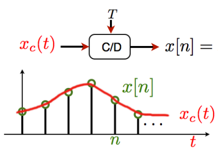


- We'll first look at these building blocks from a functional, "black box" perspective
  - Later refine and look at implementation

DSP System



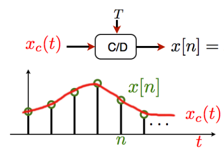
Ideal Sampling Model



Discrete and Continuous

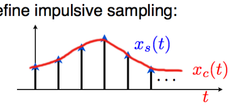
- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - f_s=1/T is the sampling frequency
  - Ω_s=2 π /T

Ideal Sampling Model



Discrete and Continuous

define impulsive sampling:



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Dirac delta function, δ (t)
  - Infinitely high and thin, area of 1
  - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- Three signals. How are they related? In time? In frequency?

Frequency Domain Analysis

- How is x[n] related to x_s(t) in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis

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$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$x[n]$:D.T

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}$$

Frequency Domain Analysis

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$x_s(t)$:C.T

$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

$x[n]$:D.T

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}$$

$$\omega = \Omega T$$

Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

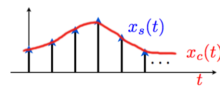
$x[n] = x_c(nT)$ $\qquad$ $x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$x_s(t)$:C.T $\qquad$ $X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$

$x[n]$:D.T $\qquad$ $X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}$ $\qquad$ $\omega = \Omega T$

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T} \qquad X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T}$$

Frequency Domain Analysis



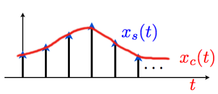
Continuous

$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$

$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{\triangleq s(t)}$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis



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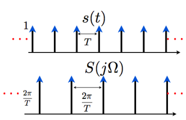
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$s(t) \leftrightarrow S(j\Omega)$

$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$ $\quad$ $\frac{2\pi}{T} = \Omega_s$



Frequency Domain Analysis

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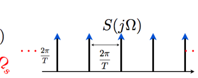

Frequency Domain Analysis

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$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T}$$

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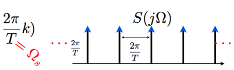
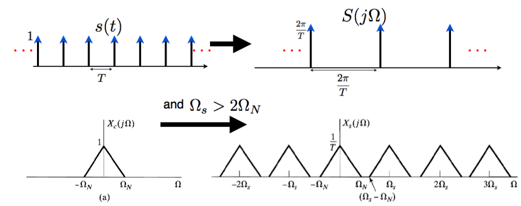
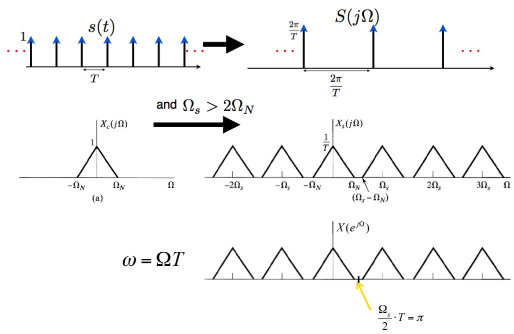


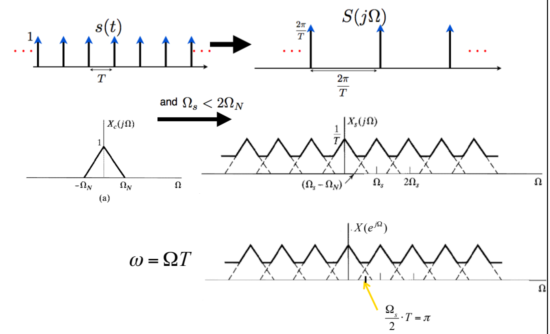
Frequency Domain Analysis



Frequency Domain Analysis



Frequency Domain Analysis

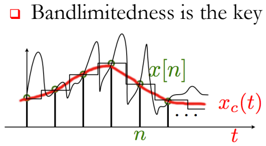


Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

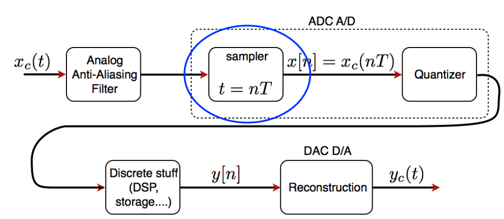
$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness

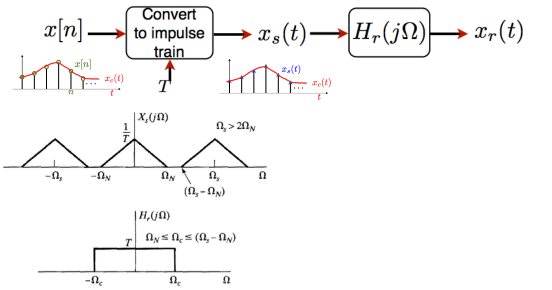


Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

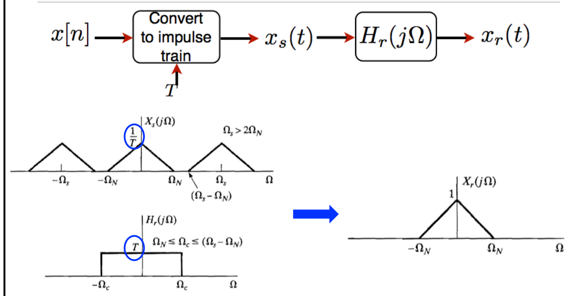
DSP System



Reconstruction in Frequency Domain



Reconstruction in Frequency Domain



Example: Cosine Input

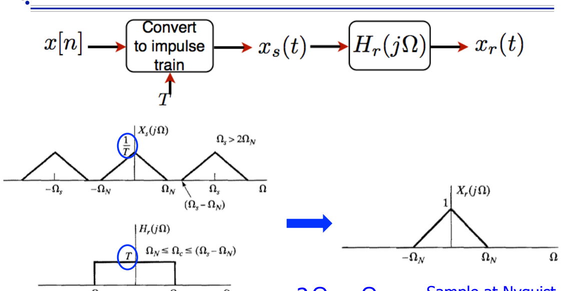
- Sample the continuous-time signal $x_c(t)=\cos(4000\pi t)$ with sampling period $T=1/6000$ (f_s=6kHz)

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T}$$

Example: Cosine Input

- Sample the continuous-time signal $x_c(t)=\cos(16000\pi t)$ with sampling period $T=1/6000$ (f_s=6kHz)

Reconstruction in Frequency Domain



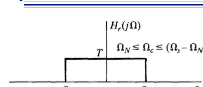
$2\Omega_N = \Omega_s$

$\Omega_N = \Omega_C = \Omega_s/2$

Sample at Nyquist and filter at signal badnwidth

Reconstruction in Time Domain

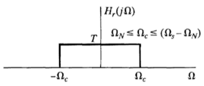
$\Omega_N = \Omega_C = \Omega_s/2$



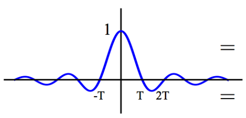
$$h_r(t) = \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Reconstruction in Time Domain

$\Omega_N = \Omega_C = \Omega_s/2$

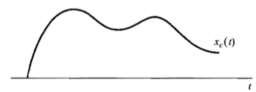


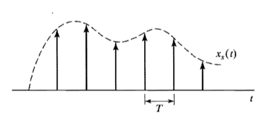
$$\begin{aligned} h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \\ &= \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\ &= \frac{T}{\pi t}\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\ &= \frac{T}{\pi t}\sin(\frac{\Omega_s}{2}t) = \frac{T}{\pi t}\sin(\frac{\pi}{T}t) \\ &= \mathrm{sinc}(\frac{t}{T}) \end{aligned}$$

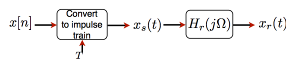


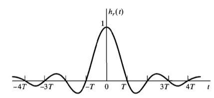
Reconstruction in Time Domain

$$\begin{aligned} x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t) \\ &= \sum_n x[n] h_r(t-nT) \end{aligned}$$









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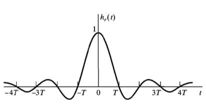

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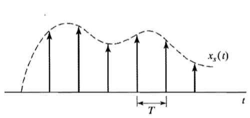

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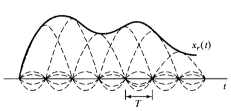
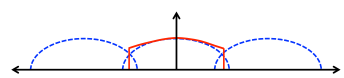


The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

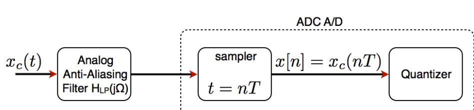
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

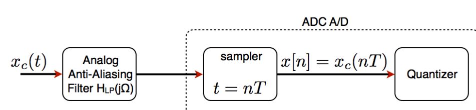


$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } |\Omega| \le \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

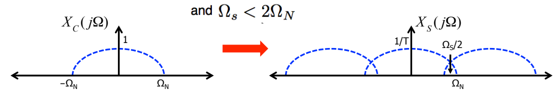
Anti-Aliasing Filter



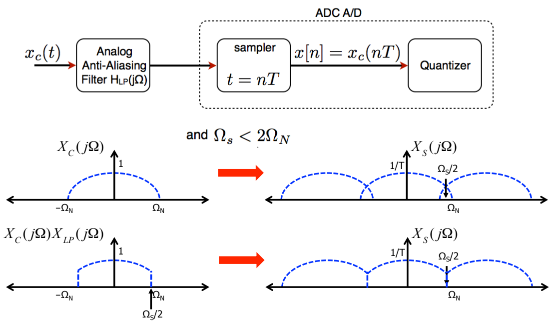
Anti-Aliasing Filter



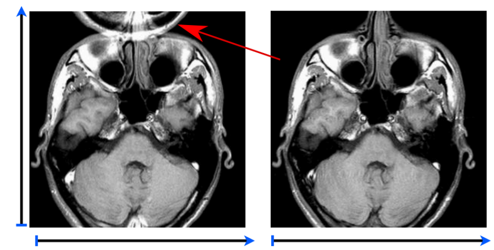
and $\Omega_s < 2\Omega_N$



Anti-Aliasing Filter



MRI aliasing example



MRI anti-aliasing example

MRI anti-aliasing example

Big Ideas

- Sampling
  - Ideal sampling modeled as impulsive sampling
  - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- Frequency Response of Sampled Signal
  - Sampled signal is period replicated input CT signal
- Reconstruction
  - Low pass/reconstruction filter results in sum of sincs
- Anti-aliasing filtering
  - Force input signal to be bandlimited

Admin

- HW 2 due tomorrow at midnight
- HW 3 posted after class
  - Due 2/9 at midnight