ESE 531: Digital Signal Processing

Lec 10: February 19, 2019

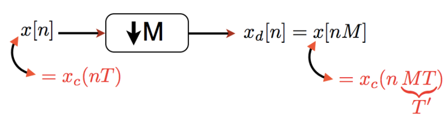
Non-Integer and Multi-rate Sampling

Lecture Outline

- Review: Downsampling/Upsampling
- Interpolation
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations

Downsampling

- Definition: Reducing the sampling rate by an integer number



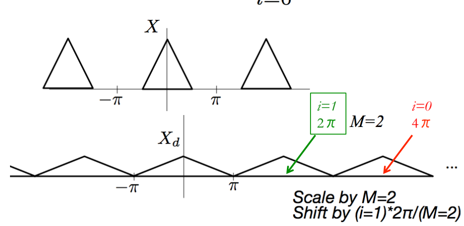
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$

stretch by M

replicate

Example

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{w}{M} - \frac{2\pi}{M}i)}\right)$$



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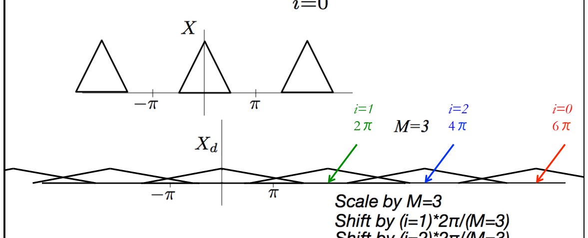
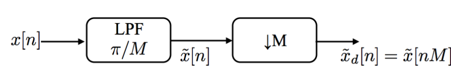
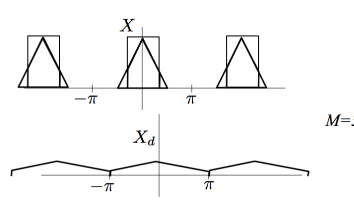


Example





Upsampling

- Definition: Increasing the sampling rate by an integer number

$x[n] = x_c(nT)$

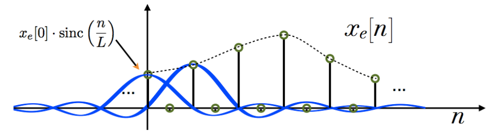
$x_i[n] = x_c(nT')$ where $T' = \frac{T}{L}$ $\quad L$ integer

Obtain $x_i[n]$ from $x[n]$ in two steps:

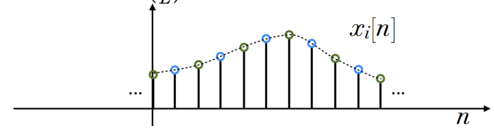
(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:

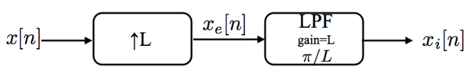


$x_i[n] = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$

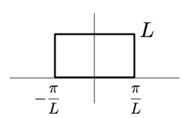


Frequency Domain Interpretation

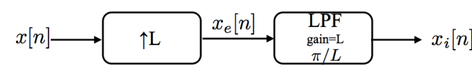
$$x_i[n] = x_e[n] * \mathrm{sinc}(n/L)$$



$\mathrm{sinc}(n/L)$ DTFT ⇒



Frequency Domain Interpretation

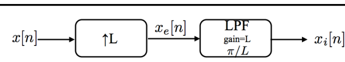


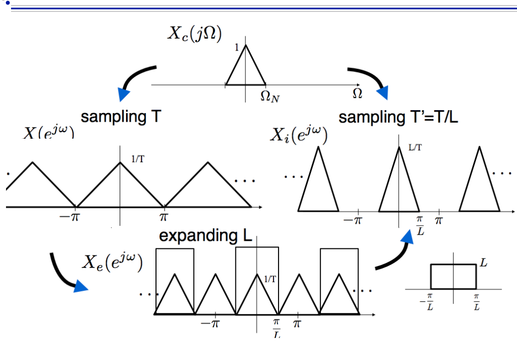
$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for } n=mL \text{ (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} = X(e^{j\omega L})$$

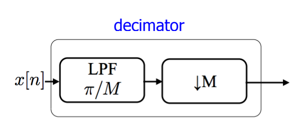
Compress DTFT by a factor of L!

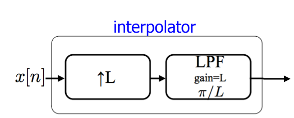
Example





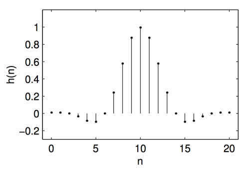
Interpolation and Decimation

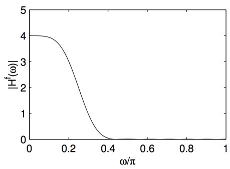




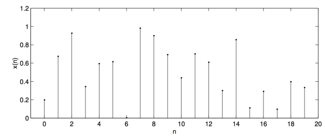
Interpolation Filter Example 1

- In this example, we interpolate a signal x(n) by a factor of 4.
- We use a linear phase Type I FIR lowpass filter of length 21.





Interpolation Filter Example 1



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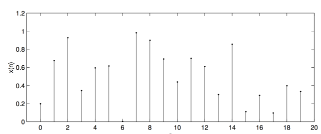

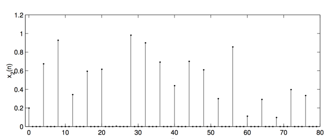

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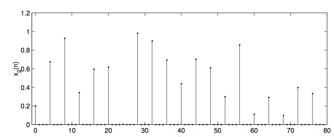
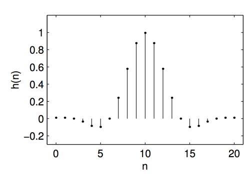
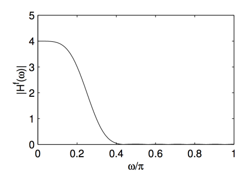


∗
(convolve)





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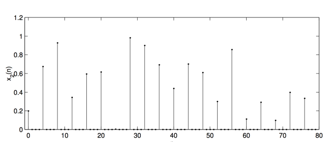

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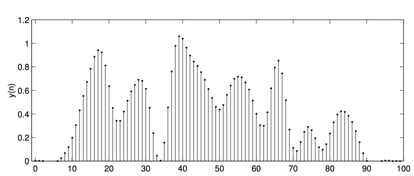
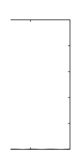
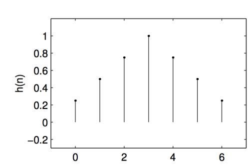
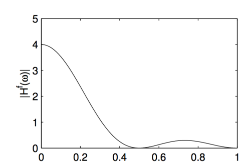




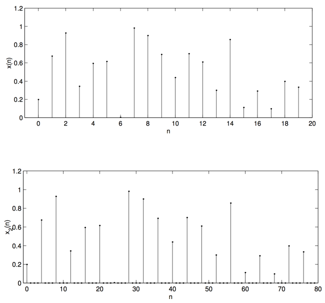
Interpolation Filter Example 2

- This time we use a filter of length 7 with the effect of linear interpolation

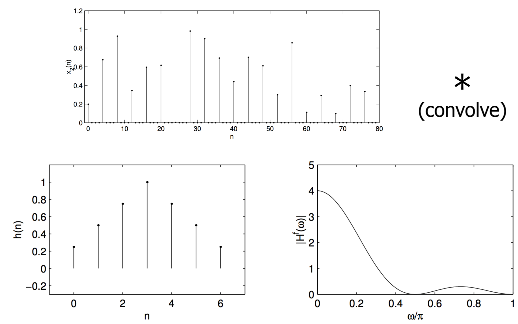




Interpolation Filter Example 2



Interpolation Filter Example 2



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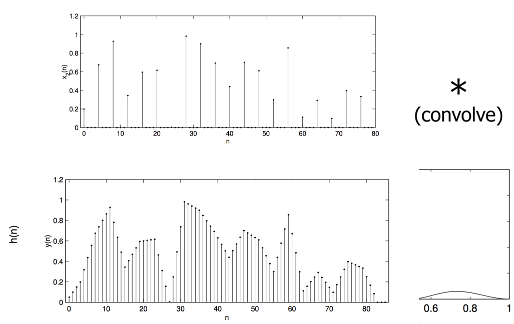

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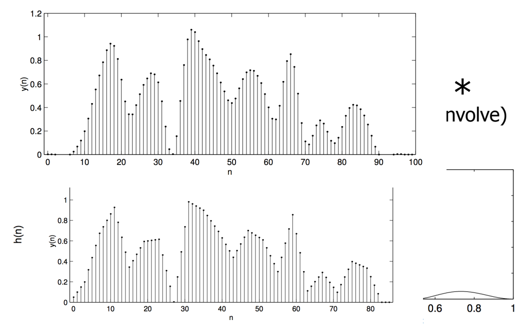

Interpolation Filter Example 3

- When interpolating a signal x(n), the interpolation filter h(n) will in general change the samples of x(n) in addition to filling in the zeros.
- Can a filter be designed so as to preserve the original samples x(n)?

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- Can a filter be designed so as to preserve the original samples x(n)?
- To be precise, if $y(n) = h(n) * [\uparrow 2]\, x(n)$ then can we design h(n) so that $y(2n) = x(n)$?

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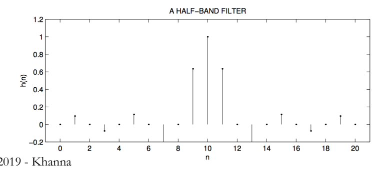
- To be precise, if y(n) = h(n) * [↑ 2] x(n) then can we design h(n) so that y(2n) = x(n)?
- Or more generally, so that $y(2n + n_o) = x(n)$?

Interpolation Filter Example 3

- When interpolating by a factor of 2, if h(n) is a half-band filter, then it will not change the samples x(n).
- A n_o-centered half-band filter h(n) is a filter that satisfies:

$$h(n) = \begin{cases} 1, & \text{for } n = n_o \\ 0, & \text{for } n = n_o \pm 2, 4, 6, \ldots \end{cases}$$

- That means, every second value of h(n) is zero, except for one such value, as shown in the figure.



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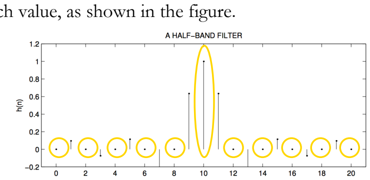

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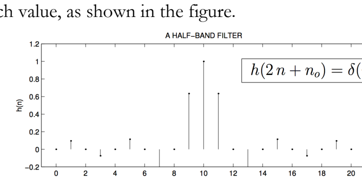

$$h(2n + n_o) = \delta(n),$$

Interpolation Filter Example 4

- When interpolating a signal x(n) by a factor L, the original samples of x(n) are preserved if h(n) is a Nyquist-L filter.
- A Nyquist-L filter simply generalizes the notion of the halfband filter to L > 2.

Interpolation Filter Example 4

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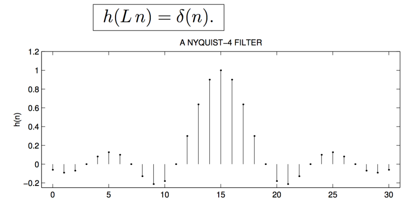
- A (0-centered) Nyquist-L filter h(n) is one for which

$$h(L\,n) = \delta(n).$$



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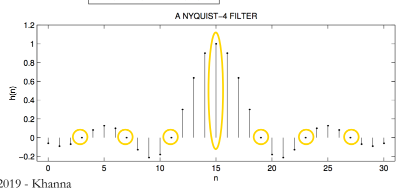

Non-integer Resampling

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- T'=TM/L

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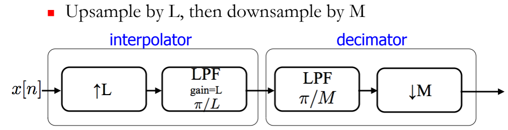
  - Upsample by L, then downsample by M



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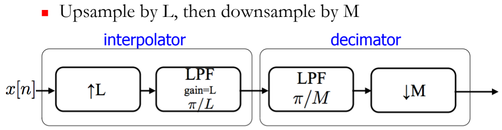
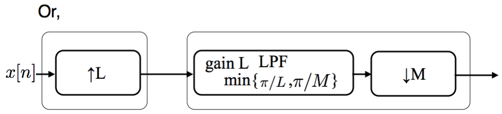
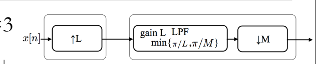


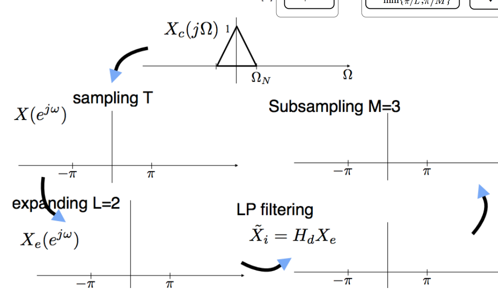
Or,



Example

- T'=3/2T → L=2, M=3





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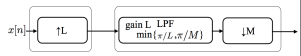
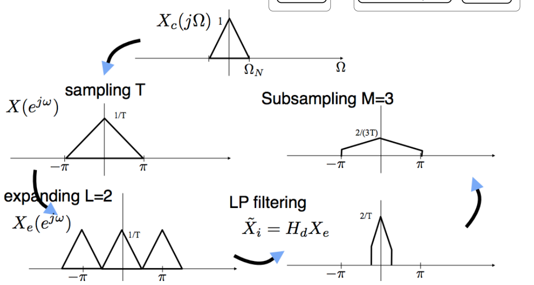
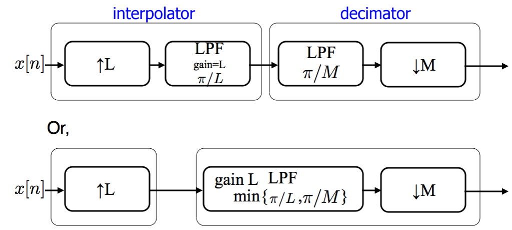




Non-integer Sampling

- T'=TM/L
  - Downsample by M, then upsample by L?



Example

- What if we want to resample by 1.01T?

Example

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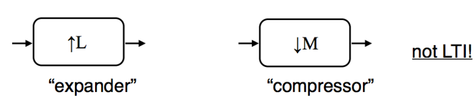
  - Upsample by L=100
  - Filter π /101
  - Downsample by M=101

Example

- What if we want to resample by 1.01T?
  - Upsample by L=100
  - Filter π /101 ($$$$$)
  - Downsample by M=101
- Fortunately there are ways around it!
  - Called multi-rate signal processing
  - Uses compressors, expanders and filtering

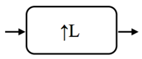
Interchanging Operations



Upsampling
-expanding in time
-compressing in frequency

Downsampling
-compressing in time
-expanding in frequency

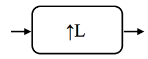
Interchanging Operations - Expander



"expander"

Upsampling
-expanding in time
-compressing in frequency

Interchanging Operations - Expander



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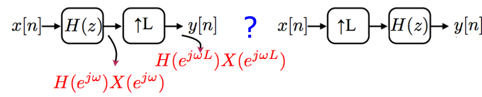

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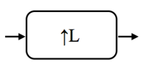

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Upsampling
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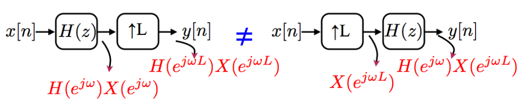

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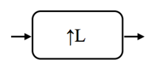

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Upsampling
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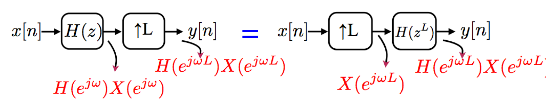
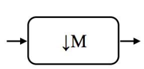


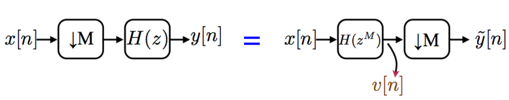
Interchanging Operations - Compressor



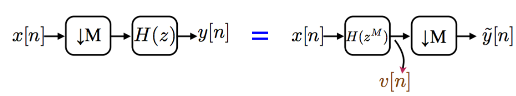
not LTI!

"compressor"

Downsampling
-compressing in time
-expanding in frequency



Interchanging Operations - Compressor

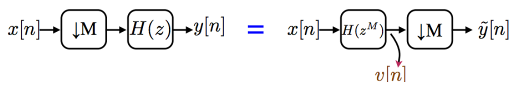


$$Y(e^{j\omega}) = H(e^{j\omega})\left(\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{H\left(e^{j(\omega-2\pi i)}\right)}_{H(e^{j\omega})} X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM(\frac{\omega}{M}-\frac{2\pi i}{M})}\right) X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

Interchanging Operations - Compressor



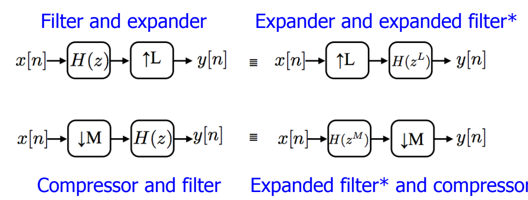
$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} H\left(e^{jM(\frac{\omega}{M}-\frac{2\pi i}{M})}\right) X\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

$$V(e^{j\omega}) = H(e^{j\omega M})X(e^{j\omega})$$

=

$$\tilde{Y}(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} V\left(e^{j(\frac{\omega}{M}-\frac{2\pi i}{M})}\right)$$

Interchanging Operations - Summary



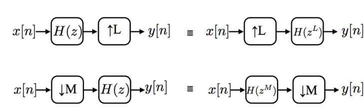
*Expanded filter = expanded impulse response, compressed freq response

Multi-Rate Signal Processing

- What if we want to resample by 1.01T?
  - Expand by L=100
  - Filter π /101 ($$$$$)
  - Compress by M=101
- Fortunately there are ways around it!
  - Called multi-rate
  - Uses compressors, expanders and filtering

Big Ideas

- Downsampling/Upsampling
- Practical Interpolation
- Non-integer Resampling
- Multi-Rate Processing
  - Interchanging Operations



Admin

- HW 4 due Sunday
  - Typo, homework problem 4.28 removed, homework handout fixed.