

ESE 531: Digital Signal Processing

Lec 8: February 6th, 2018
Sampling and Reconstruction



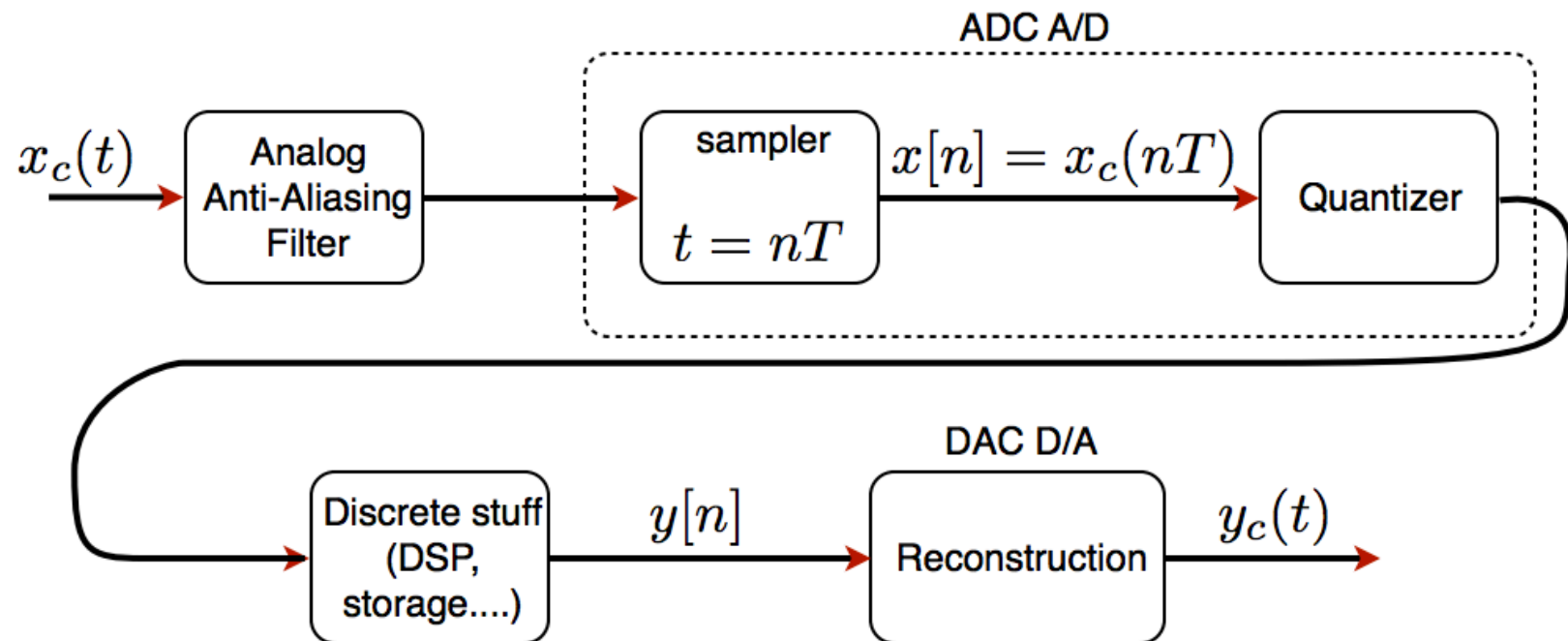
Lecture Outline

- ❑ Review
 - Ideal sampling
 - Frequency response of sampled signal
 - Reconstruction
 - Anti-aliasing filtering
- ❑ DT processing of CT signals
 - Impulse Invariance
- ❑ CT processing of DT signals (why??)

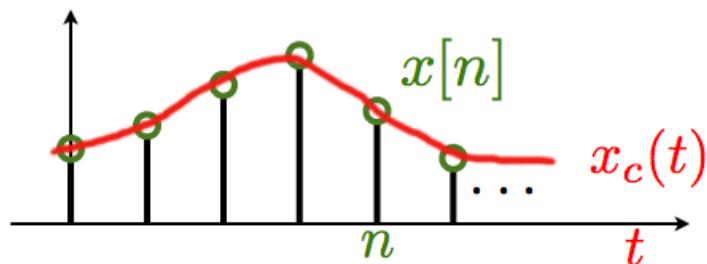
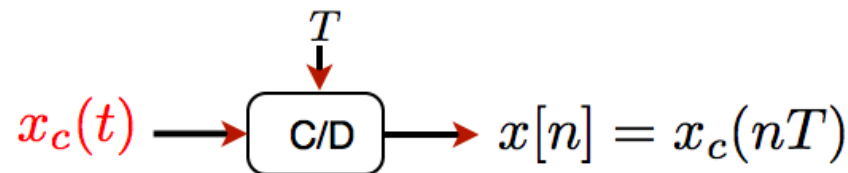
Last Time...

Sampling, Frequency Response of Sampled
Signal, Reconstruction, Anti-aliasing filtering

DSP System

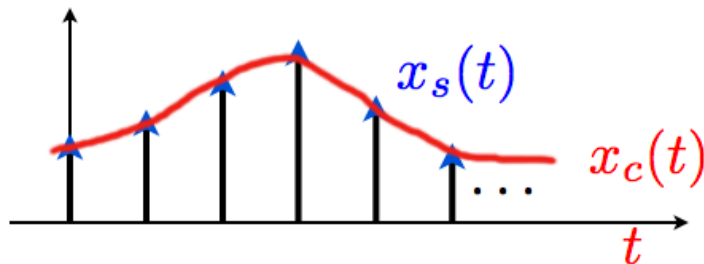


Ideal Sampling Model



Discrete and Continuous

define impulsive sampling:



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t - T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis

- How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) \quad : \text{C.T.}$$

$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

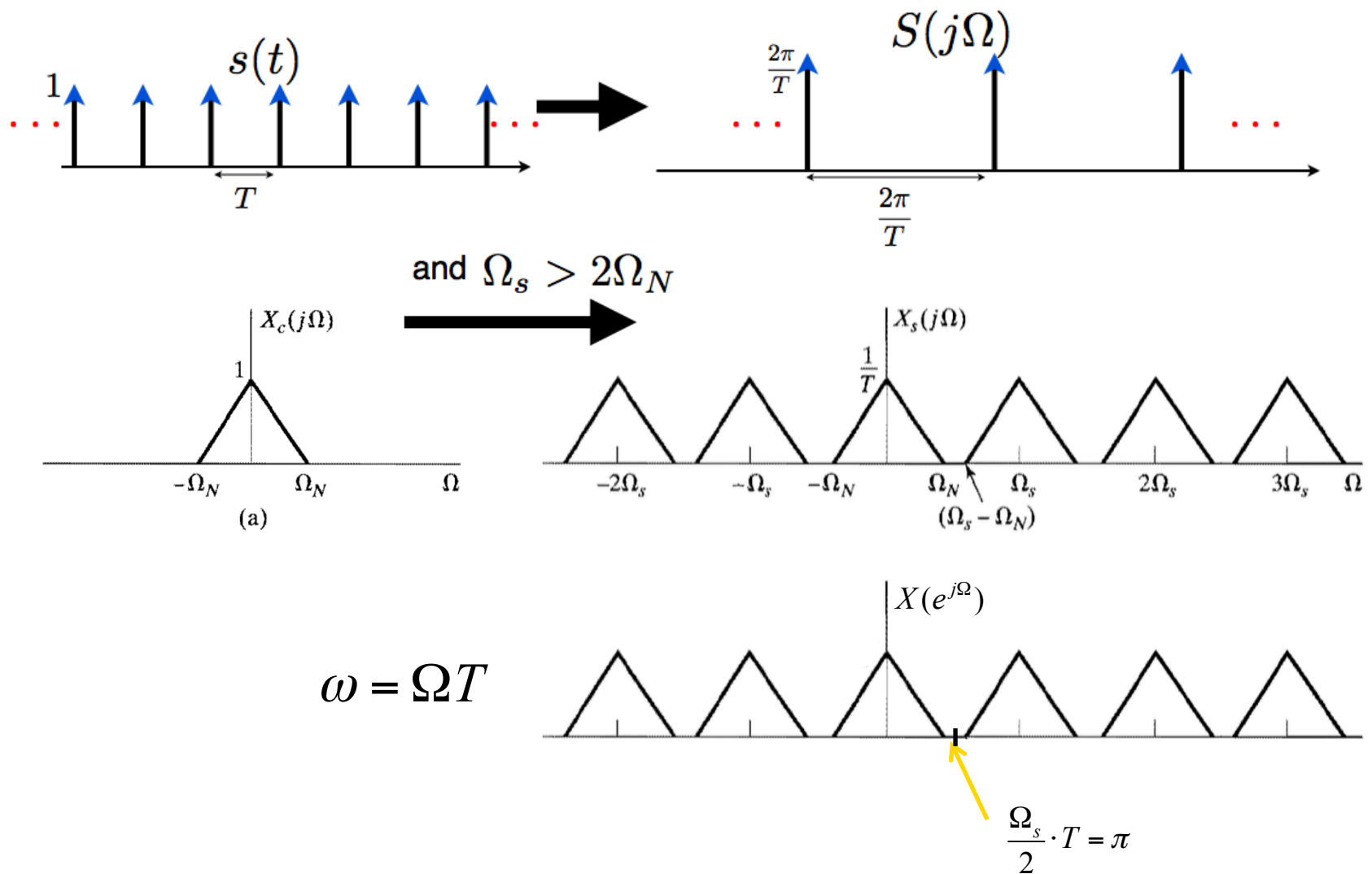
$$x[n] \quad : \text{D.T.}$$

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \quad \omega = \Omega T$$

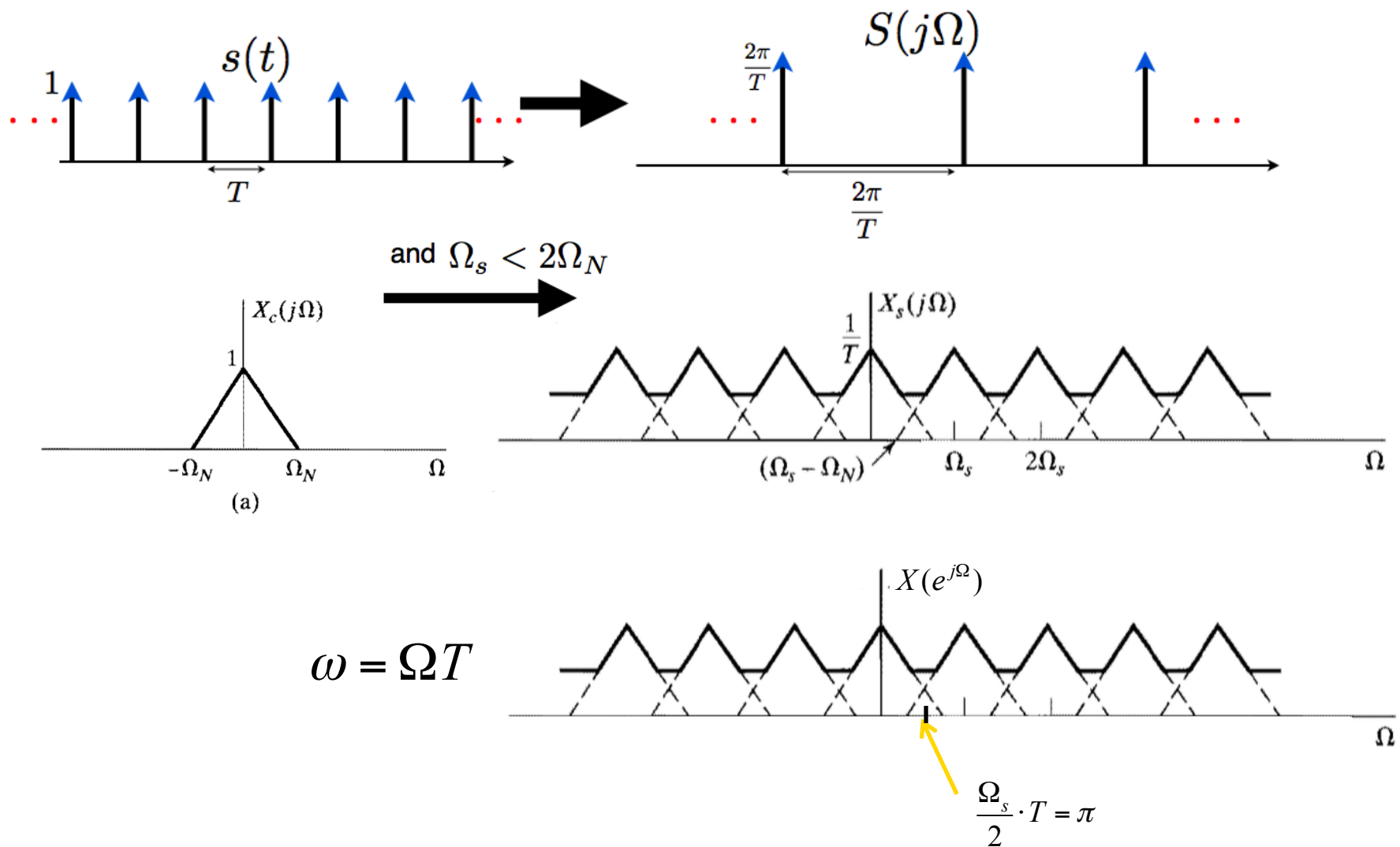
$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$$

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T}$$

Frequency Domain Analysis



Frequency Domain Analysis

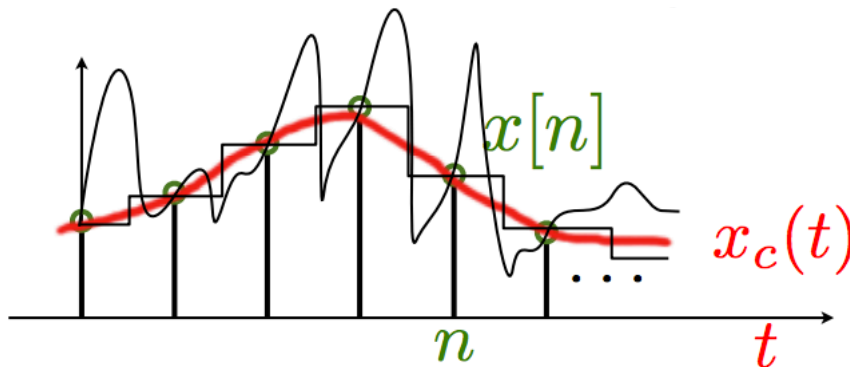


Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

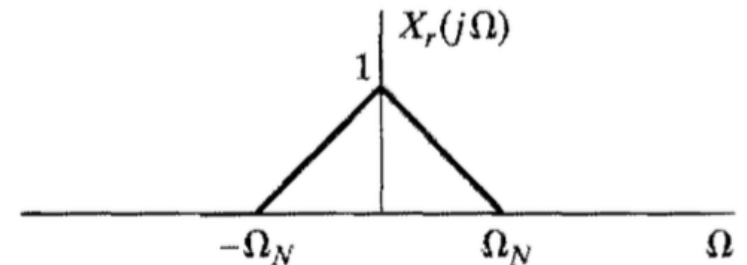
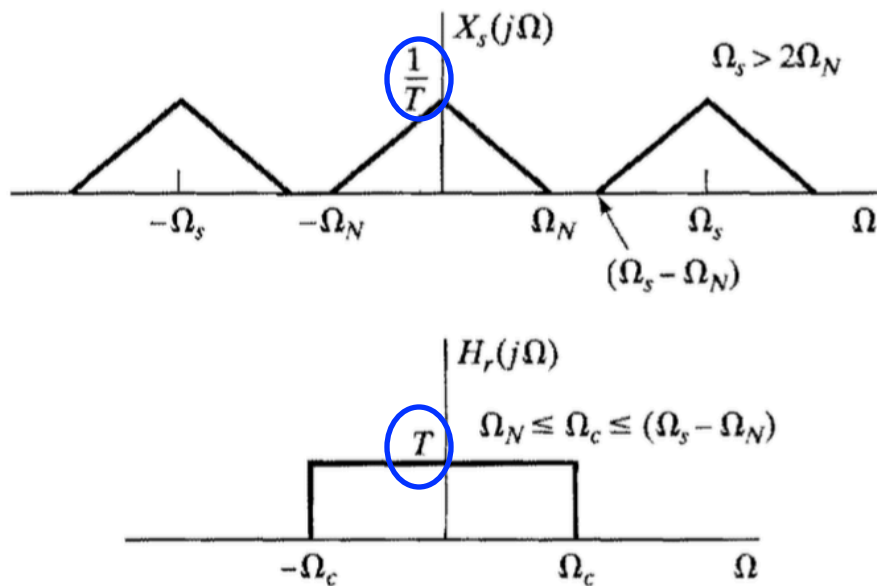
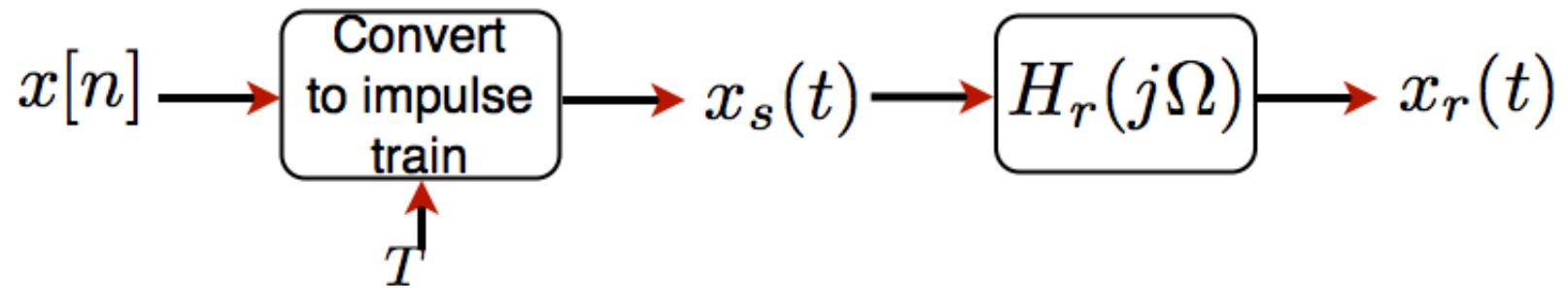
$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$
- Bandlimitedness is the key to uniqueness



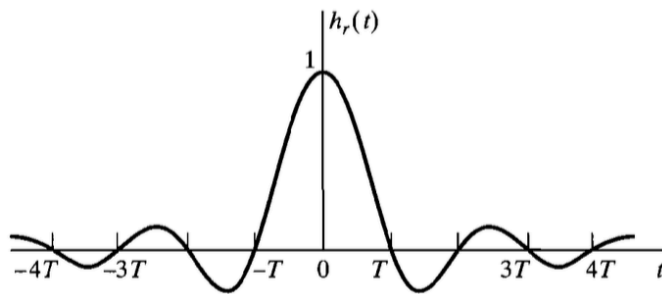
Multiple signals go through the samples, but only one is bandlimited within our sampling band

Reconstruction in Frequency Domain

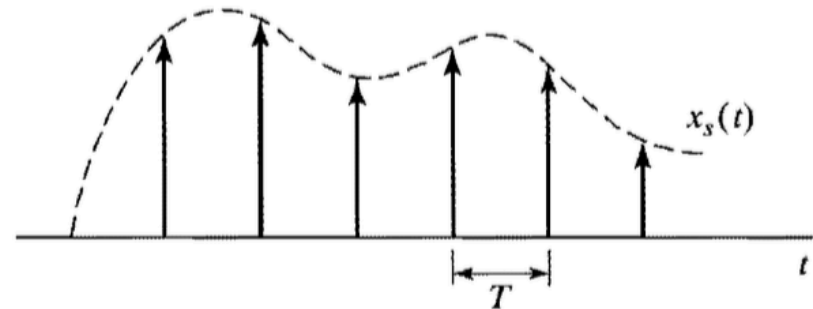


Reconstruction in Time Domain

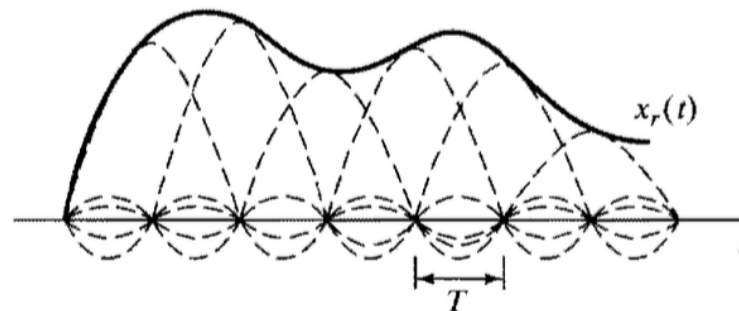
$$\begin{aligned}
 x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n] \delta(t - nT) \right) * h_r(t) \\
 &= \sum_n x[n] h_r(t - nT)
 \end{aligned}$$



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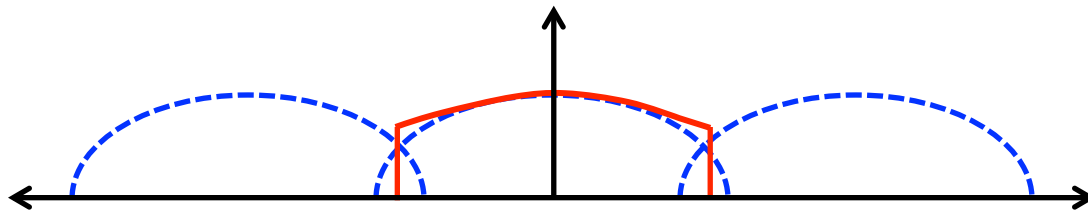
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The sum of "sincs" gives $x_r(t) \rightarrow$ unique signal that is bandlimited by sampling bandwidth

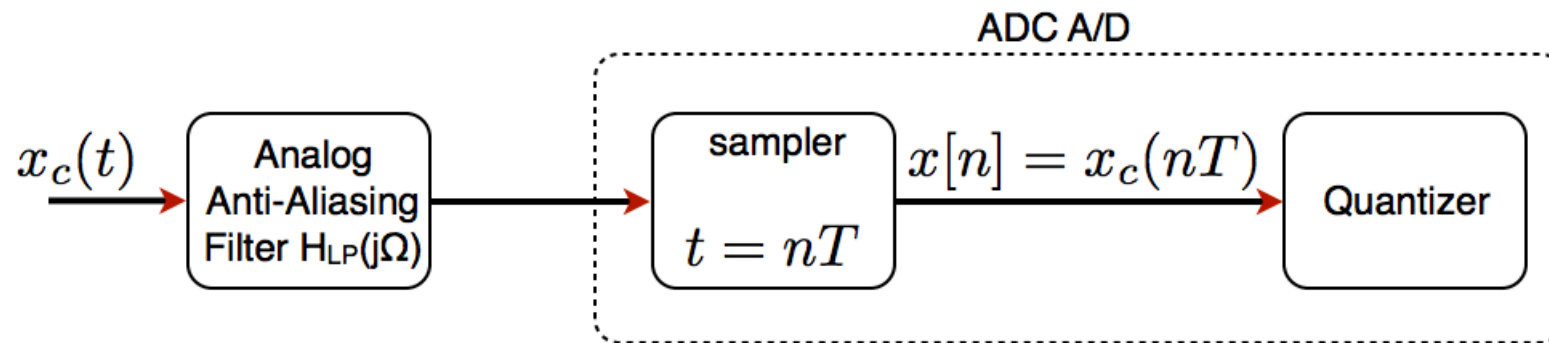
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

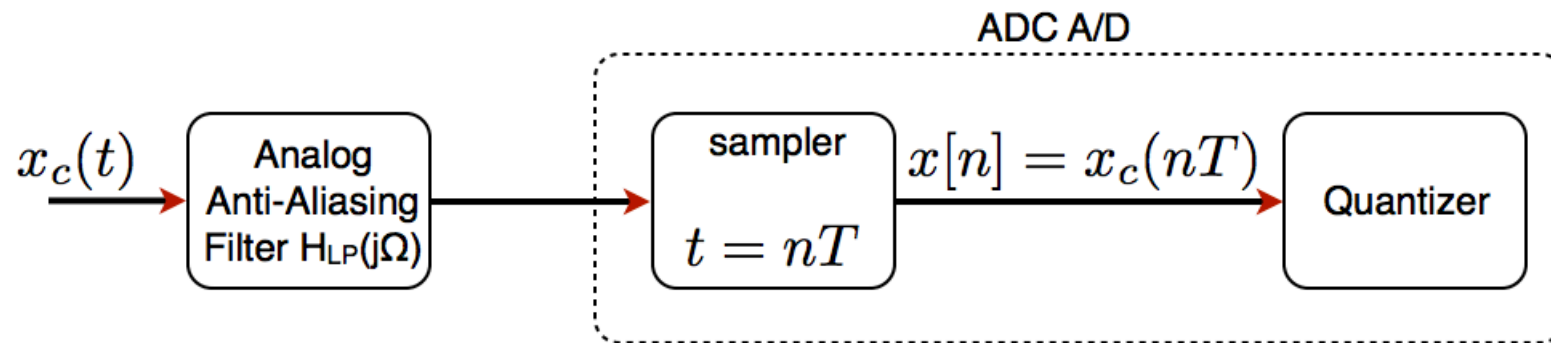


$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

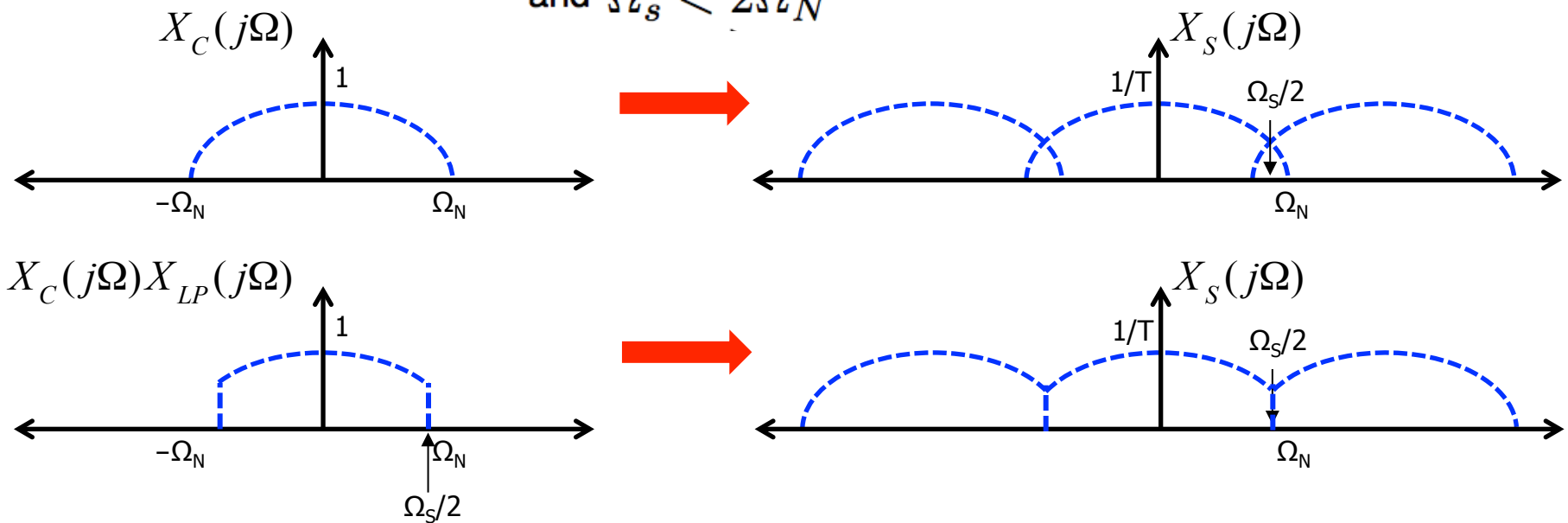
Anti-Aliasing Filter



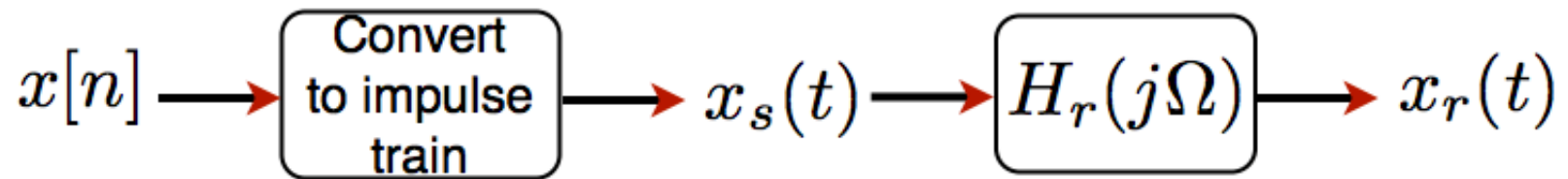
Anti-Aliasing Filter



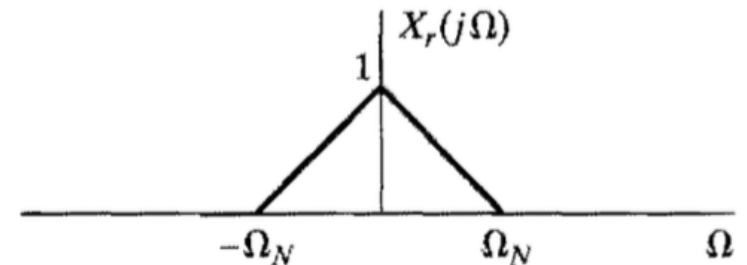
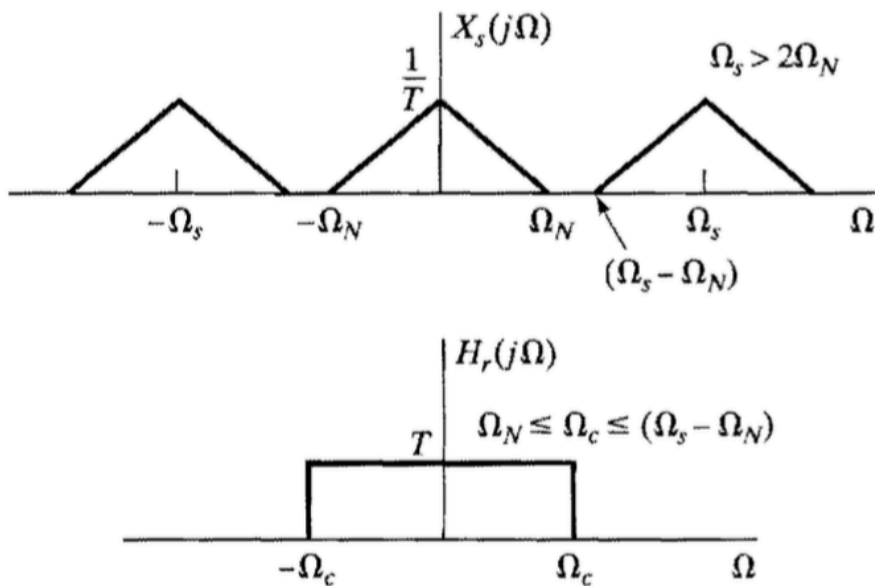
and $\Omega_s < 2\Omega_N$



Reconstruction in Frequency Domain

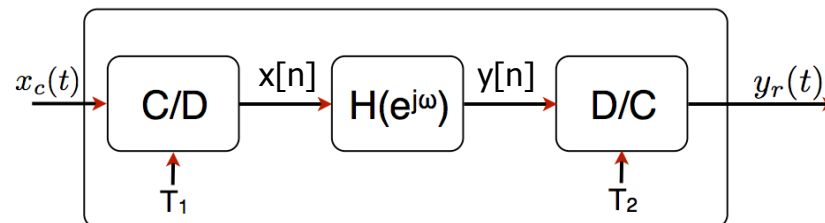


T Different T ?

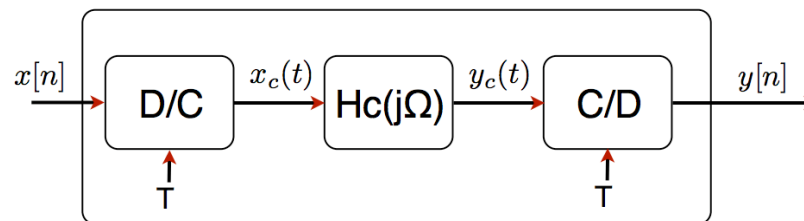


Signal Processing

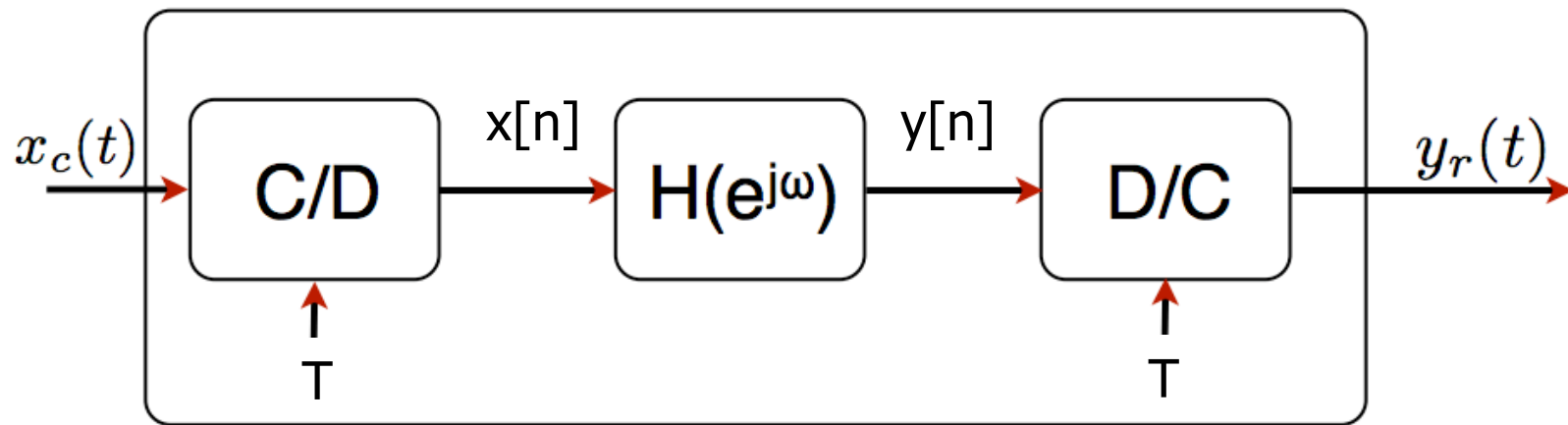
- ❑ Use theory of sampling (C/D) and reconstruction (D/C) to implement signal processing
- ❑ Two cases:
 - Discrete-time processing of continuous-time signals



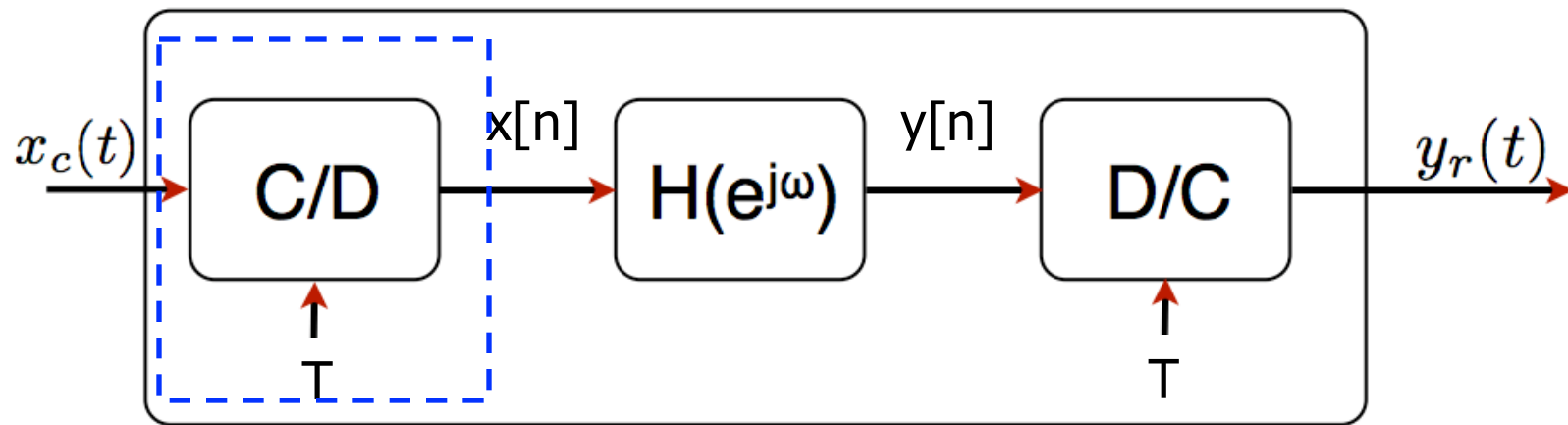
- Continuous-time processing of discrete-time signals



Discrete-Time Processing of Continuous Time



Discrete-Time Processing of Continuous Time

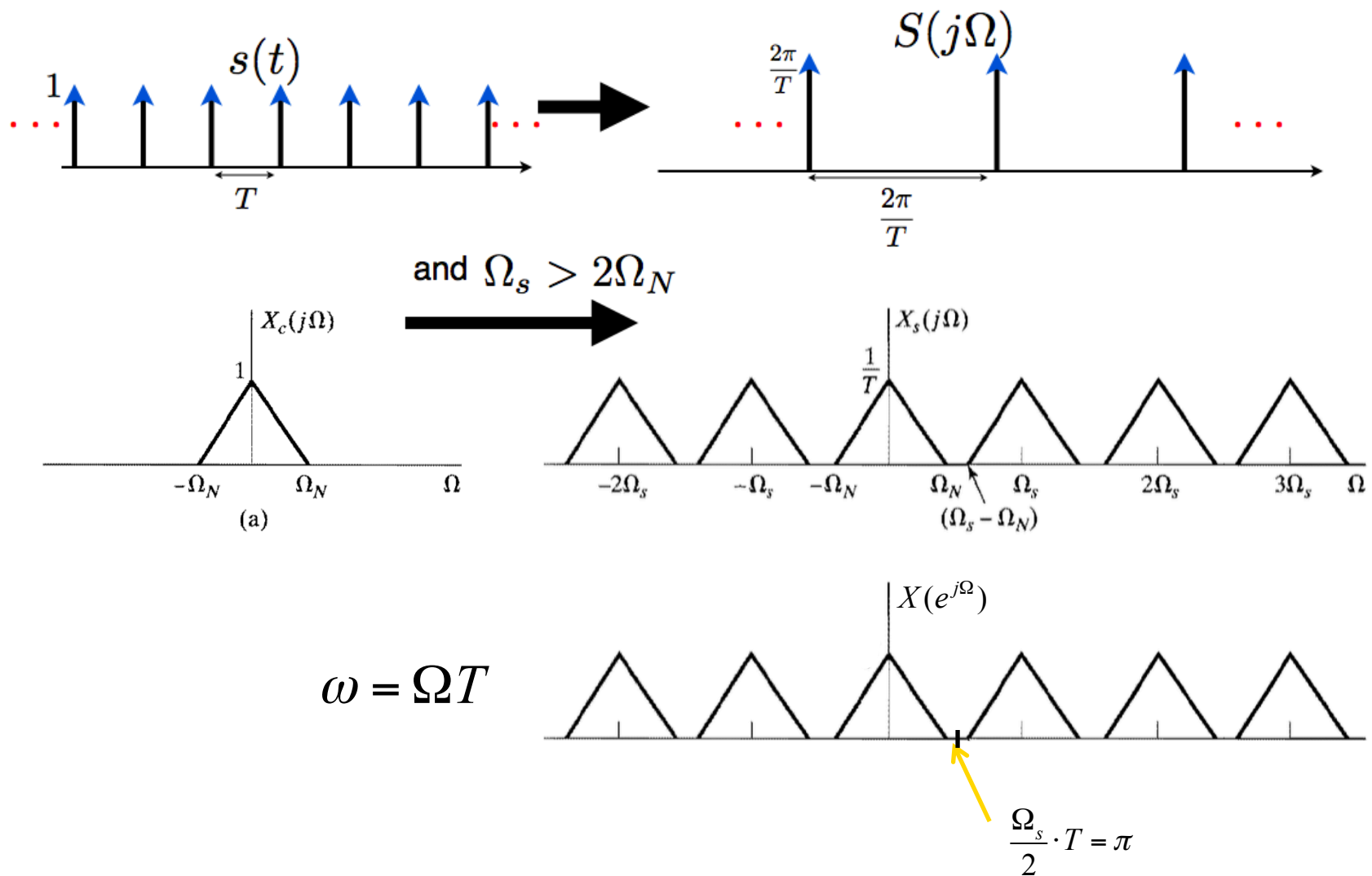


$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Omega_s = \frac{2\pi}{T} \end{aligned}$$

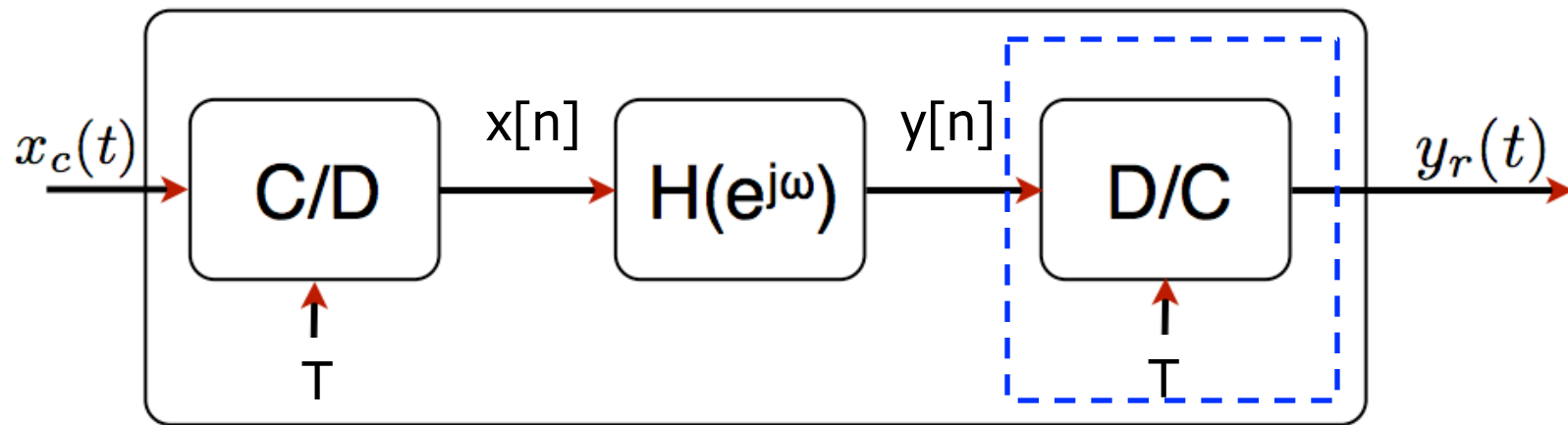
$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$$

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

Frequency Domain Analysis



Discrete-Time Processing of Continuous Time

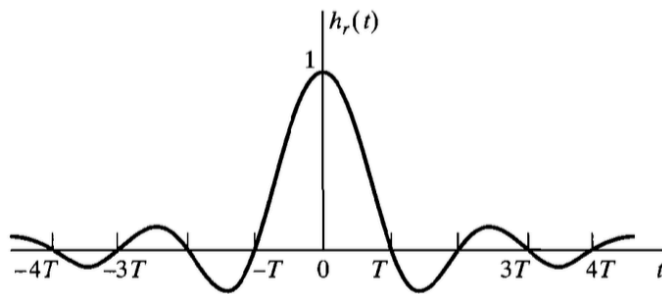


$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

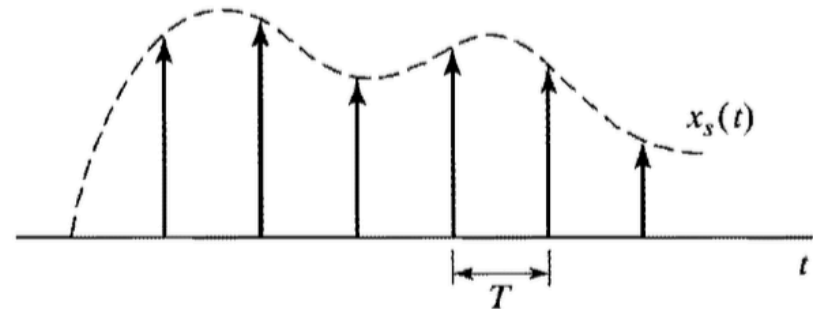
$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT) / T]}{\pi(t - nT) / T}$$

Reconstruction in Time Domain

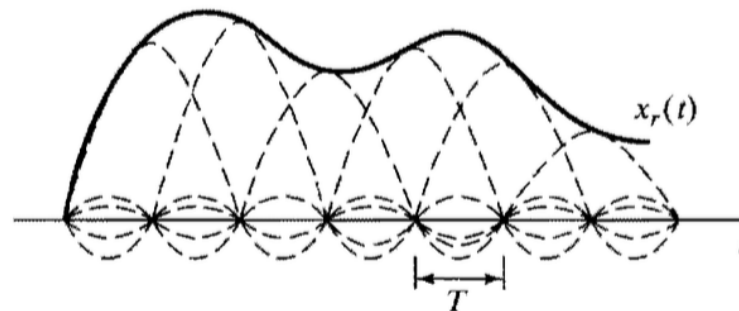
$$\begin{aligned}x_r(t) &= x_s(t) * h_r(t) = \left(\sum_n x[n] \delta(t - nT) \right) * h_r(t) \\&= \sum_n x[n] h_r(t - nT)\end{aligned}$$



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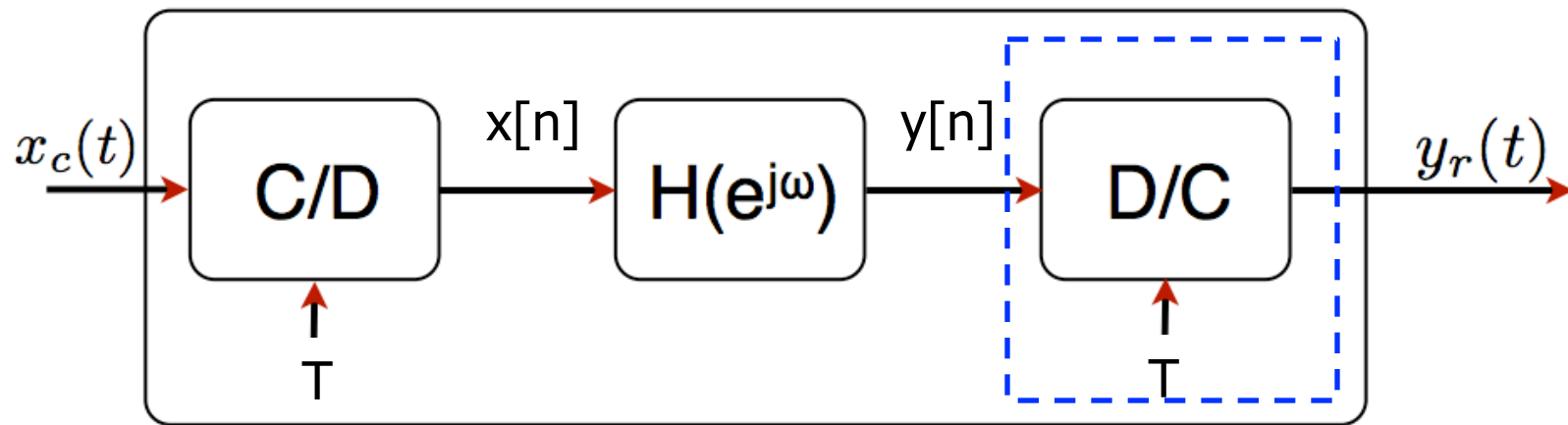


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Discrete-Time Processing of Continuous Time

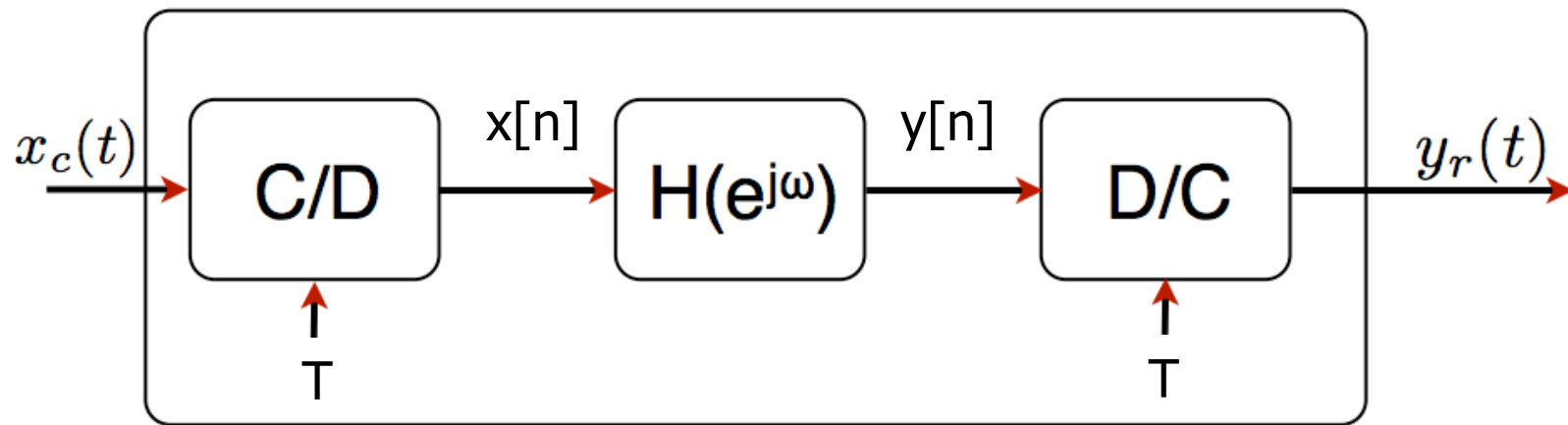


$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

Sum of scaled
shifted sincs

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT) / T]}{\pi(t - nT) / T}$$

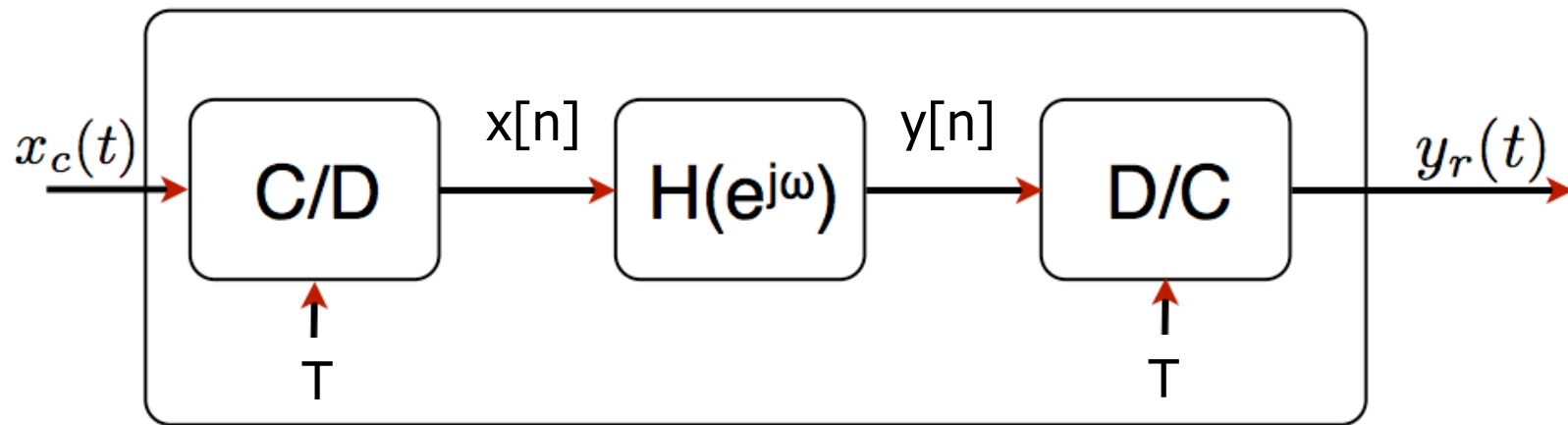
Discrete-Time Processing of Continuous Time



$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right] \quad y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

- If $h[n]$ is LTI, $H(e^{j\omega})$ exists
 - Is the whole system from $x_c(t) \rightarrow y_r(t)$ LTI?

Discrete-Time Processing of Continuous Time



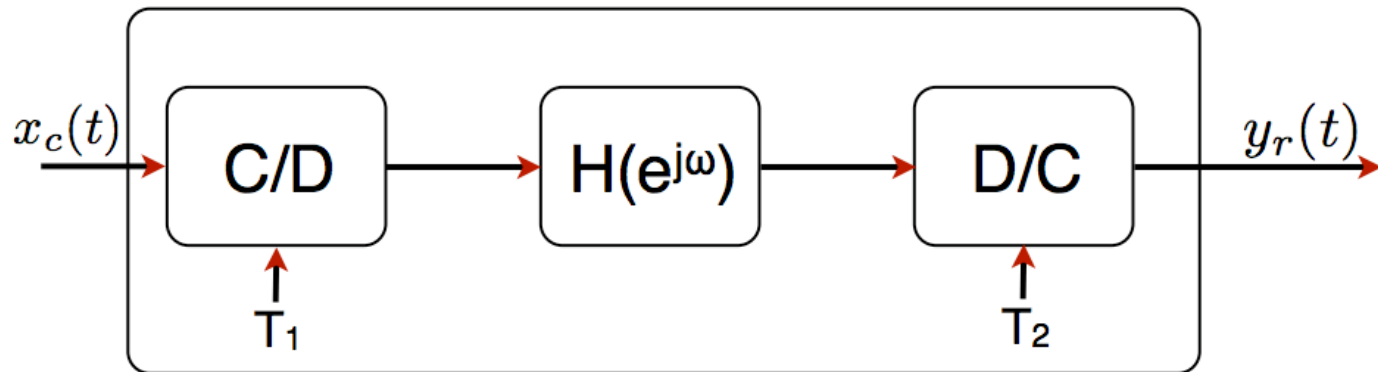
$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right] \quad y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

□ If $x_c(t)$ is bandlimited by $\Omega_s/T = \pi/T$, then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{eff}(j\Omega) = \begin{cases} H(e^{j\omega}) \Big|_{\omega=\Omega T} & |\Omega| < \Omega_s / T \\ 0 & else \end{cases}$$

Example 1

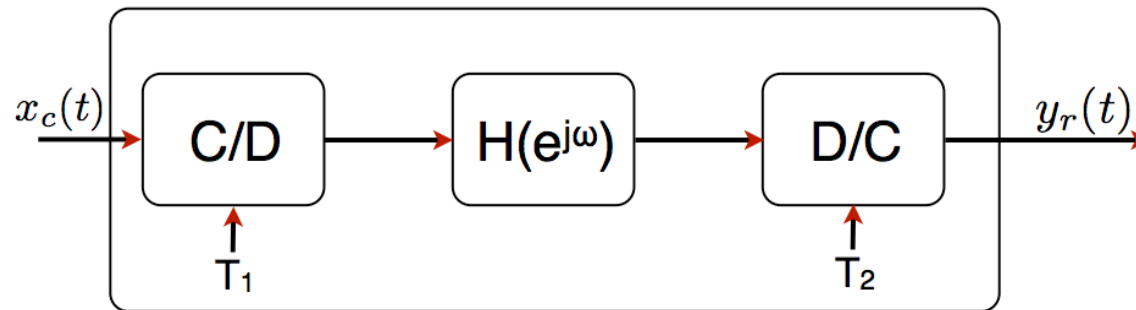
- Consider the following system



- Where
$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$
- What is the effective frequency response of the system? What happens to a signal bandlimited by Ω_N ?

Example 2

- DT implementation of an ideal CT bandlimited differentiator



- The ideal CT differentiator is defined by

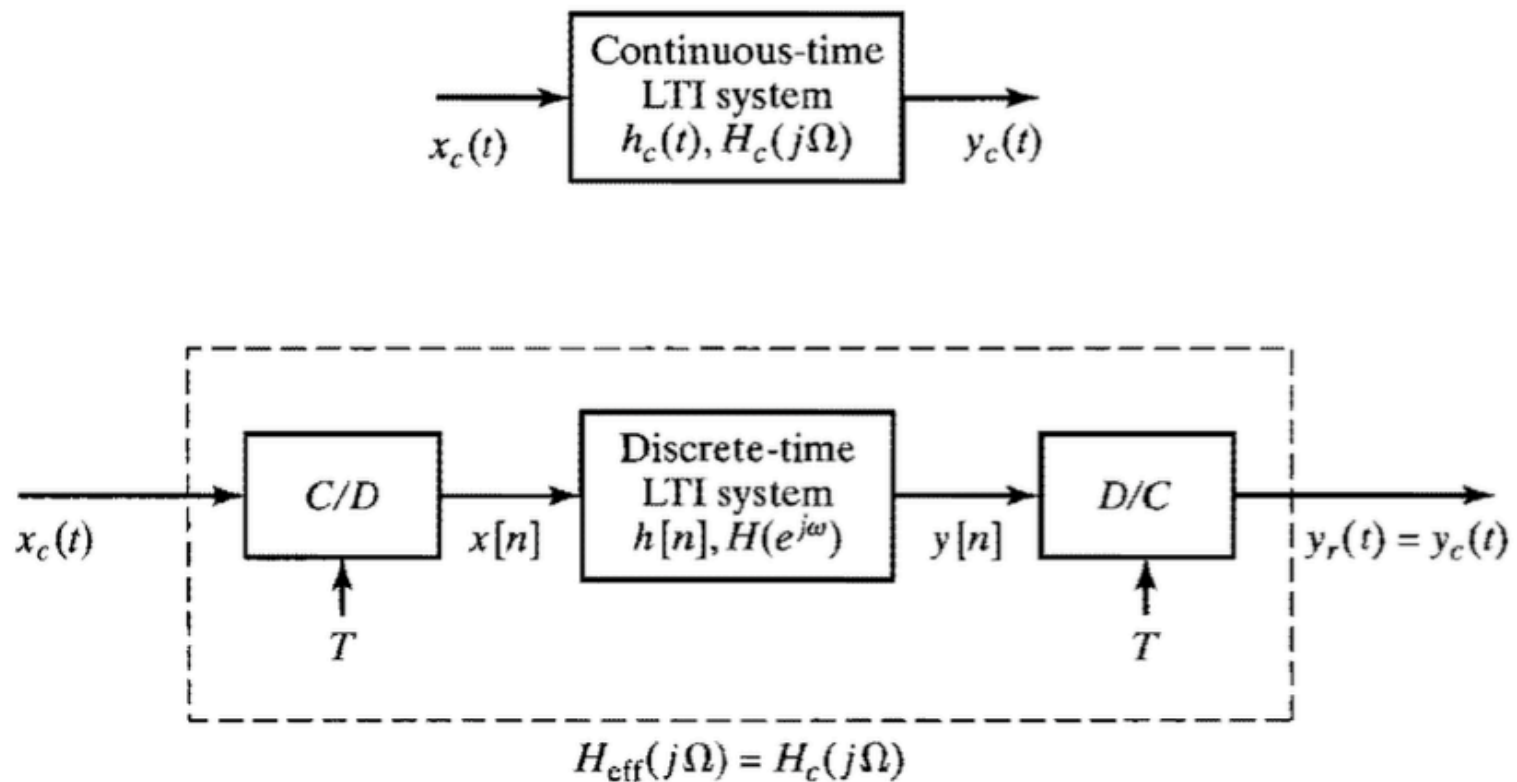
$$y_C(t) = \frac{d}{dt}[x_C(t)]$$

- With corresponding

$$H_C(j\Omega) = j\Omega$$

Impulse Invariance

- Want to implement continuous-time system in discrete-time





Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

Impulse Invariance

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$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad |\omega| < \pi$$

- With the further requirement that T be chosen such that

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$$h[n] = Th_c(nT)$$



Impulse Invariance

□ Let,

$$h[n] = h_c(nT)$$

□ If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

Impulse Invariance

□ Let,

$$h[n] = h_c(nT)$$

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

□ If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

$$H(e^{j\omega}) = \frac{1}{T} H_c \left(j \frac{\omega}{T} \right), \quad |\omega| < \pi$$

Impulse Invariance

□ Let,

$$h[n] = T h_c(nT)$$

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

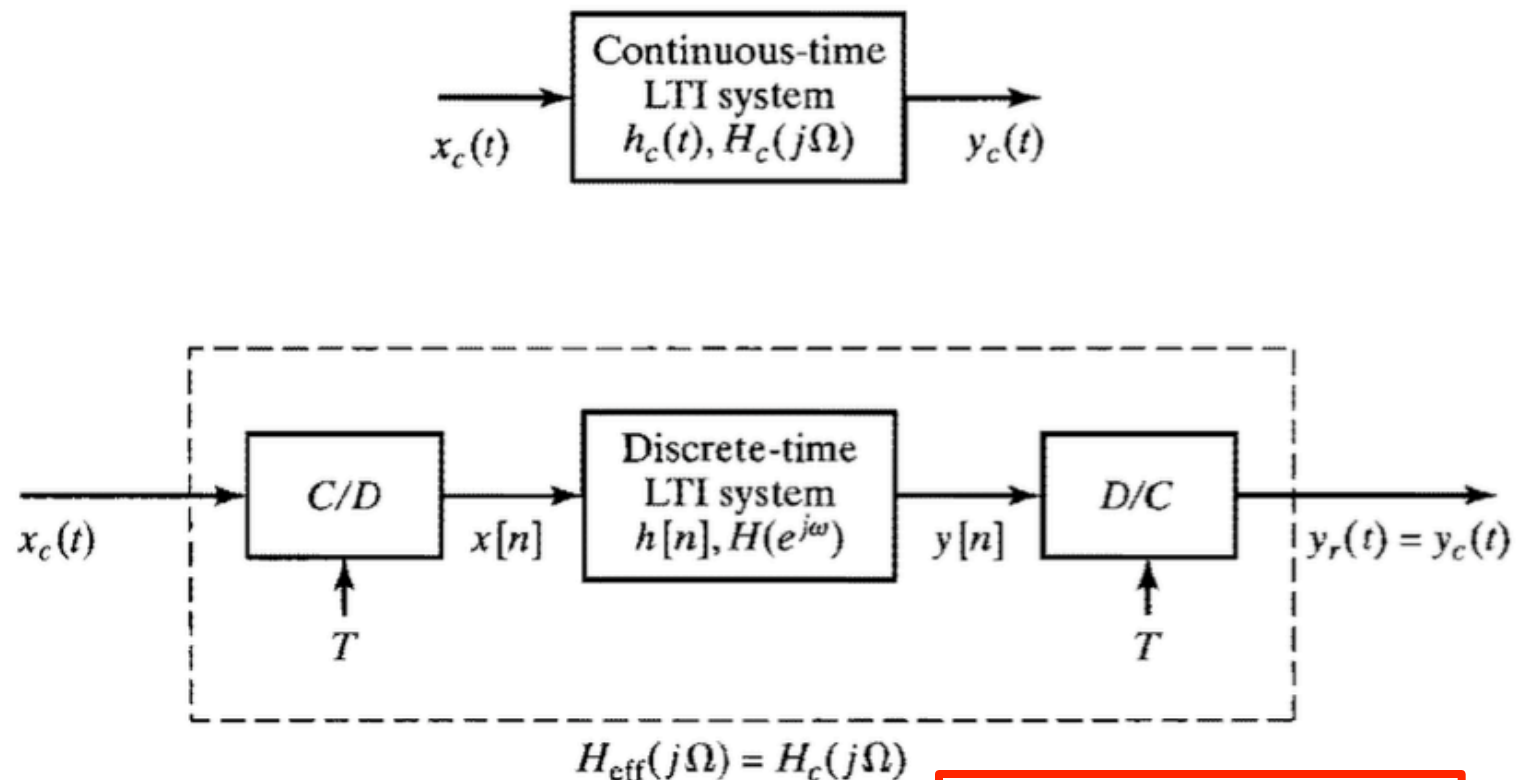
□ If sampling at Nyquist Rate then

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$$H(e^{j\omega}) = \frac{T}{T} H_c \left(j \frac{\omega}{T} \right), \quad |\omega| < \pi$$

Impulse Invariance

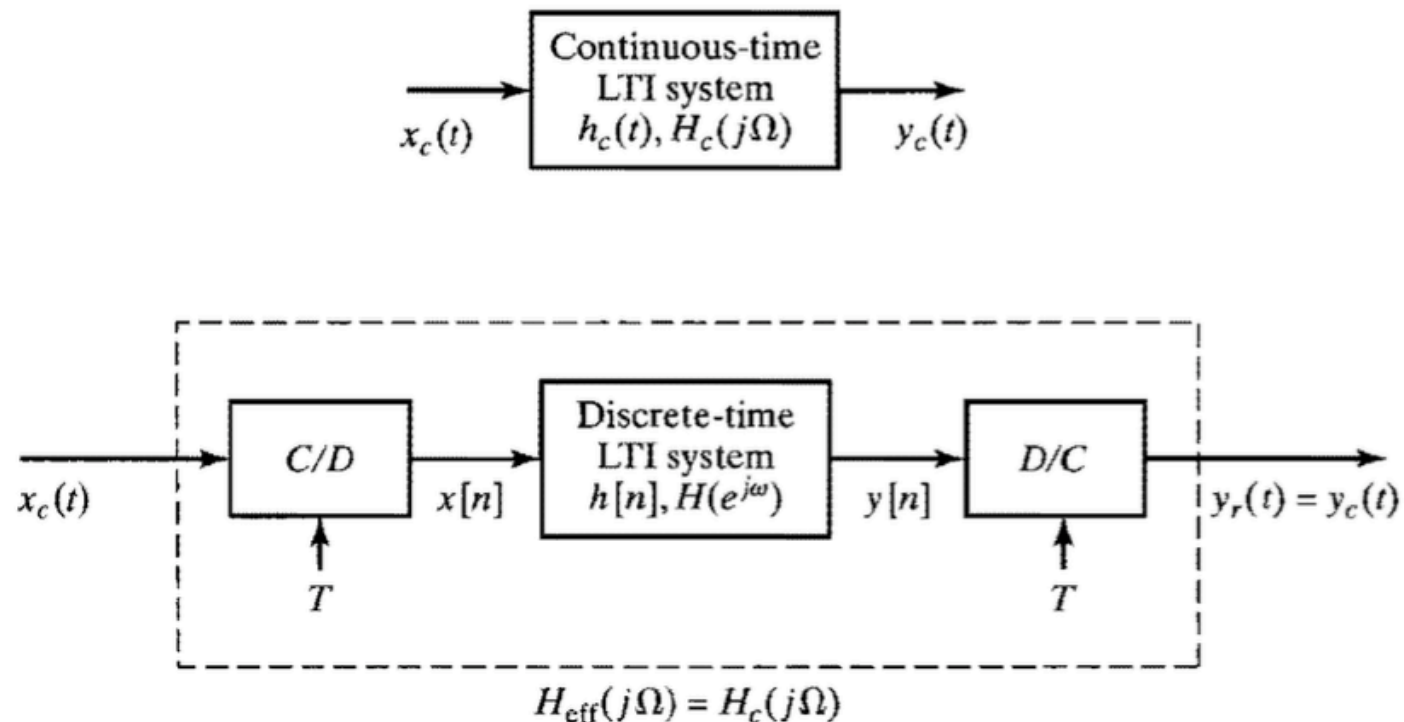
- Want to implement continuous-time system in discrete-time



$$h[n] = Th_c(nT)$$

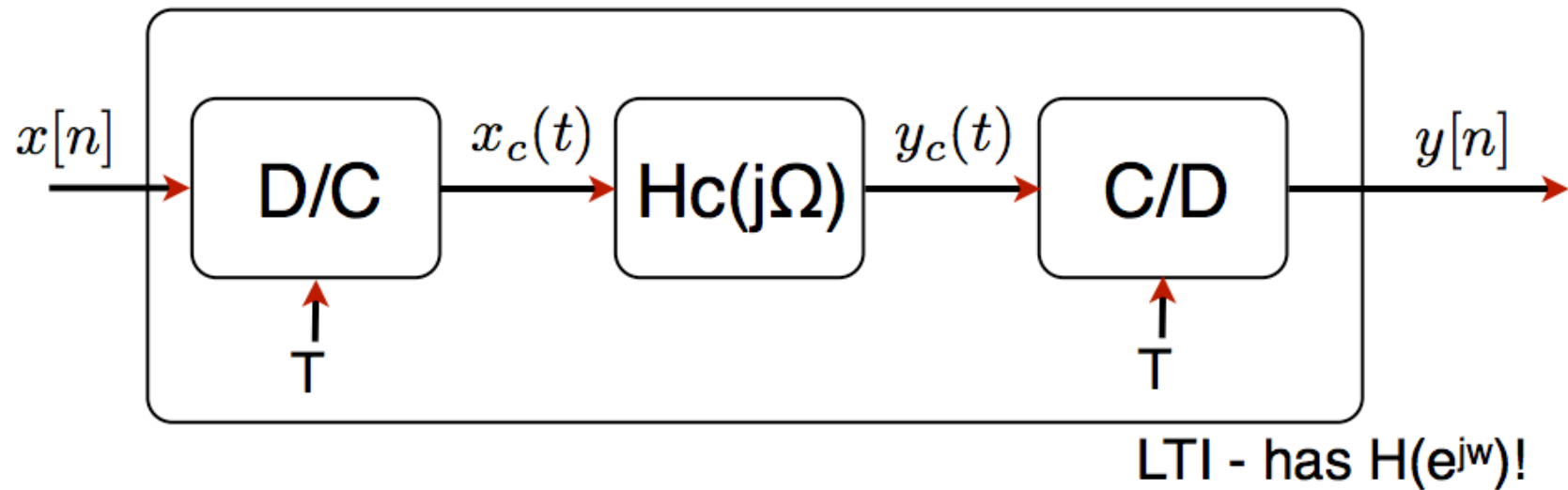
Example 3: DT Lowpass Filter

- We wish to implement a lowpass filter with cutoff frequency Ω_c on continuous time signal in discrete time with the following system

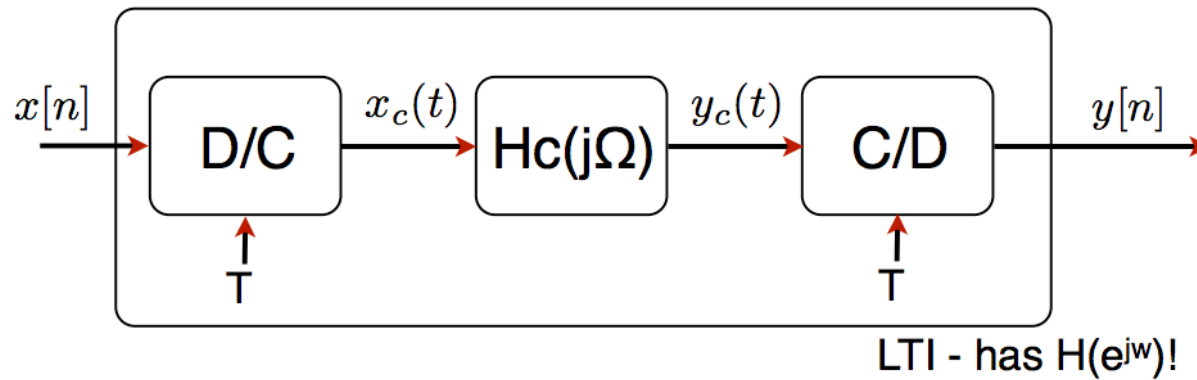


Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time

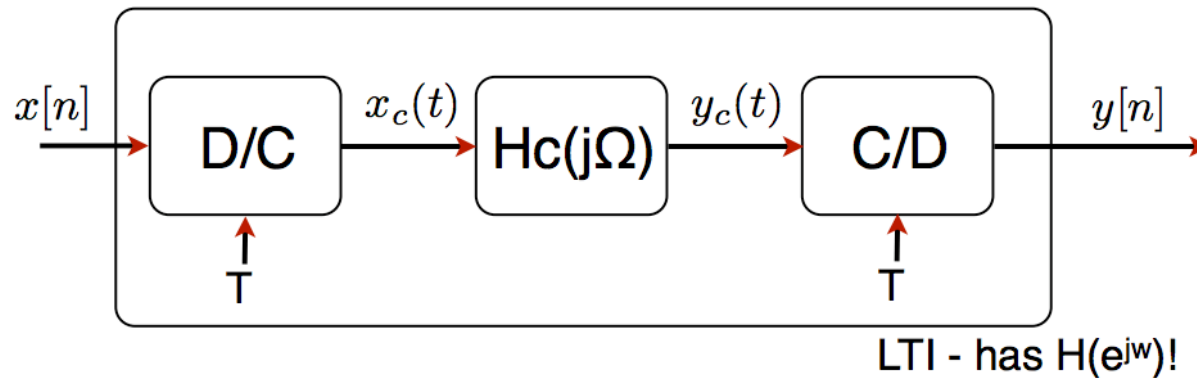


Continuous-Time Processing of Discrete-Time



$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & \text{else} \end{cases}$$

Continuous-Time Processing of Discrete-Time

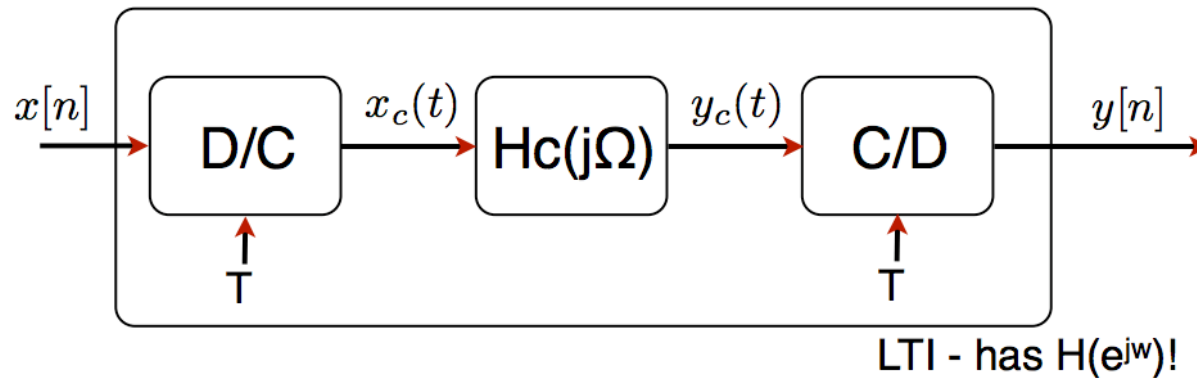


$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & \text{else} \end{cases}$$

Also bandlimited

→ $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$

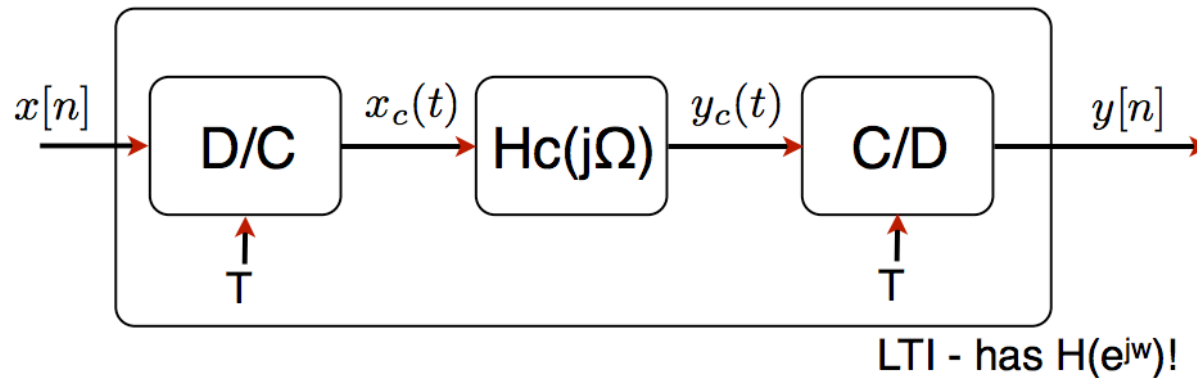
Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Y_c \left[j(\Omega - k\Omega_s) \right] \Big|_{\Omega=\omega/T} = \frac{1}{T} Y_c(j\Omega) \Big|_{\Omega=\omega/T}$$

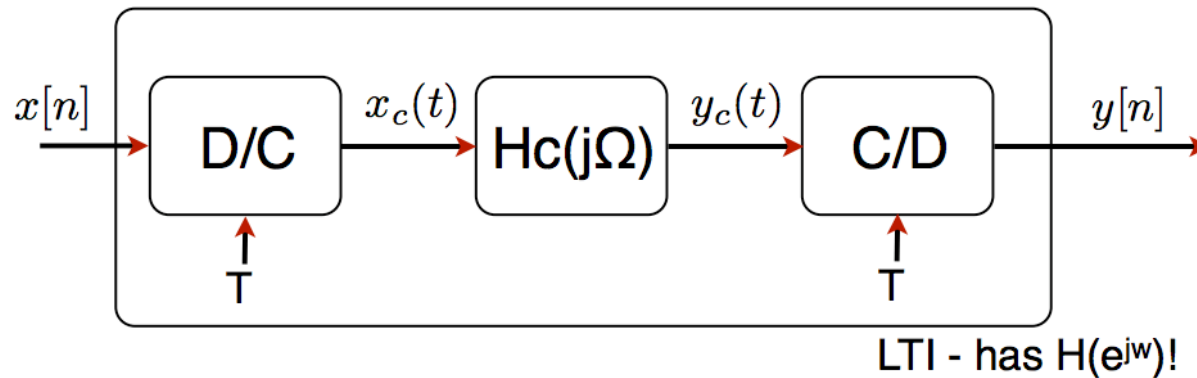
Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

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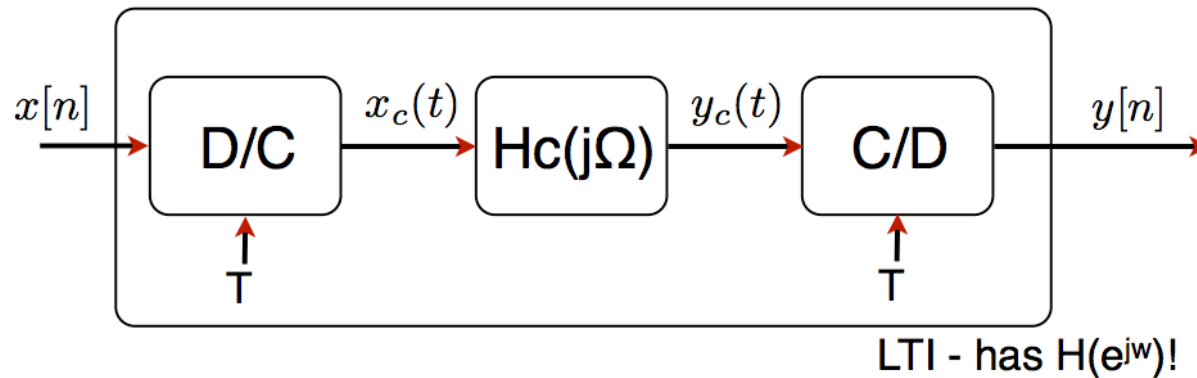
Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \qquad Y(e^{j\omega}) = \frac{1}{T} Y_c(j\Omega) \Big|_{\Omega=\omega/T}$$

$$Y(e^{j\omega}) = \frac{1}{T} H_c(j\Omega) \Big|_{\Omega=\omega/T} X_c(j\Omega) \Big|_{\Omega=\omega/T}$$

Continuous-Time Processing of Discrete-Time



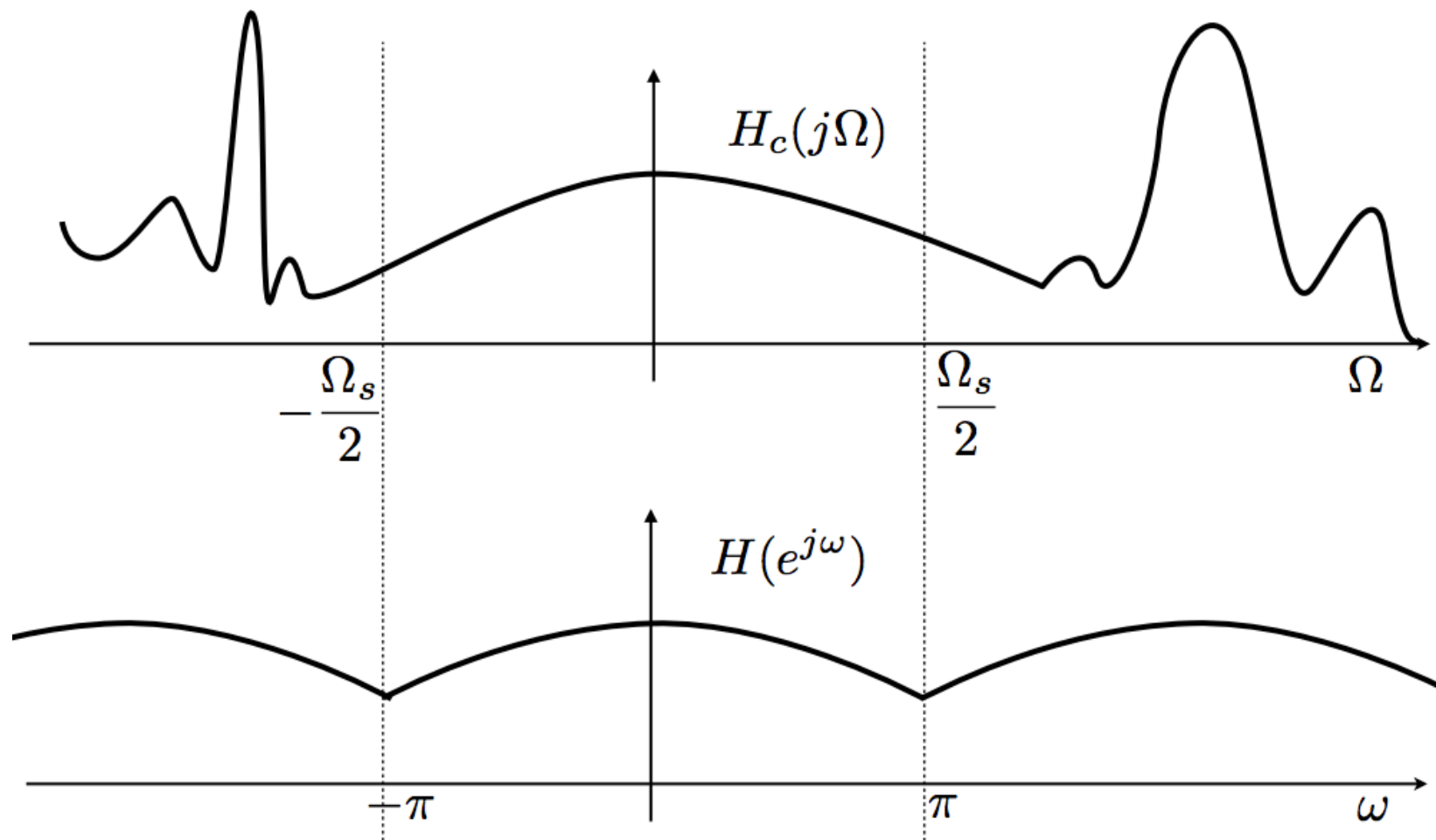
$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \qquad Y(e^{j\omega}) = \frac{1}{T} Y_c(j\Omega) \Big|_{\Omega=\omega/T}$$

$$Y(e^{j\omega}) = \frac{1}{T} H_c(j\Omega) \Big|_{\Omega=\omega/T} X_c(j\Omega) \Big|_{\Omega=\omega/T}$$

$$= \frac{1}{T} H_c(j\Omega) \Big|_{\Omega=\omega/T} (TX(e^{j\omega})) \qquad |\omega| < \pi$$

$$H(e^{j\omega})$$

Example



Example: Non-integer Delay

- What is the time domain operation when Δ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

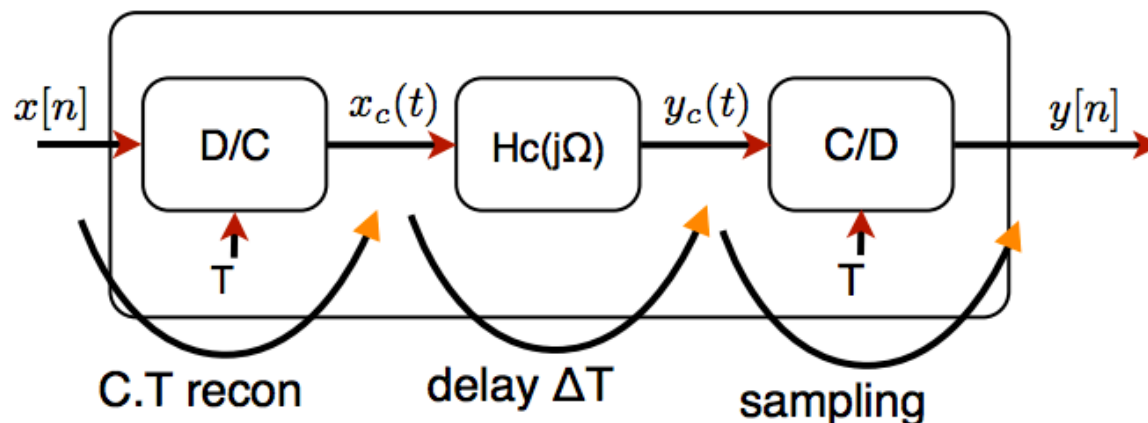
$$\begin{aligned}\delta[n] &\leftrightarrow 1 \\ \delta[n - n_d] &\leftrightarrow e^{-j\omega n_d}\end{aligned}$$

Example: Non-integer Delay

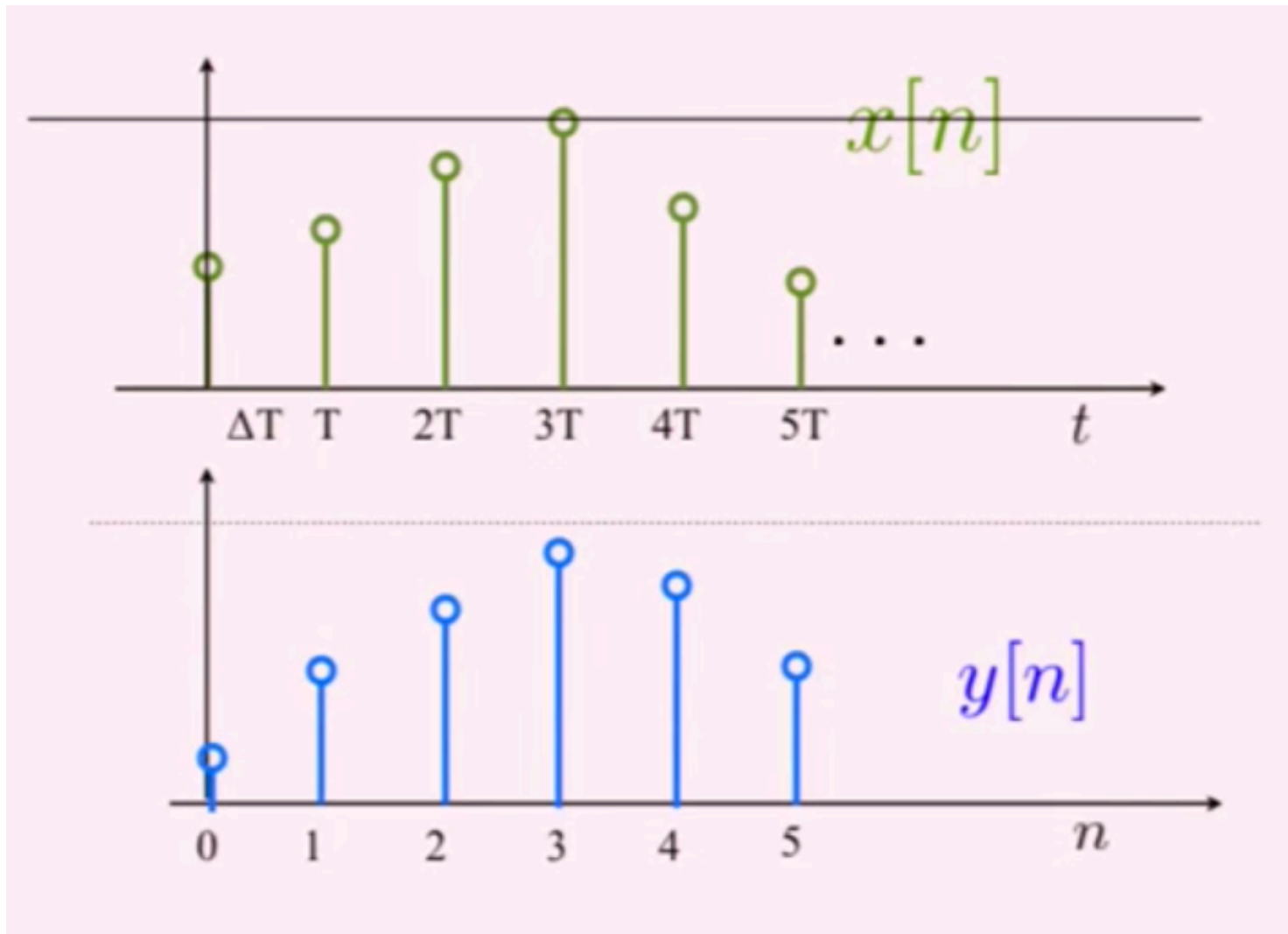
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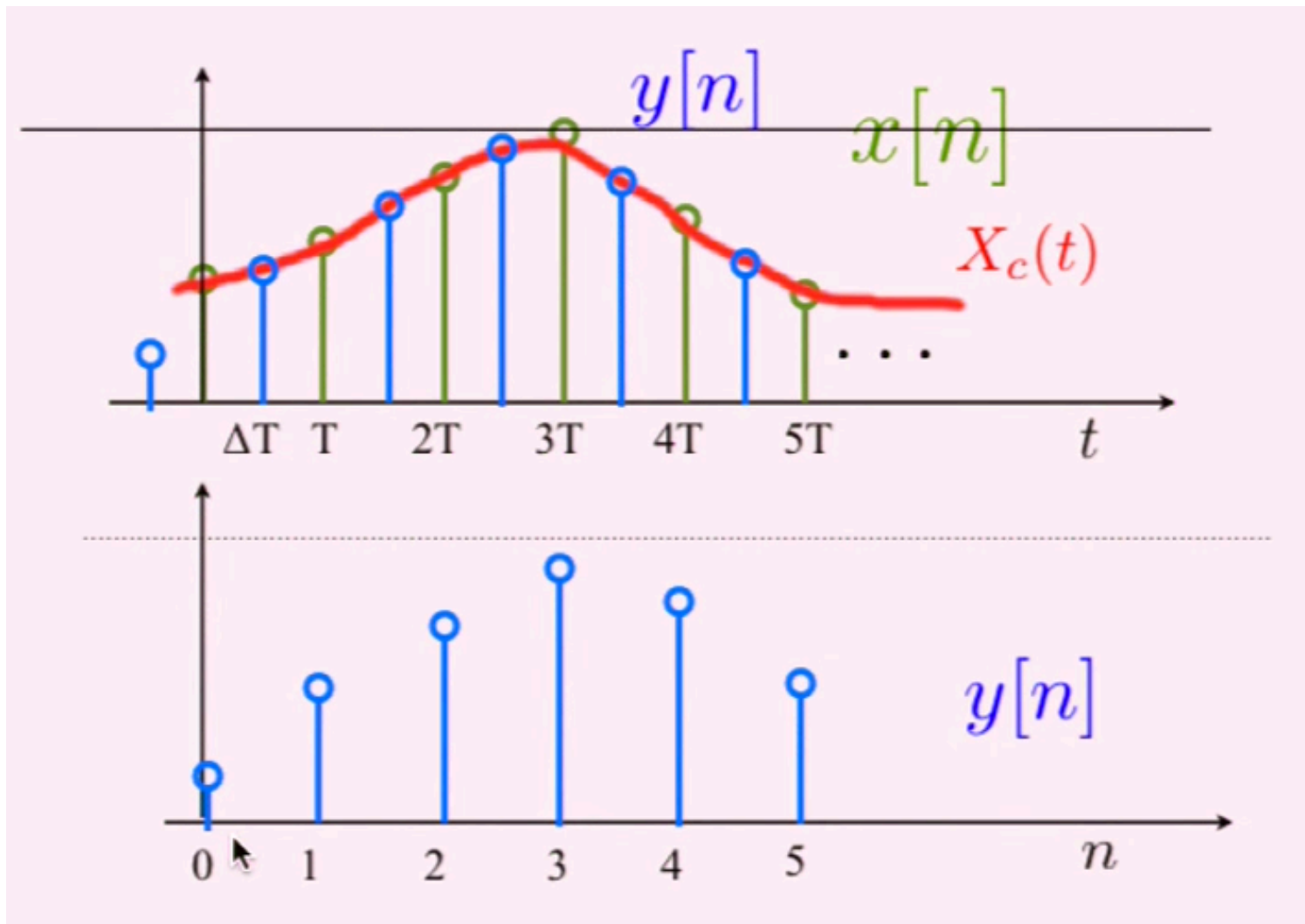
Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ delay of ΔT in time



Example: Non-integer Delay

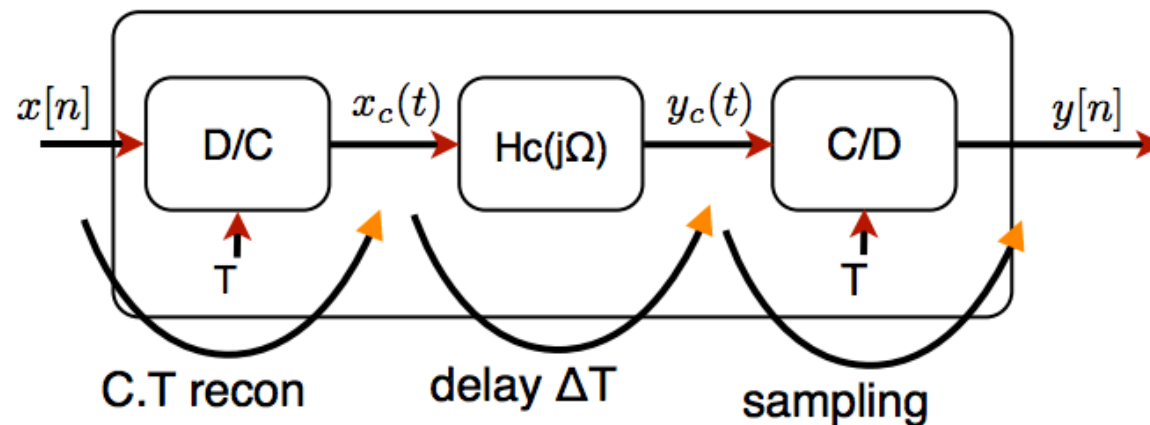


Example: Non-integer Delay



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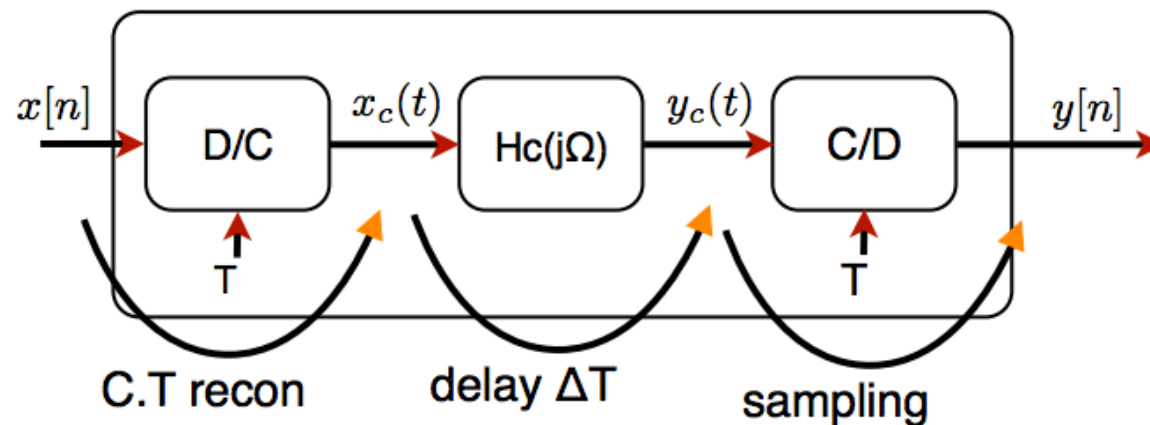
- ❑ The block diagram is for interpretation/analysis only



$$y_c(t) = x_c(t - T\Delta)$$

Example: Non-integer Delay

- The block diagram is for interpretation/analysis only



$$y_c(t) = x_c(t - T\Delta)$$

$$y[n] = y_c(nT) = x_c(nT - T\Delta)$$

$$= \sum_k x[k] \text{sinc} \left(\frac{t - kT - T\Delta}{T} \right) \Bigg|_{t=nT}$$

$$= \sum_k x[k] \text{sinc}(n - k - \Delta)$$



Example: Non-integer Delay

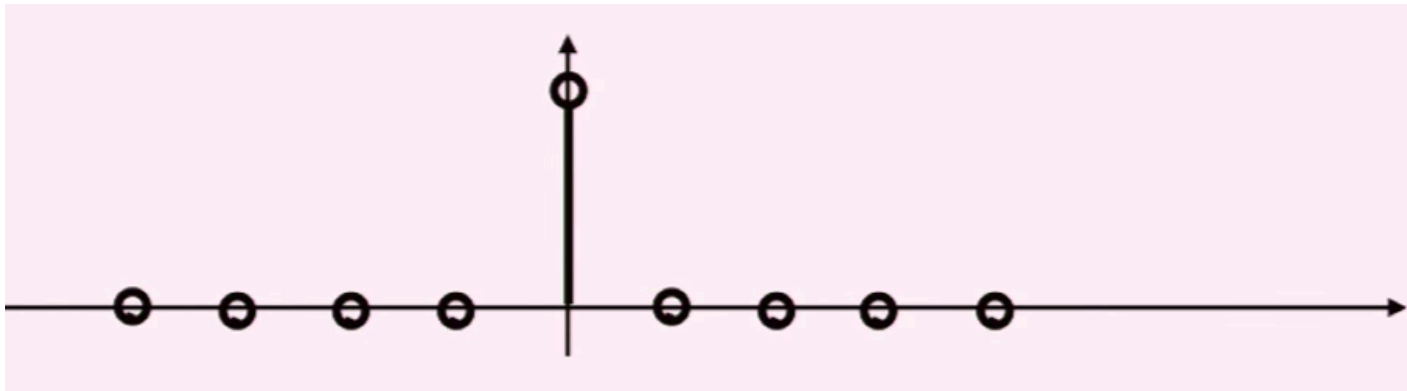
- ❑ My delay system has an impulse response of a sinc with a continuous time delay

$$h[n] = \text{sinc}(n - \Delta)$$

Example: Non-integer Delay

- My delay system has an impulse response of a sinc with a continuous time delay

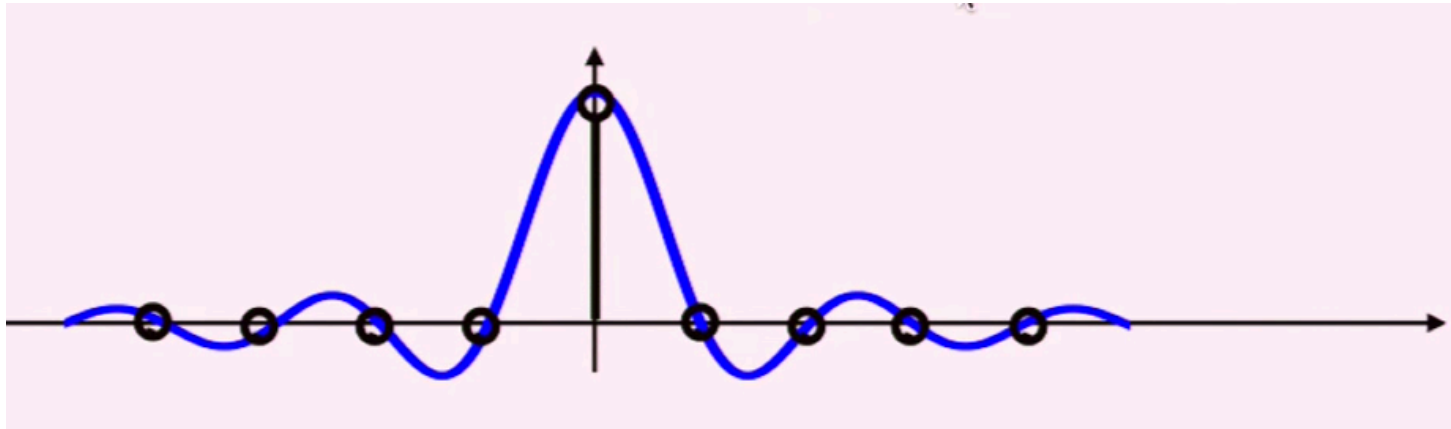
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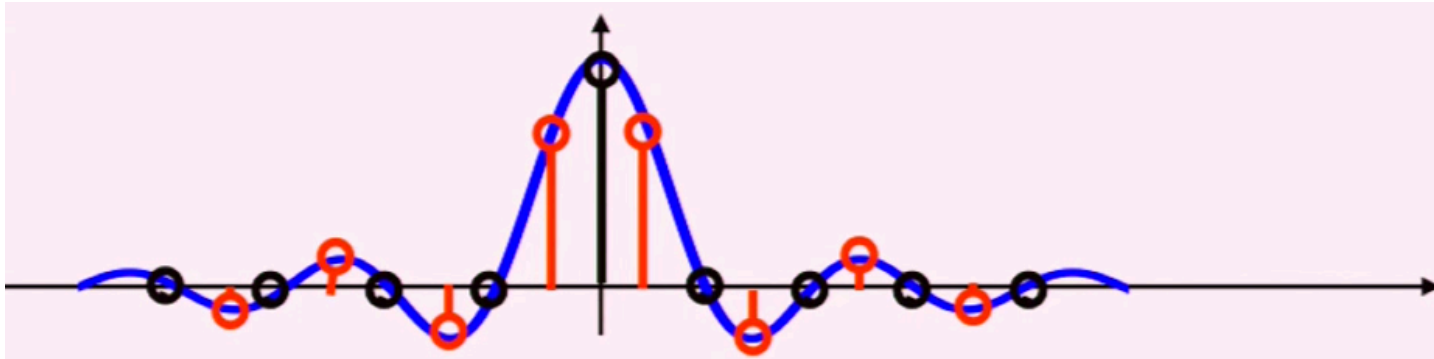
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Big Ideas

- ❑ Sampling and reconstruction
 - Rely on bandlimitedness for unique reconstruction
- ❑ CT processing of DT
 - Effectively LTI if no aliasing
- ❑ DT processing of CT
 - Always LTI
 - Useful for interpretation
- ❑ Changing the sampling rates next time
 - Upsampling, downsampling



Admin

- ❑ HW 3 due Friday