University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2019 | Final | Monday, May 13

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed.
- Two two-sided 8.5x11 cheat sheet allowed.
- Sign Code of Academic Integrity statement at back of exam book.

Name:

Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Total	Mean: 69.1, Stdev: 13.8

Transform Pairs/Properties and Formulas

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega}) \quad$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

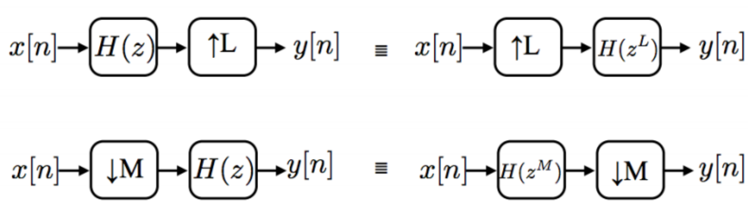
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M}-\frac{2\pi}{M}i)})$

Interchange Identities:



DFT Equations:

N-point DFT of $\{x[n], n = 0, 1, ..., N-1\}$ is $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, for $k = 0, 1, ..., N-1$

N-point IDFT of $\{X[k], k = 0, 1, ..., N-1\}$ is $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$, for $n = 0, 1, ..., N-1$

1. (30 points) Consider the stable LTI system with transfer function:

$$H(z) = \frac{1 - 4z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} \tag{1}$$

The system function $H(z)$ can be factored such that:

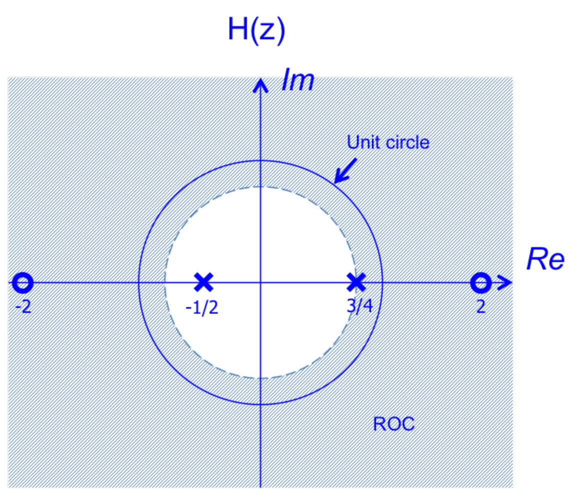
$$H(z) = H_{min}(z)H_{ap}(z) \tag{2}$$

where $H_{min}(z)$ is a minimum phase system, and $H_{ap}(z)$ is an all pass system, i.e.,

$$|H_{ap}(z)| = 1 \tag{3}$$

(a) Draw the pole-zero diagram for $H(z)$, and indicate the region of convergence on your diagram.

$$H(z) = \frac{\left(1 - 2z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 - \frac{3}{4}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$$

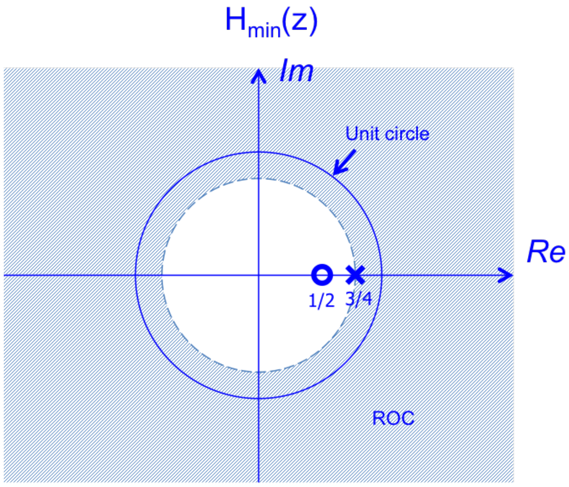


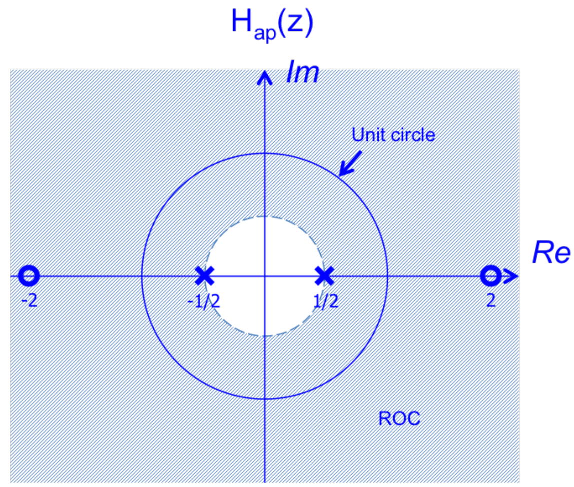
(b) For both the minimum phase system, $H_{min}(z)$, and all pass system, $H_{ap}(z)$, write expressions for the transfer functions including the region of convergence for each transfer function.

$H_{min}(z)$: $\boxed{H_{min}(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{3}{4}z^{-1}}}$ ROC: $\boxed{|z| > \frac{3}{4}}$

$H_{ap}(z)$: $\boxed{H_{ap}(z) = \frac{\left(1-2z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}}$ ROC: $\boxed{|z| > \frac{1}{2}}$

(c) Draw the pole-zero diagrams for both the minimum phase system, $H_{min}(z)$, and all pass system, $H_{ap}(z)$. Indicate the regions of convergence on your diagrams.





2. (20 points)

(a) Determine the impulse response $h[n]$ in Figure 1(b) so that the I/O relationship of the system in Figure 1(b) is exactly the same as the I/O relationship of the system in Figure 1(a). Hint: Analyze the system of Figure 1(a) in the frequency domain with the interchange identities.

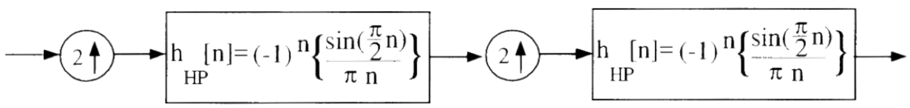


Figure 1(a)

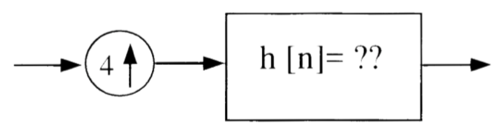
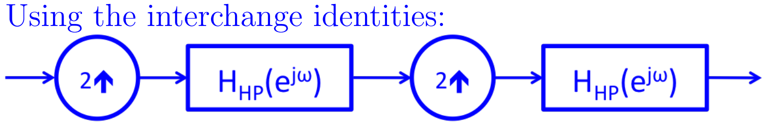
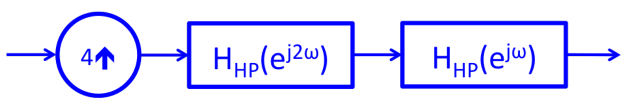


Figure 1(b)

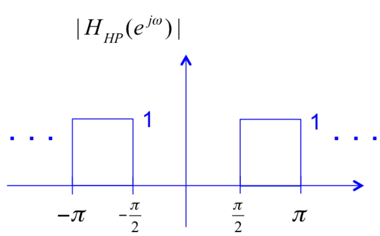
$h[n]$: $\boxed{h_{eq}[n] = 2\frac{sin\left(\frac{\pi}{8}n\right)}{\pi n}cos\left(\frac{5\pi}{8}n\right)}$

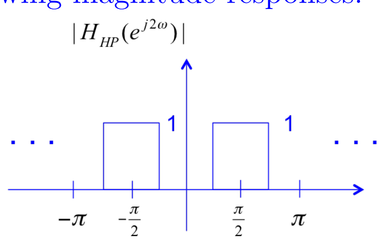
Using the interchange identities:



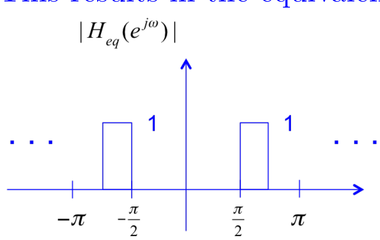


The two filters have the following magnitude responses:





This results in the equivalent filter having the following magnitude response:



(b) Determine the the impulse response $h_{eq}[n]$ in Figure 2(b) so that the I/O relationship of the system in Figure 2(b) is exactly the same as the I/O relationship of the system in Figure 2(a). Hint: Analyze the system of Figure 2(a) in the time domain with the interchange identities.

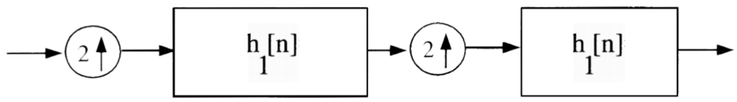


Figure 2(a)

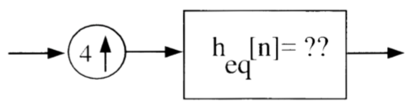
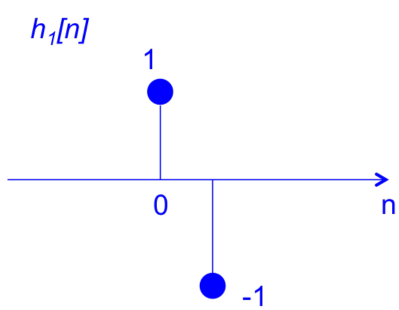


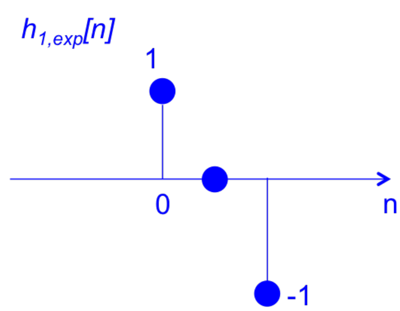
Figure 2(b)

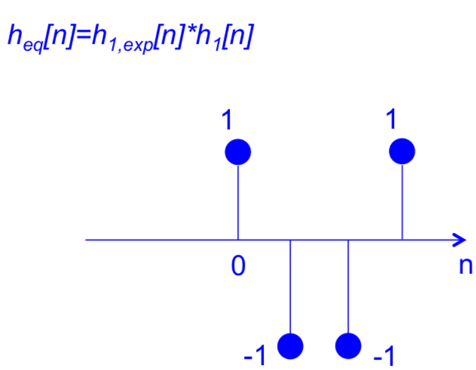
where $h_1[n] = \delta[n] - \delta[n-1]$.

$h_{eq}[n]$: $\boxed{h_{eq}[n] = \delta[n] - \delta[n-1] - \delta[n-2] + \delta[n-3]}$

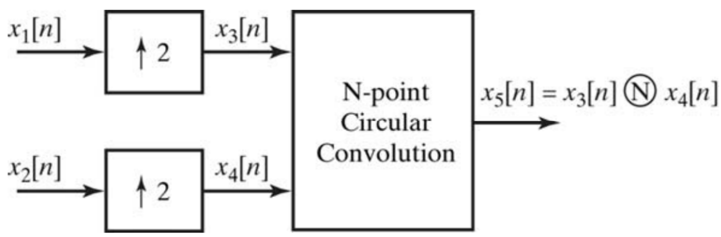
We have the same interchange identity as above, where we interchange the first filter, $h_1[n]$, and upsampling block with the upsampling block and expanded filter, $h_{1,exp}[n]$. The expanded filter has an upsampled impulse response and therefore the impulse responses can be drawn as:







3. (20 points) In the system shown in the figure below, $x_1[n]$ and $x_2[n]$ are both causal, 32-point sequences (that is, they are both zero outside the interval $0 \leq n \leq 31$). $y[n]$ denotes the linear convolution of $x_1[n]$ and $x_2[n]$, such that $y[n] = x_1[n] * x_2[n]$.



(a) Determine **all values** of N for which all the values of $y[n]$ can be completely recovered from $x_5[n]$. Explain your answer. (Hint: Think carefully how long $y[n]$, $x_3[n]$ and $x_4[n]$ are.)

The signals $x_3[n]$ and $x_4[n]$ are each length 63. In that sense, the linear convolution has length $63 + 63 - 1 = 125$, and can be obtained with $N \geq 125$. Every other sample of that linear convolution is 0, as can be seen using the flip-and-slide interpretation of the linear convolution of $x_3[n]$ and $x_4[n]$, which both have values of 0 for every other sample. The nonzero values correspond to values of $y[n]$.

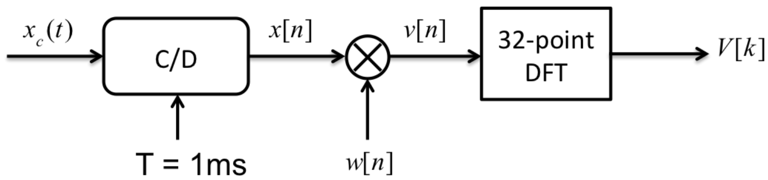
However, since $y[n]$ is only length $32+32-1 = 63$, we may be able to use smaller values of N in the circular convolution, though we need $N \geq 63$. Suppose we make N an odd number such that $63 \leq N < 125$. Then first, the aliased samples from circular convolution will fall in the positions of the 0 values in $x_5[n]$. Second, when these values are nonzero, they correspond to values of $y[n]$. Third, since N is larger than 63 we get all samples of $y[n]$.

Therefore we can use any odd number for N such that $63 \leq N < 125$, and any odd or even N such that $N \geq 125$.

(b) Specify explicitly how you can recover $y[n]$ from $x_5[n]$ for the **smallest** value of N which you determined in part (a).

For $N = 63$, start with $x_5[0]$ and select every other sample. These are $y[0]$ through $y[31]$. Then start with $x_5[1]$ and select every other sample. These are $y[32]$ through $y[62]$. This interleaving is described as:
$\{x_5[0], x_5[1], x_5[2], x_5[3], x_5[4], ...x_5[62]\} = \{y[0], y[32], y[1], y[33], y[2], ...x_5[62]\}$

4. (30 points) A system for the discrete-time spectral analysis of continuous-time signals is shown below



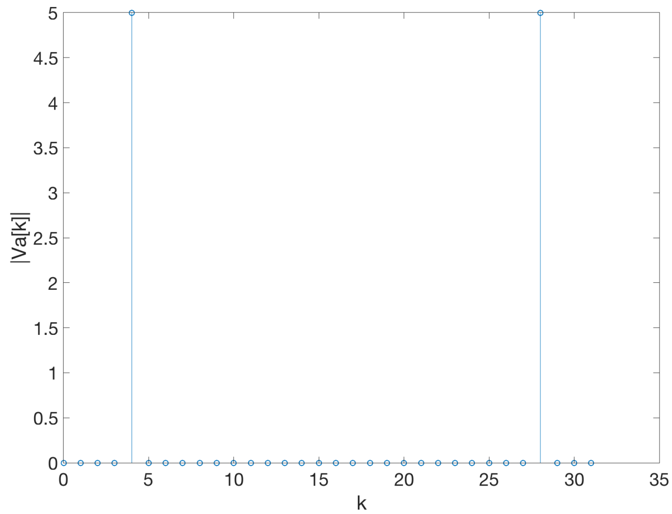
$w[n]$ is a rectangular window of length 32:

$$w[n] = \begin{cases} 1/32 & \text{if } 0 \leq n \leq 31 \\ 0 & \text{else} \end{cases} \tag{4}$$

Listed below are ten continuous-time signals which could possibly be $x_c(t)$.

$x_1(t) = 1000cos(230\pi t)$	$x_6(t) = 1000e^{j(250)\pi t}$
$x_2(t) = 1000cos(115\pi t)$	$x_7(t) = 10cos(250\pi t)$
$x_3(t) = 10e^{j(460)\pi t}$	$x_8(t) = 1000cos(218.75\pi t)$
$x_4(t) = 1000e^{j(230)\pi t}$	$x_9(t) = 10e^{j(200)\pi t}$
$x_5(t) = 10e^{j(230)\pi t}$	$x_{10}(t) = 1000e^{j(187.5)\pi t}$

(a) Indicate which signal(s) above could have been the input $x_c(t)$ for the plot of $|V_a[k]|$. Explain your reasoning for full credit.



The graph shows two peaks, one corresponding to positive frequencies and one to negative frequencies (i.e in the upper half of the DFT). This means the input signal is a cosine signal. The two peaks occur at $k = 4$ and $k = 28$:

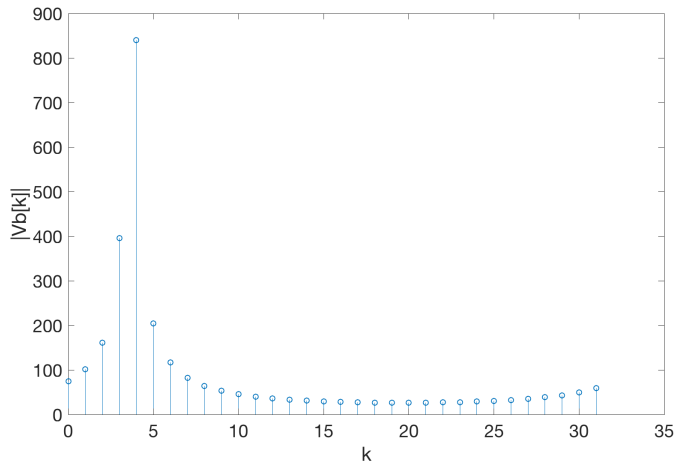
$$\omega|_{k=4} = \frac{2\pi}{N}(k) = \frac{2\pi}{32}(4) = \frac{\pi}{4}$$

$$\omega|_{k=28} = \frac{2\pi}{N}(k) = \frac{2\pi}{32}(28) = \frac{7\pi}{4} = -\frac{\pi}{4}$$

Additionally with the magnitude being less than 5, we know that $x_7(t)$ is the only candidate that fits. But we can check that the continuous time frequency matches as well:

$$\Omega = \frac{\omega}{T} \rightarrow \Omega|_{k=4} = \frac{\pi}{4} \cdot \frac{1}{10^{-3}} = 250\pi$$

(b) Indicate which signal(s) above could have been the input $x_c(t)$ for the plot of $|V_b[k]|$. Explain your reasoning for full credit.



The graph shows just one peak, so we can eliminate any cosine signals. This means the input signal is a complex sinusoid at just one frequency. Additionally the scale of the magnitude is close to 1000, so we can eliminate all but signals $x_4(t)$, $x_6(t)$, and $x_{10}(t)$. The continuous time frequency does not correspond to a freqeuncy $\omega_k = \frac{2\pi k}{32}$ because if it did we would only have one non-zero DFT value.

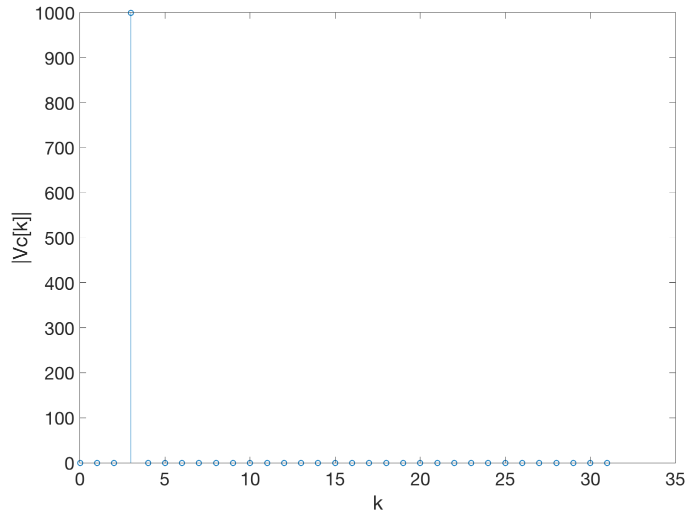
$$x_6(t) : \Omega = 250\pi \rightarrow \omega = \Omega T = 250\pi \cdot 10^{-3} = \frac{\pi}{4} \rightarrow k = 4$$

$$x_{10}(t) : \Omega = 187.5\pi \rightarrow \omega = \Omega T = 187.5\pi \cdot 10^{-3} = \frac{3\pi}{16} \rightarrow k = 3$$

The only signal left is $x_4(t)$ and we can see that the continuous time frequency does not correspond to a DT frequency that is an integer multiple of $\frac{2\pi}{32}$:

$$x_4(t) : \Omega = 230\pi \rightarrow \omega = \Omega T = 230\pi \cdot 10^{-3} = \frac{23\pi}{100} \rightarrow k \notin \mathbb{Z}$$

(c) Indicate which signal(s) above could have been the input $x_c(t)$ for the plot of $|V_c[k]|$. Explain your reasoning for full credit.



The graph shows just one peak, so we can eliminate any cosine signals. This means the input signal is a complex sinusoid at just one frequency. Additionally the scale of the magnitude is close to 1000, so we can eliminate all but signals $x_4(t)$, $x_6(t)$, and $x_{10}(t)$. This time continuous time frequency does correspond to a freqeuncy $\omega_k = \frac{2\pi k}{32}$ because we only have one non-zero DFT value at $k = 3$, which we know corresponds to $x_{10}(t)$

$$x_{10}(t) : \Omega = 187.5\pi \rightarrow \omega = \Omega T = 187.5\pi \cdot 10^{-3} = \frac{3\pi}{16} \rightarrow k = 3$$