University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2019 | Final | Monday, May 13

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed.
- Two two-sided 8.5x11 cheat sheet allowed.
- Sign Code of Academic Integrity statement at back of exam book.

Name:

Grade:

Q1	
Q2	
Q3	
Q4	
Total	

Transform Pairs/Properties and Formulas

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k) + \pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega}) \quad$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1}\cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

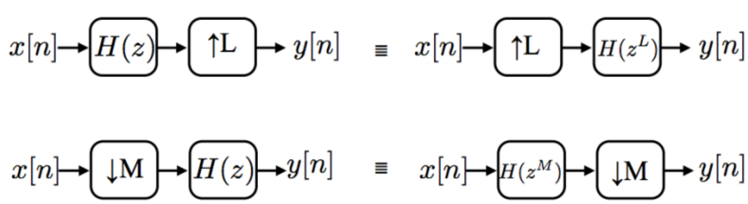
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M}-\frac{2\pi}{M}i)})$

Interchange Identities:



DFT Equations:

N-point DFT of $\{x[n], n = 0, 1, ..., N-1\}$ is $X[k] = \sum\limits_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, for $k = 0, 1, ..., N-1$

N-point IDFT of $\{X[k], k = 0, 1, ..., N-1\}$ is $x[n] = \frac{1}{N}\sum\limits_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$, for $n = 0, 1, ..., N-1$

1. (30 points) Consider the stable LTI system with transfer function:

$$H(z) = \frac{1 - 4z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} \tag{1}$$

The system function $H(z)$ can be factored such that:

$$H(z) = H_{min}(z)H_{ap}(z) \tag{2}$$

where $H_{min}(z)$ is a minimum phase system, and $H_{ap}(z)$ is an all pass system, i.e.,

$$|H_{ap}(z)| = 1 \tag{3}$$

(a) Draw the pole-zero diagram for $H(z)$, and indicate the region of convergence on your diagram.

(b) For both the minimum phase system, $H_{min}(z)$, and all pass system, $H_{ap}(z)$, write expressions for the transfer functions including the region of convergence for each transfer function.

$H_{min}(z)$: $\boxed{}$ ROC: $\boxed{}$

$H_{ap}(z)$: $\boxed{}$ ROC: $\boxed{}$

(c) Draw the pole-zero diagrams for both the minimum phase system, $H_{min}(z)$, and all pass system, $H_{ap}(z)$. Indicate the regions of convergence on your diagrams.

2. (20 points)

(a) Determine the impulse response $h[n]$ in Figure 1(b) so that the I/O relationship of the system in Figure 1(b) is exactly the same as the I/O relationship of the system in Figure 1(a). Hint: Analyze the system of Figure 1(a) in the frequency domain with the interchange identities.

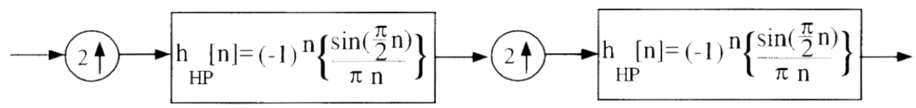


Figure 1(a)

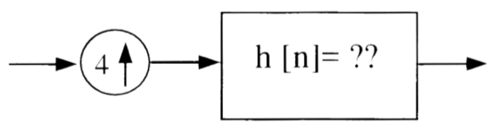


Figure 1(b)

$h[n]$:

(b) Determine the the impulse response $h_{eq}[n]$ in Figure 2(b) so that the I/O relationship of the system in Figure 2(b) is exactly the same as the I/O relationship of the system in Figure 2(a). Hint: Analyze the system of Figure 2(a) in the time domain with the interchange identities.

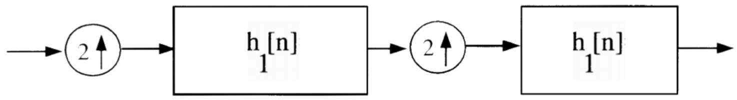


Figure 2(a)

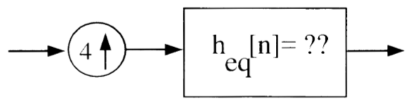
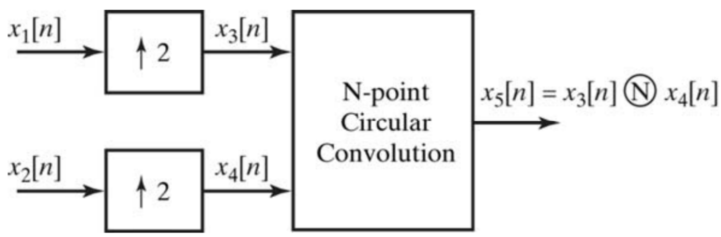


Figure 2(b)

where $h_1[n] = \delta[n] - \delta[n-1]$.

$h_{eq}[n]$:

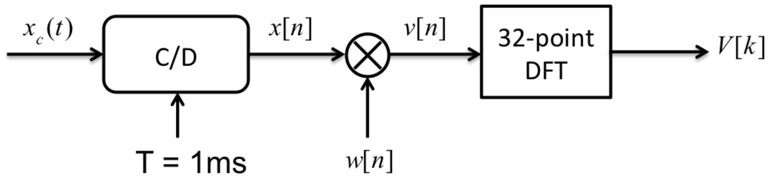
3. (20 points) In the system shown in the figure below, $x_1[n]$ and $x_2[n]$ are both causal, 32-point sequences (that is, they are both zero outside the interval $0 \leq n \leq 31$). $y[n]$ denotes the linear convolution of $x_1[n]$ and $x_2[n]$, such that $y[n] = x_1[n] * x_2[n]$.



(a) Determine **all values** of N for which all the values of $y[n]$ can be completely recovered from $x_5[n]$. Explain your answer. (Hint: Think carefully how long $y[n]$, $x_3[n]$ and $x_4[n]$ are.)

(b) Specify explicitly how you can recover $y[n]$ from $x_5[n]$ for the **smallest** value of N which you determined in part (a).

4. (30 points) A system for the discrete-time spectral analysis of continuous-time signals is shown below



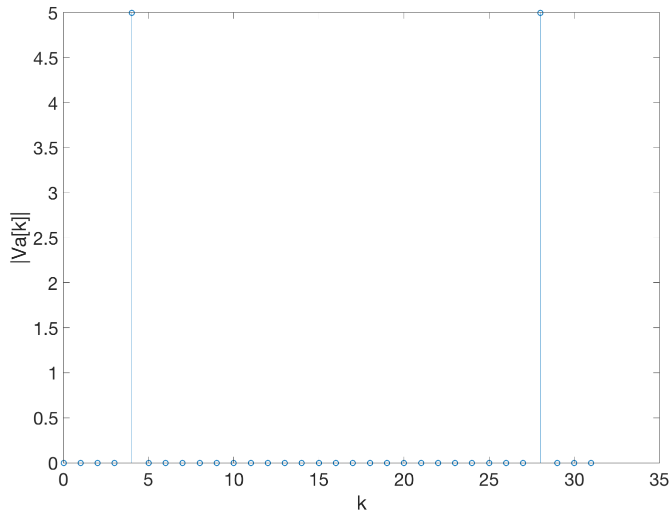
$w[n]$ is a rectangular window of length 32:

$$w[n] = \begin{cases} 1/32 & \text{if } 0 \leq n \leq 31 \\ 0 & \text{else} \end{cases} \tag{4}$$

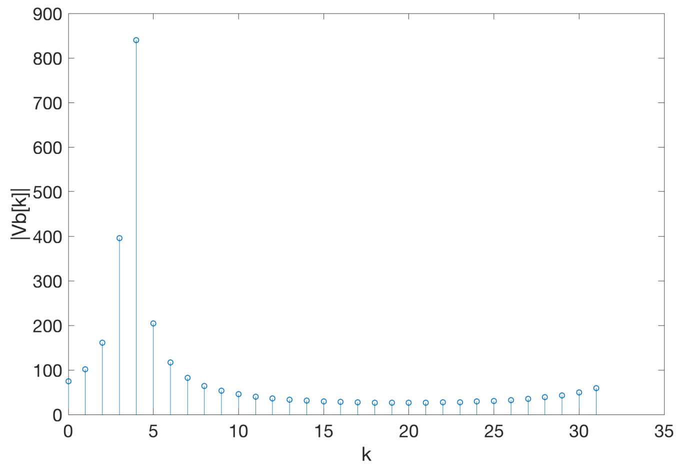
Listed below are ten continuous-time signals which could possibly be $x_c(t)$.

$x_1(t) = 1000cos(230\pi t)$	$x_6(t) = 1000e^{j(250)\pi t}$
$x_2(t) = 1000cos(115\pi t)$	$x_7(t) = 10cos(250\pi t)$
$x_3(t) = 10e^{j(460)\pi t}$	$x_8(t) = 1000cos(218.75\pi t)$
$x_4(t) = 1000e^{j(230)\pi t}$	$x_9(t) = 10e^{j(200)\pi t}$
$x_5(t) = 10e^{j(230)\pi t}$	$x_{10}(t) = 1000e^{j(187.5)\pi t}$

(a) Indicate which signal(s) above could have been the input $x_c(t)$ for the plot of $|V_a[k]|$. Explain your reasoning for full credit.



(b) Indicate which signal(s) above could have been the input $x_c(t)$ for the plot of $|V_b[k]|$. Explain your reasoning for full credit.



(c) Indicate which signal(s) above could have been the input $x_c(t)$ for the plot of $|V_c[k]|$. Explain your reasoning for full credit.