University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2018 | Midterm | Tuesday, March 13

- 3 Problems with point weightings shown. All 3 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Total	

Mean: 68.2, Stdev:16.6

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

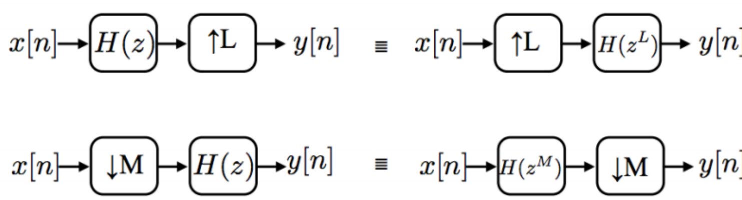
Z-Transform Equations:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$
Downsampling by M (↓M): $X_{down} = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

Interchange Identities:



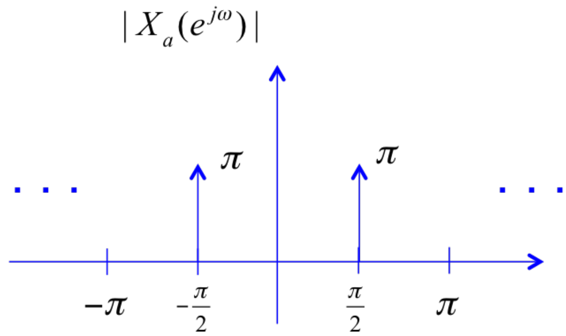
1. (20pts) Sampling.

   (a) Consider the continuous-time signal $x_a(t)$ given below. A discrete-time signal is created by sampling $x_a(t)$ according to $x_a[n] = x_a(nT_s)$ with $F_s = \frac{1}{T_s} = 4W$. Plot the magnitude of the DTFT of $x_a[n]$, $|X_a(e^{j\omega})|$, over $-\pi < \omega < \pi$. Show all work.

$$x_a(t) = cos(2\pi W t) \tag{1}$$

$$x_a[n] = x_a(nT_s) = cos(2\pi W \cdot n \cdot \frac{1}{4W}) \tag{2}$$

$$x_a[n] = cos(\frac{\pi}{2}n) \tag{3}$$

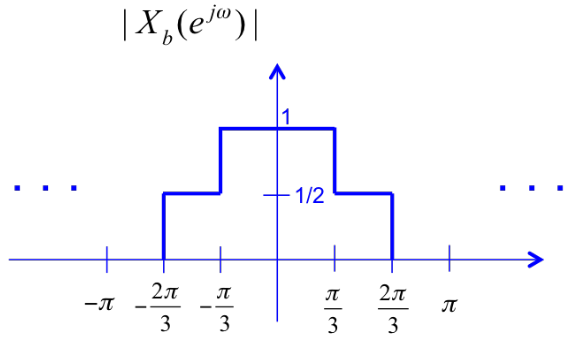


(b) Consider the continuous-time signal $x_b(t)$ given below. A discrete-time signal is created by sampling $x_b(t)$ according to $x_b[n] = x_b(nT_s)$ with $F_s = \frac{1}{T_s} = 3W$. Plot the magnitude of the DTFT of $x_b[n]$, $|X_b(e^{j\omega})|$, over $-\pi < \omega < \pi$. Show all work.

$$x_b(t) = T_s \frac{1}{2}\left(\frac{sin(2\pi W t)}{\pi t} + \frac{sin(2\pi \frac{W}{2} t)}{\pi t}\right) \tag{4}$$

$$x_b[n] = x_b(nT_s) = \frac{1}{3W} \cdot \frac{1}{2}\left(\frac{sin(2\pi W \frac{n}{3W})}{\pi \frac{n}{3W}} + \frac{sin(2\pi \frac{W}{2} \frac{n}{3W})}{\pi \frac{n}{3W}}\right) \tag{5}$$

$$x_b[n] = \frac{1}{2}\left(\frac{sin(\frac{2\pi}{3} n)}{\pi n} + \frac{sin(\frac{\pi}{3} n)}{\pi n}\right) \tag{6}$$



2. (50pts) Discrete-time Systems. Consider a discrete-time system characterized by the difference equation below where $j = e^{j\frac{\pi}{2}}$.

$$y[n] = j \cdot y[n-1] + x[n] - x[n-4] \tag{7}$$

(a) Is this an LTI system?
Yes. It's a linear constant coefficient difference equation.

(b) Find the transfer function, $H(z) = \frac{Y(z)}{X(z)}$, of the system.

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1-z^{-4}}{1-jz^{-1}} = \frac{z^4-1}{z^3(z-j)} && (8) \\ &= \frac{(z-1)(z+1)(z-j)(z+j)}{z^3(z-j)} && (9) \\ &= \frac{(z-1)(z+1)(z+j)}{z^3} && (10) \end{aligned}$$

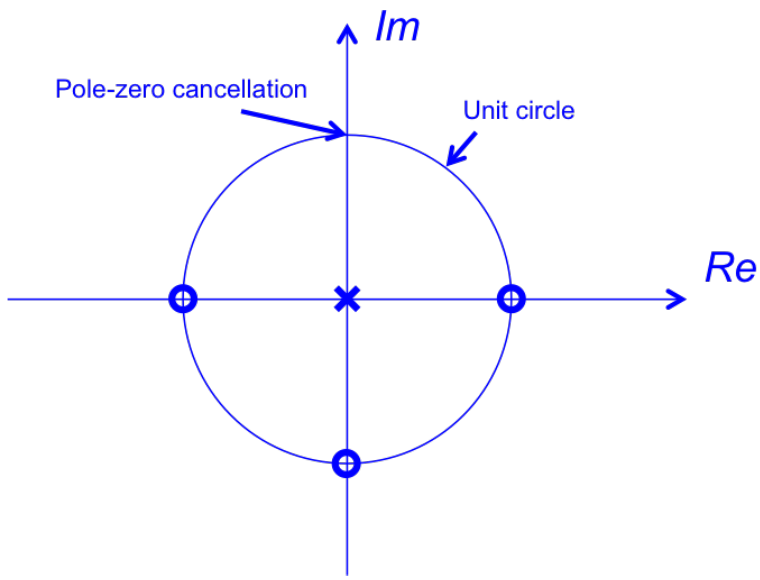
(c) Find the impulse response, $h[n]$. HINT: This is a FIR system.

$$\begin{aligned} H(z) &= \frac{(z-1)(z+1)(z+j)}{z^3} && (11) \\ &= z^{-3}(z-1)(z+1)(z+j) && (12) \\ &= 1 + jz^{-1} - z^{-2} - jz^{-3} && (13) \\ & && (14) \end{aligned}$$

h[0]=1, h[1]=j, h[2]=-1, h[3]=-j, h[n]=0 for all other n.

$$h[n] = e^{j\frac{\pi}{2}n}(u[n] - u[n-4]) \tag{15}$$

(d) Draw the pole zero diagram for this system.



(e) For this same system, write an equivalent difference equation that only depends on the current input and a finite number of past inputs.
Use $h[n] = \{1, j, -1, -j\}$

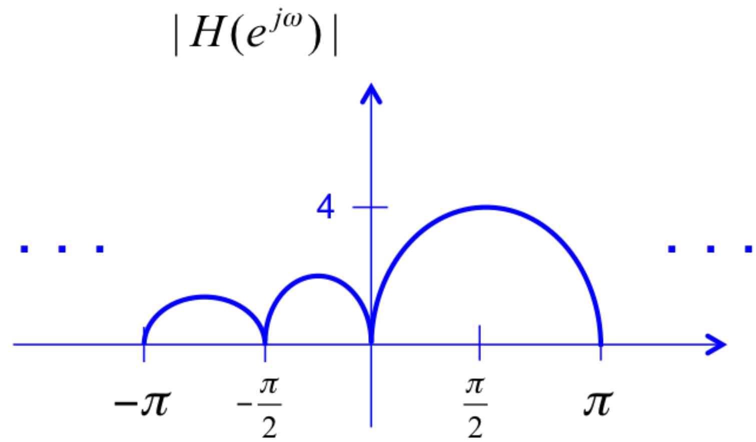
$$\begin{aligned} y[n] &= x[n] * h[n] && (16) \\ &= \sum_{m=-\infty}^{\infty} x[n-m]h[m] && (17) \\ &= x[n] + jx[n-1] - x[n-2] - jx[n-3] && (18) \end{aligned}$$

(f) Plot a rough sketch of the magnitude of the frequency response, $|H(e^{j\omega})|$, over $-\pi < \omega < \pi$.

Using the impulse response and Table 2.3 entries 9 and 10 with $M = 3$ and $\omega_0 = \frac{\pi}{2}$:

$$H(e^{j\omega}) = \frac{sin((\omega - \frac{\pi}{2})\frac{4}{2})}{sin((\omega - \frac{\pi}{2})\frac{1}{2})} e^{-j(\omega - \frac{\pi}{2})\frac{3}{2}} \tag{19}$$

Periodic sinc function with zero crossings at multiples of $\pi/2$ and main/center lobe at $\omega = \frac{\pi}{2}$ with magnitude of 4.



(g) Determine the overall output $y[n]$ of the system when the input is the sum of infinite-length sine waves, $x[n] = 3 + 2^{j\frac{\pi}{2}n} + \sqrt{2}e^{-j\frac{\pi}{2}n} + e^{j\pi}$.

There was a typo on the original exam. Should be $x[n] = 3 + 2e^{j\frac{\pi}{2}n} + \sqrt{2}e^{-j\frac{\pi}{2}n} + e^{j\pi n}$

Since this is an LTI system:

$$\begin{aligned} y[n] &= H(e^{j\cdot 0})3 + H(e^{j\cdot\frac{\pi}{2}})2e^{j\frac{\pi}{2}n} + H(e^{-j\cdot\frac{\pi}{2}})\sqrt{2}e^{-j\frac{\pi}{2}n} + H(e^{j\cdot\pi})e^{j\pi n} && (20) \\ y[n] &= H(e^{j\cdot\frac{\pi}{2}})2e^{j\frac{\pi}{2}n} && (21) \\ y[n] &= 4 \cdot 2e^{j\frac{\pi}{2}n} && (22) \\ & && (23) \end{aligned}$$

3. (30 pts) Resampling and Filtering. Determine the impulse response $h_{eq}[n]$ in Figure 1(b) so that the I/O relationship of the system in Figure 1(b) is exactly the same as the I/O relationship of the system in Figure 1(a). You should work in the frequency domain and show all work. Plot the magnitude of the DTFT of $h_{eq}[n]$, $|H_{eq}(e^{j\omega})|$, over $-\pi < \omega < \pi$.

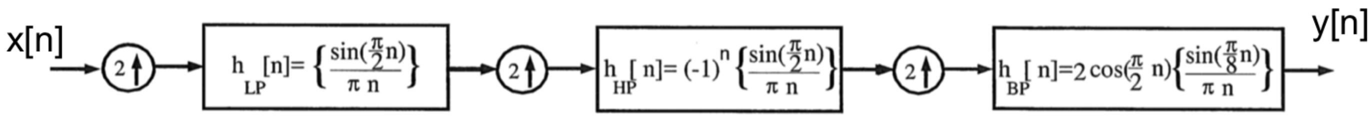


Figure 1(a).

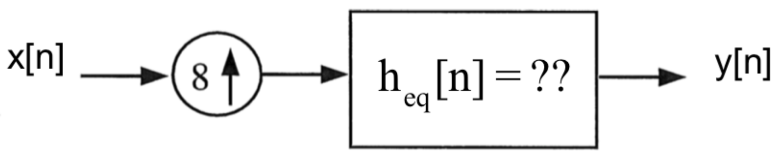
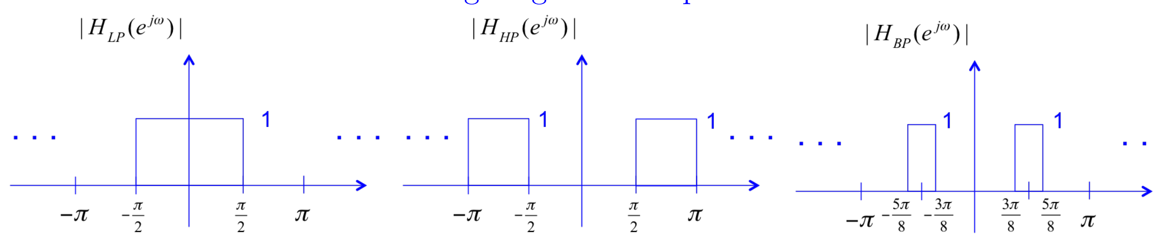
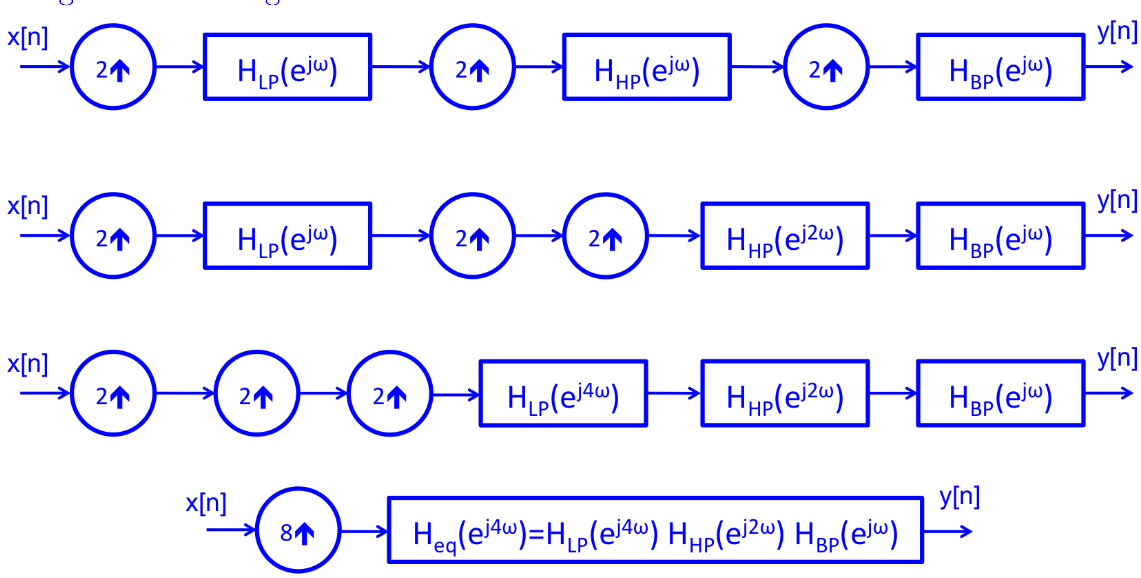


Figure 1(b).

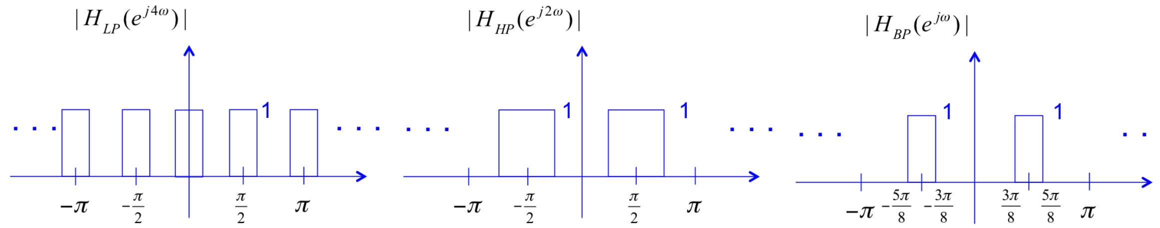
The three filters have the following magnitude responses:



Using the interchange identities:



The new expanded filters and the original bandpass have the following responses:



Therefore the equivalent filter is just the same as the bandpass filter: