ESE 531: Digital Signal Processing

Lec 9: February 13th, 2018

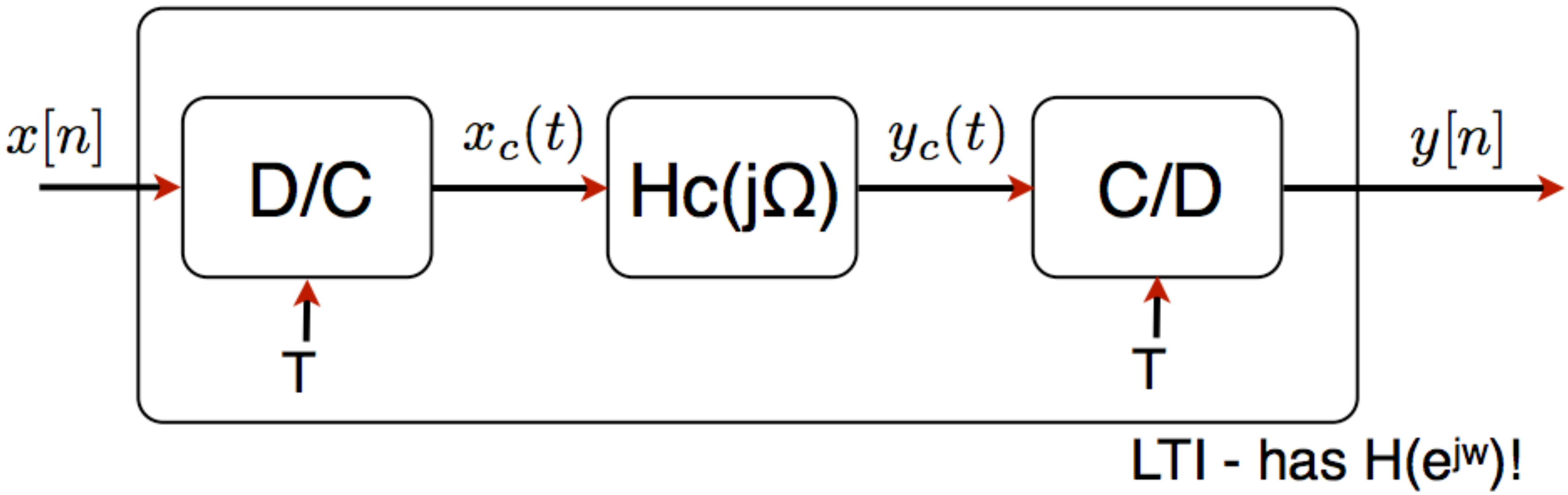
Downsampling/Upsampling and Practical Interpolation

Lecture Outline

- CT processing of DT signals
- Downsampling
- Upsampling

Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time



Example: Non-integer Delay

- What is the time domain operation when Δ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

$$\boxed{\begin{array}{c} \delta[n] \leftrightarrow 1 \\ \delta[n-n_d] \leftrightarrow e^{-j\omega n_d} \end{array}}$$

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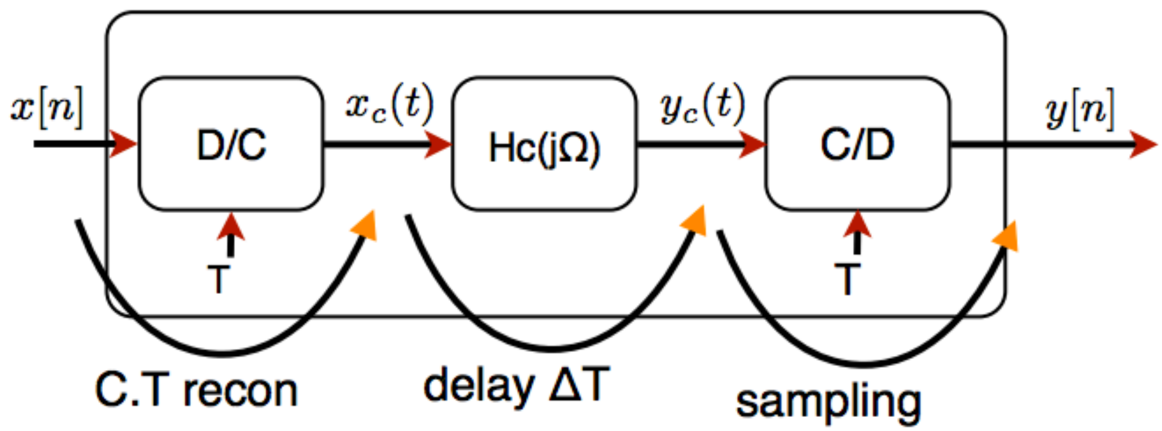
Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ delay of ΔT in time



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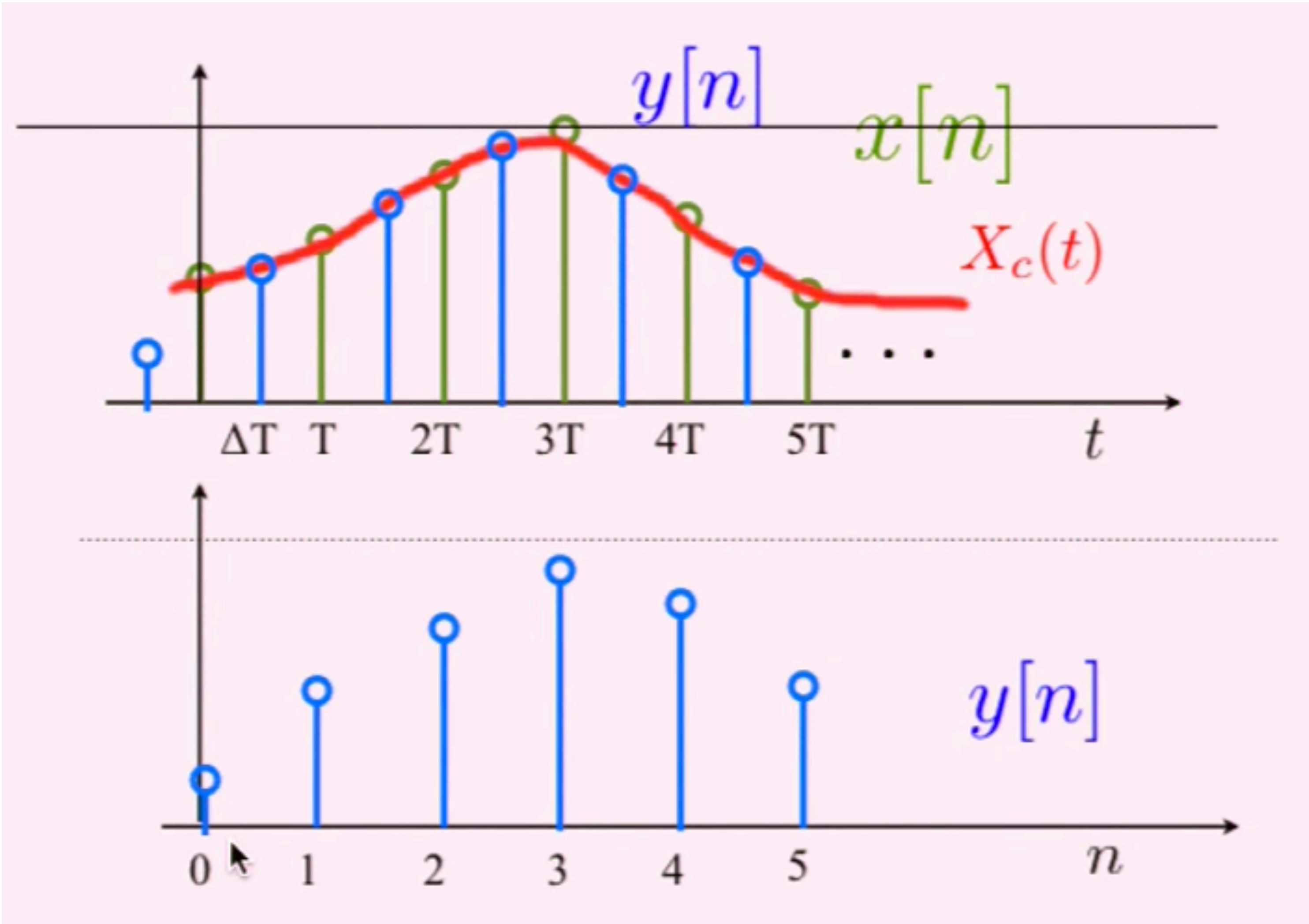
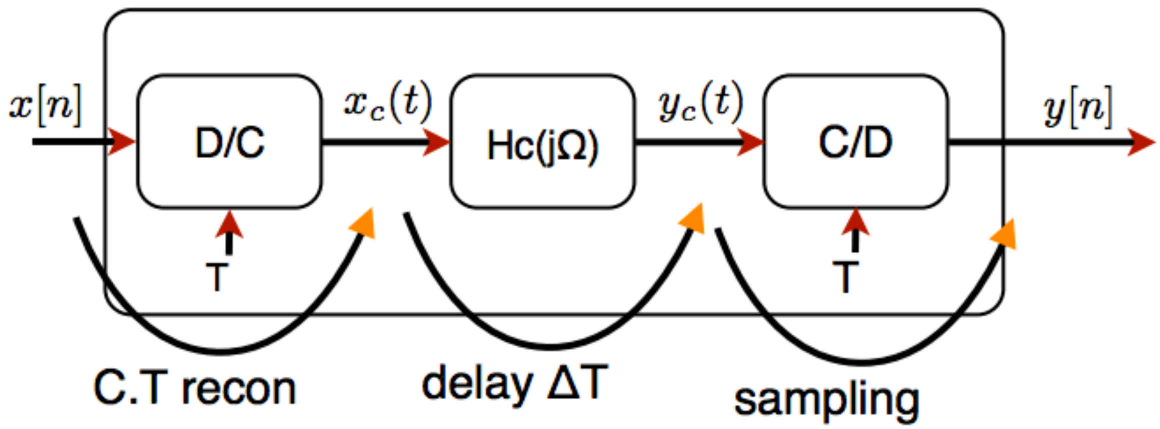


Example: Non-integer Delay

- The block diagram is for interpretation/analysis only



$$y_c(t) = x_c(t - T \cdot \Delta)$$

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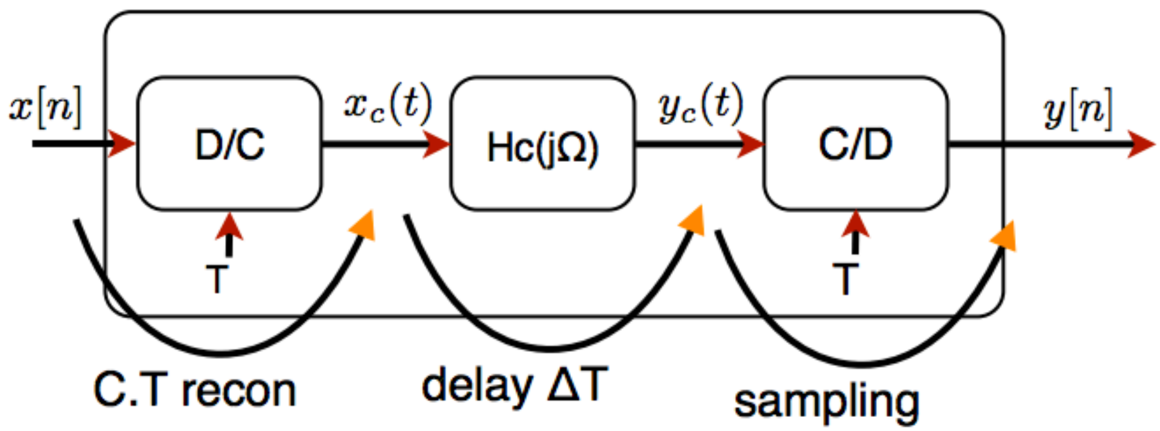

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$$y[n] = y_c(nT) = x_c(nT - T \cdot \Delta)$$

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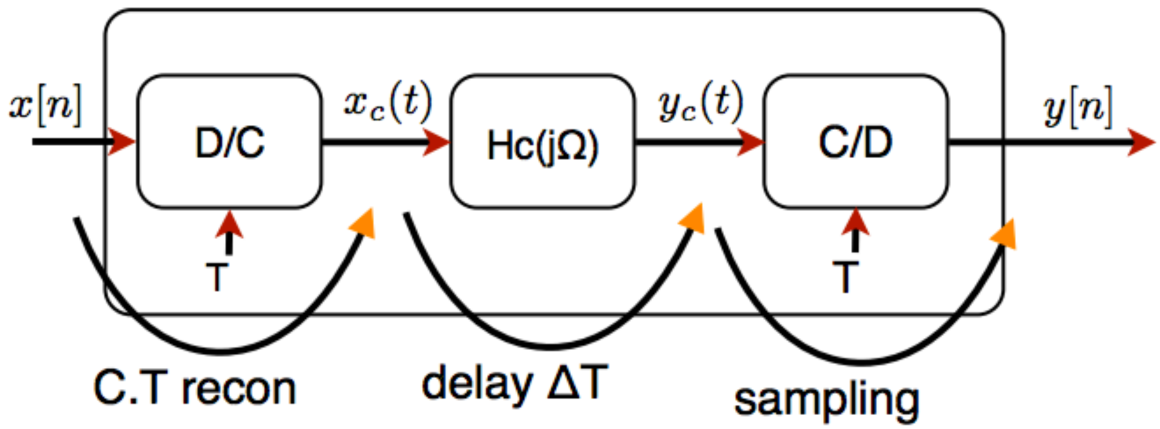


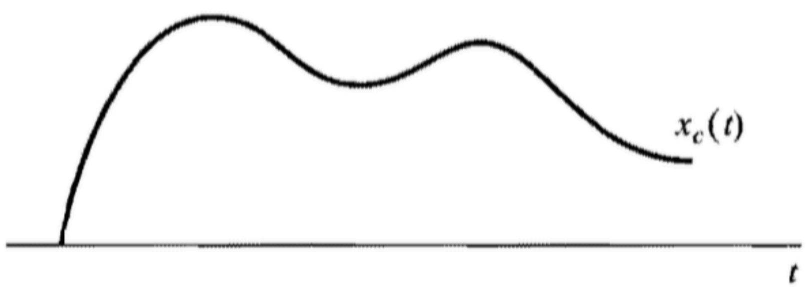
$$y_c(t) = x_c(t - T \cdot \Delta)$$

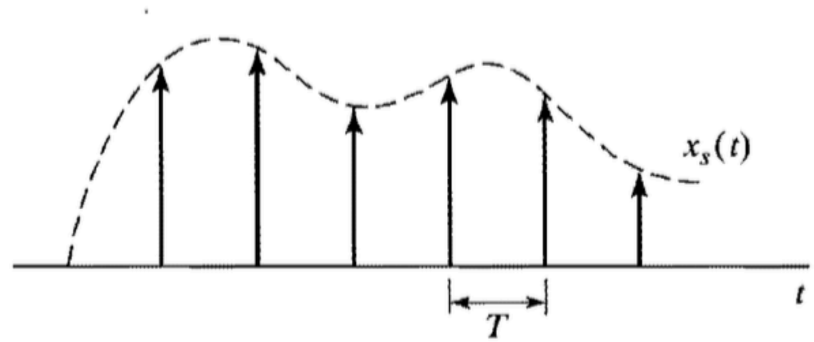
$$x_c(t) = \sum_k x[k] h_r(t - kT)$$

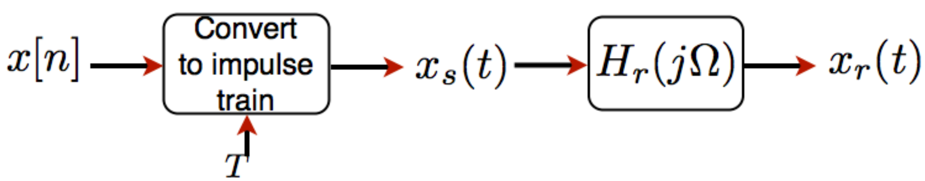
$$y[n] = y_c(nT) = x_c(nT - T \cdot \Delta)$$

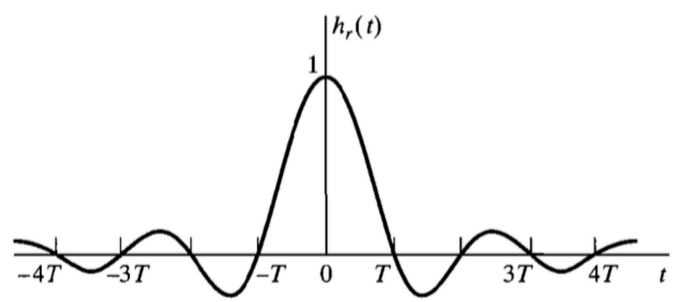
Reminder: Reconstruction in Time Domain

$$
\begin{aligned}
x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t) \\
&= \sum_n x[n] h_r(t - nT)
\end{aligned}
$$









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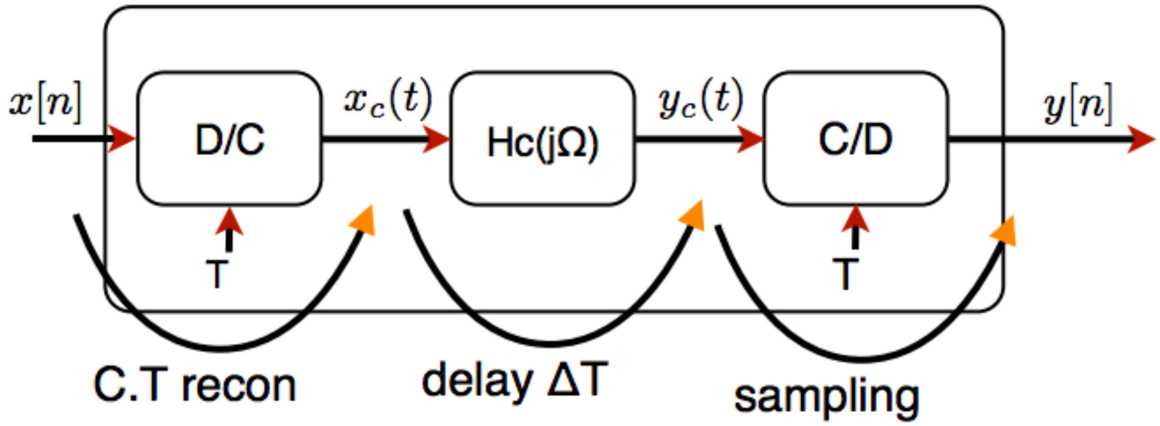


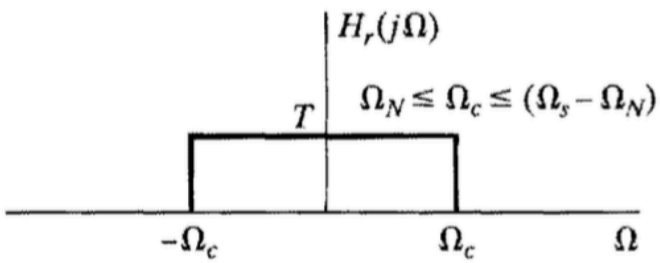
$$y_c(t) = x_c(t - T \cdot \Delta)$$

$$y[n] = y_c(nT) = x_c(nT - T \cdot \Delta)$$

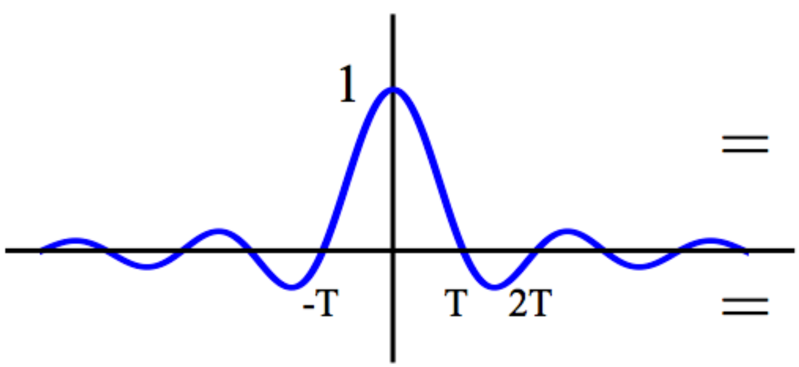
$$x_c(t) = \sum_k x[k] h_r(t - kT)$$

$$= \sum_k x[k] \text{sinc}\left(\frac{t - kT}{T}\right)$$

Reminder: Reconstruction in Time Domain



$$
\begin{aligned}
h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \\
&= \frac{T}{2\pi}\frac{1}{jt} s^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\
&= \frac{T}{\pi t}\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\
&= \frac{T}{\pi t}\sin\left(\frac{\Omega_s}{2}t\right) = \frac{T}{\pi t}\sin\left(\frac{\pi}{T}t\right) \\
&= \mathrm{sinc}\left(\frac{t}{T}\right)
\end{aligned}
$$



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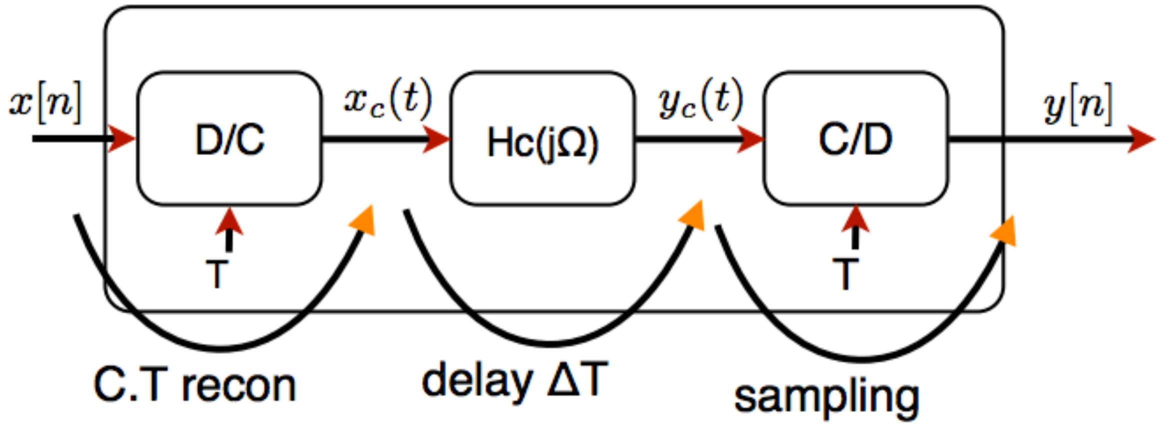

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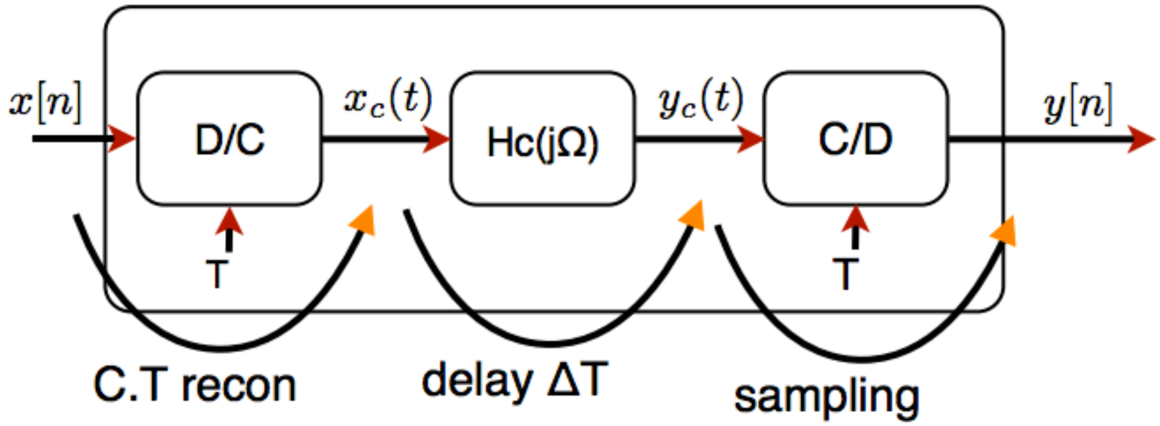

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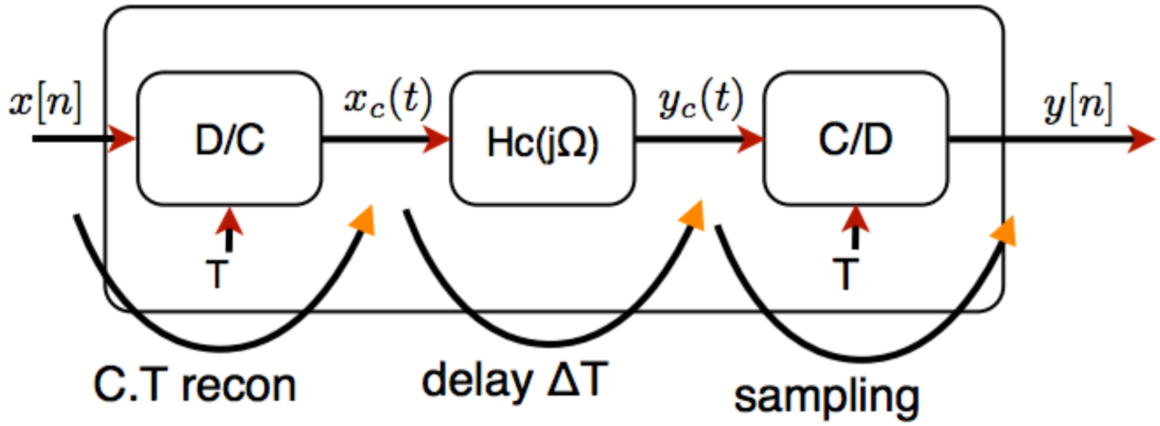

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$$x_c(nT - T\cdot\Delta) = \sum_k x[k]\text{sinc}\left(\frac{nT - T\cdot\Delta - kT}{T}\right) = \sum_k x[k]\text{sinc}\left(n - \Delta - k\right)$$

Example: Non-integer Delay

- Delay system has an impulse response of a sinc with a continuous time delay

$$y[n] = \sum_k x[k]\,\mathrm{sinc}(n - \Delta - k)$$
$$= x[n] * \mathrm{sinc}(n - \Delta)$$

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$$\boxed{\Rightarrow h[n] = \mathrm{sinc}(n - \Delta)}$$

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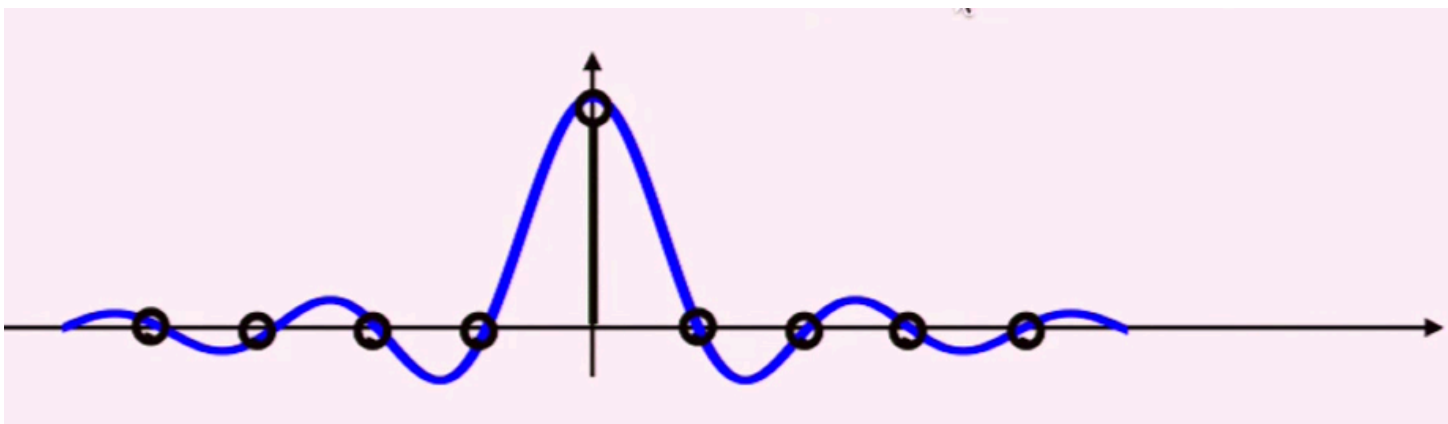

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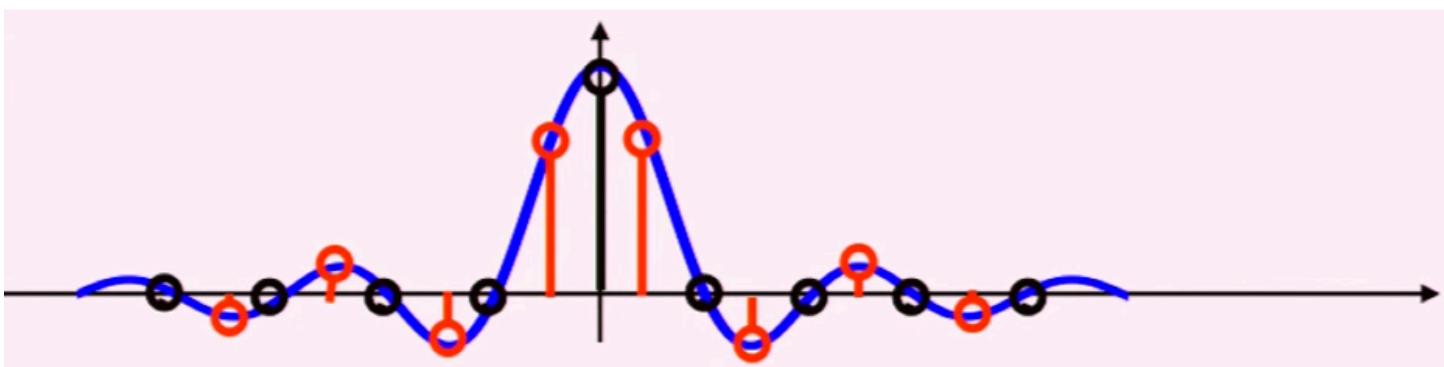

Downsampling

- Definition: Reducing the sampling rate by an integer number

$$x[n] \longrightarrow \boxed{\downarrow \mathrm{M}} \longrightarrow x_d[n] = x[nM]$$

$$x[n] = x_c(nT)$$

$$x[nM] = x_c(n\underbrace{MT}_{T'})$$

Downsampling

- Similar to C/D conversion
  - Need to worry about aliasing
  - Use anti-aliasing filter to mitigate effects

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- If your discrete time signal is finely sampled almost like a CT signal
  - Downsampling is just like sampling (C/D conversion)

Downsampling

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x[nM]$$

$$= x_c(nT) \qquad\qquad = x_c(n\underbrace{MT}_{T'})$$

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

Downsampling

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- Want to relate $X_d(e^{j\omega})$ to $X(e^{j\omega})$ not $X_c(j\Omega)$
- Separate sum into two sums—fine sum and coarse sum (i.e like counting minutes within hours)

Downsampling

The DTFT:

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$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

- k=rM+i
  - i = 0, 1, …, M-1
  - r = -∞, ..., ∞

Downsampling

$$k = rM + i$$

$$\begin{aligned} X_d(e^{j\omega}) &= \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right) \\ &= \frac{1}{M}\sum_{i=0}^{M-1}\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}i - \frac{2\pi}{T}r\right)\right) \end{aligned}$$

Downsampling

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$$

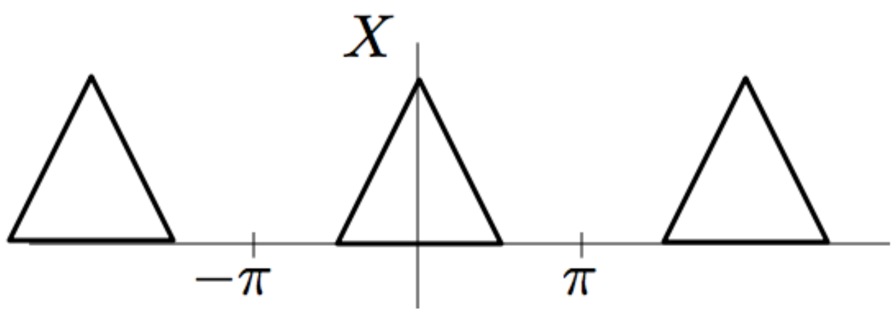
$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}-\underbrace{\frac{2\pi}{T}}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi}{M}i\right)}\right)$$

stretch by M   replicate

Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



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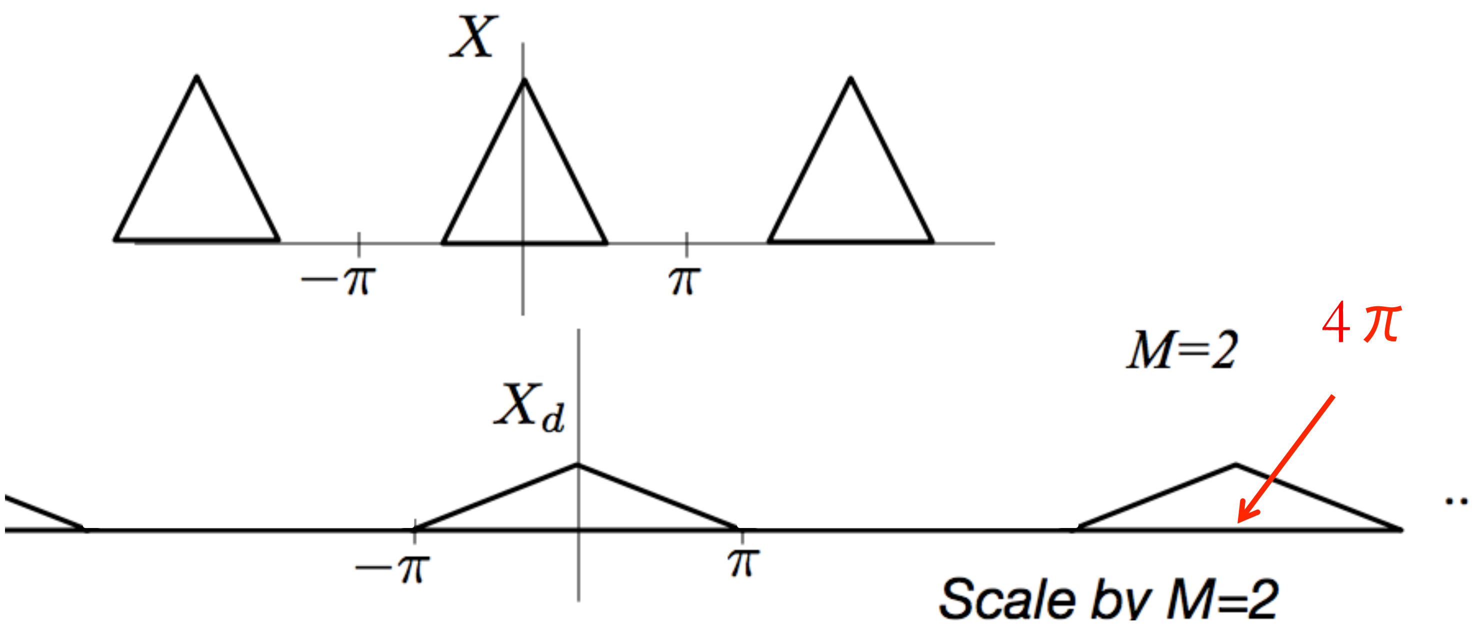

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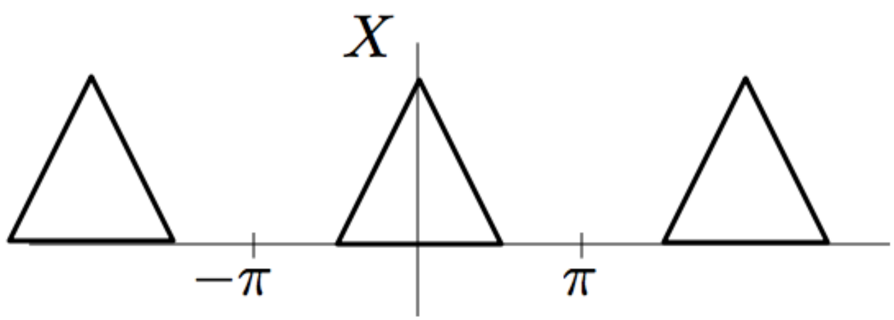

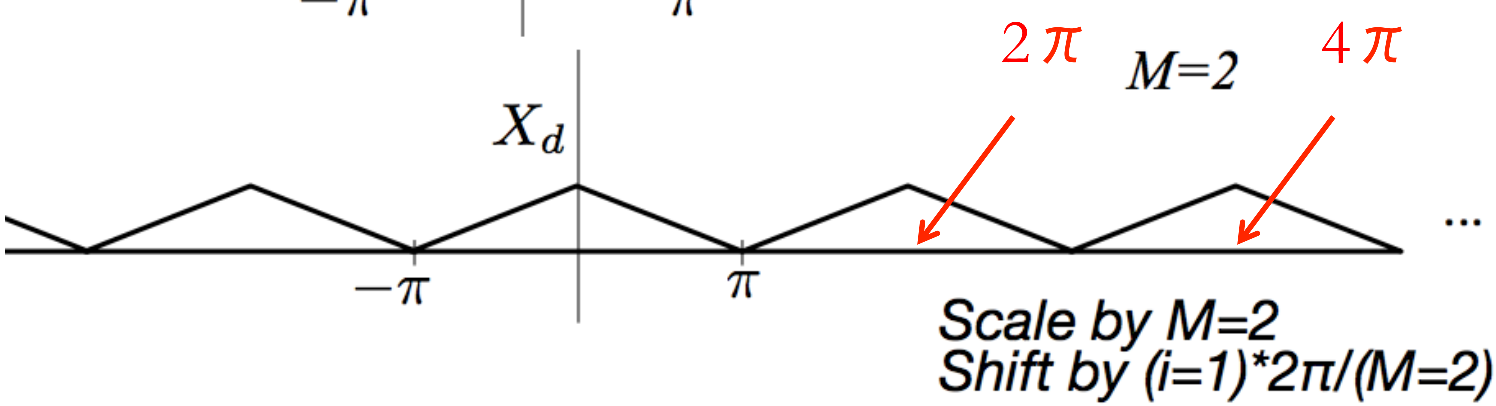

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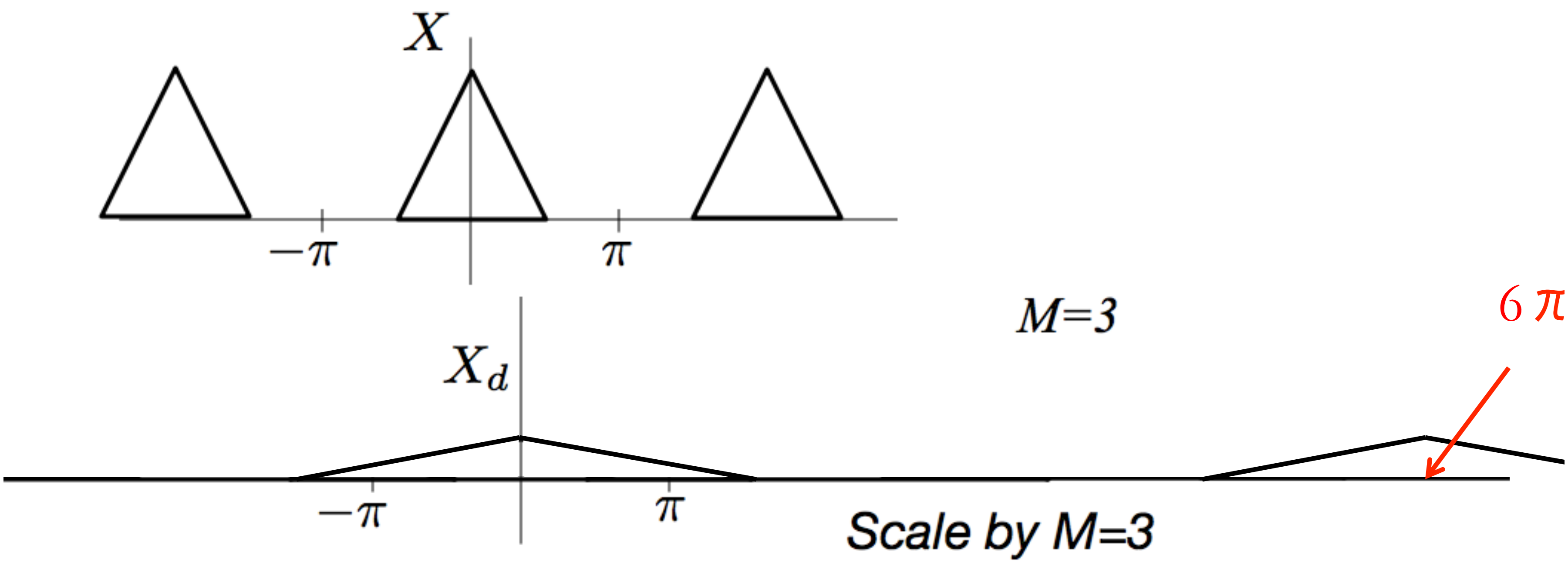

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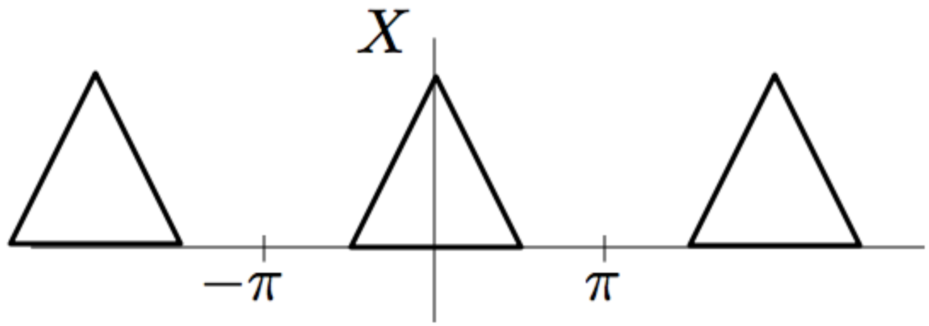

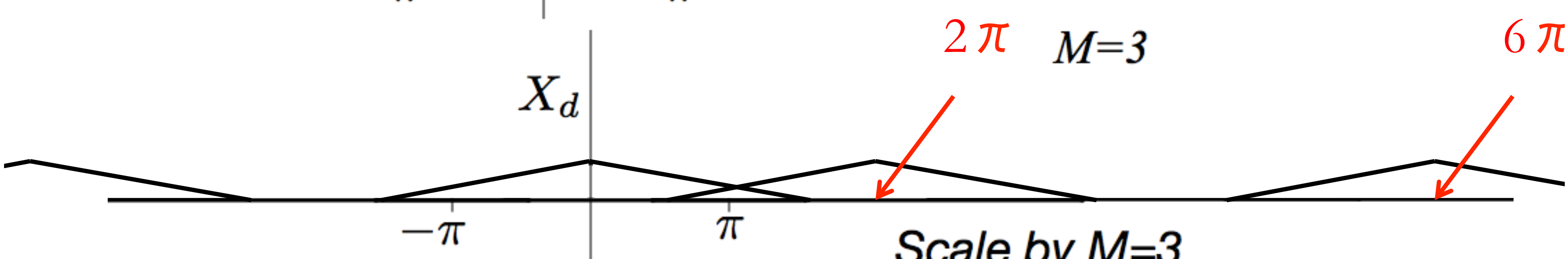

Scale by M=3
*Shift by (i=1)*2π/(M=3)*

Example

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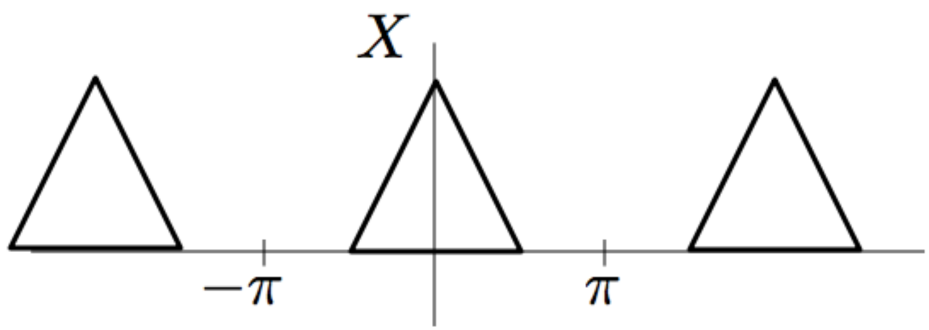
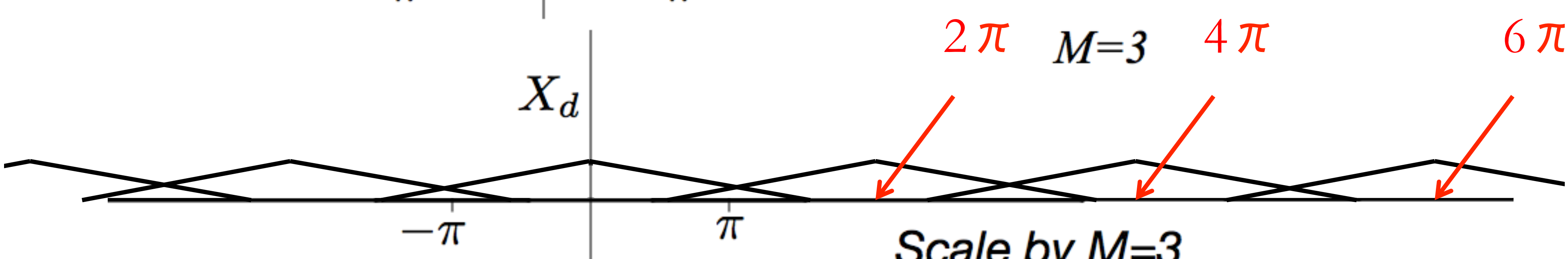
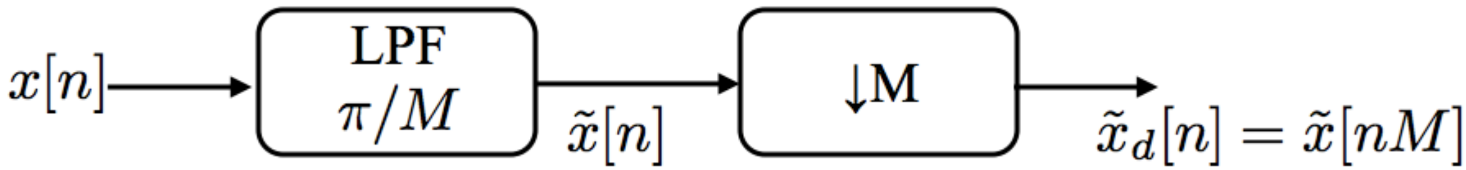
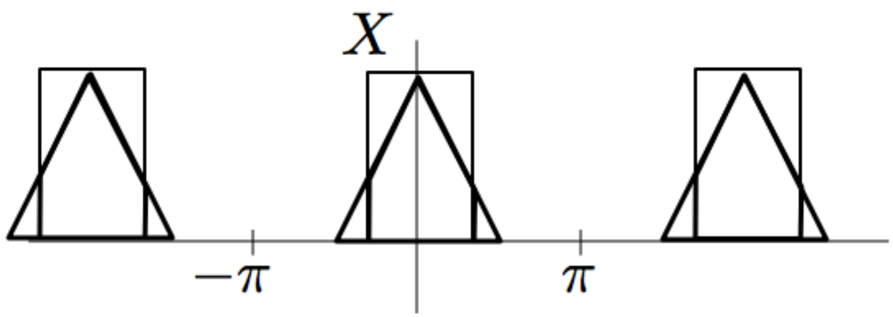




Scale by M=3
Shift by (i=1)*2π/(M=3)
Shift by (i=2)*2π/(M=3)

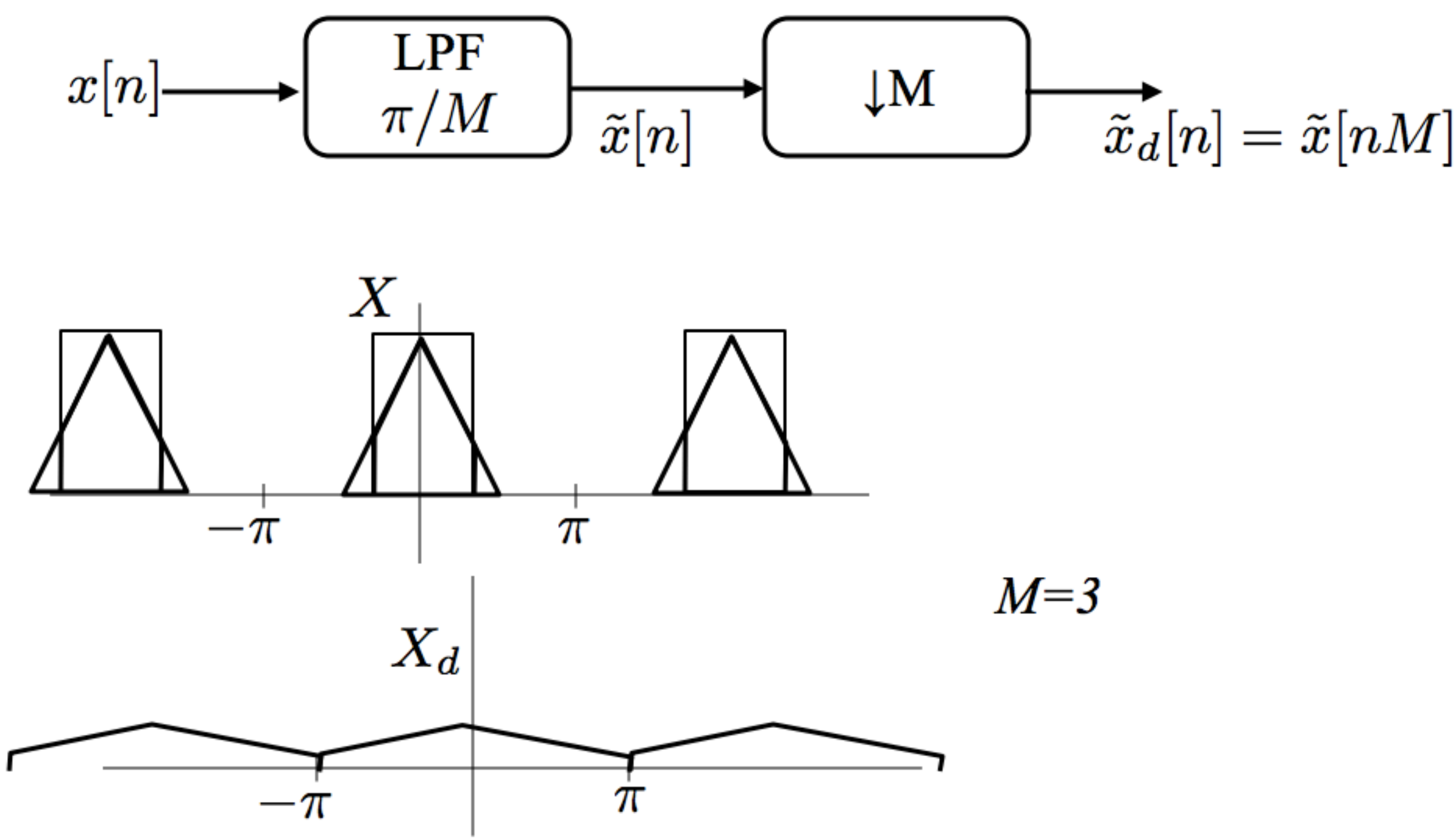
Example





M=3

Example



Upsampling

- Much like D/C converter
- Upsample by A LOT → almost continuous

- Intuition:
  - Recall our D/C model: $x[n] \rightarrow x_s(t) \rightarrow x_c(t)$
  - Approximate "$x_s(t)$" by placing zeros between samples
  - Convolve with a sinc to obtain "$x_c(t)$"

Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

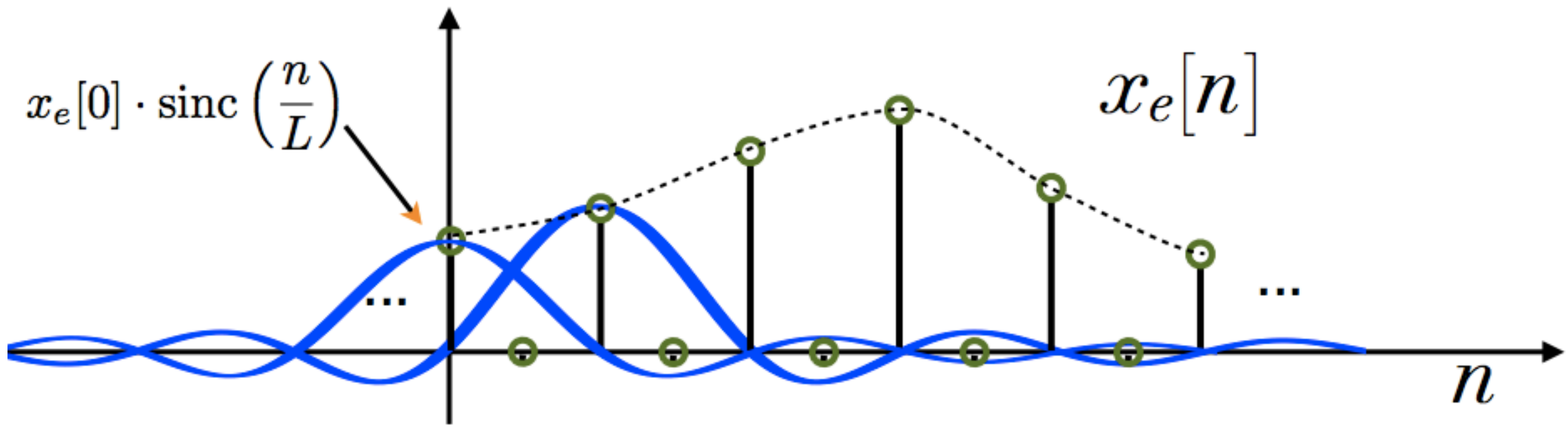
$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

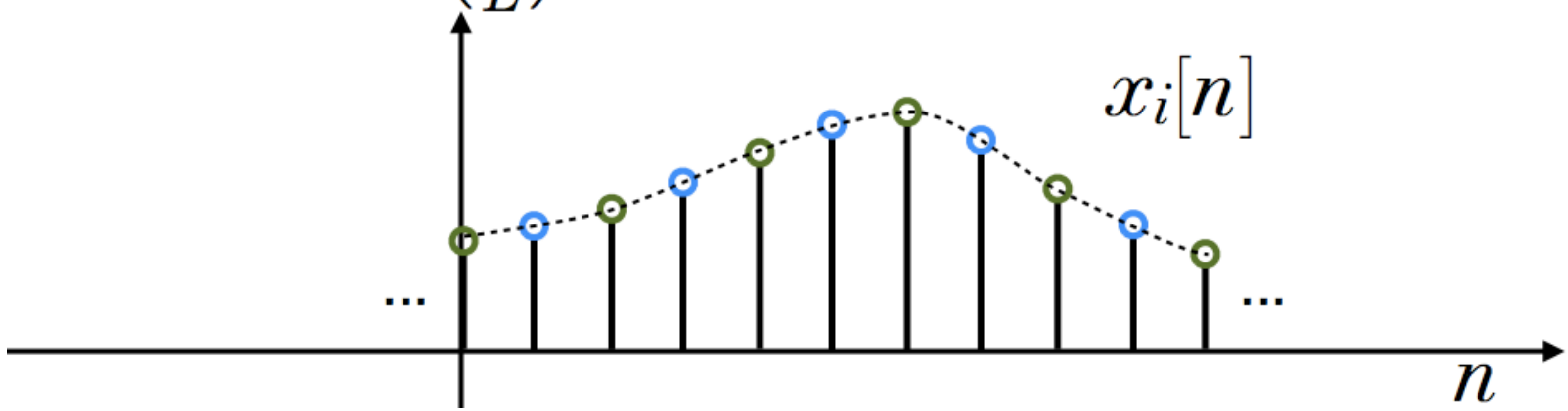
(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:



$$x_i[n] = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$$



Upsampling

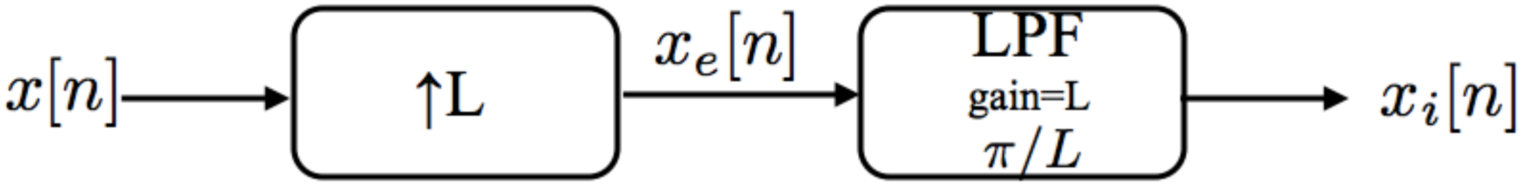
$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL]$$

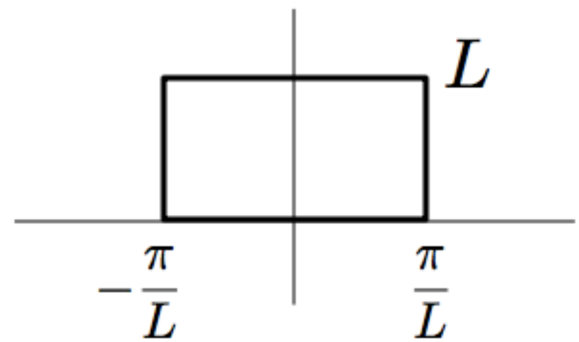
$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\text{sinc}(\frac{n - kL}{L})$$

Frequency Domain Interpretation

$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$



$\text{sinc}(n/L)$ **DTFT** $\Rightarrow$



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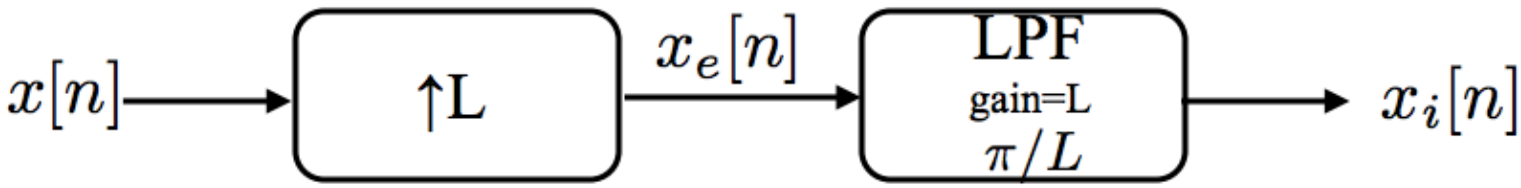
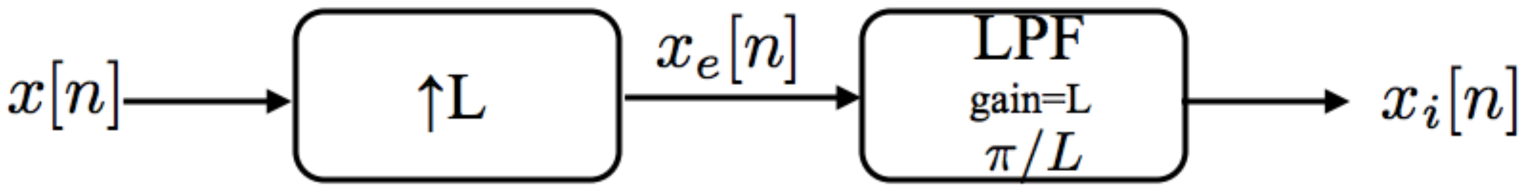


$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\substack{\neq 0 \text{ only for n=mL} \\ \text{(integer m)}}} e^{-j\omega n}$$

Frequency Domain Interpretation



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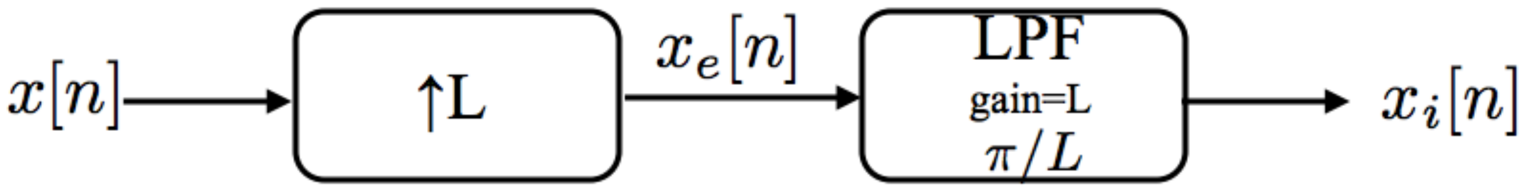
$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL}$$

Compress DTFT by a factor of L!

Frequency Domain Interpretation

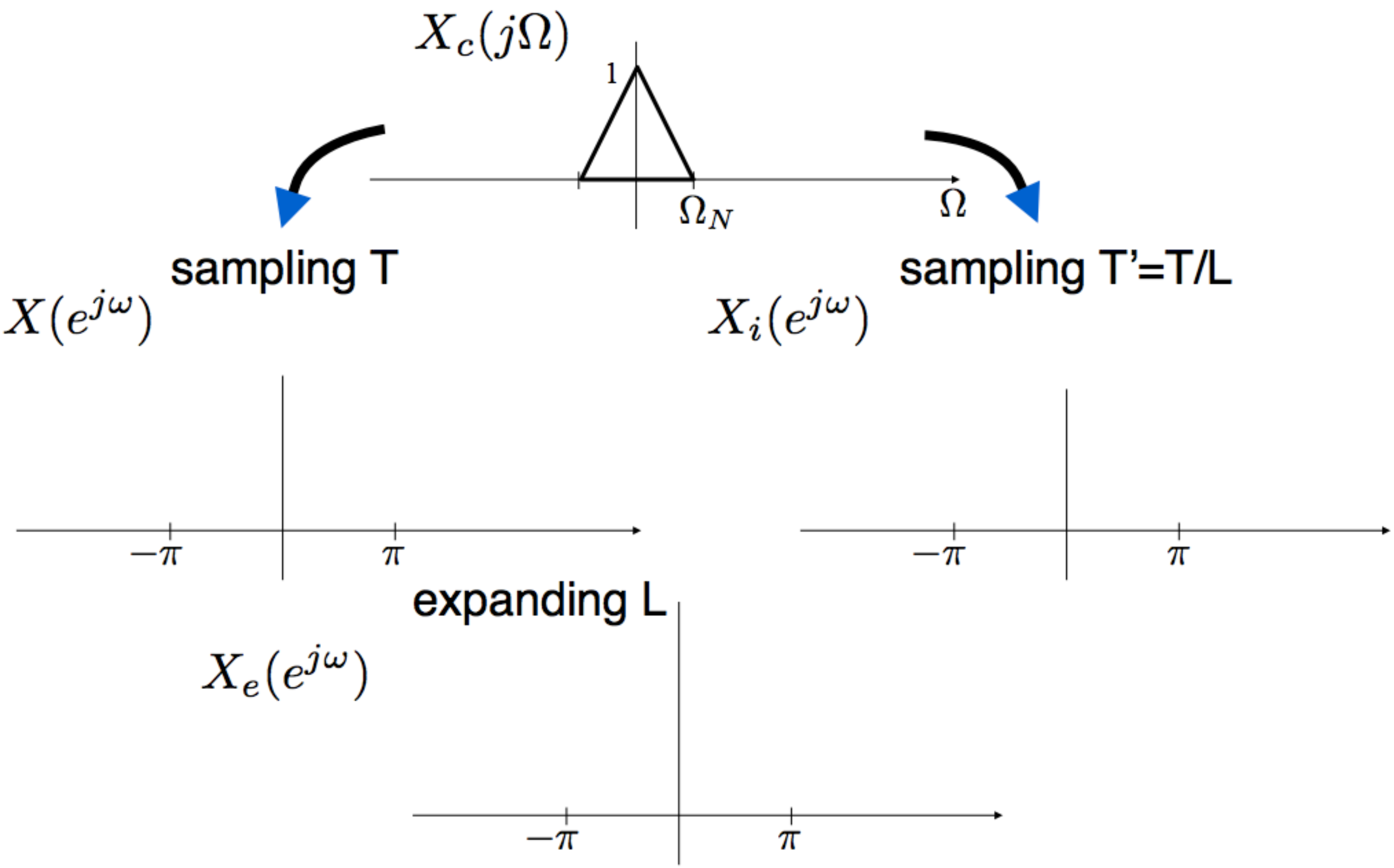


$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} \qquad = X(e^{j\omega L})$$

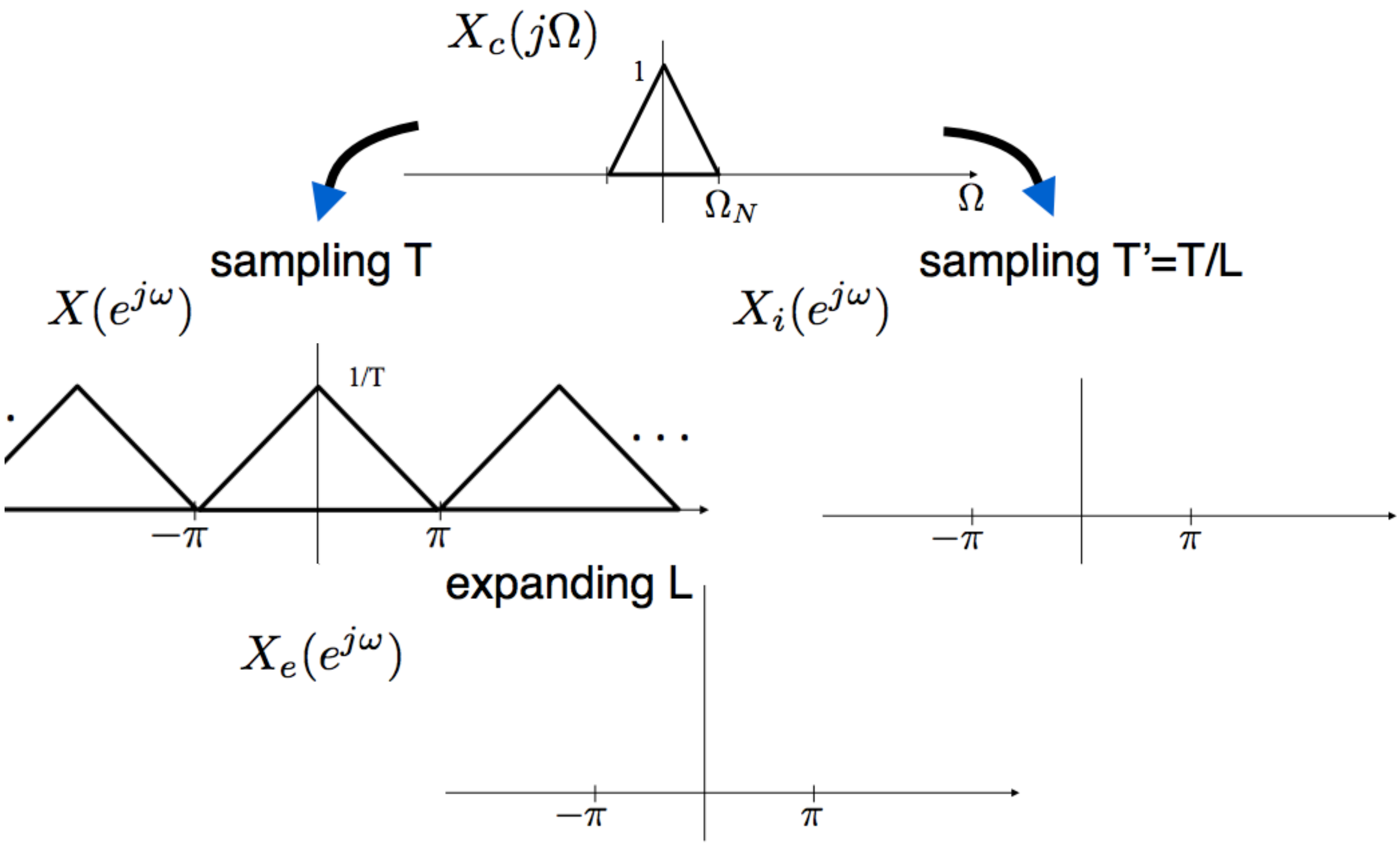
Compress DTFT by a factor of L!

Example



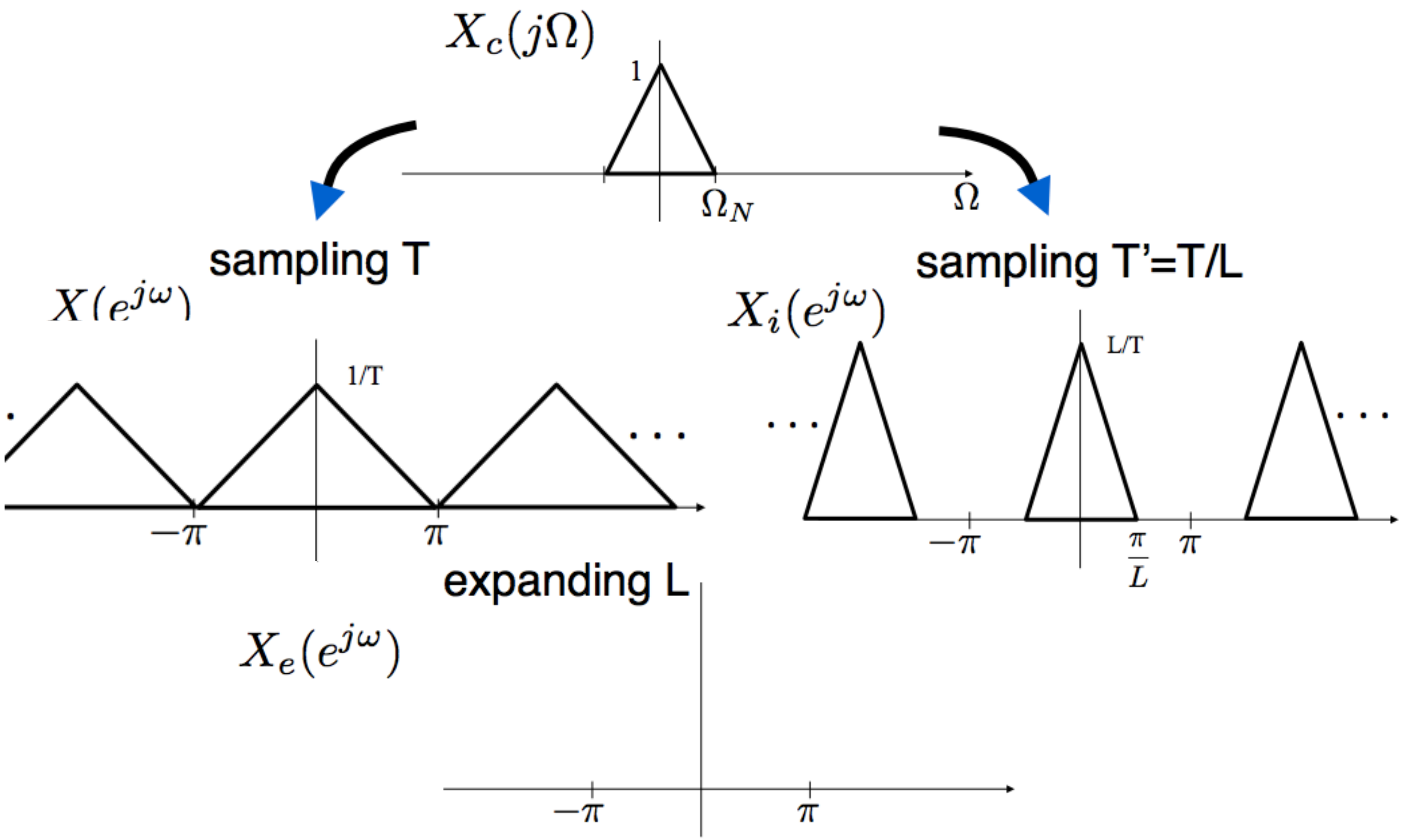
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example



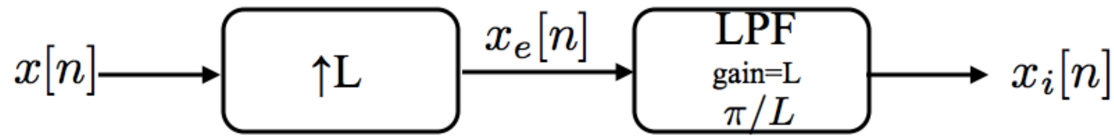
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

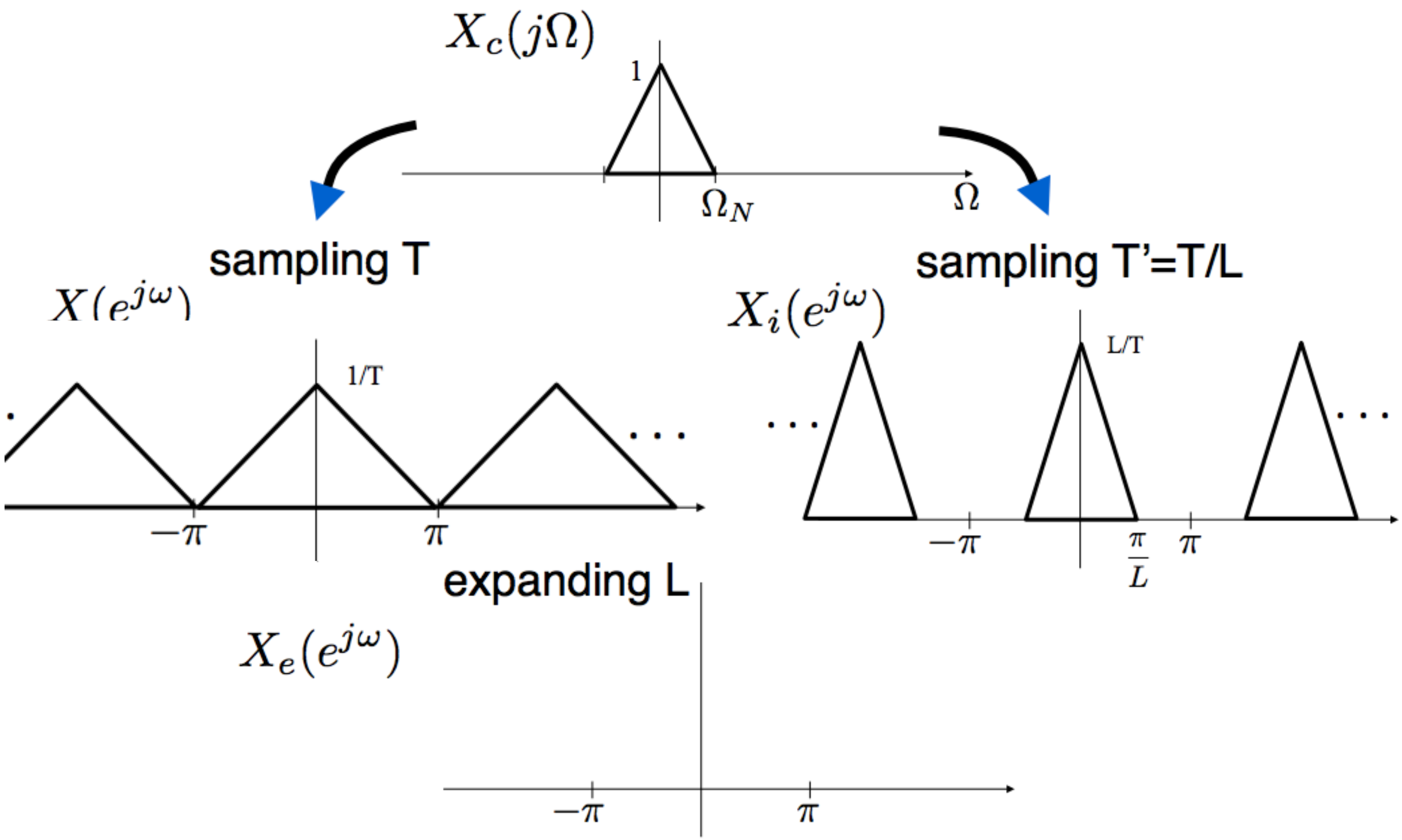
Example



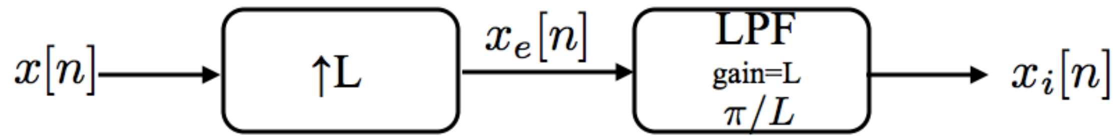
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example

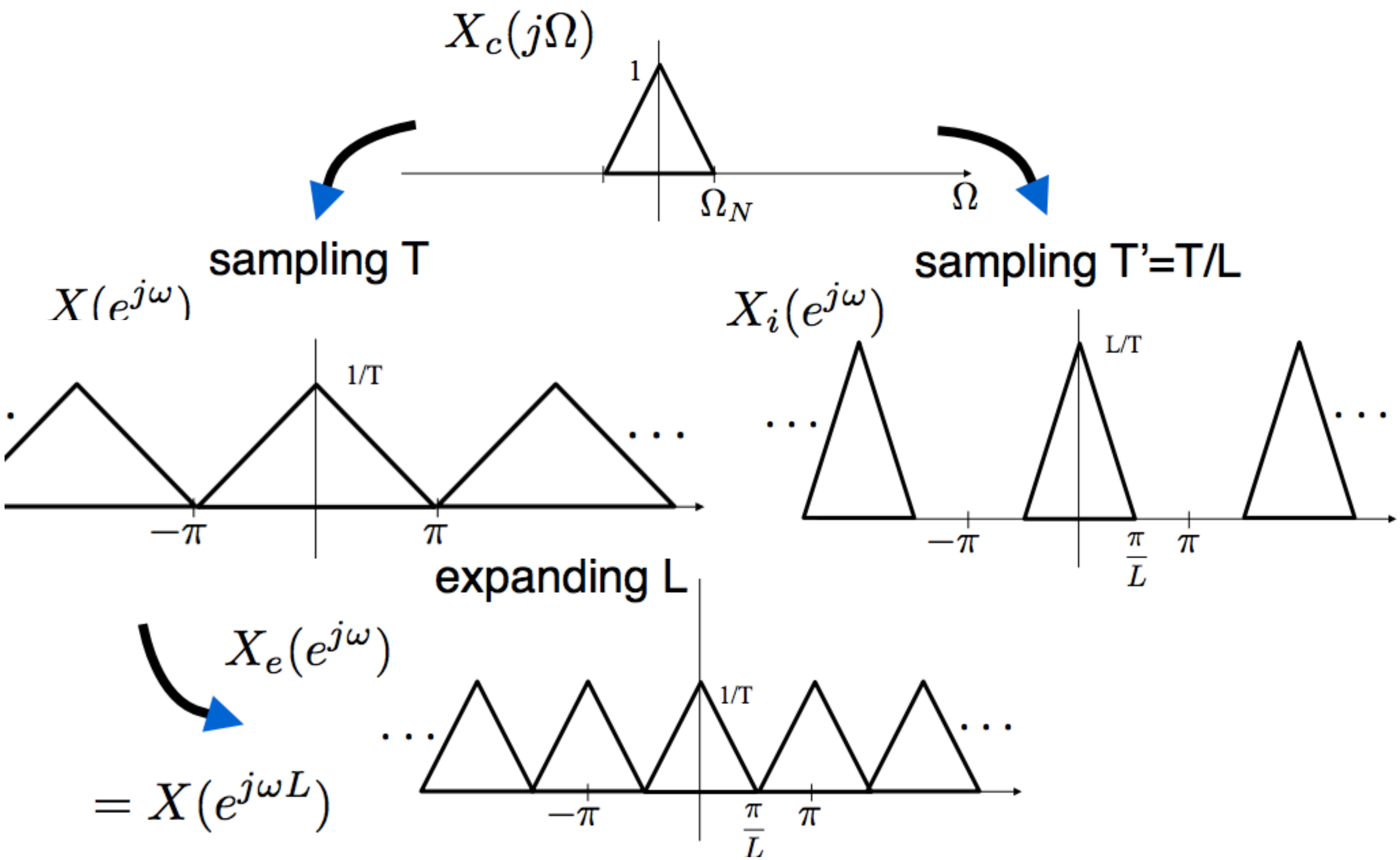

$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
sampling T'=T/L
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example

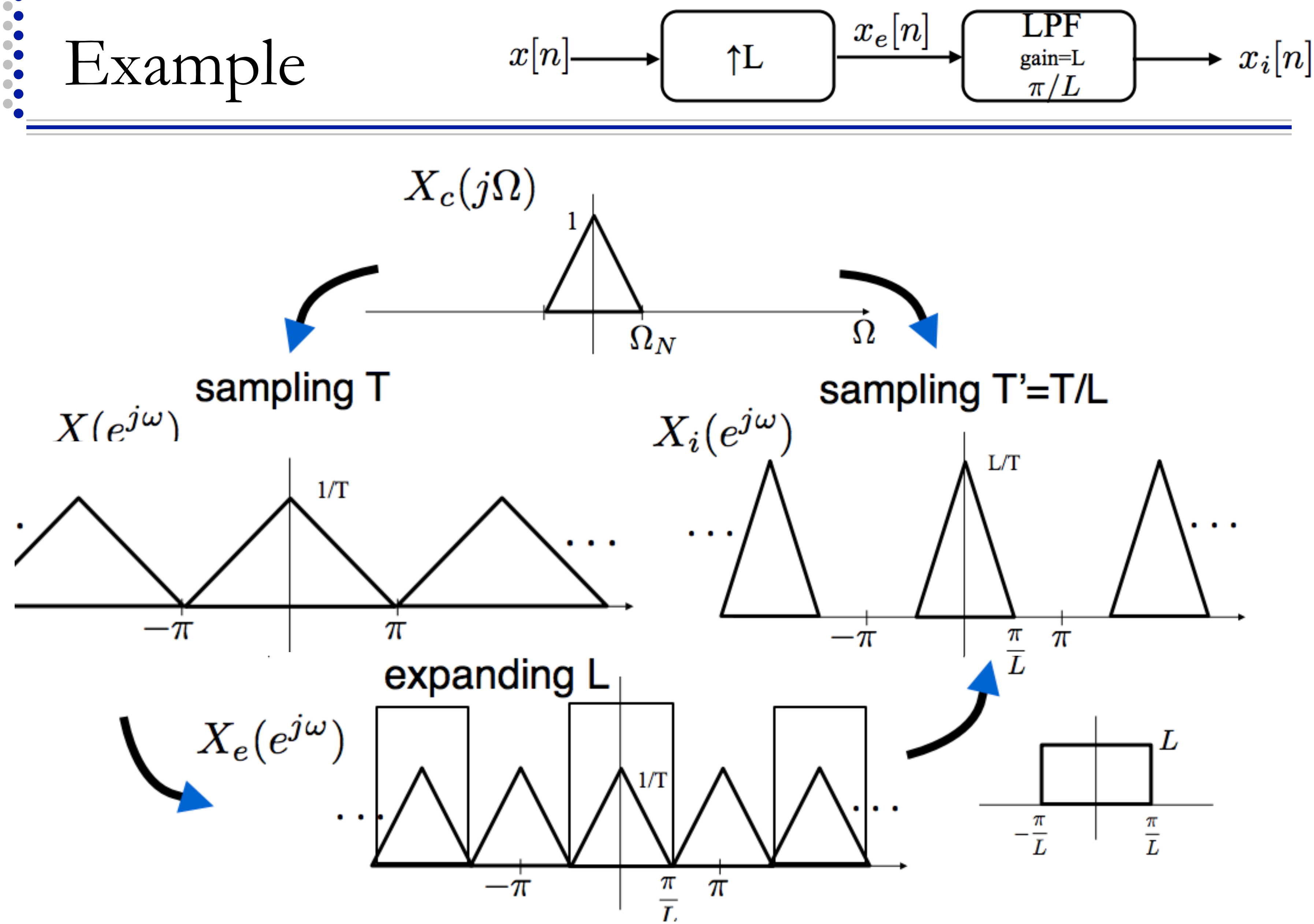


$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$



$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$\frac{\pi}{L}$
expanding L
$X_e(e^{j\omega})$
$= X(e^{j\omega L})$

Example



$x[n]$
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$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$
$X_c(j\Omega)$
1
Ω_N
Ω
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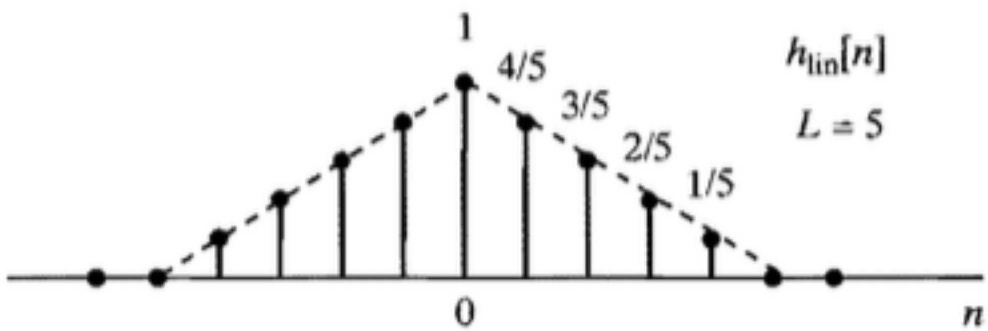
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Practical Interpolation

- Interpolate with simple, practical filters
  - Linear interpolation – samples between original samples fall on a straight line connecting the samples
    - Convolve with triangle instead of sinc

$$h_{\text{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \le L, \\ 0, & \text{otherwise,} \end{cases}$$



Practical Interpolation

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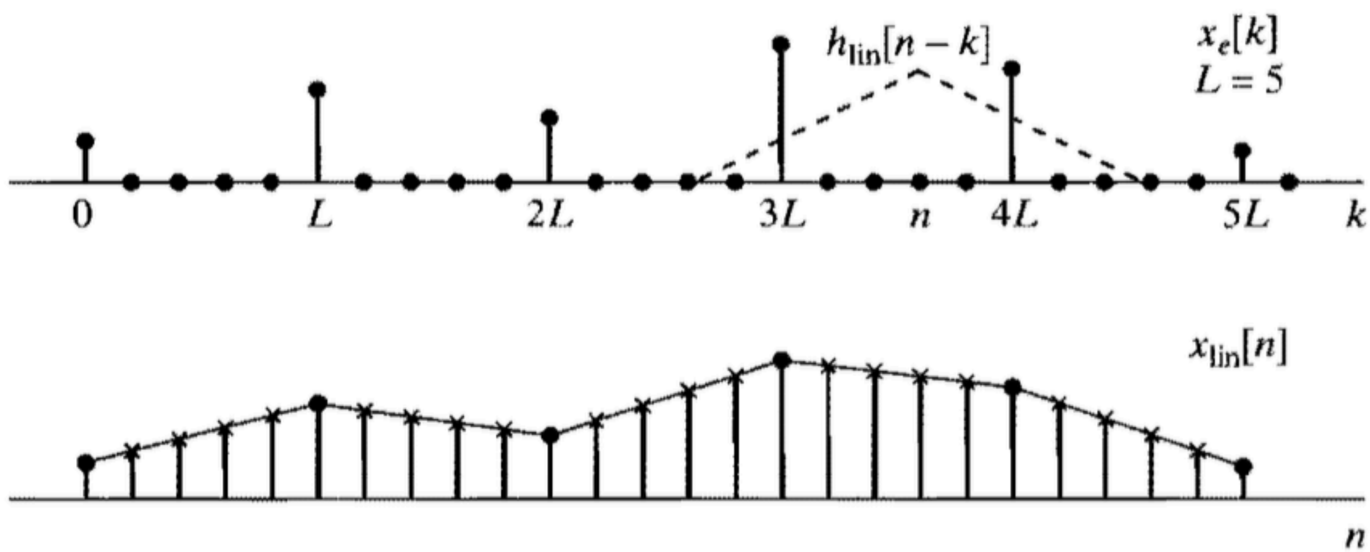
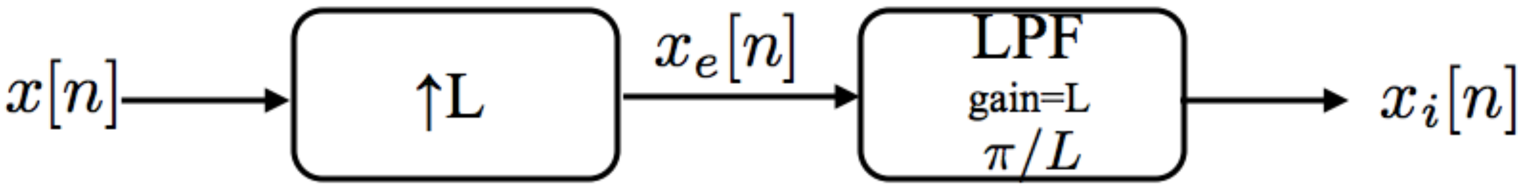
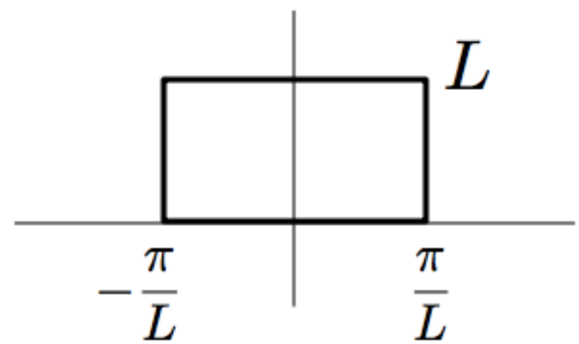


Frequency Domain Interpretation

$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$

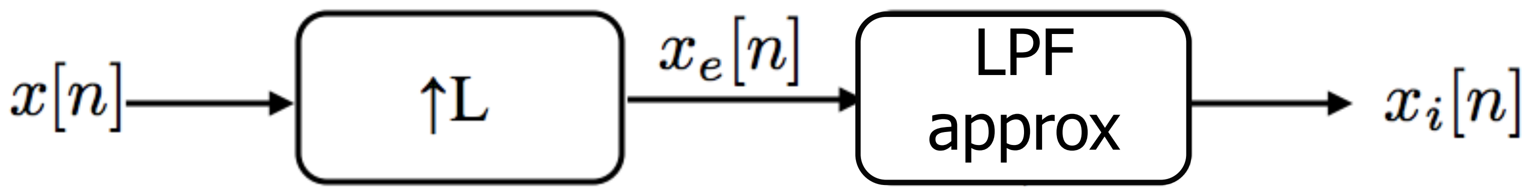


sinc(n/L) **DTFT** ⇒

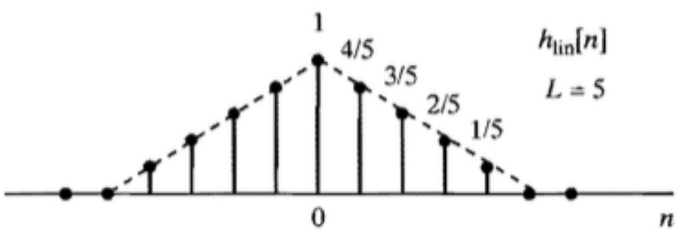


Linear Interpolation -- Frequency Domain

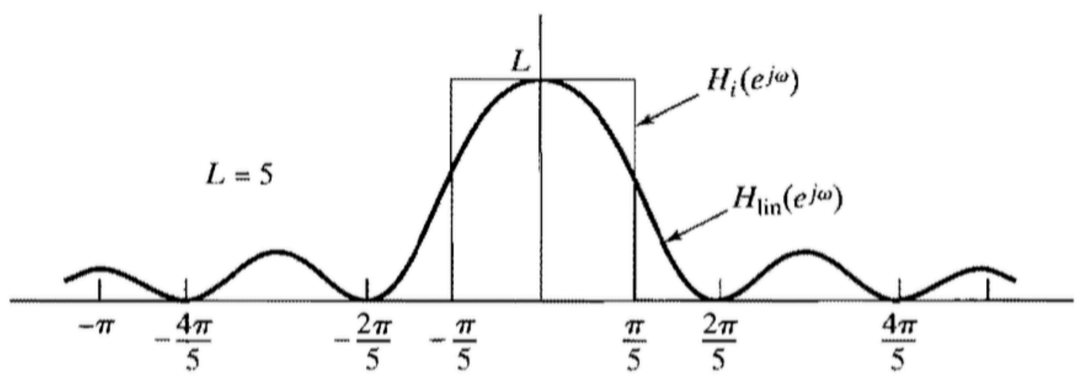
$$x_i[n] = x_e[n] * h_{lin}[n]$$



$$h_{\text{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \le L, \\ 0, & \text{otherwise}, \end{cases}$$



DTFT ⇒



Linear Interpolation -- Frequency Domain

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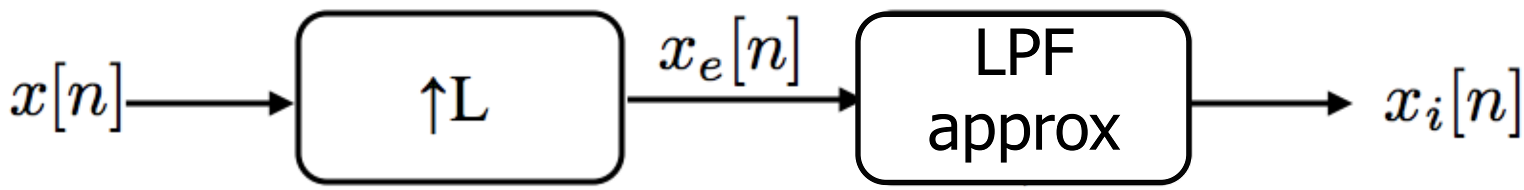

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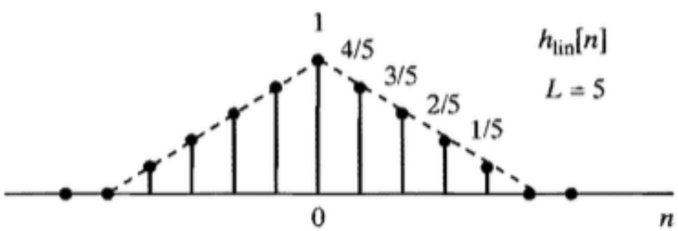
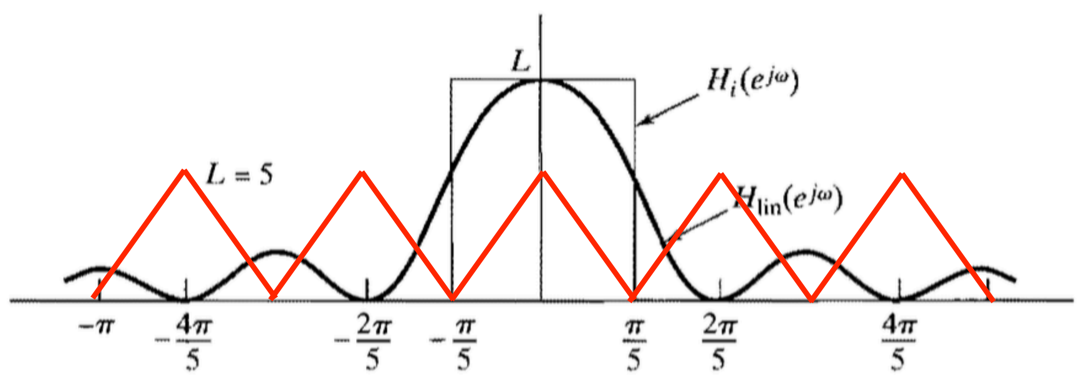


DTFT ⇒



Linear Interpolation -- Frequency Domain

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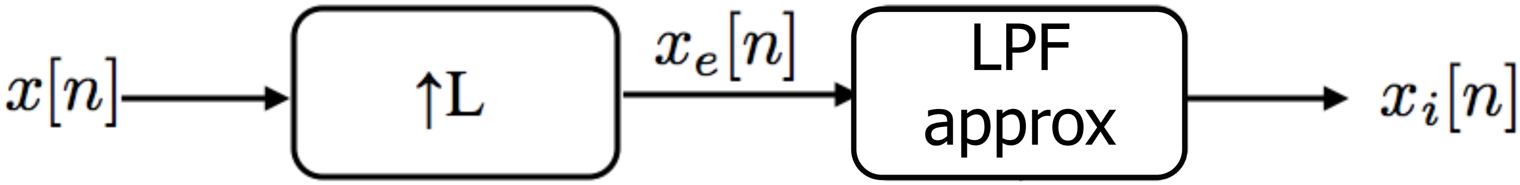

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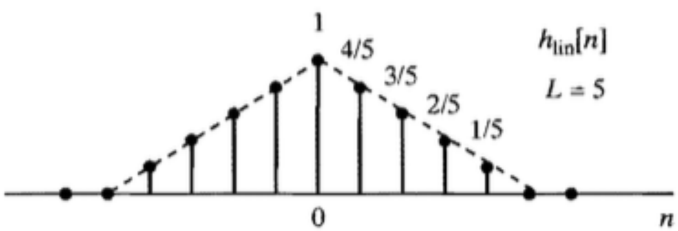
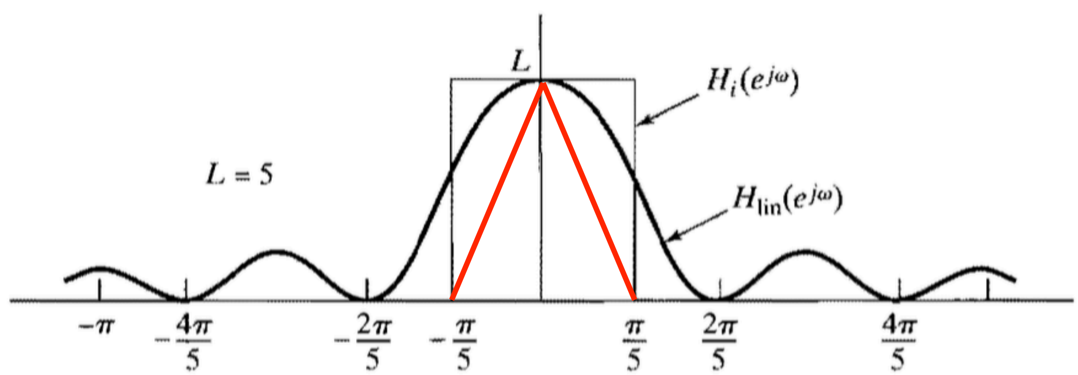


DTFT ⇒



Linear Interpolation -- Frequency Domain

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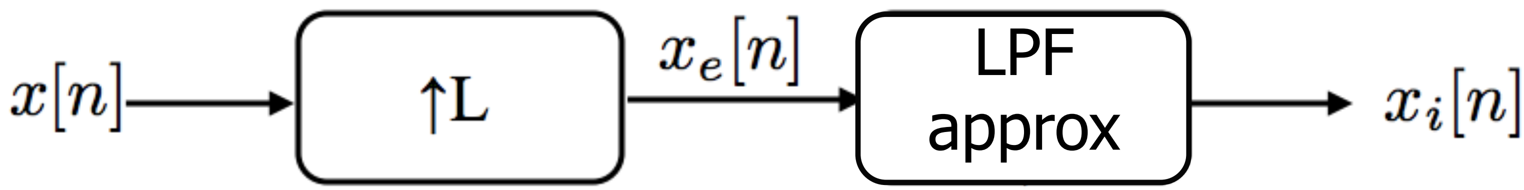

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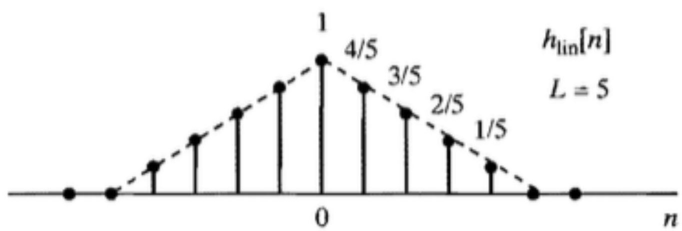
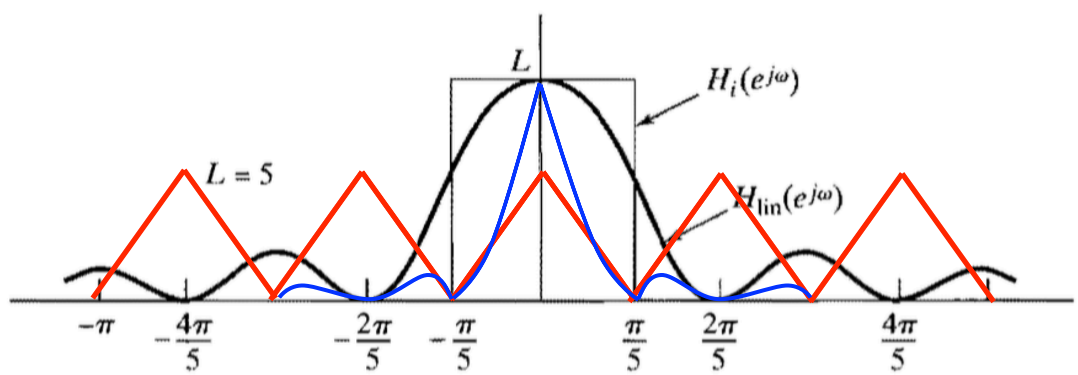


DTFT ⇒



Big Ideas

- CT processing of DT signals
  - Allows for interpretation of DT systems
- Downsampling
  - Like a C/D converter
- Upsampling
  - Like a D/C converter
- Practical Interpolation
  - Linear interpolation
  - Approximate sinc function with triangle

Admin

- HW 4 due Friday