ESE 531: Digital Signal Processing

Lec 10: February 19, 2019

Non-Integer and Multi-rate Sampling

Lecture Outline

- Review: Downsampling/Upsampling
- Interpolation
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations

Downsampling

- Definition: Reducing the sampling rate by an integer number

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x[nM]$$

$$x[n] = x_c(nT) \qquad x_d[n] = x_c(n\underbrace{MT}_{T'})$$

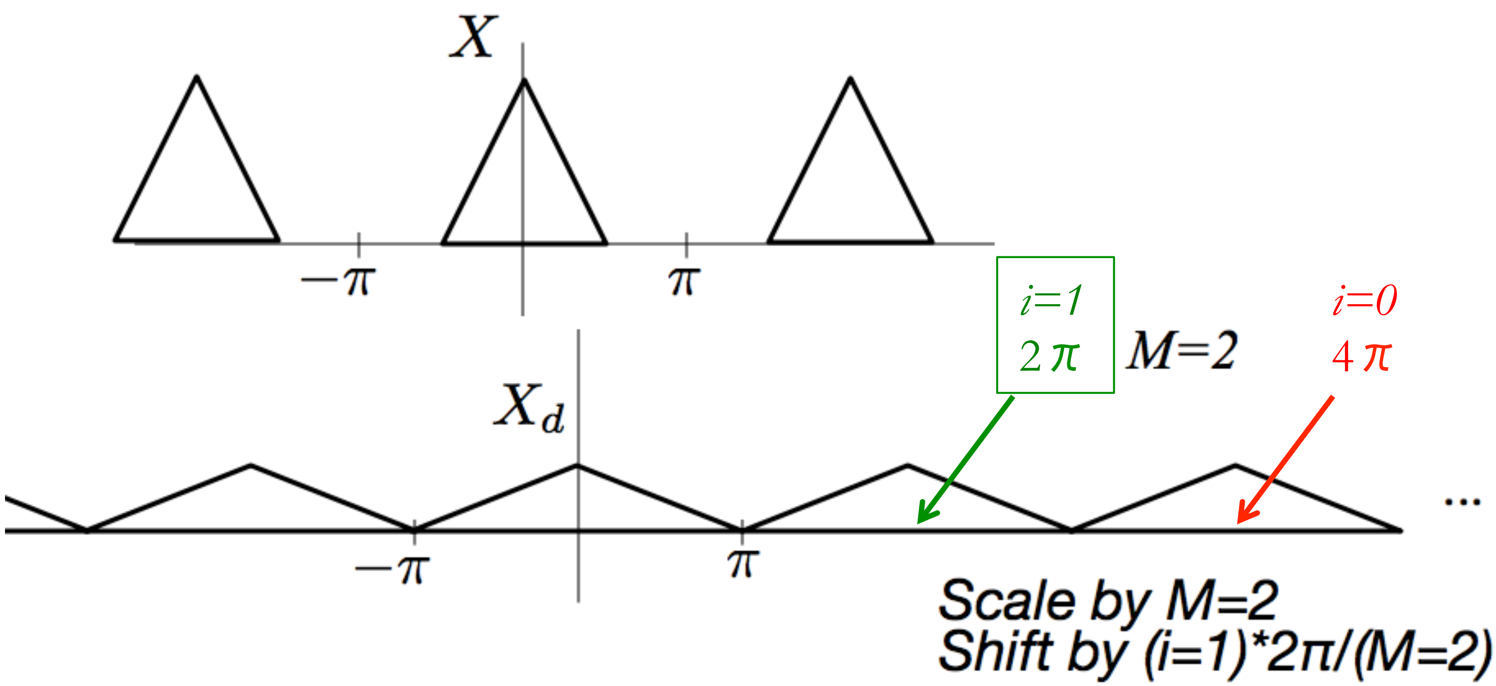
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$

stretch by M

replicate

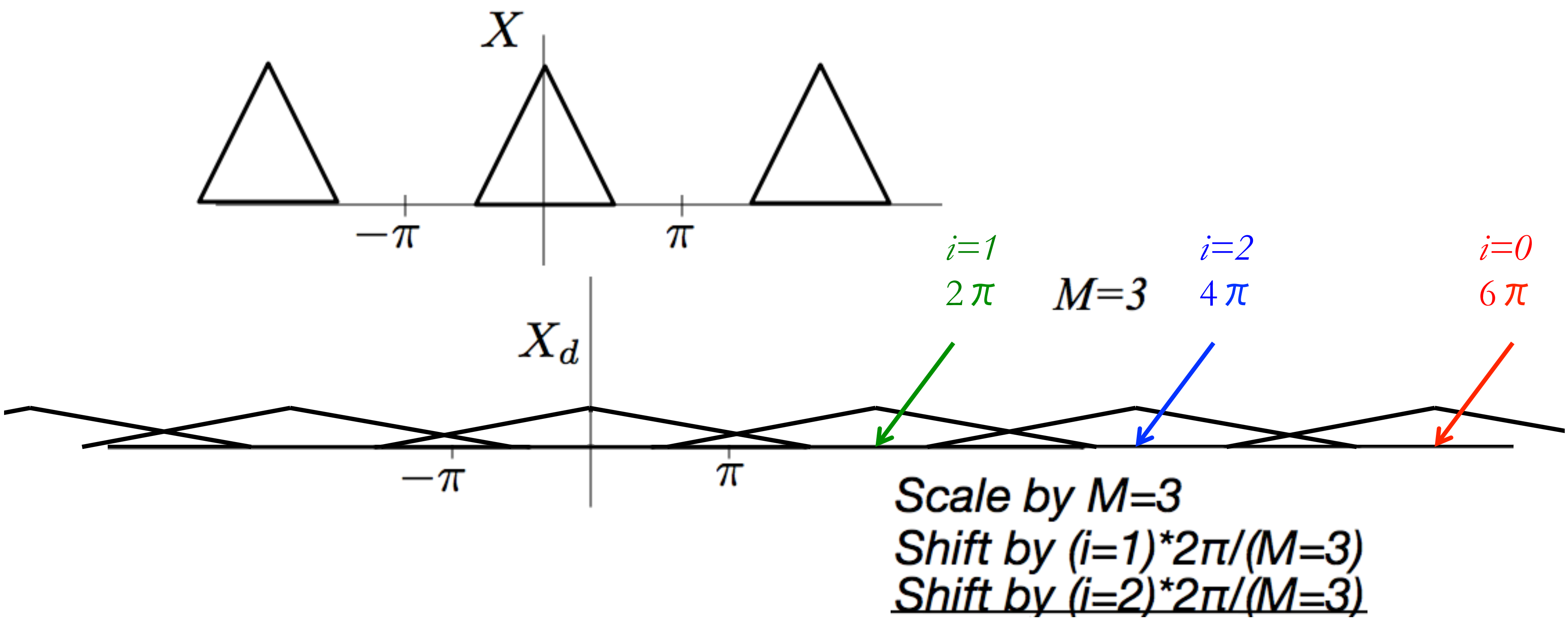
Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M} i\right)}\right)$$

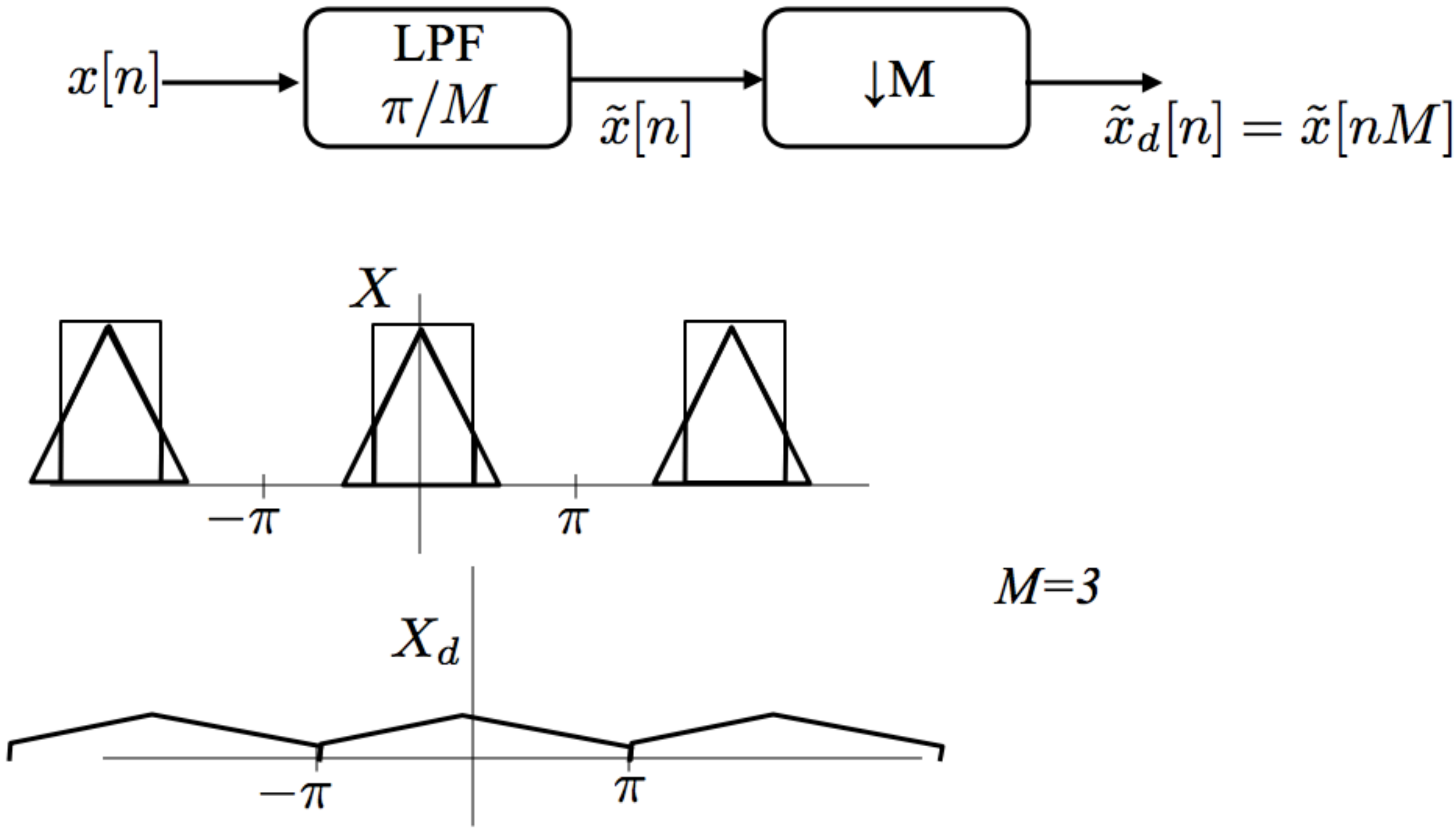


Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M} i\right)}\right)$$



Example



$x[n]$
LPF
π/M
$\tilde{x}[n]$
$\downarrow$M
$\tilde{x}_d[n] = \tilde{x}[nM]$
X
$-\pi$
π
X_d
$-\pi$
π
M=3

Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

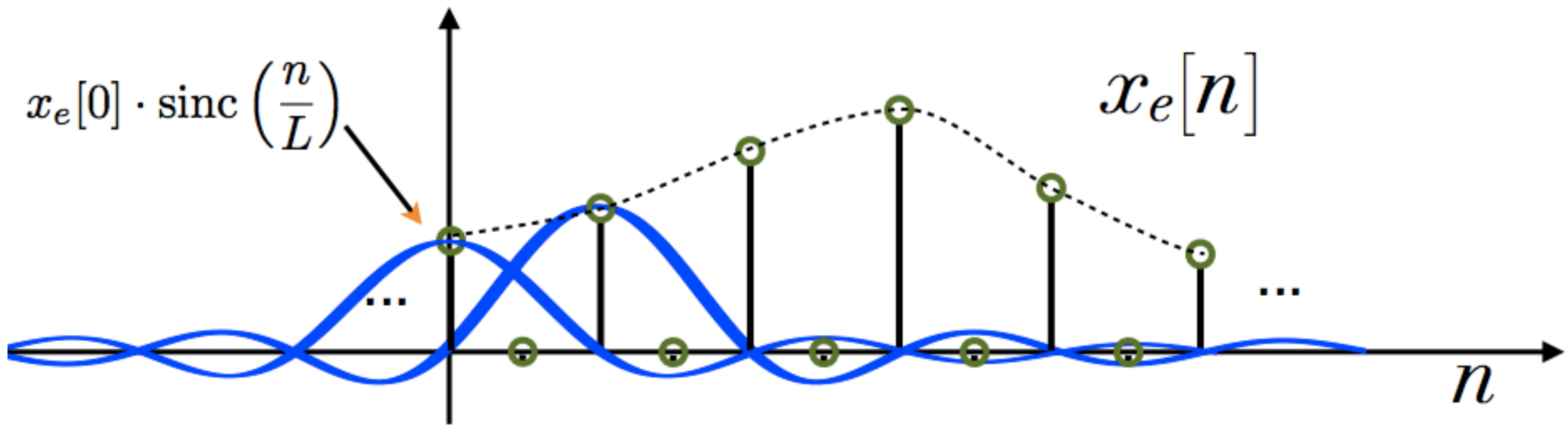
$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

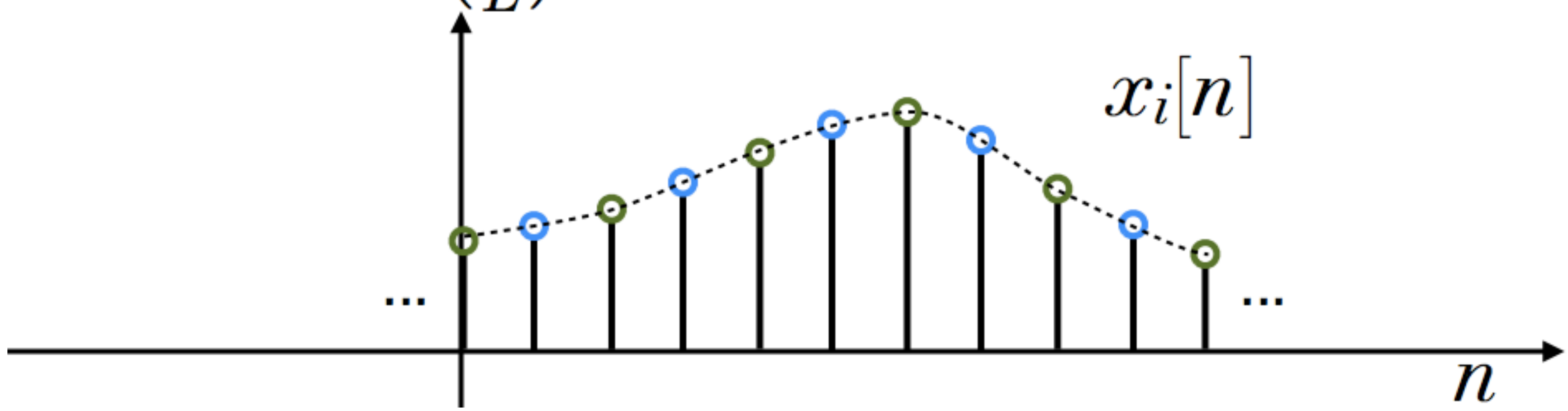
(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:

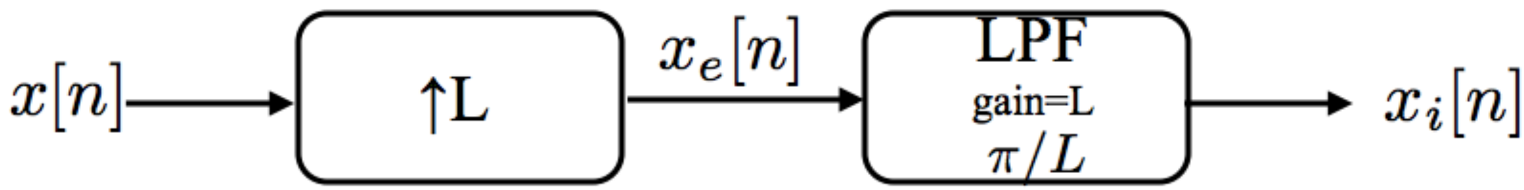


$$x_i[n] = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$$

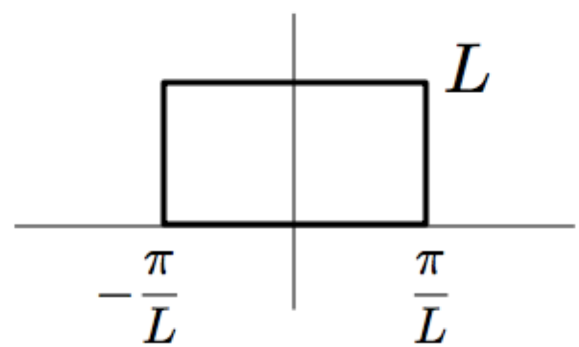


Frequency Domain Interpretation

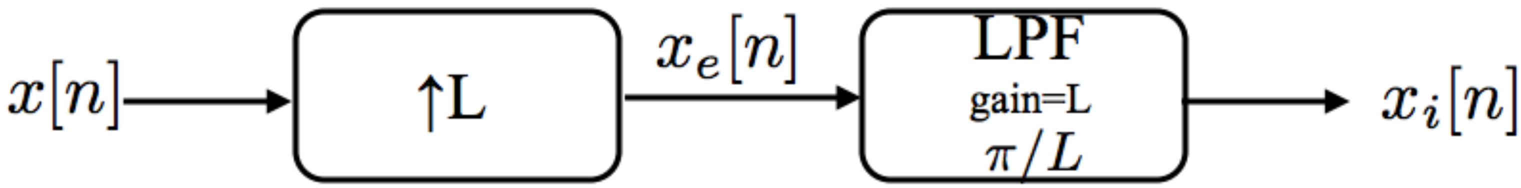
$$x_i[n] = x_e[n] * \mathrm{sinc}(n/L)$$



$\mathrm{sinc}(n/L)$ **DTFT ⇒**



Frequency Domain Interpretation

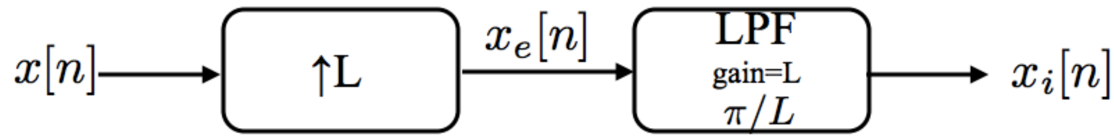


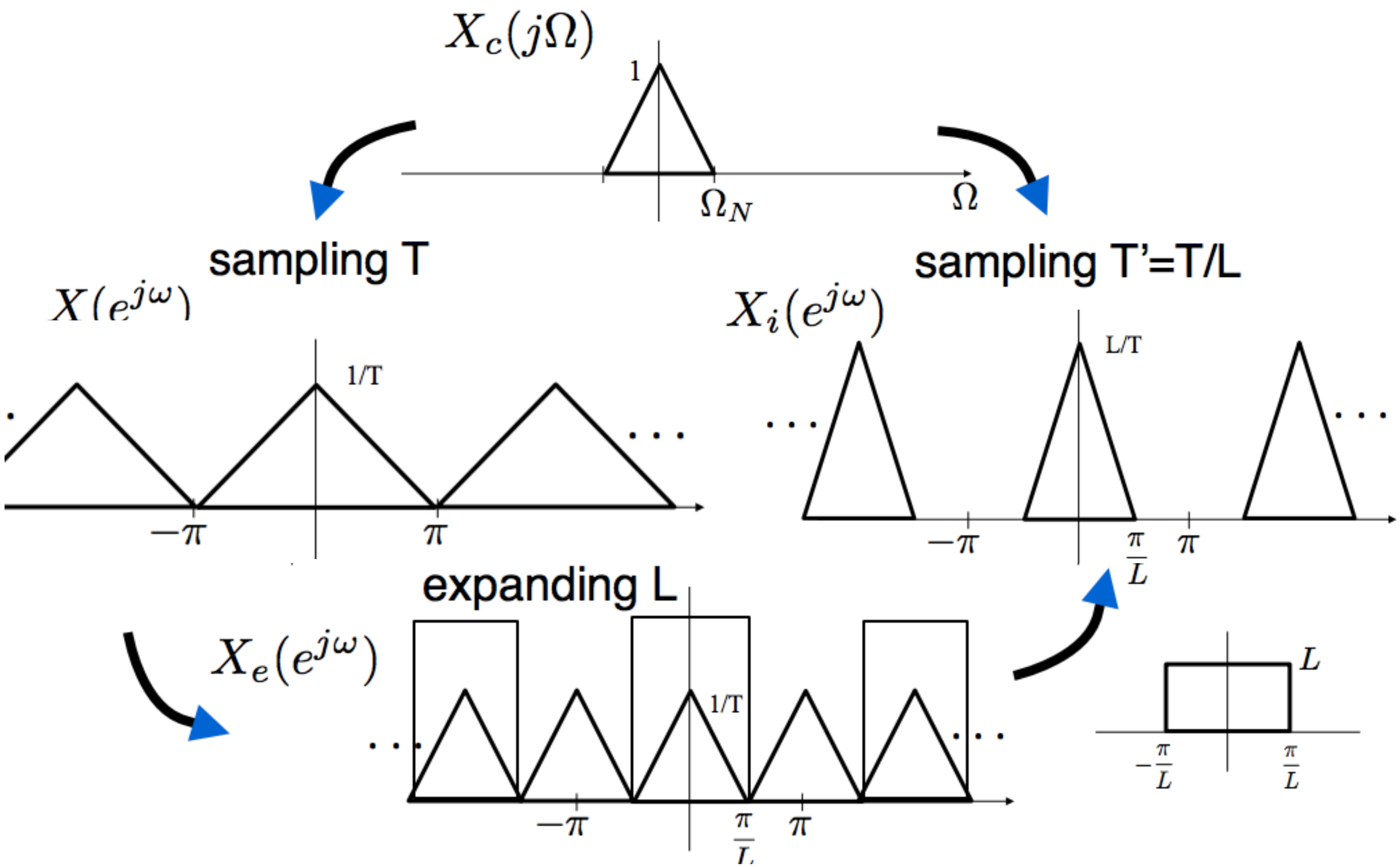
$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\substack{\neq 0 \text{ only for n=mL} \\ \text{(integer m)}}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} = \boxed{X(e^{j\omega L})}$$

Compress DTFT by a factor of L!

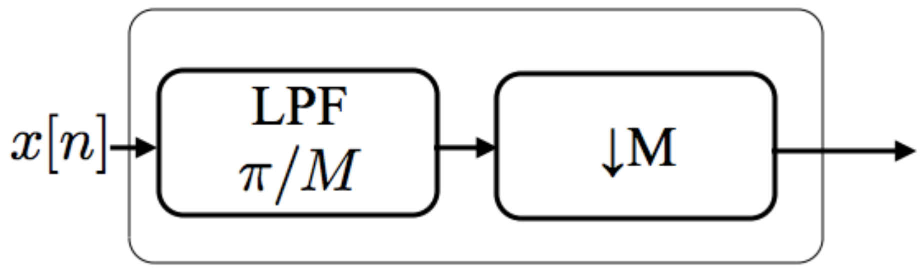
Example


$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$

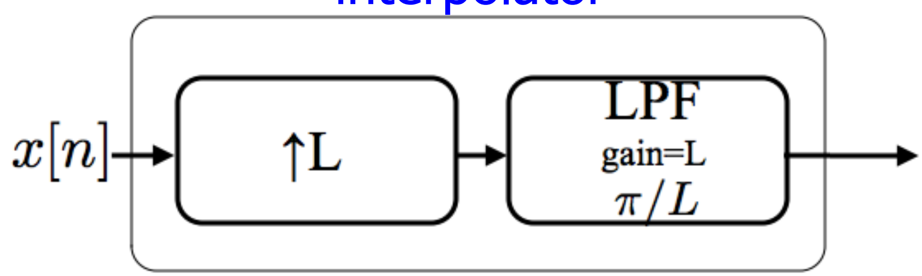

$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
sampling T'=T/L
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Interpolation and Decimation

decimator

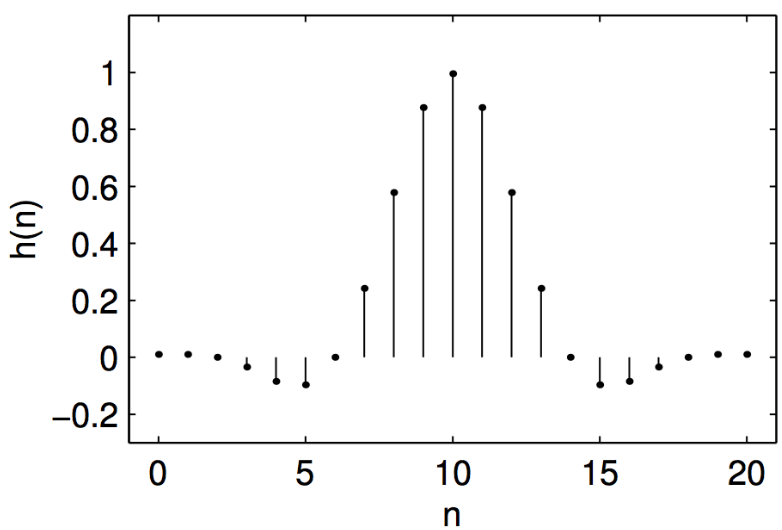


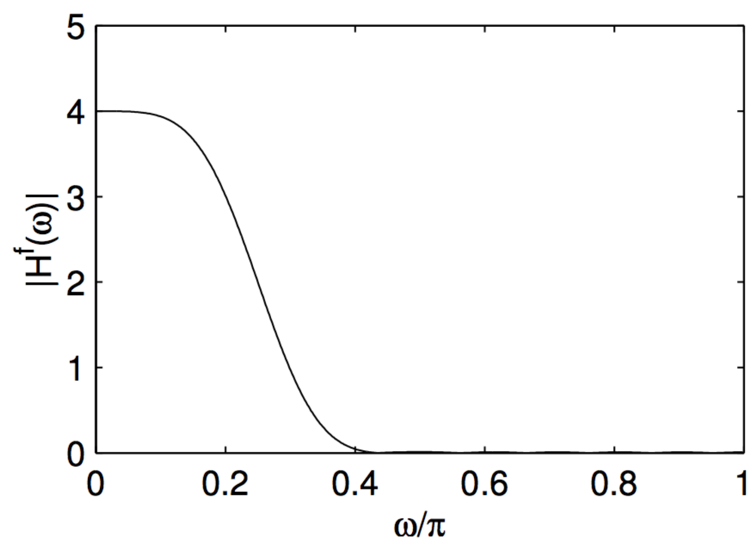
interpolator



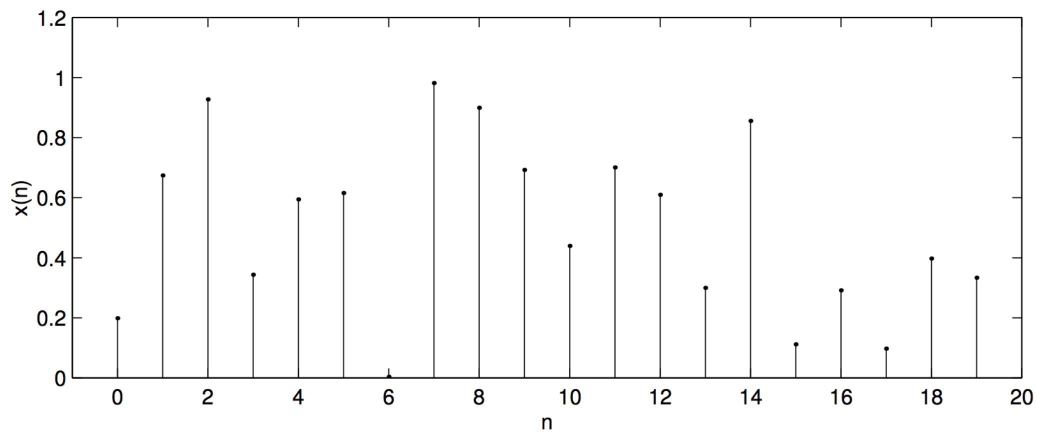
Interpolation Filter Example 1

- In this example, we interpolate a signal x(n) by a factor of 4.
- We use a linear phase Type I FIR lowpass filter of length 21.

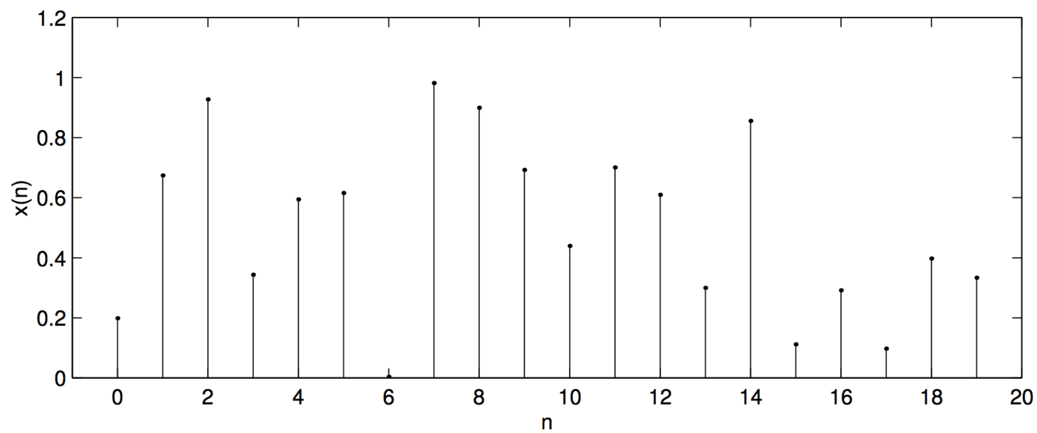


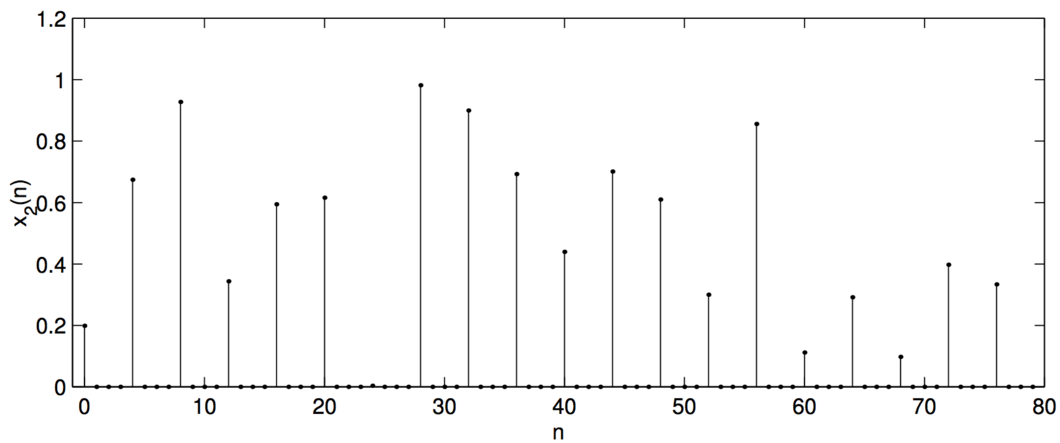


Interpolation Filter Example 1

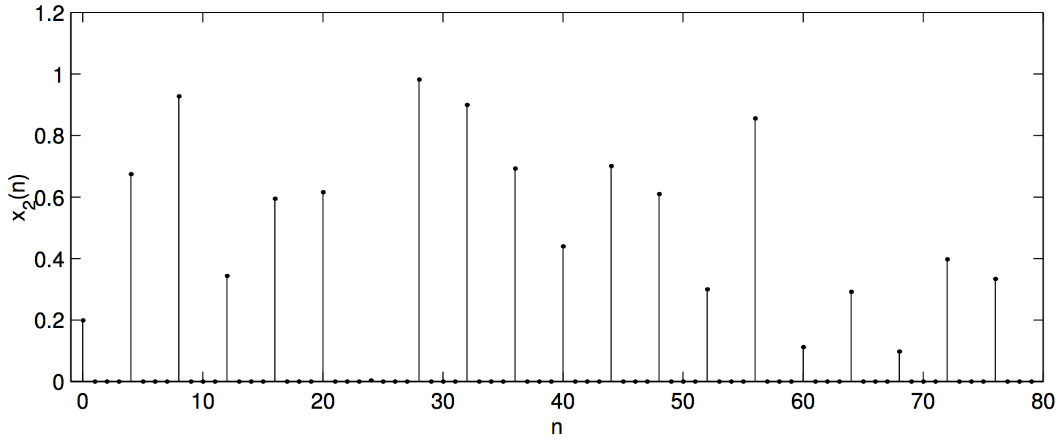


Interpolation Filter Example 1

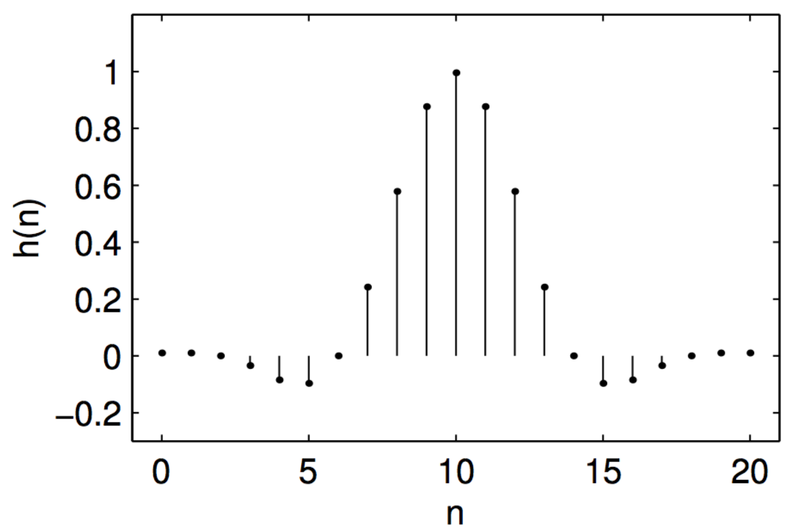


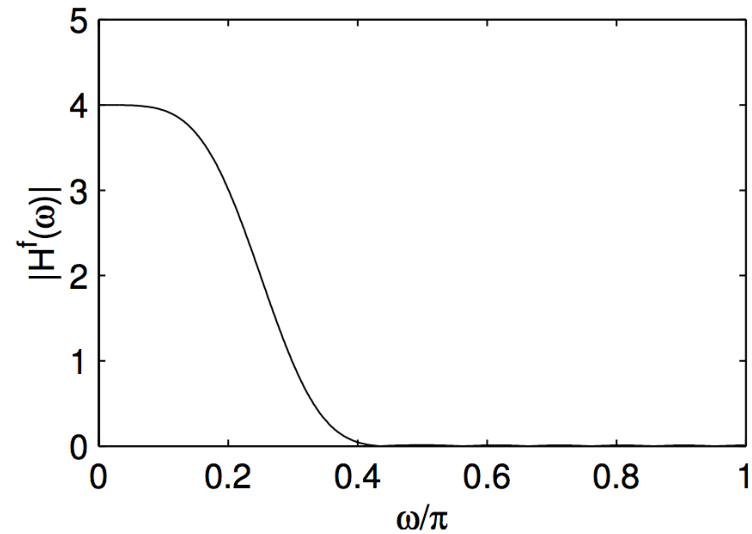


Interpolation Filter Example 1

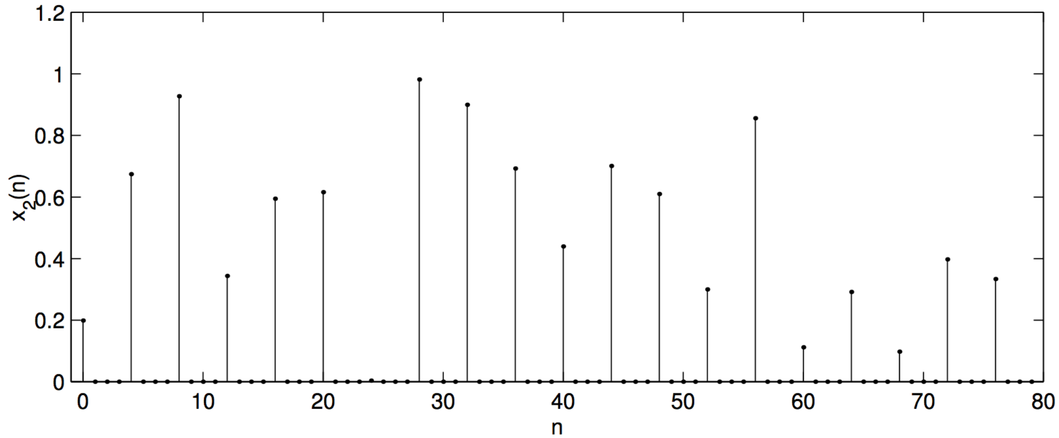


*
(convolve)

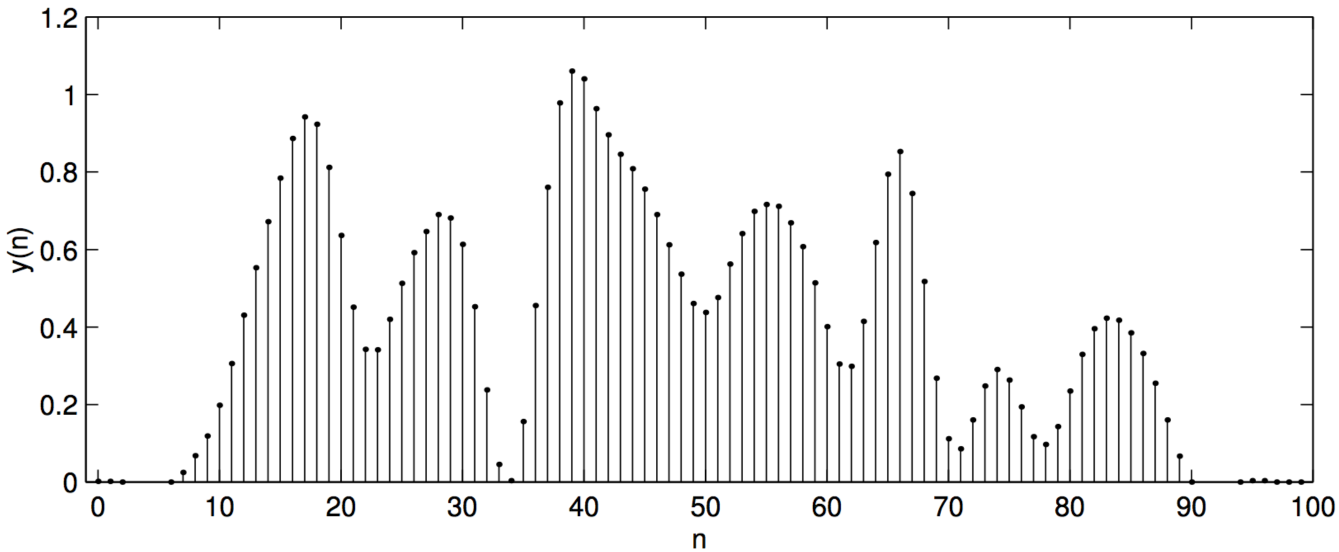


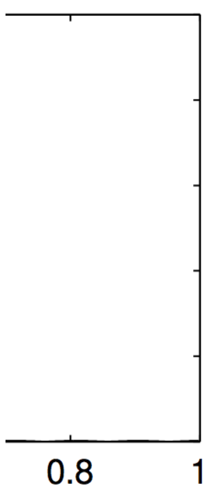


Interpolation Filter Example 1



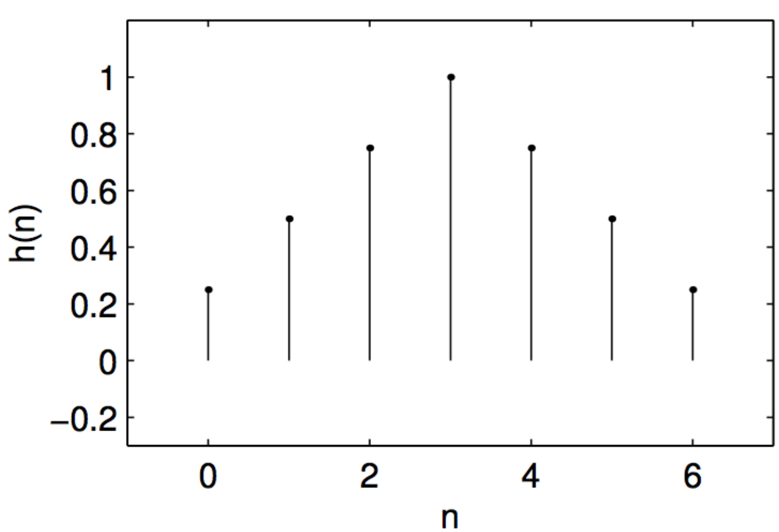
∗
(convolve)

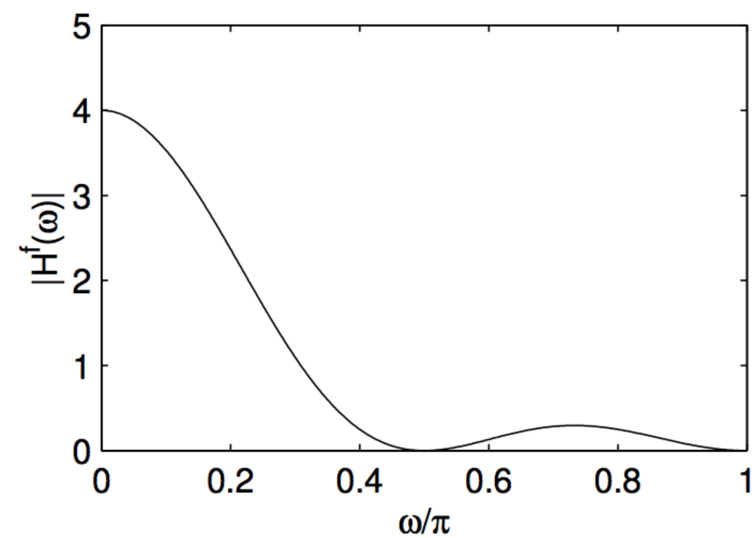




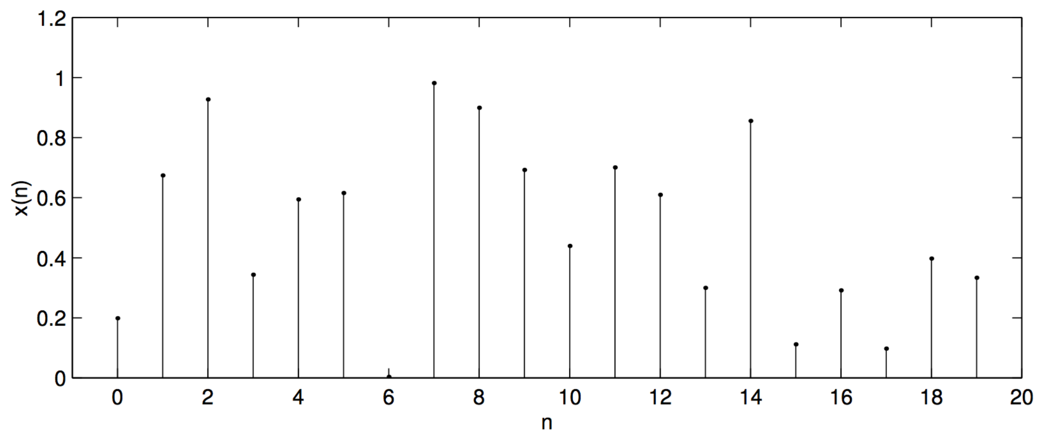
Interpolation Filter Example 2

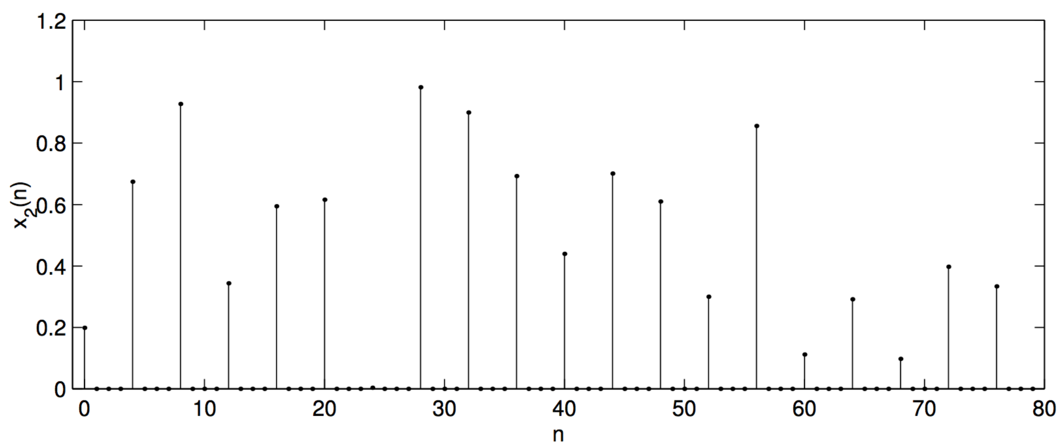
- This time we use a filter of length 7 with the effect of linear interpolation



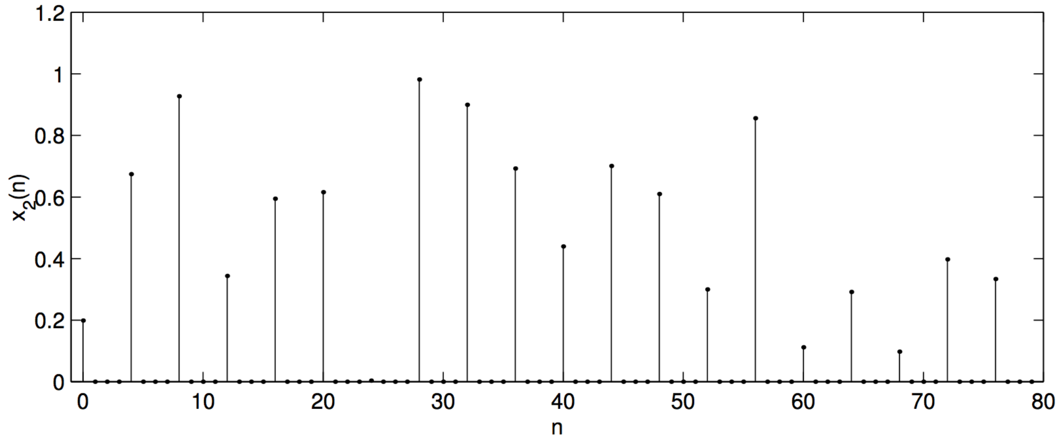


Interpolation Filter Example 2



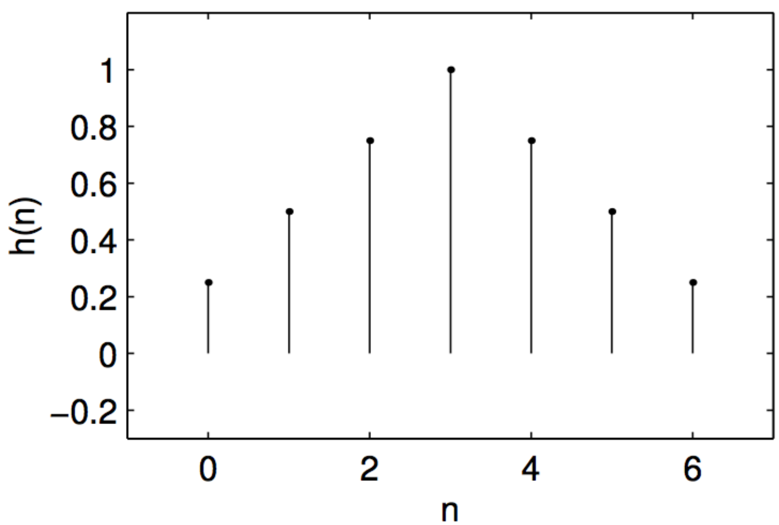


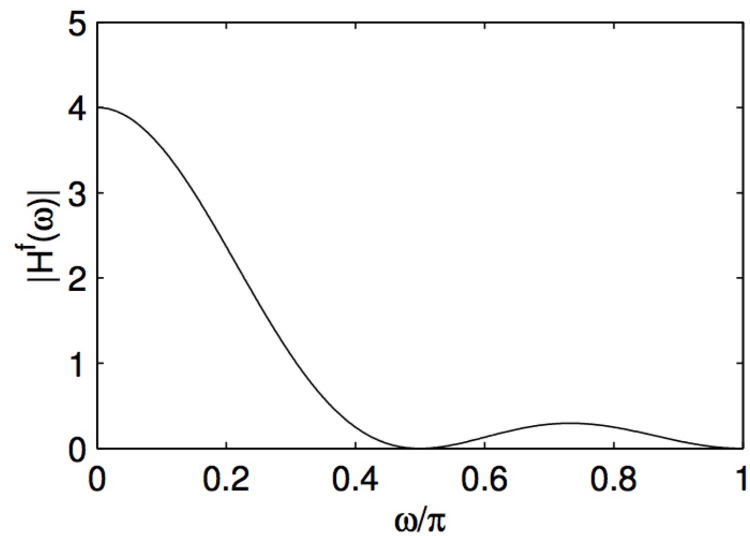
Interpolation Filter Example 2



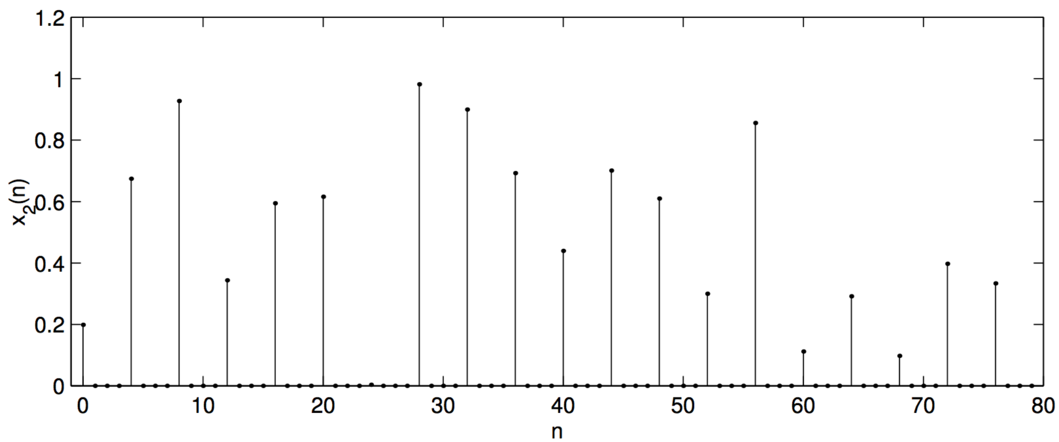
$*$

(convolve)





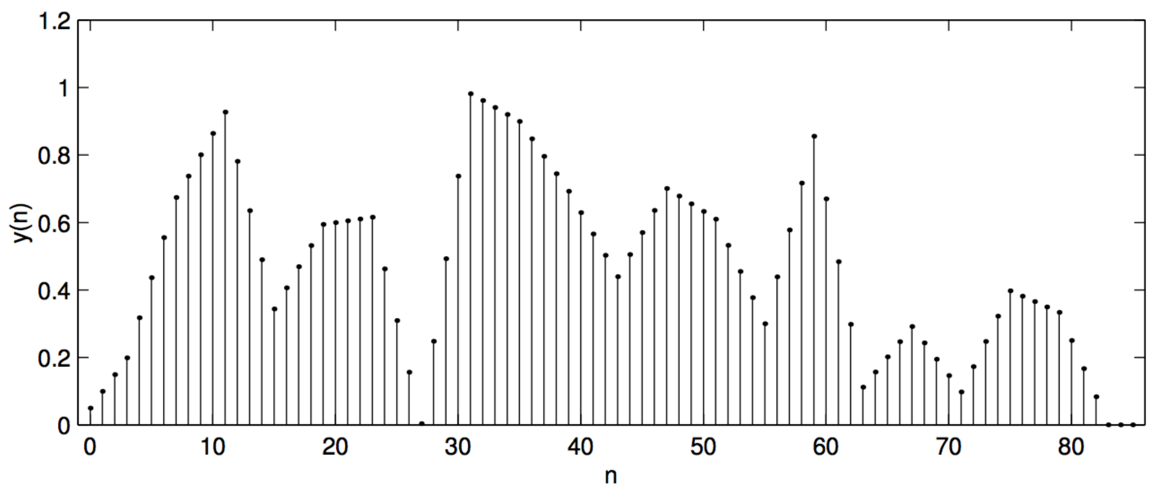
Interpolation Filter Example 2

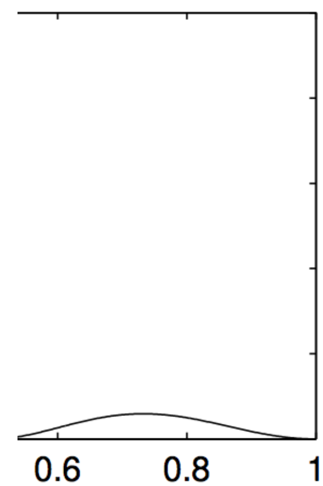


$*$

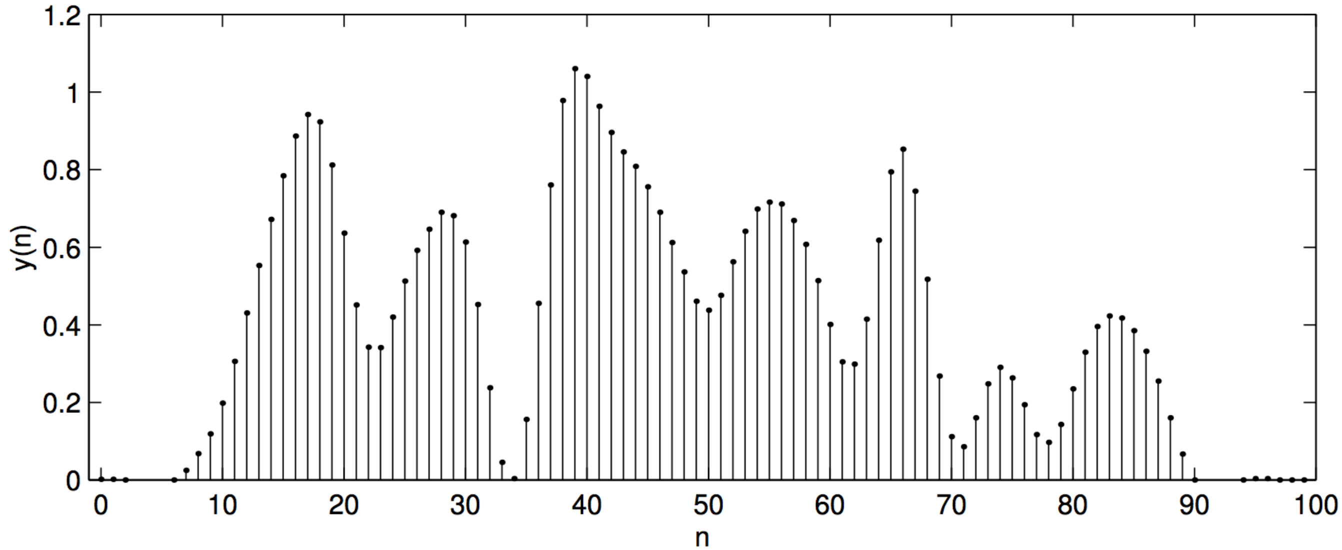
(convolve)

h(n)





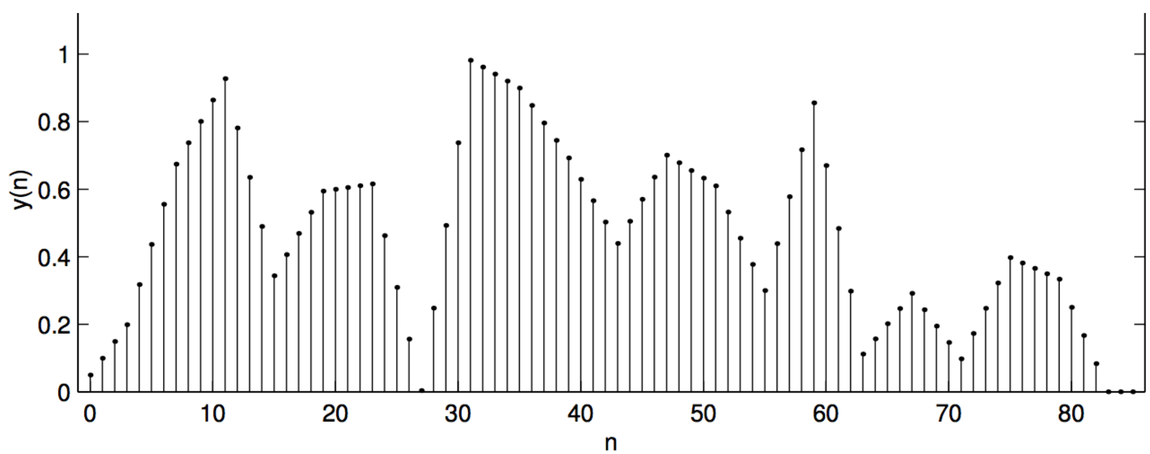
Interpolation Filter Example 2

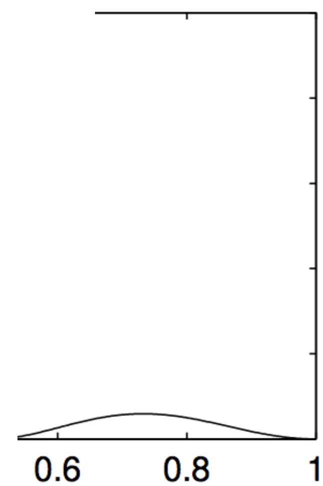


*

nvolve)

h(n)





Interpolation Filter Example 3

- When interpolating a signal x(n), the interpolation filter h(n) will in general change the samples of x(n) in addition to filling in the zeros.
- Can a filter be designed so as to preserve the original samples x(n)?

Interpolation Filter Example 3

- When interpolating a signal x(n), the interpolation filter h(n) will in general change the samples of x(n) in addition to filling in the zeros.
- Can a filter be designed so as to preserve the original samples x(n)?
- To be precise, if $y(n) = h(n) * [\uparrow 2]\, x(n)$ then can we design h(n) so that $y(2n) = x(n)$?

Interpolation Filter Example 3

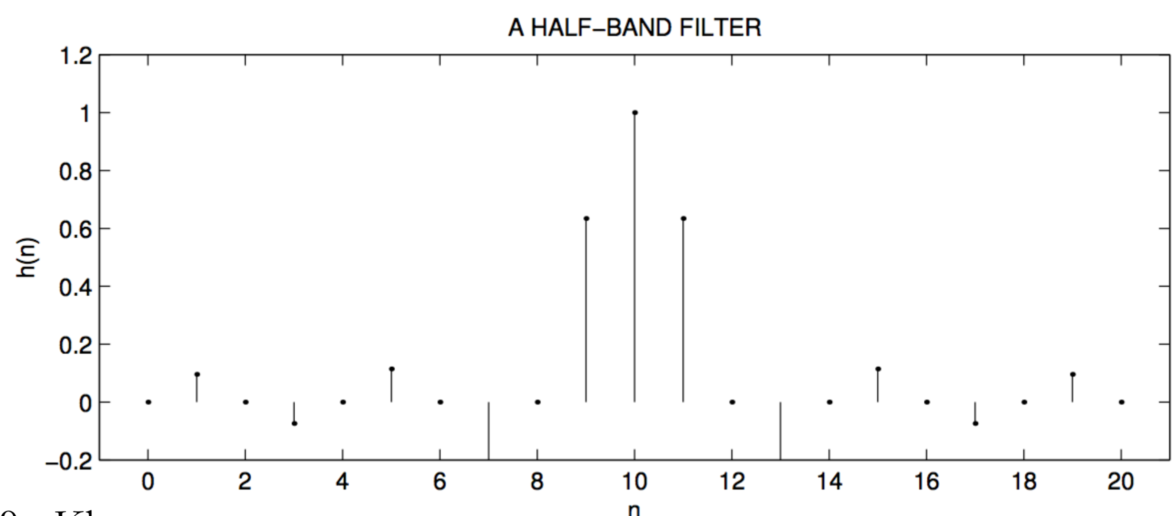
- When interpolating a signal $x(n)$, the interpolation filter $h(n)$ will in general change the samples of $x(n)$ in addition to filling in the zeros.
- Can a filter be designed so as to preserve the original samples $x(n)$?
- To be precise, if $y(n) = h(n) * [\uparrow 2]\, x(n)$ then can we design $h(n)$ so that $y(2n) = x(n)$?
- Or more generally, so that $y(2n + n_o) = x(n)$?

Interpolation Filter Example 3

- When interpolating by a factor of 2, if h(n) is a half-band filter, then it will not change the samples x(n).
- A n_o-centered half-band filter h(n) is a filter that satisfies:

$$h(n) = \begin{cases} 1, & \text{for } n = n_o \\ 0, & \text{for } n = n_o \pm 2, 4, 6, \ldots \end{cases}$$

- That means, every second value of h(n) is zero, except for one such value, as shown in the figure.

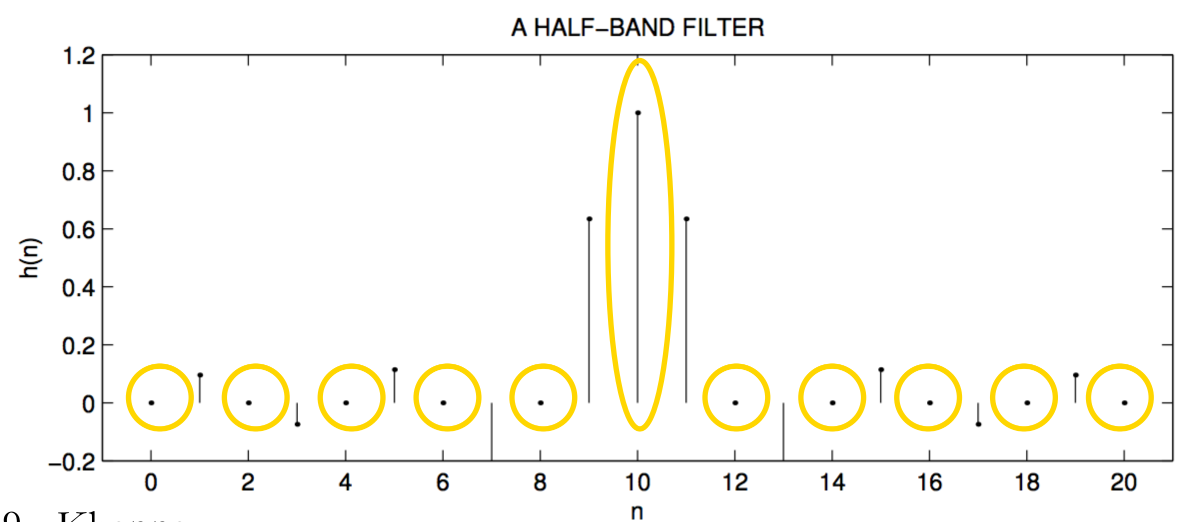


Interpolation Filter Example 3

- When interpolating by a factor of 2, if h(n) is a half-band filter, then it will not change the samples x(n).
- A n_o-centered half-band filter h(n) is a filter that satisfies:

$$h(n) = \begin{cases} 1, & \text{for } n = n_o \\ 0, & \text{for } n = n_o \pm 2, 4, 6, \dots \end{cases}$$

- That means, every second value of h(n) is zero, except for one such value, as shown in the figure.

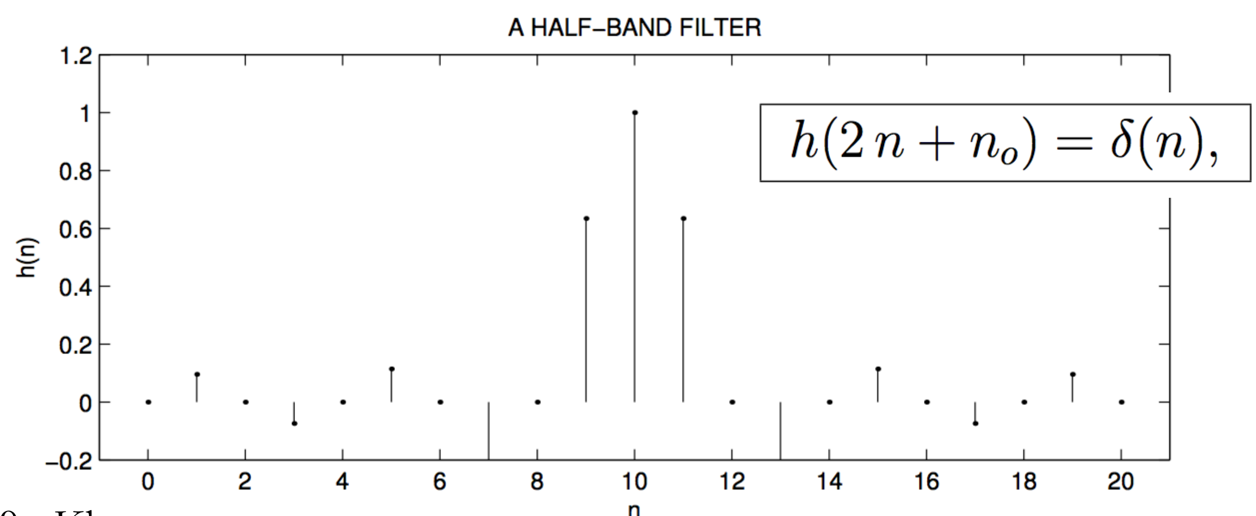


Interpolation Filter Example 3

- When interpolating by a factor of 2, if h(n) is a half-band filter, then it will not change the samples x(n).
- A n_o-centered half-band filter h(n) is a filter that satisfies:

$$h(n) = \begin{cases} 1, & \text{for } n = n_o \\ 0, & \text{for } n = n_o \pm 2, 4, 6, \dots \end{cases}$$

- That means, every second value of h(n) is zero, except for one such value, as shown in the figure.



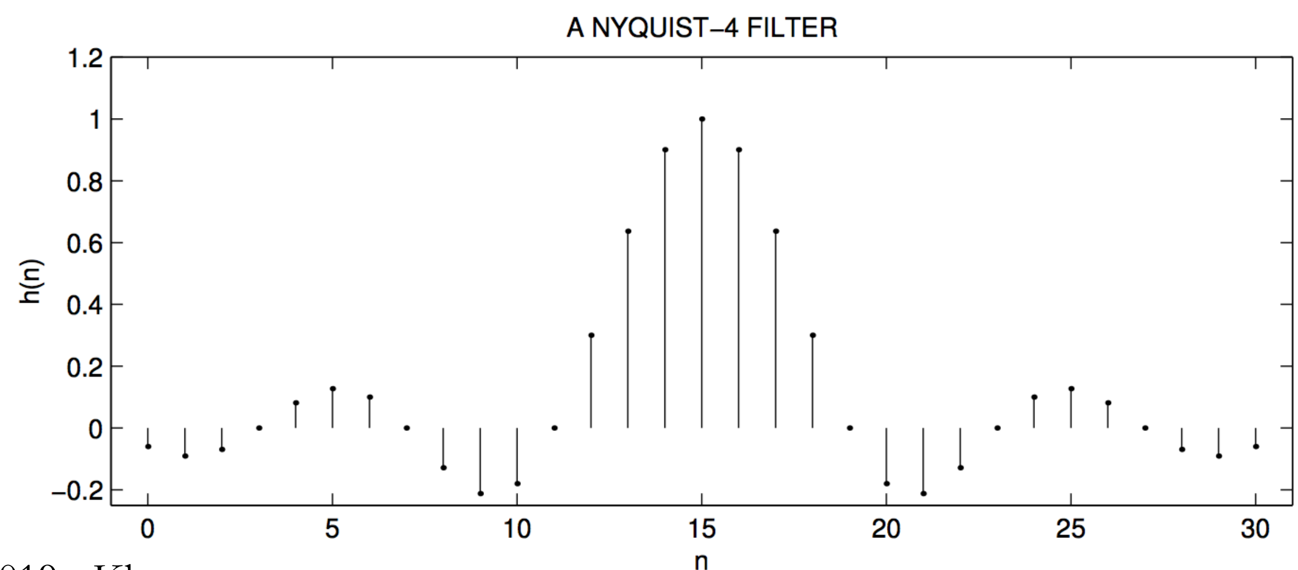
Interpolation Filter Example 4

- When interpolating a signal x(n) by a factor L, the original samples of x(n) are preserved if h(n) is a Nyquist-L filter.
- A Nyquist-L filter simply generalizes the notion of the halfband filter to $L > 2$.

Interpolation Filter Example 4

- When interpolating a signal x(n) by a factor L, the original samples of x(n) are preserved if h(n) is a Nyquist-L filter.
- A Nyquist-L filter simply generalizes the notion of the halfband filter to L > 2.
- A (0-centered) Nyquist-L filter h(n) is one for which

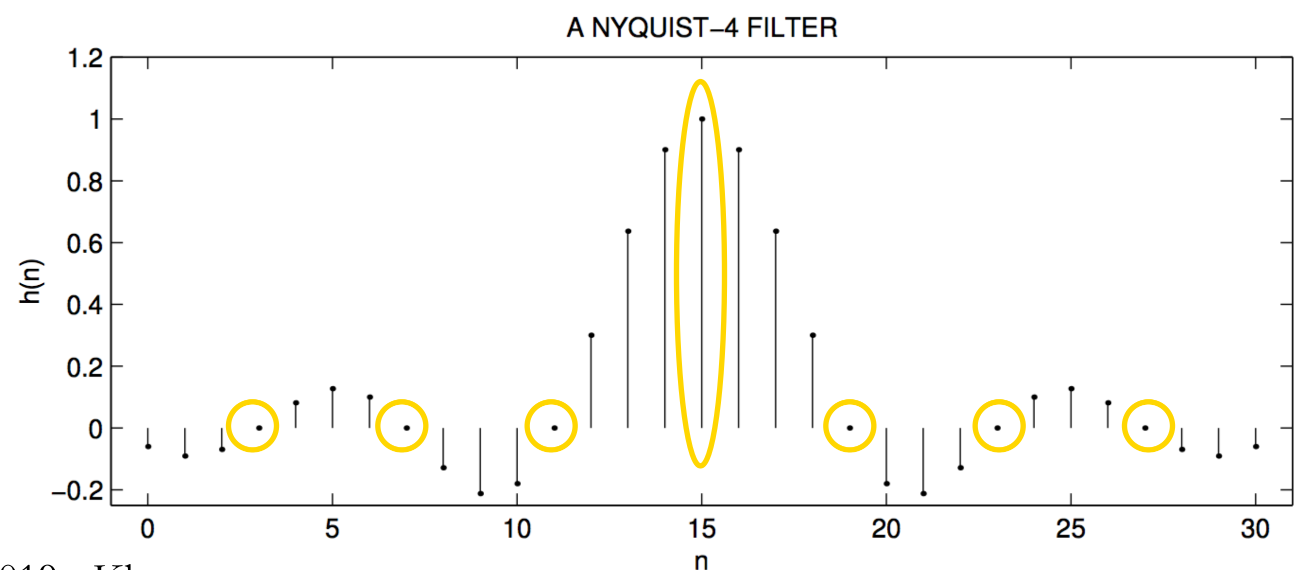
$$\boxed{h(L\,n) = \delta(n).}$$



Interpolation Filter Example 4

- When interpolating a signal x(n) by a factor L, the original samples of x(n) are preserved if h(n) is a Nyquist-L filter.
- A Nyquist-L filter simply generalizes the notion of the halfband filter to L > 2.
- A (0-centered) Nyquist-L filter h(n) is one for which

$$\boxed{h(L\,n) = \delta(n).}$$



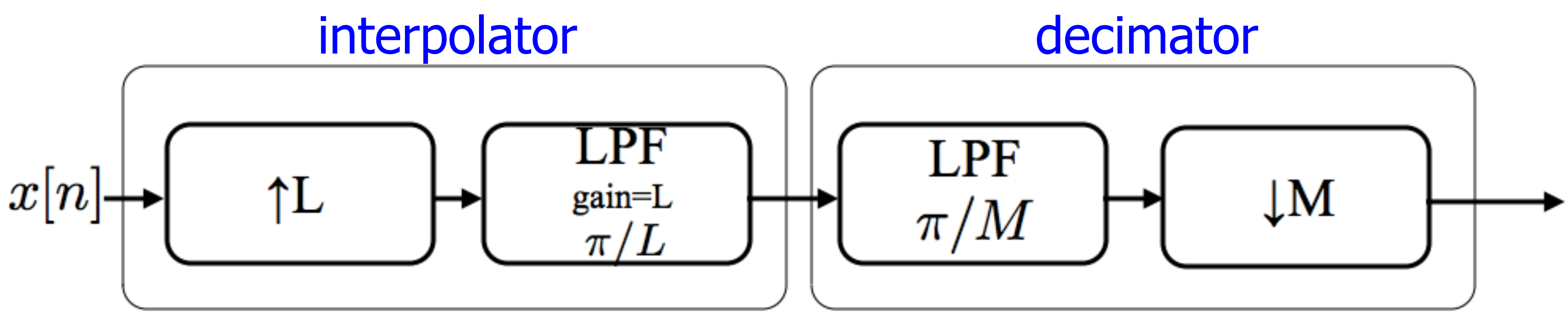
Non-integer Resampling

Non-integer Resampling

- $T' = TM/L$

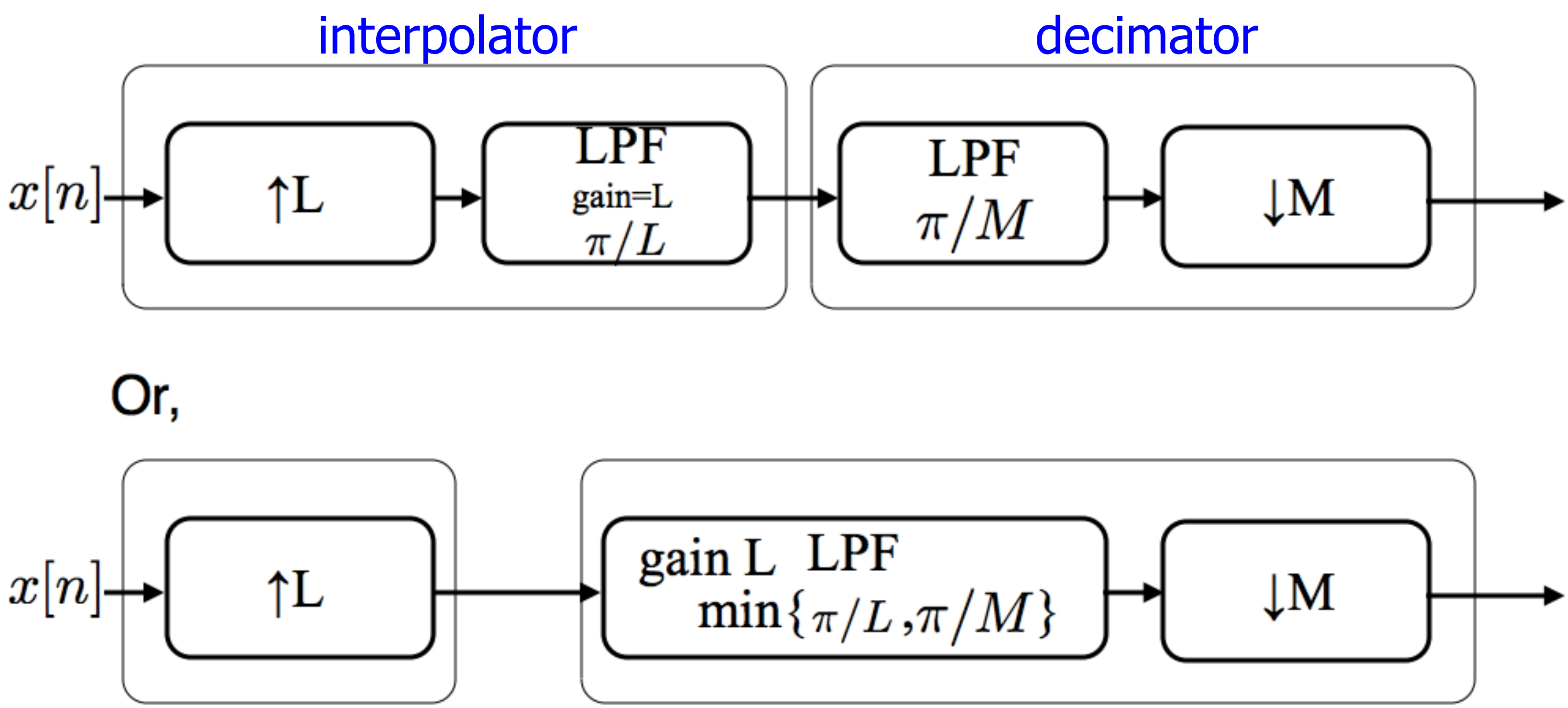
Non-integer Resampling

- T'=TM/L
  - Upsample by L, then downsample by M



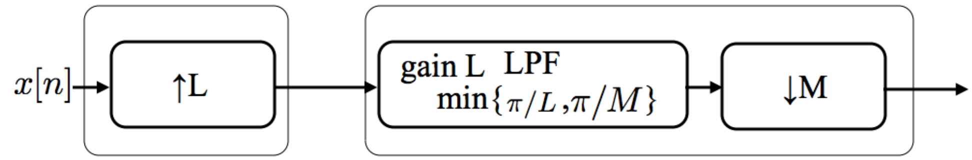
Non-integer Resampling

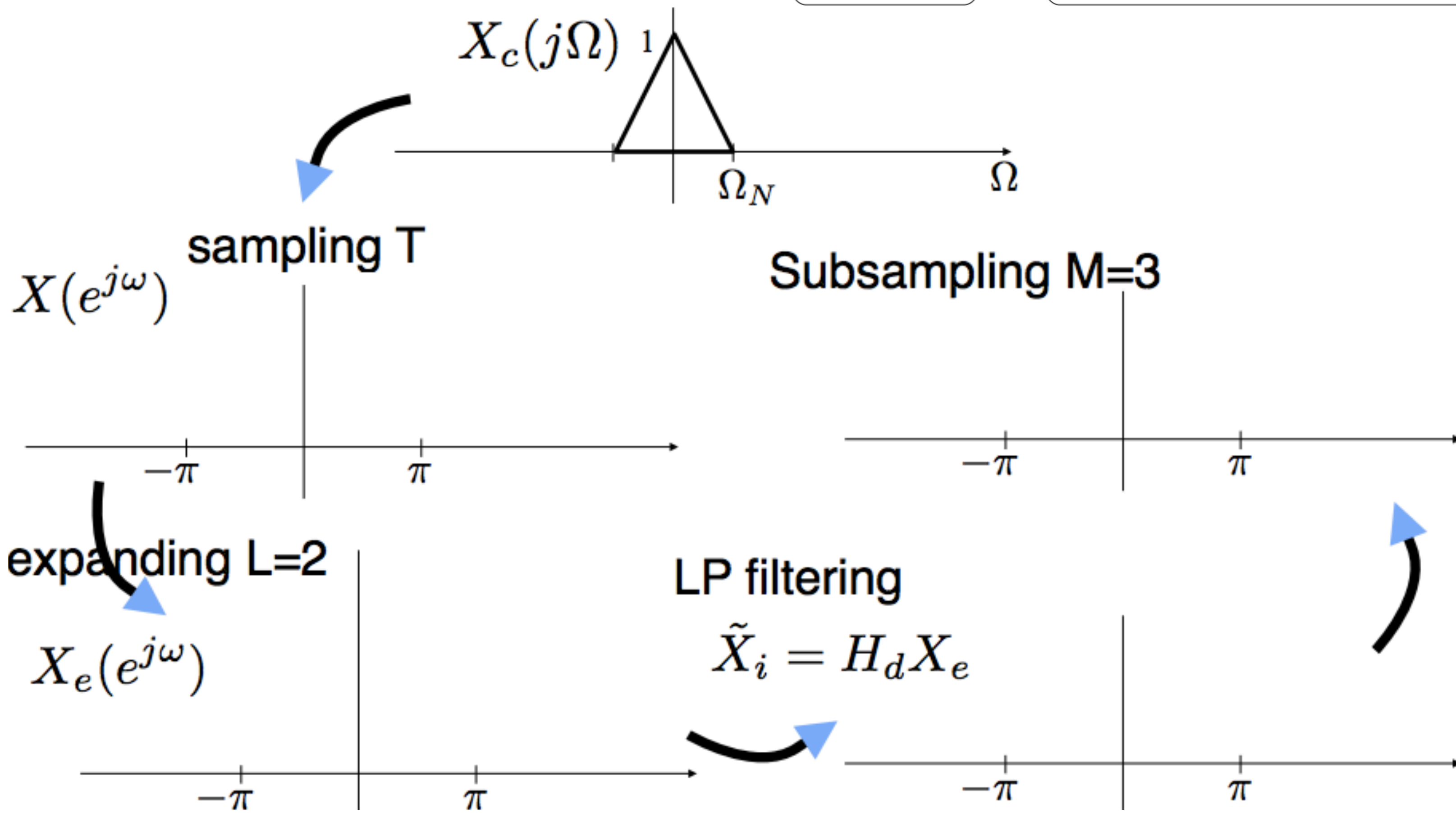
- $T' = TM/L$
  - Upsample by L, then downsample by M



Example

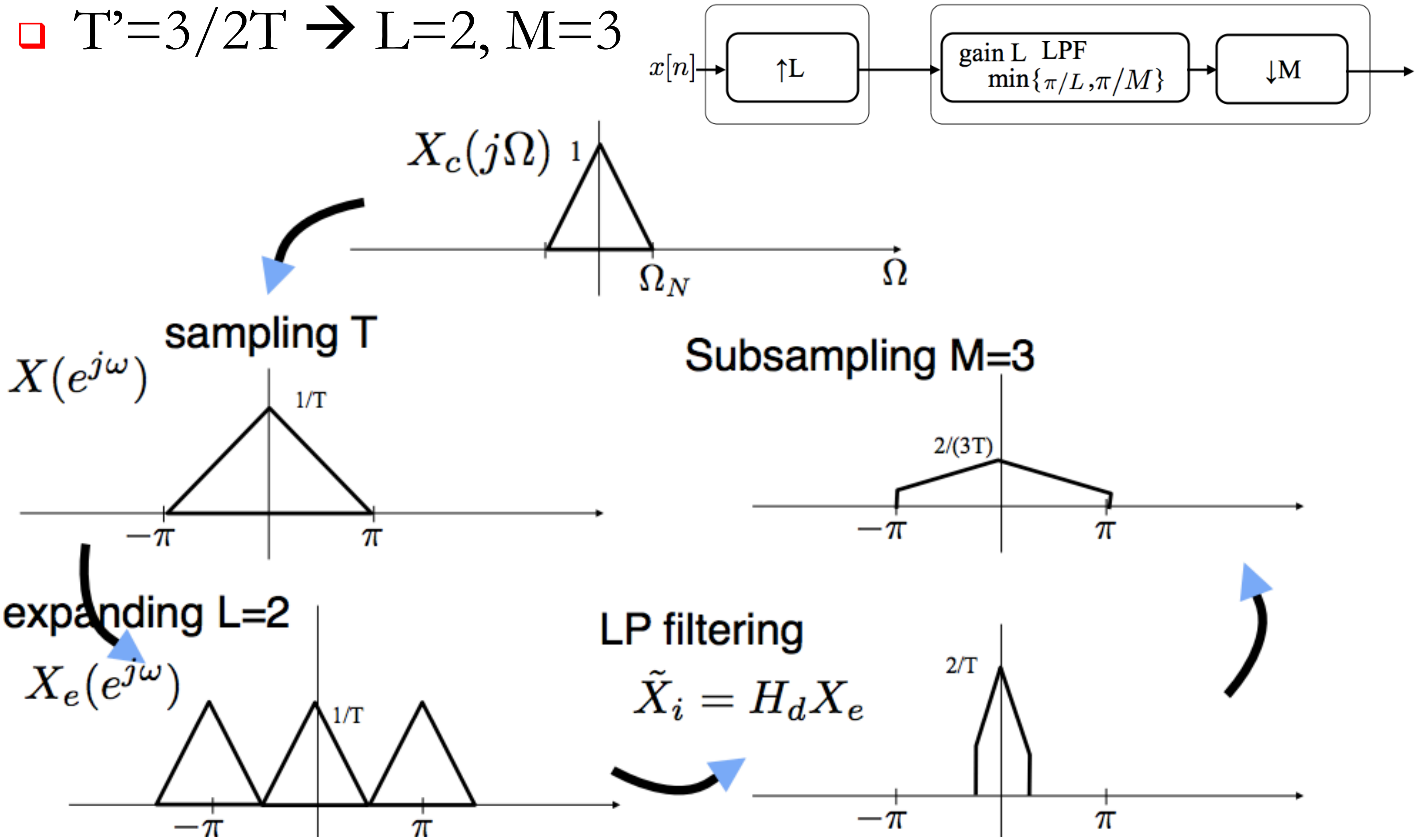
- T'=3/2T → L=2, M=3





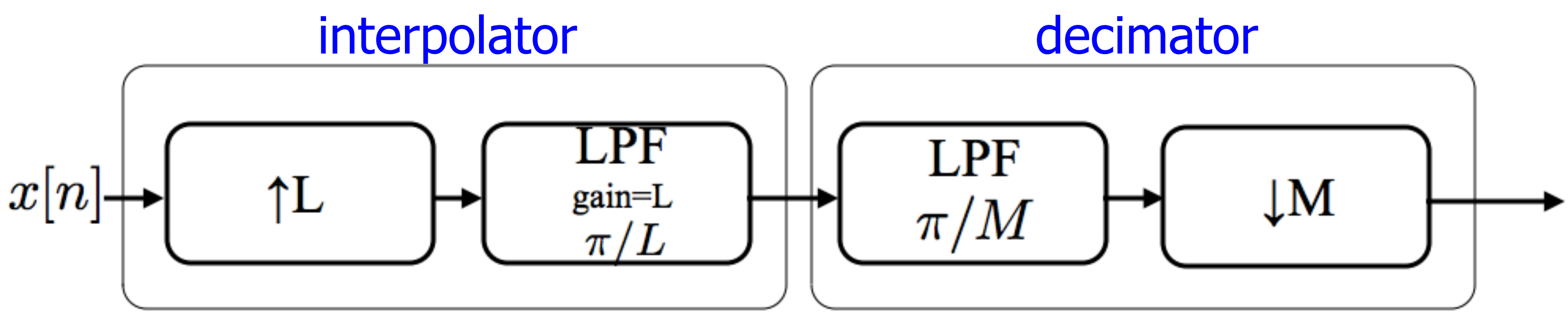
Example

- T'=3/2T → L=2, M=3

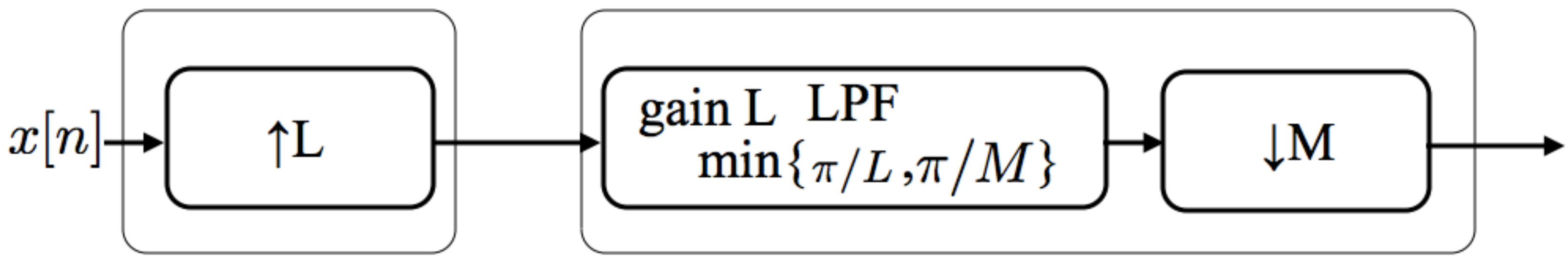


Non-integer Sampling

- T'=TM/L
  - Downsample by M, then upsample by L?



Or,



Example

- What if we want to resample by 1.01T?

Example

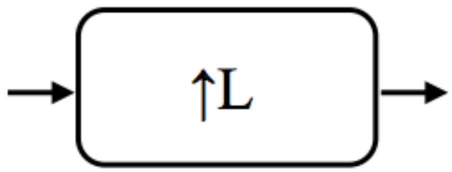
- What if we want to resample by 1.01T?
  - Upsample by L=100
  - Filter $\pi/101$
  - Downsample by M=101

Example

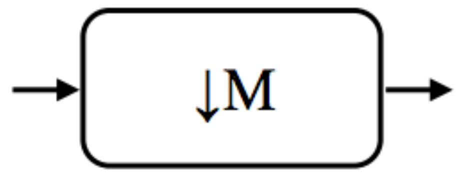
- What if we want to resample by 1.01T?
  - Upsample by L=100
  - Filter $\pi/101$ ($$$$$)
  - Downsample by M=101

- Fortunately there are ways around it!
  - Called multi-rate signal processing
  - Uses compressors, expanders and filtering

Interchanging Operations



"expander"



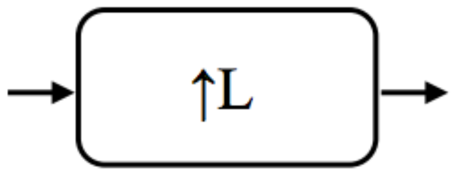
"compressor"

<u>not LTI!</u>

Upsampling
-expanding in time
-compressing in frequency

Downsampling
-compressing in time
-expanding in frequency

Interchanging Operations - Expander



"expander"

Upsampling
-expanding in time
-compressing in frequency

Interchanging Operations - Expander

$$\rightarrow \boxed{\uparrow L} \rightarrow$$

"expander"

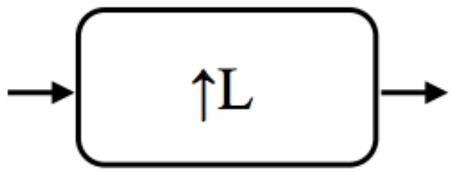
Upsampling
-expanding in time
-compressing in frequency

$$x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow y[n] \quad ? \quad x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z)} \rightarrow y[n]$$

$H(e^{j\omega L})X(e^{j\omega L})$

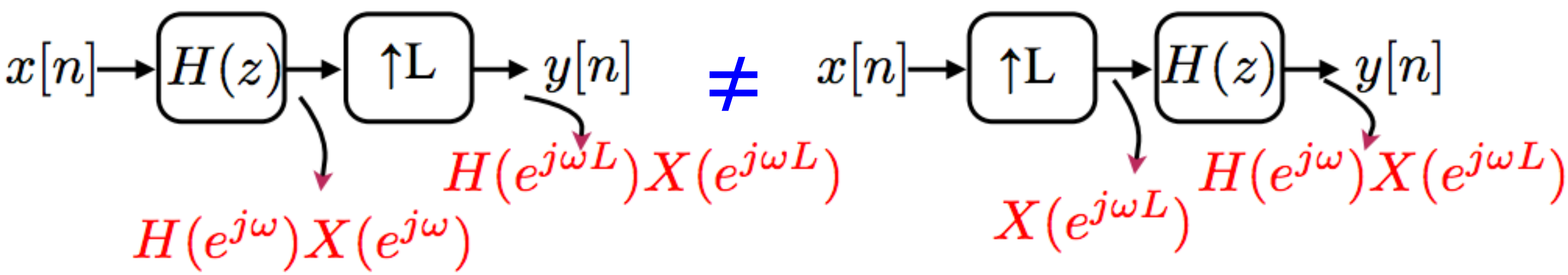
$H(e^{j\omega})X(e^{j\omega})$

Interchanging Operations - Expander

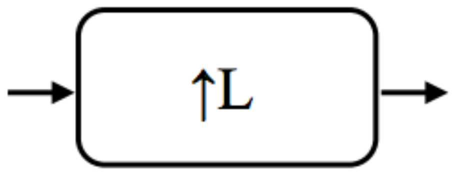


"expander"

Upsampling
-expanding in time
-compressing in frequency

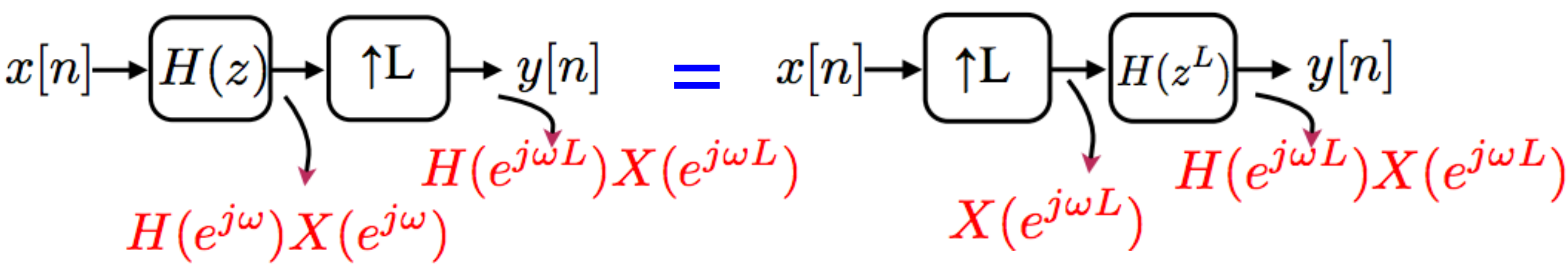


Interchanging Operations - Expander

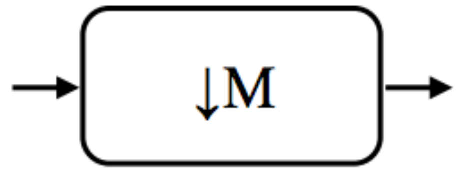


"expander"

Upsampling
-expanding in time
-compressing in frequency



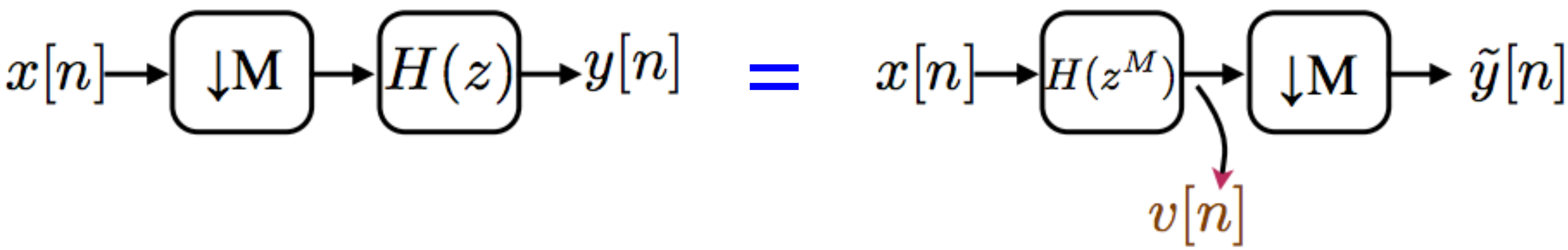
Interchanging Operations - Compressor



not LTI!

"compressor"

Downsampling
-compressing in time
-expanding in frequency

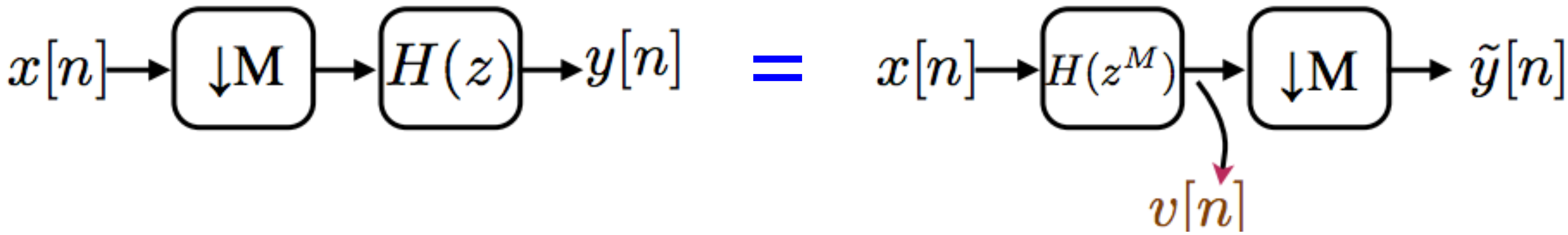


Interchanging Operations - Compressor

$$x[n] \rightarrow \boxed{\downarrow \mathrm{M}} \rightarrow \boxed{H(z)} \rightarrow y[n] \quad = \quad x[n] \rightarrow \boxed{H(z^M)} \xrightarrow[v[n]]{} \boxed{\downarrow \mathrm{M}} \rightarrow \tilde{y}[n]$$

$$\begin{aligned}
Y(e^{j\omega}) &= H(e^{j\omega})\left(\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)\right) \\
&= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{H\left(e^{j(\omega-2\pi i)}\right)}_{H(e^{j\omega})} X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right) \\
&= \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right) X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi i}{M}\right)}\right)
\end{aligned}$$

Interchanging Operations - Compressor



$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM(\frac{\omega}{M}-\frac{2\pi i}{M})}\right) X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

$$V(e^{j\omega}) = H(e^{j\omega M})X(e^{j\omega})$$

$$\tilde{Y}(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} V\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

=

Interchanging Operations - Summary

Filter and expander		Expander and expanded filter*
$x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow y[n]$	$\equiv$	$x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow y[n]$
$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow y[n]$	$\equiv$	$x[n] \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow y[n]$
Compressor and filter		Expanded filter* and compressor

*Expanded filter = expanded impulse response, compressed freq response

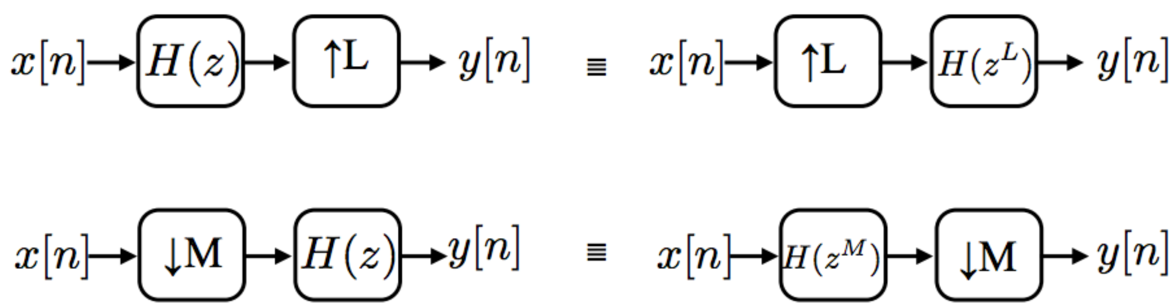
Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter $\pi/101$ ($$$$$)
  - Compress by M=101

- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Big Ideas

- Downsampling/Upsampling
- Practical Interpolation
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations



Admin

- HW 4 due Sunday
  - Typo, homework problem 4.28 removed, homework handout fixed.