ESE 531: Digital Signal Processing

Lec 13: February 28, 2018

Frequency Response of LTI Systems

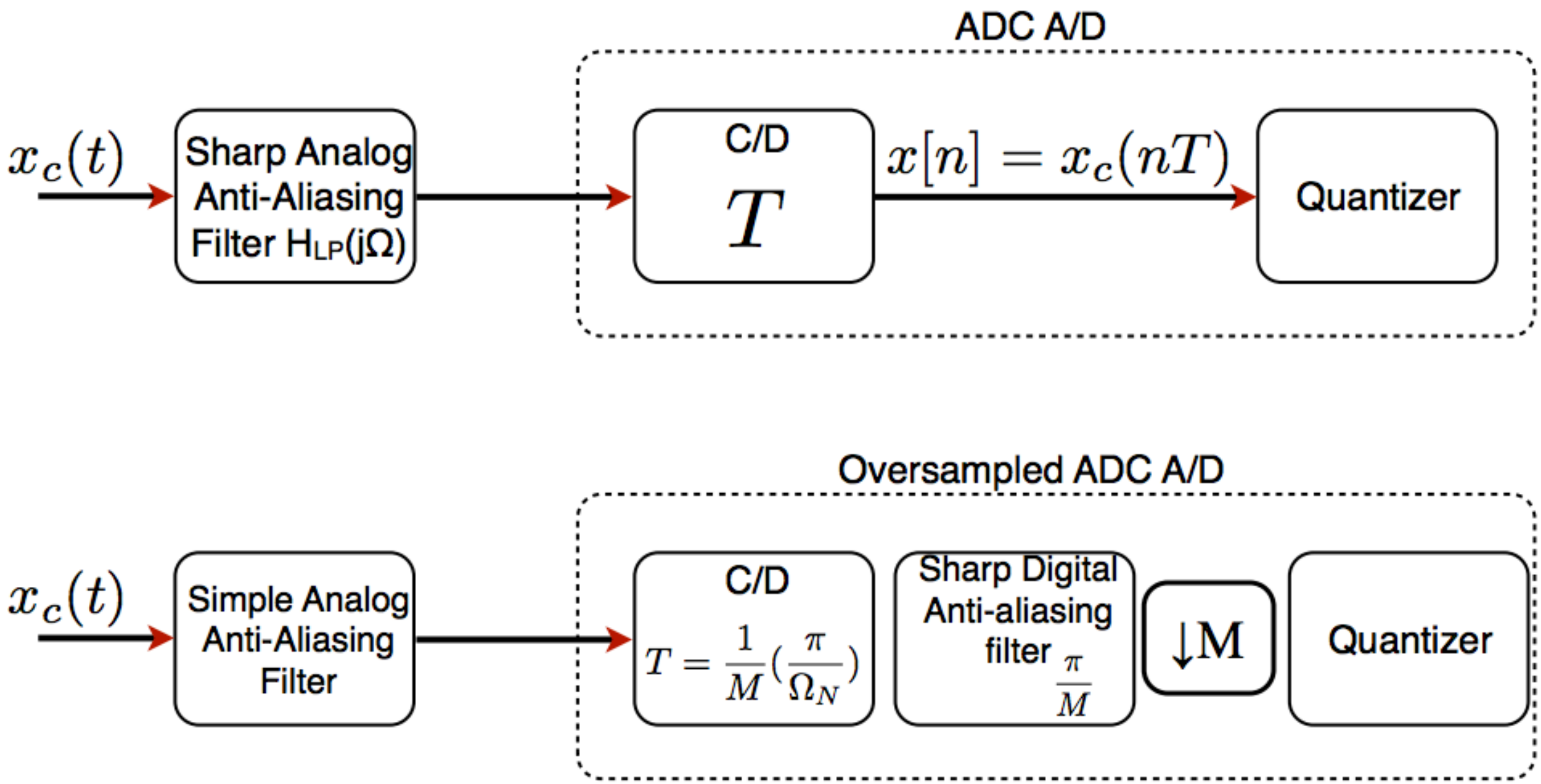
Penn ESE 531 Spring 2018 – Khanna
Adapted from M. Lustig, EECS Berkeley

Lecture Outline

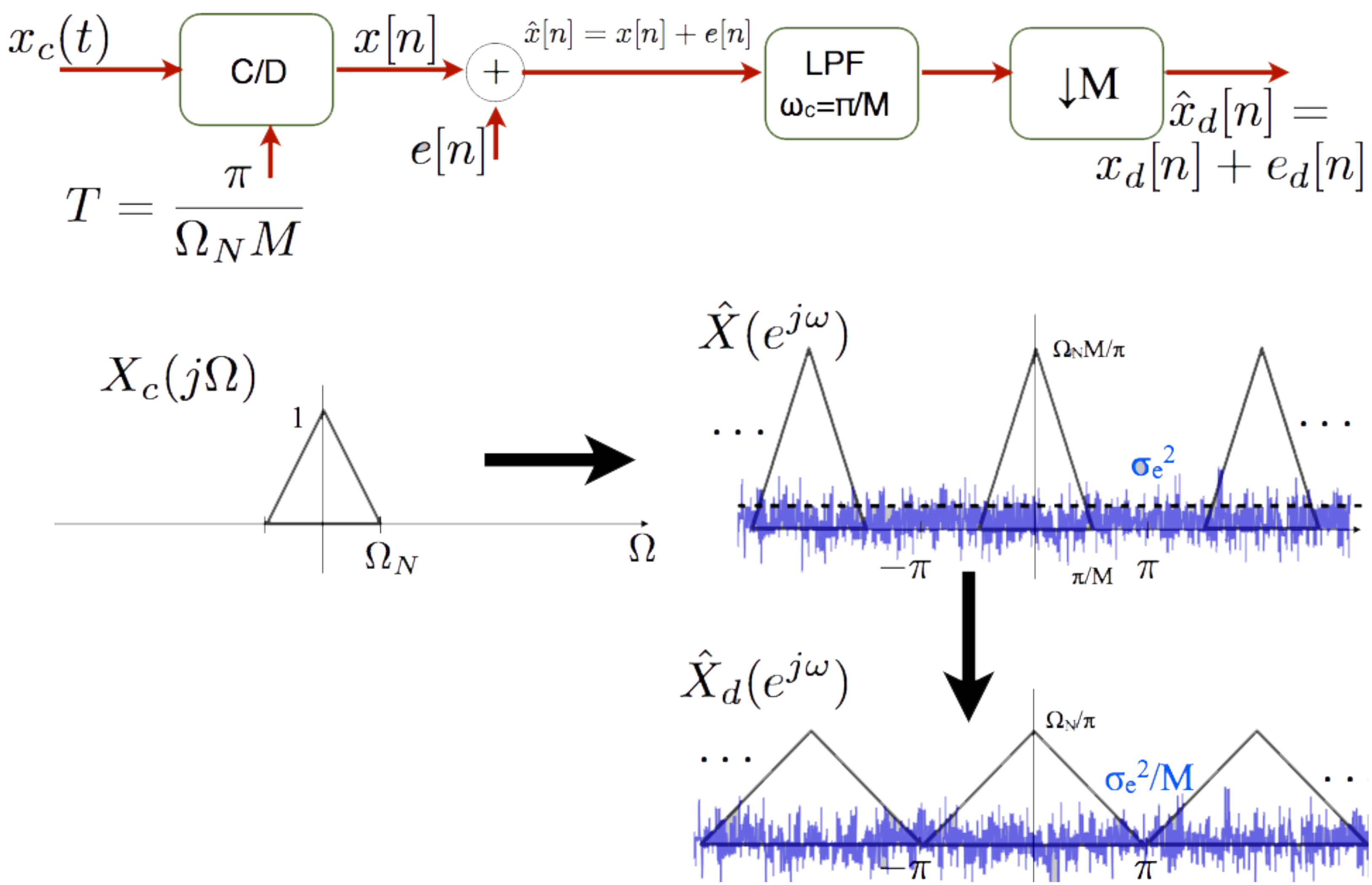
- Noise Shaping
- Frequency Response of LTI Systems
  - Magnitude Response
    - Simple Filters
  - Phase Response
    - Group Delay
  - Example: Zero on Real Axis

Noise Shaping

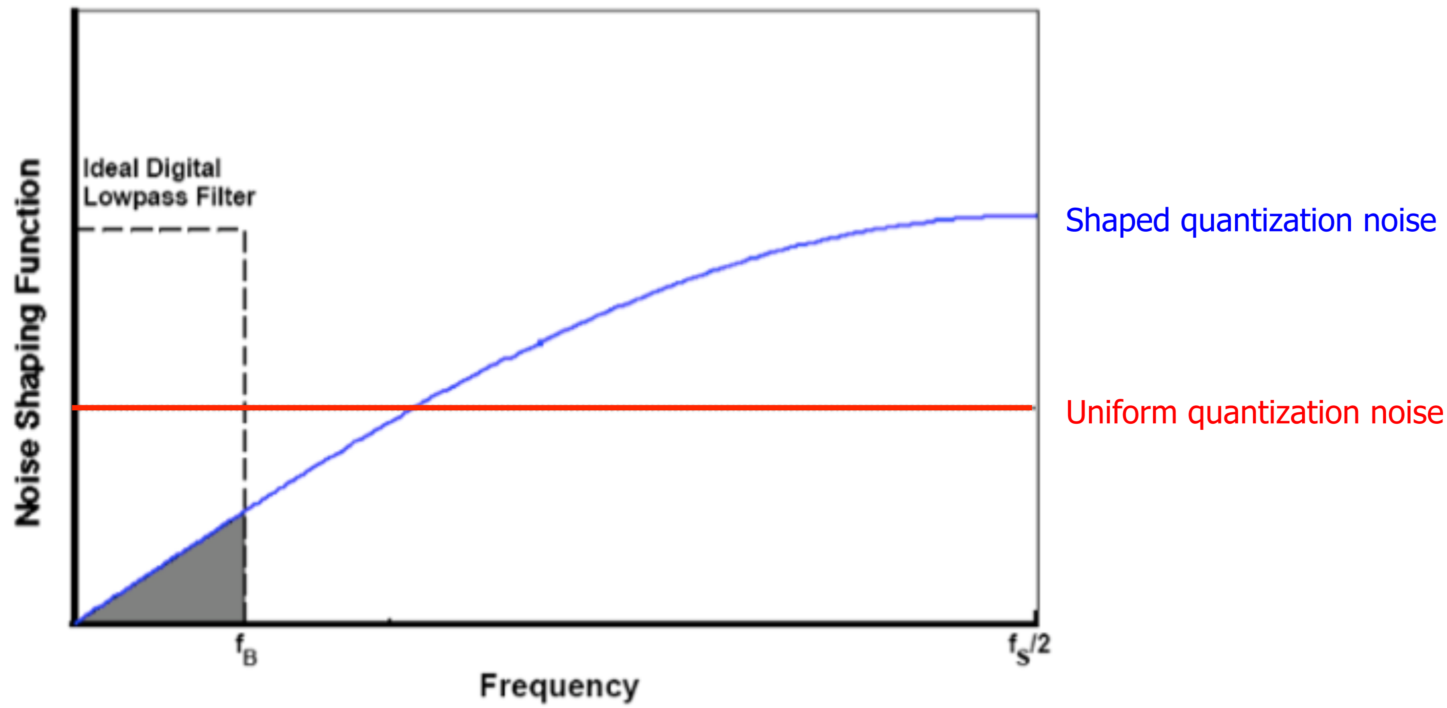
Oversampled ADC



Quantization Noise with Oversampling

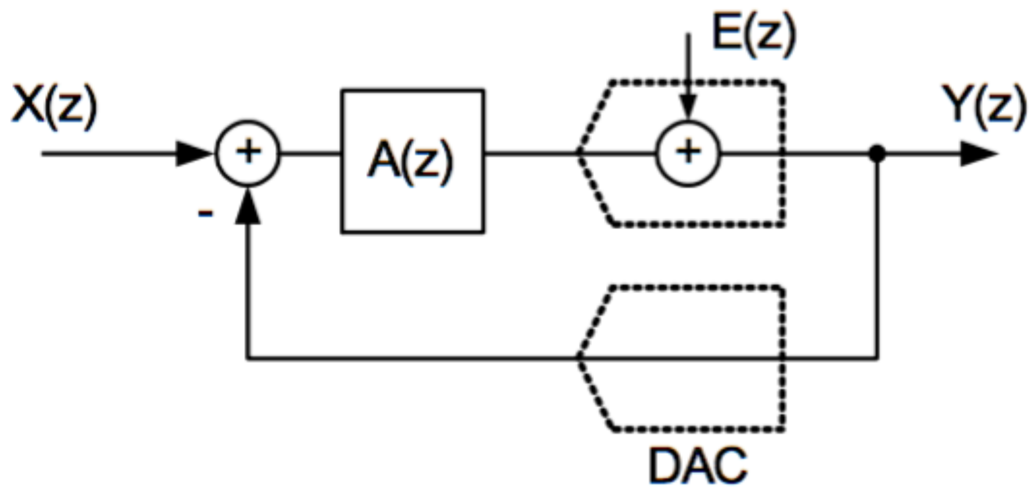


Noise Shaping



- Idea: "Somehow" build an ADC that has most of its quantization noise at high frequencies
- Key: Feedback

Noise Shaping Using Feedback



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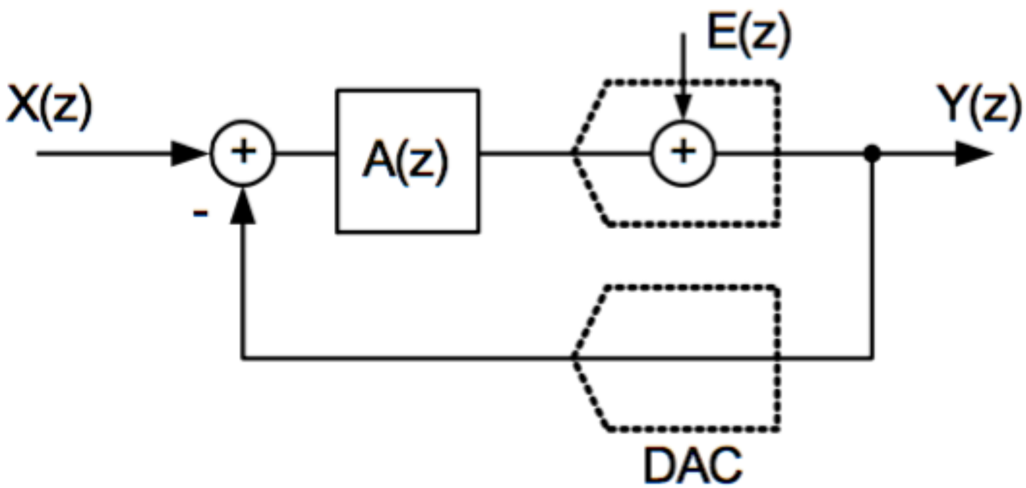


$$Y(z) = E(z) + A(z)X(z) - A(z)Y(z)$$

$$= E(z)\frac{1}{1+A(z)} + X(z)\frac{A(z)}{1+A(z)}$$

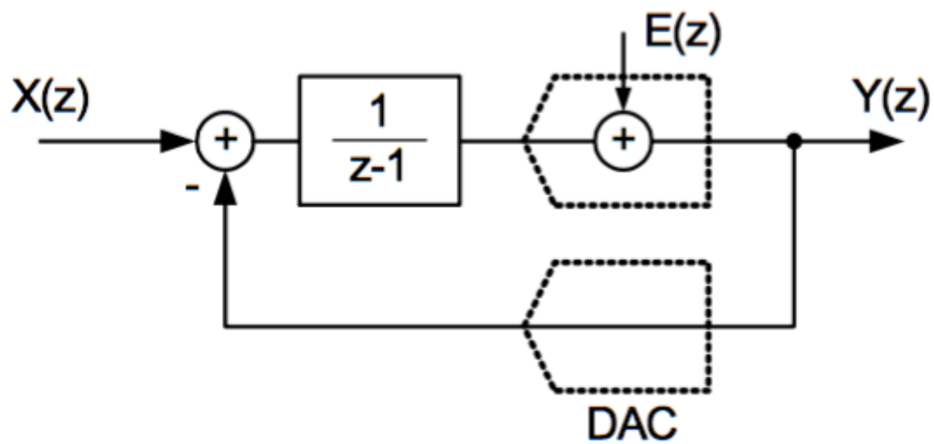
$$= E(z)\underbrace{H_E(z)}_{\substack{\text{Noise}\\\text{Transfer}\\\text{Function}}} + X(z)\underbrace{H_X(z)}_{\substack{\text{Signal}\\\text{Transfer}\\\text{Function}}}$$

Noise Shaping Using Feedback

$$Y(z) = E(z)\underbrace{\frac{1}{1+A(z)}}_{\substack{\text{Noise}\\ \text{Transfer}\\ \text{Function}}} + X(z)\underbrace{\frac{A(z)}{1+A(z)}}_{\substack{\text{Signal}\\ \text{Transfer}\\ \text{Function}}}$$

- Objective
  - Want to make STF unity in the signal frequency band
  - Want to make NTF "small" in the signal frequency band
- If the frequency band of interest is around DC ($0 \ldots f_B$) we achieve this by making $|A(z)| >> 1$ at low frequencies
  - Means that NTF << 1
  - Means that STF ≅ 1

First Order Sigma-Delta Modulator



$$Y(z) = E(z)\frac{1}{1+\frac{1}{z-1}} + X(z)\frac{\frac{1}{z-1}}{1+\frac{1}{z-1}}$$

$$= E(z)\left(1 - z^{-1}\right) + X(z)z^{-1}$$

- Output is equal to delayed input plus filtered quantization noise

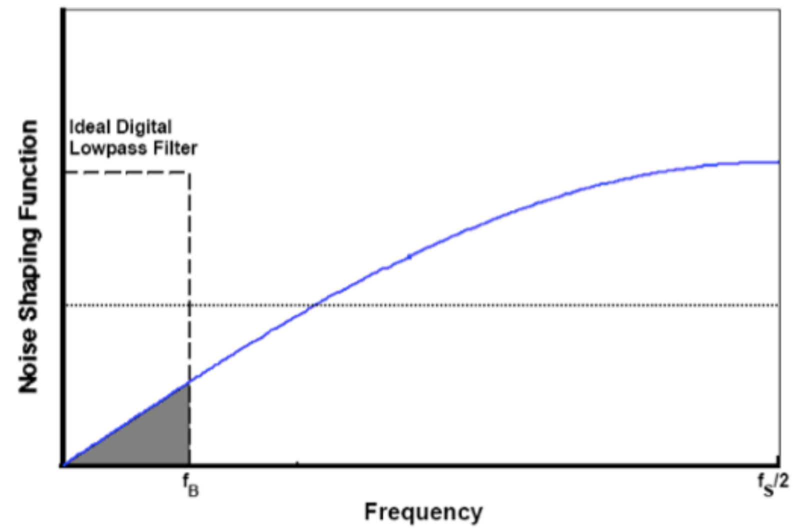
NTF Frequency Domain Analysis

$$H_e(z) = 1 - z^{-1}$$

$$H_e(j\omega) = \left(1 - e^{-j\omega T}\right) = 2e^{-j\omega T/2}\left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2}\right)$$

$$= 2e^{-j\frac{\omega T}{2}}\left(j\sin\left(\frac{\omega T}{2}\right)\right) = 2\sin\left(\frac{\omega T}{2}\right)e^{-j\frac{\omega T - \pi}{2}}$$

$$\left|H_e(f)\right| = 2\left|\sin(\pi f T)\right| = 2\left|\sin\left(\pi\frac{f}{f_s}\right)\right|$$

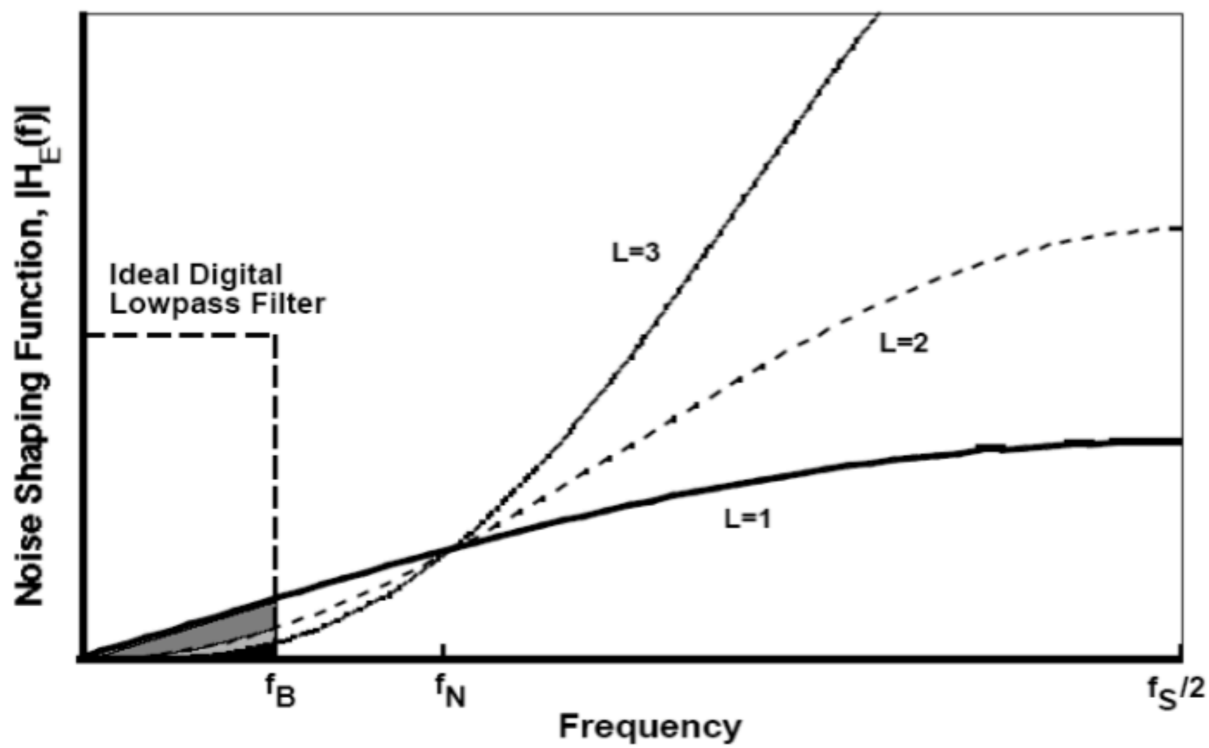


- "First order noise Shaping"
  - Quantization noise is attenuated at low frequencies, amplified at high frequencies

Higher Order Noise Shaping

- L^{th} order noise transfer function

$$H_E(z) = \left(1 - z^{-1}\right)^L$$



Frequency Response of LTI Systems

Frequency Response of LTI System

- LTI Systems are uniquely determined by their impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[k] * h[k]$$

- We can write the input-output relation also in the z-domain

$$Y(z) = H(z)X(z)$$

- Or we can define an LTI system with its frequency response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- $H(e^{j\omega})$ defines magnitude and phase change at each frequency

Frequency Response of LTI System

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$$

- We can define a magnitude response

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right| \left|X\left(e^{j\omega}\right)\right|$$

- And a phase response

$$\angle Y\left(e^{j\omega}\right) = \angle H\left(e^{j\omega}\right) + \angle X\left(e^{j\omega}\right)$$

Phase Response

- Limit the range of the phase response

$$-\pi < \mathrm{ARG}[H(e^{j\omega})] \leq \pi.$$

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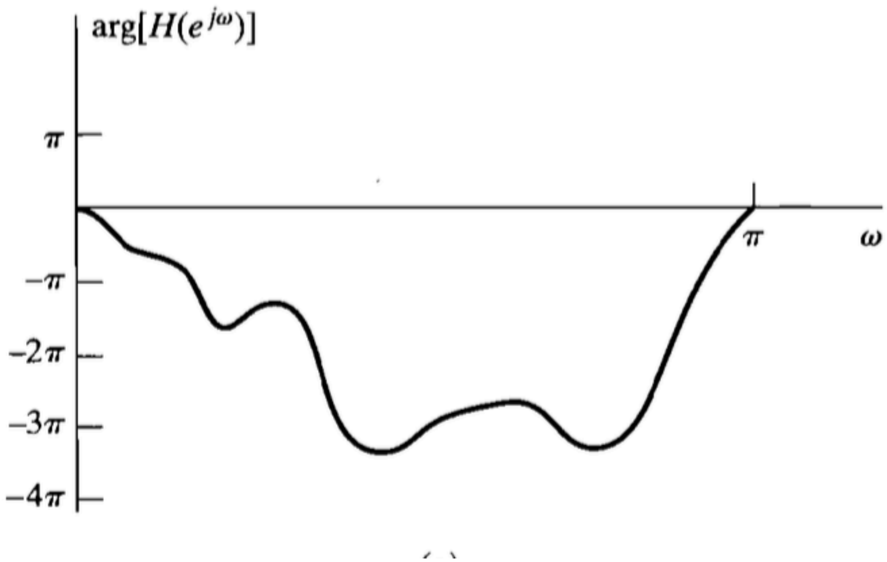

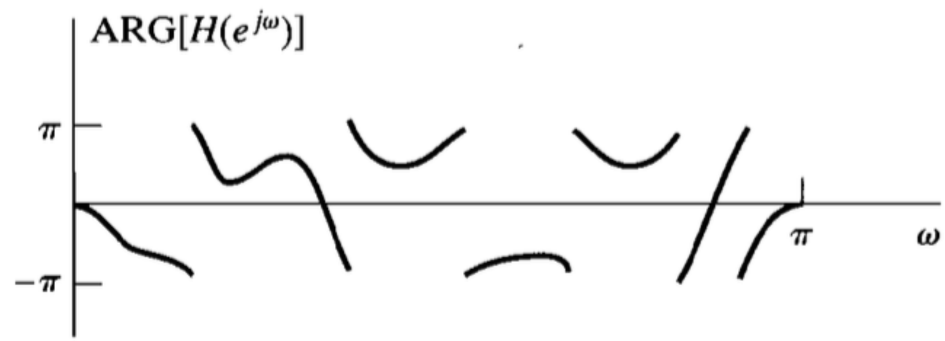

Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Unwrapped phase

- More later…

Linear Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = |\frac{b_0}{a_0}| \cdot \frac{\prod_{k=0}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=0}^{N} |1 - d_k e^{-j\omega}|}$$

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Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| \;\; = |v_1|$$

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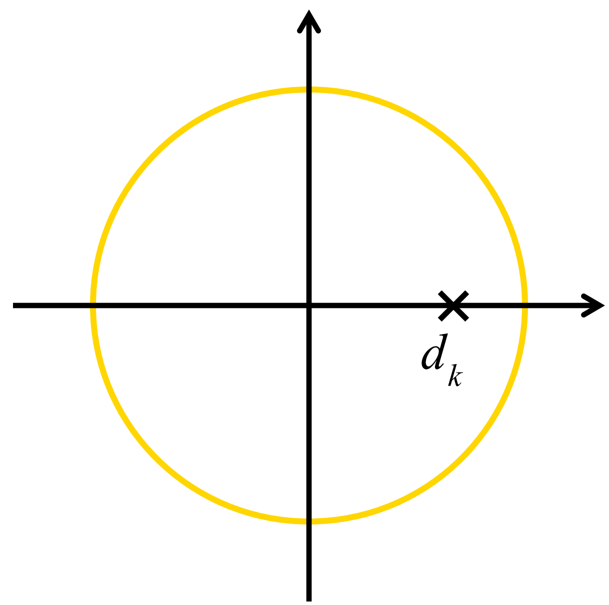

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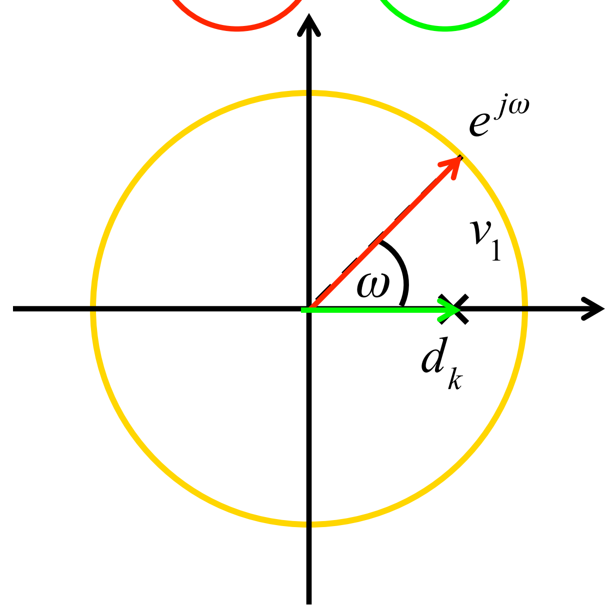

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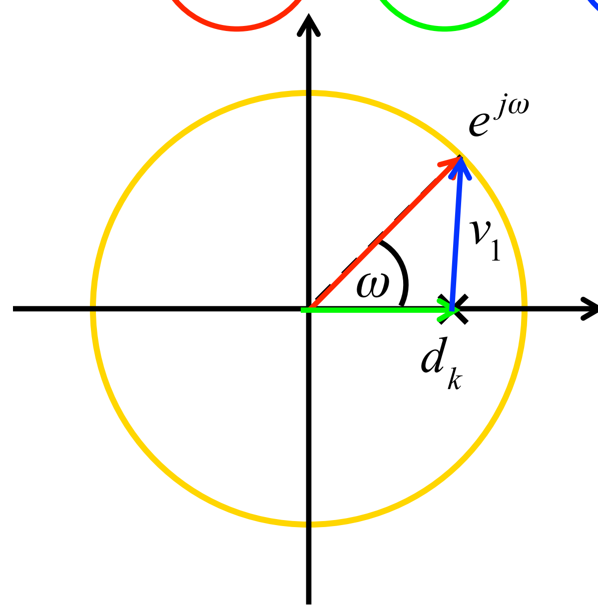

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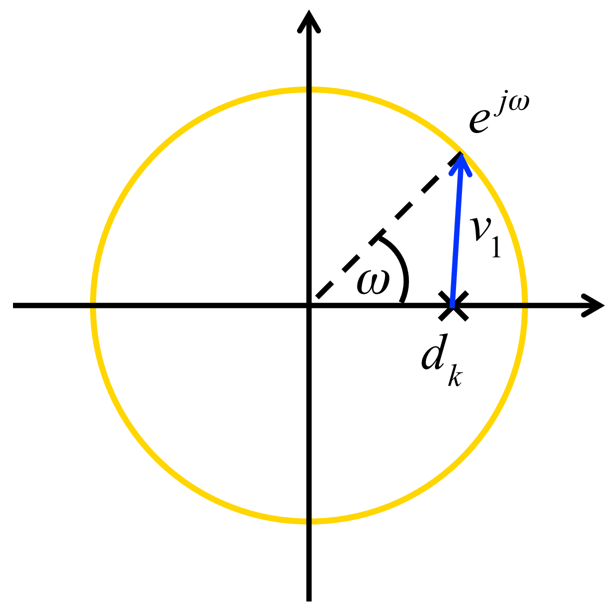

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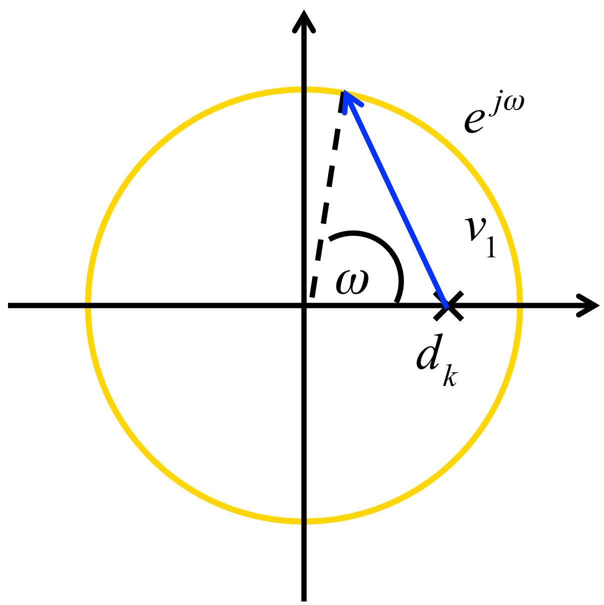

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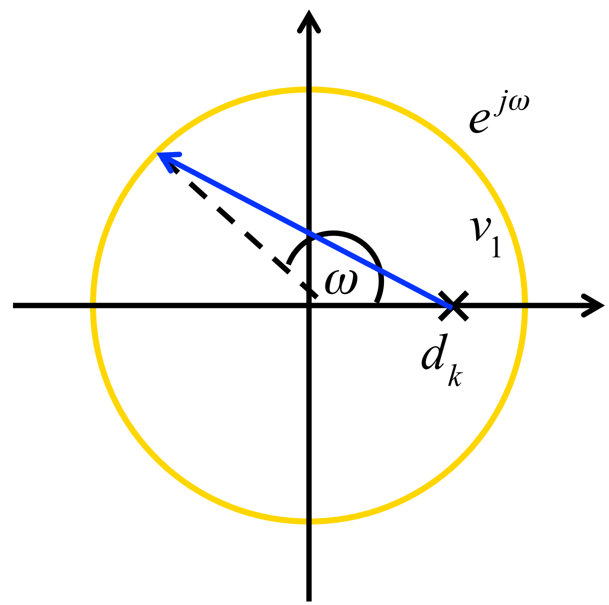

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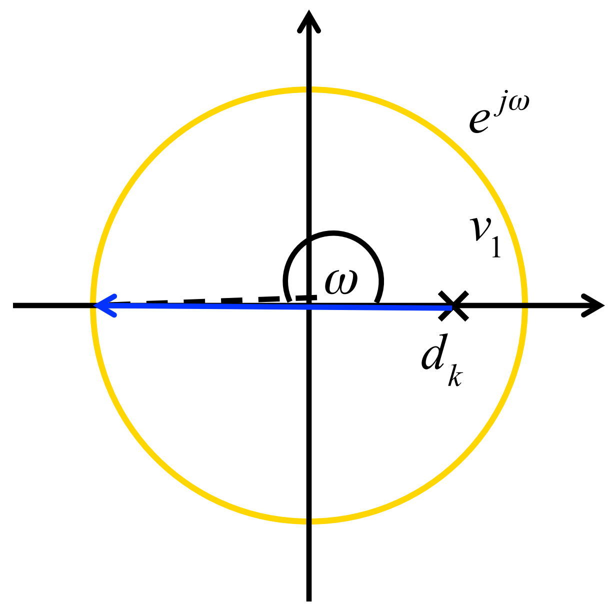

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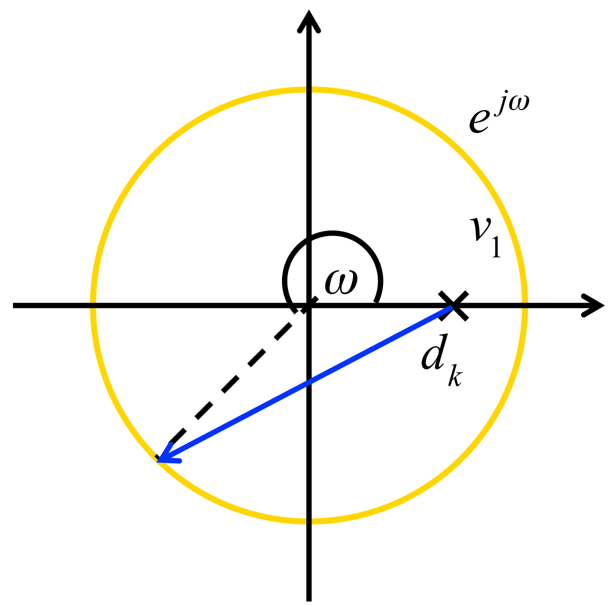

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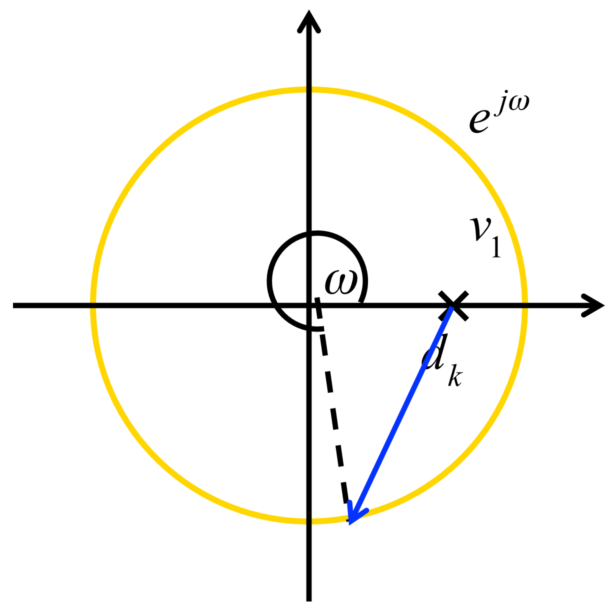

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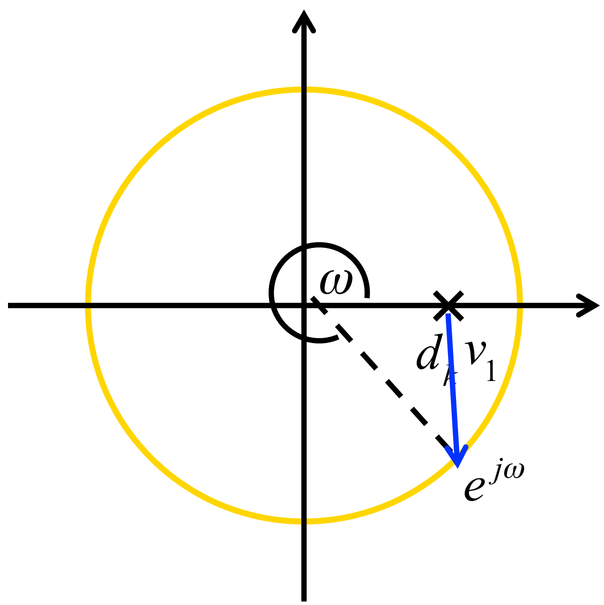

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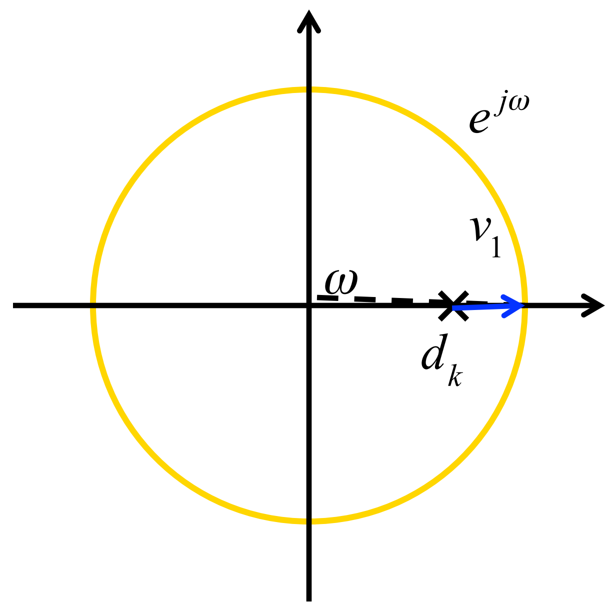
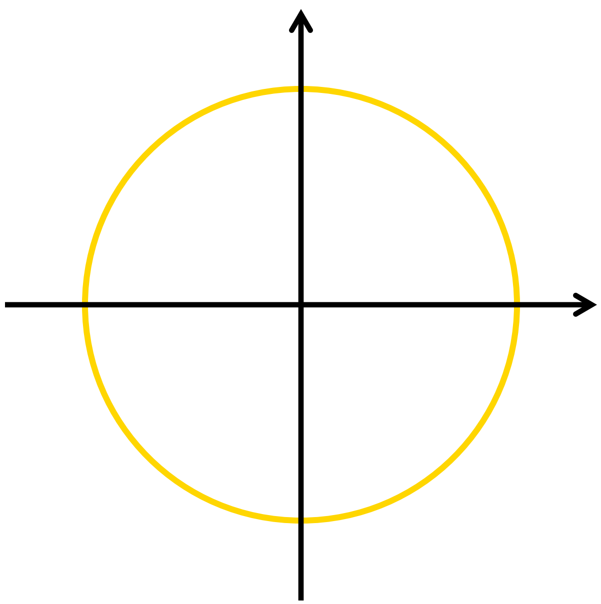


Magnitude Response Example

$$H(z) = 0.05\frac{1+z^{-1}}{1-0.9z^{-1}}$$

$$|H(z)| = 0.05\frac{|v_2|}{|v_1|}$$



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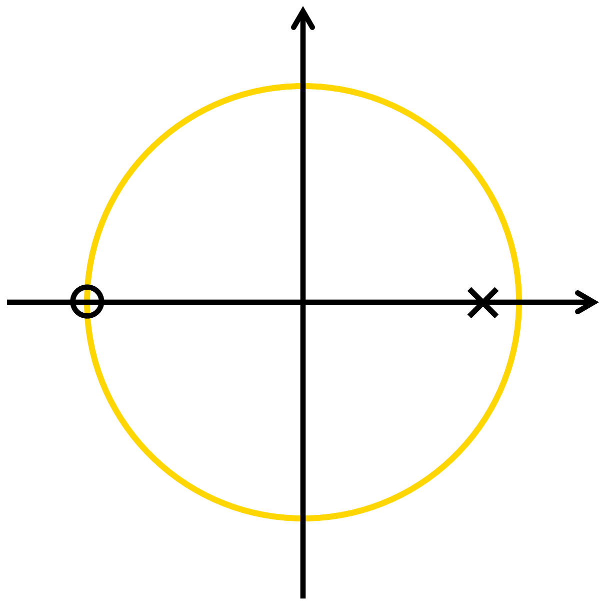

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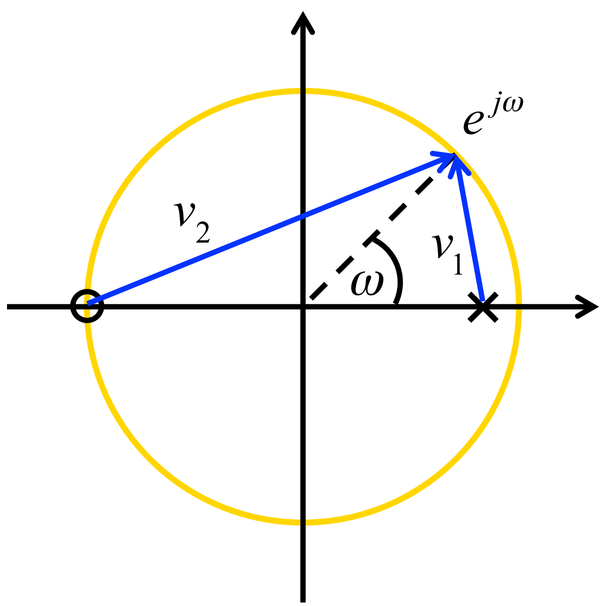

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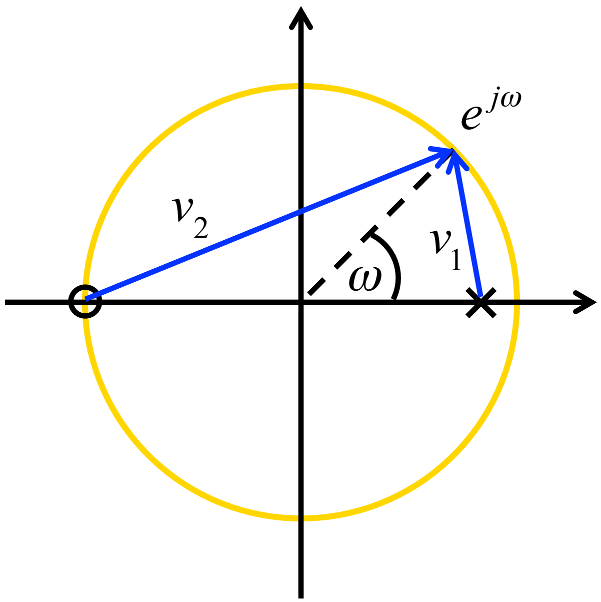

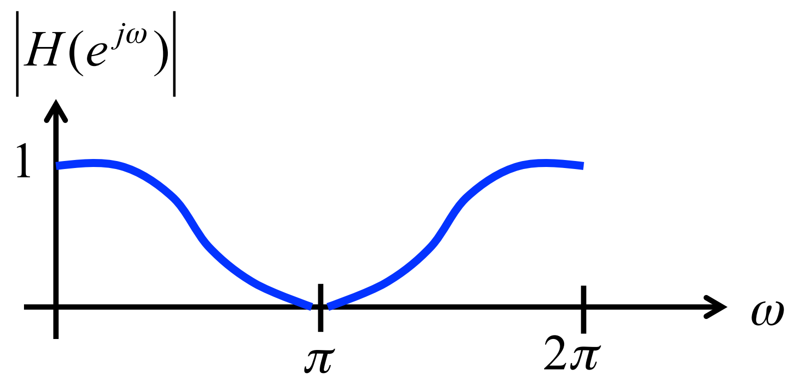

Simple Low Pass Filter

$$H_{LP}(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}} \qquad |\alpha| < 1$$

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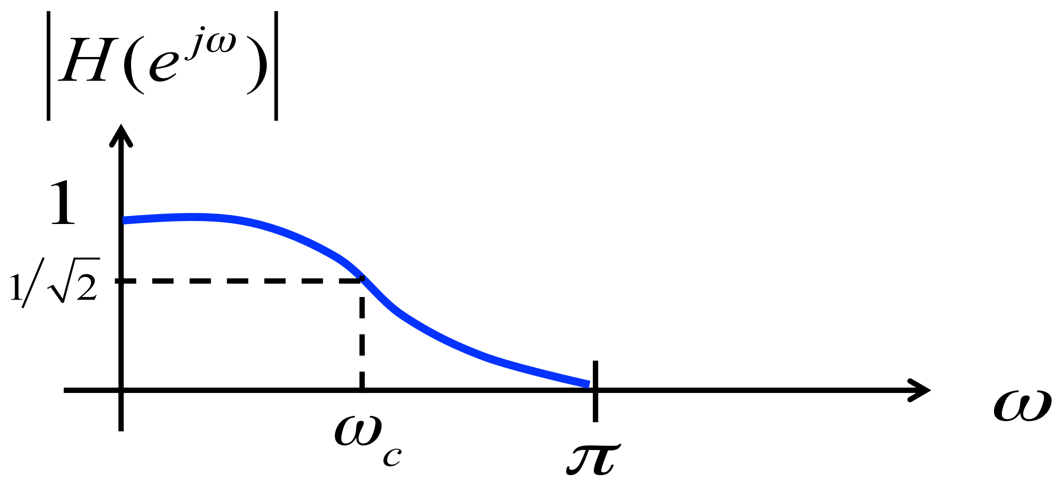

w_c is the 3dB cutoff frequency $\qquad \alpha = \frac{1-\sin(\omega_c)}{\cos(\omega_c)}$

Simple High Pass Filter

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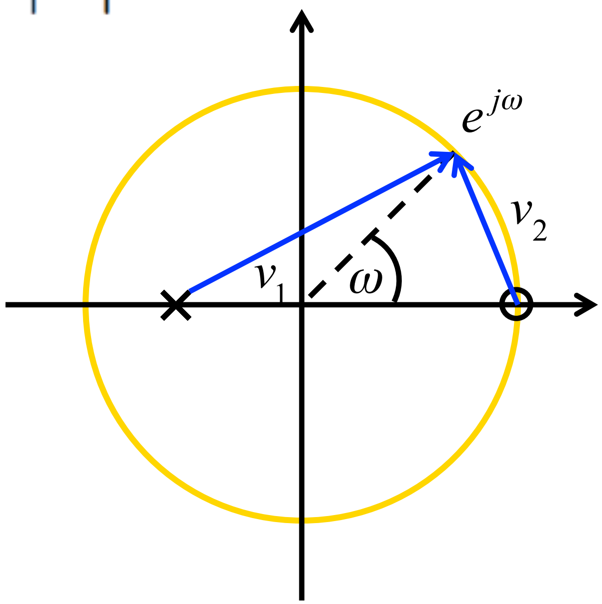

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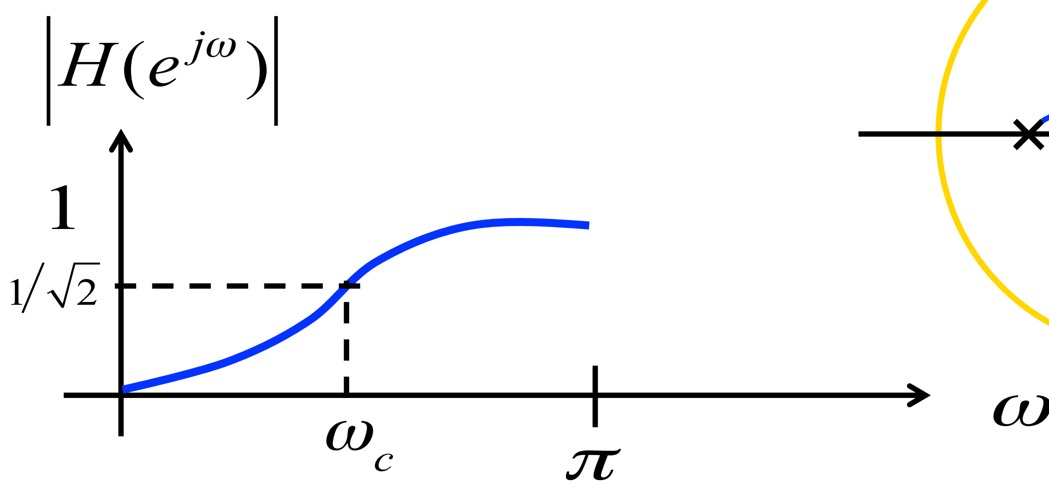

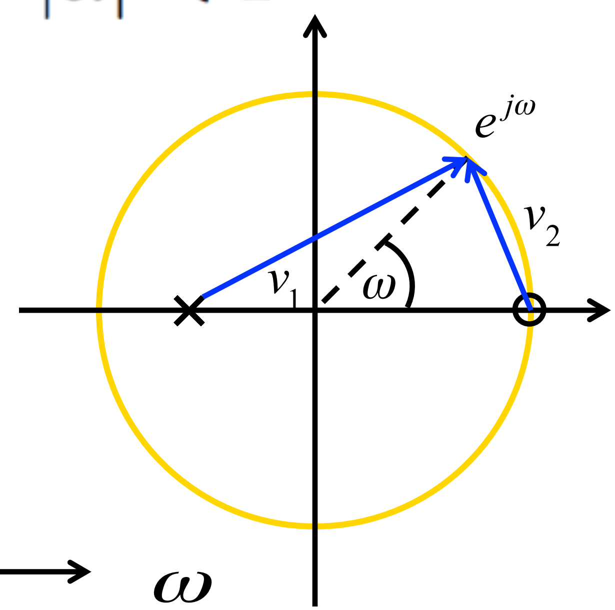

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Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \qquad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$

$\cos(\omega_0) = \beta$

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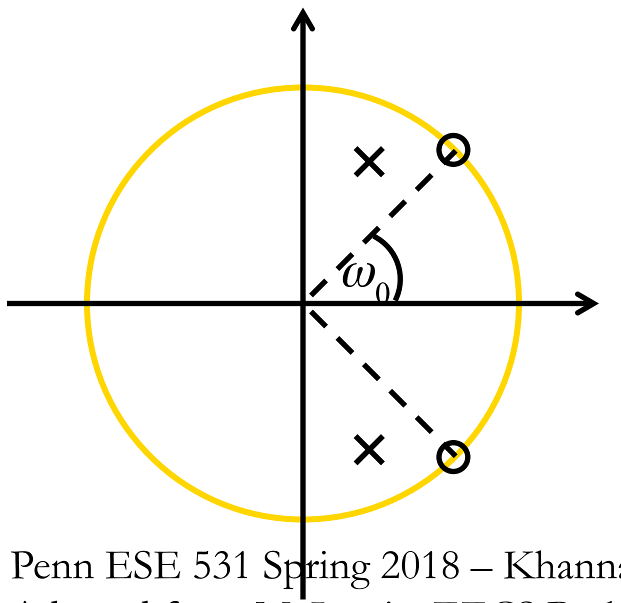

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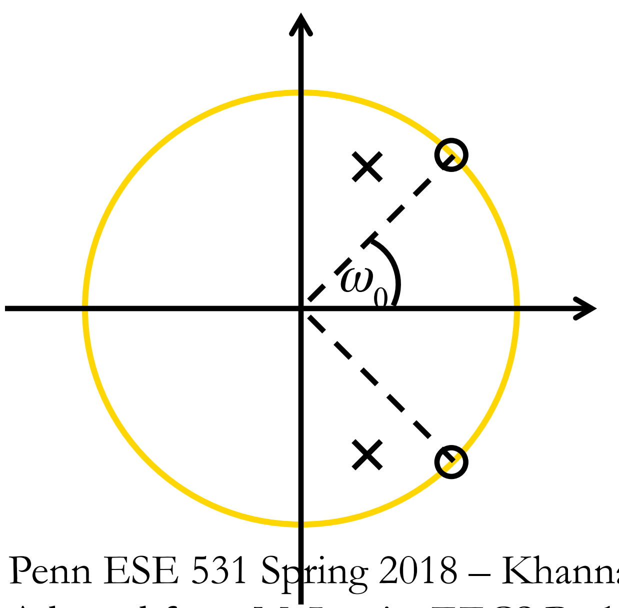

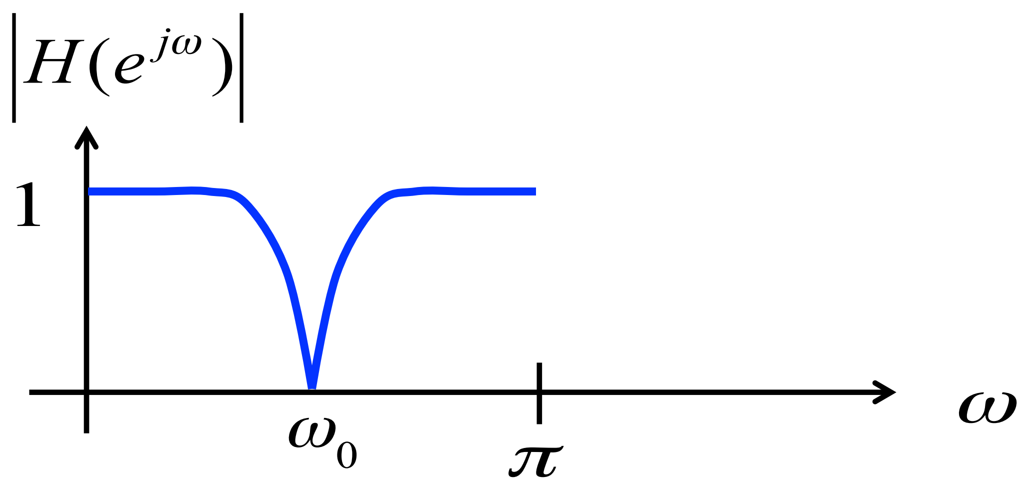

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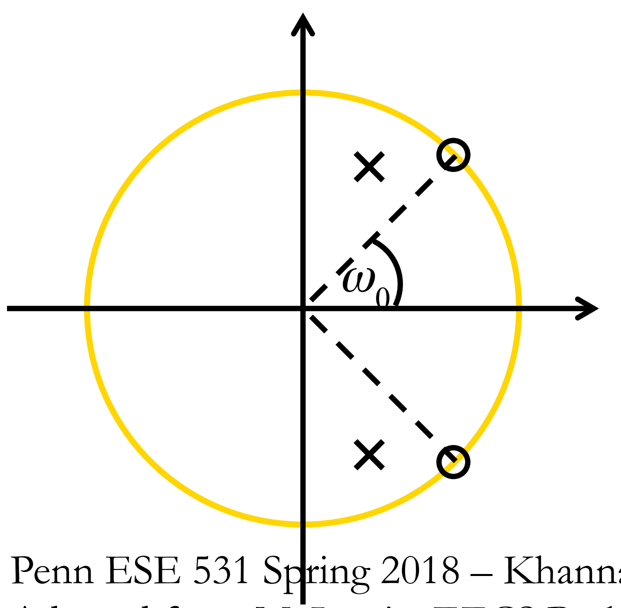
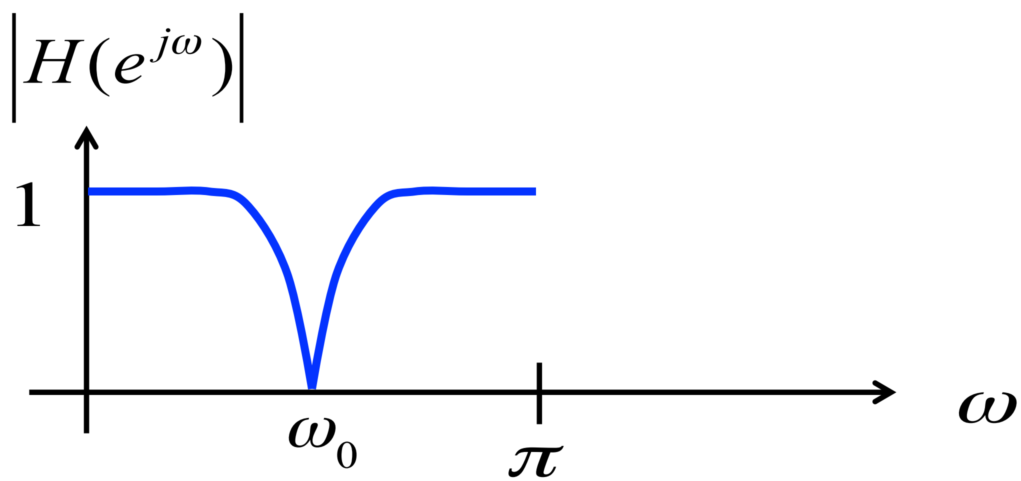
Note:

$$H_{BS}(\mp 1) = \frac{1+\alpha}{2}\frac{2\pm 2\beta}{(1+\alpha)(1\pm\beta)} = 1$$





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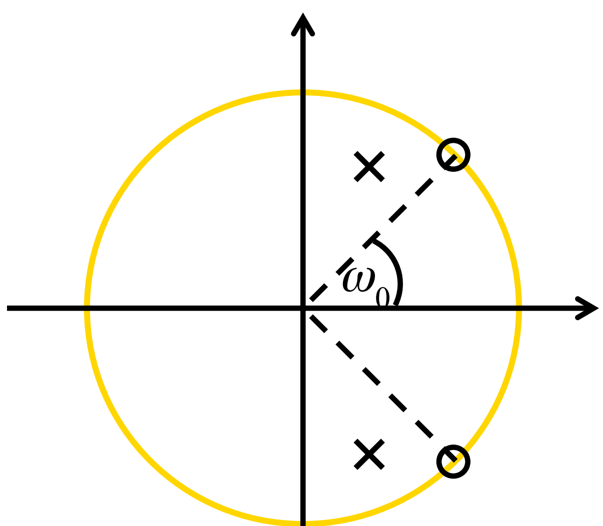
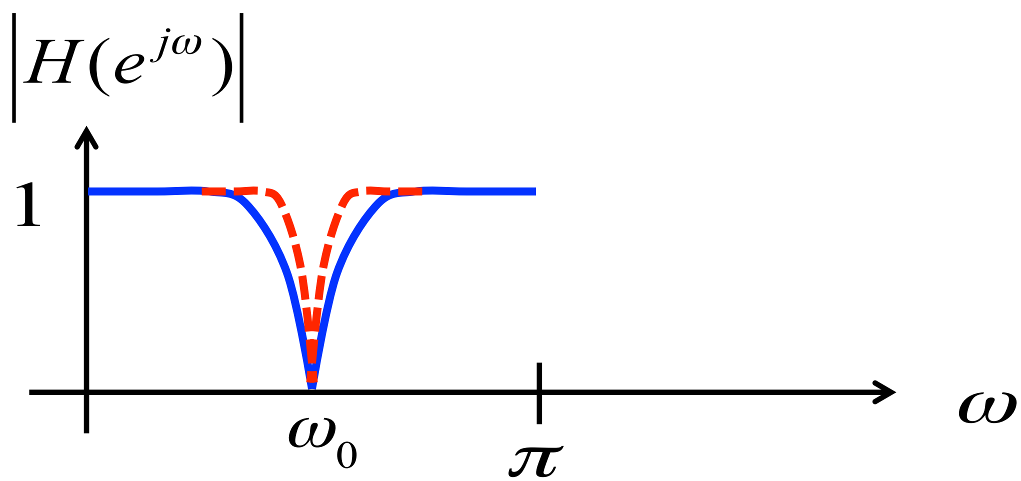
Note: As $\alpha \to 1$ poles approach zeros

$$H_{BS}(\mp 1) = \frac{1+\alpha}{2}\,\frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$$





Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \qquad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

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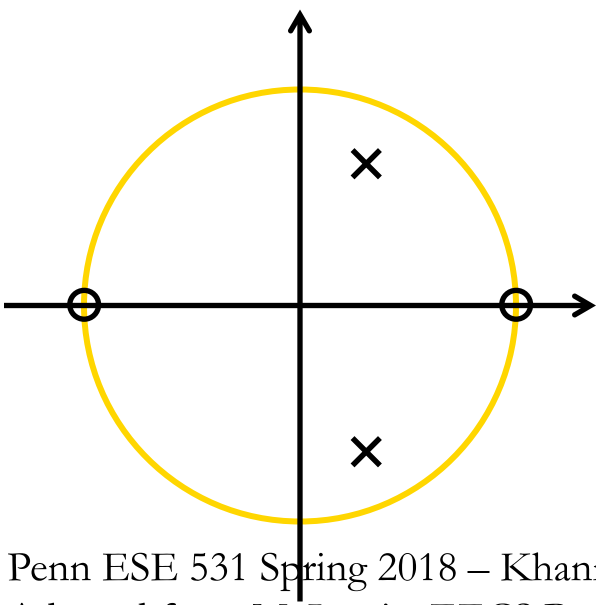

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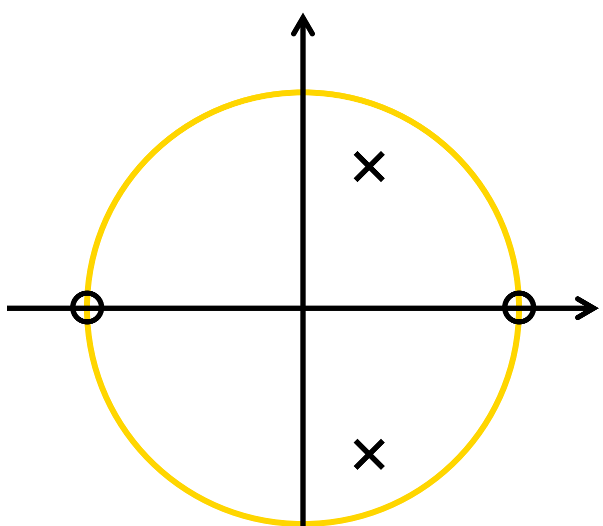

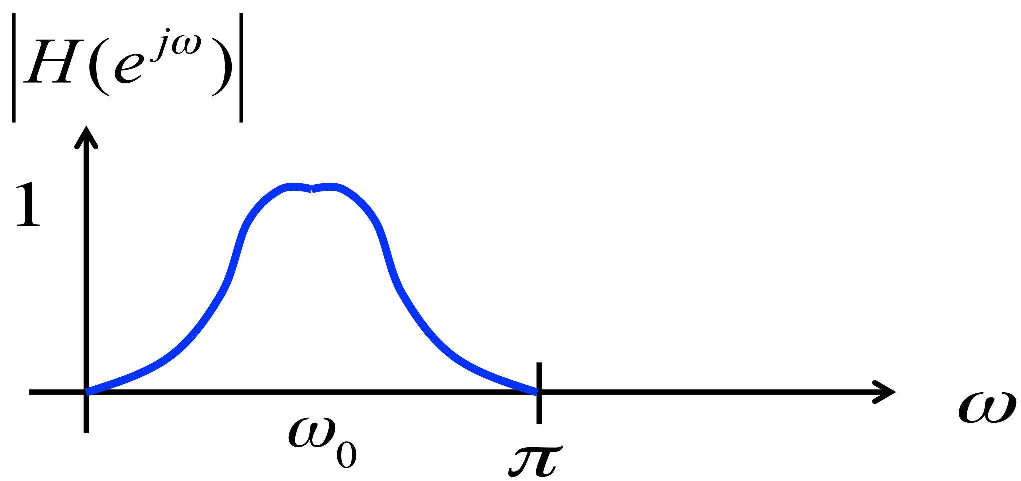

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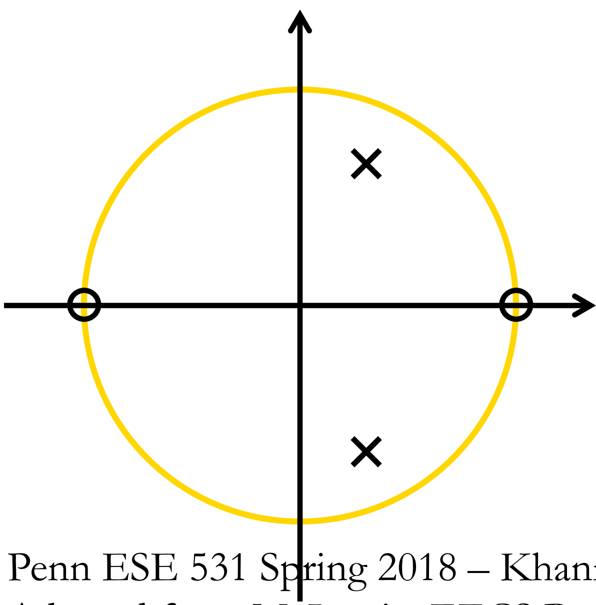
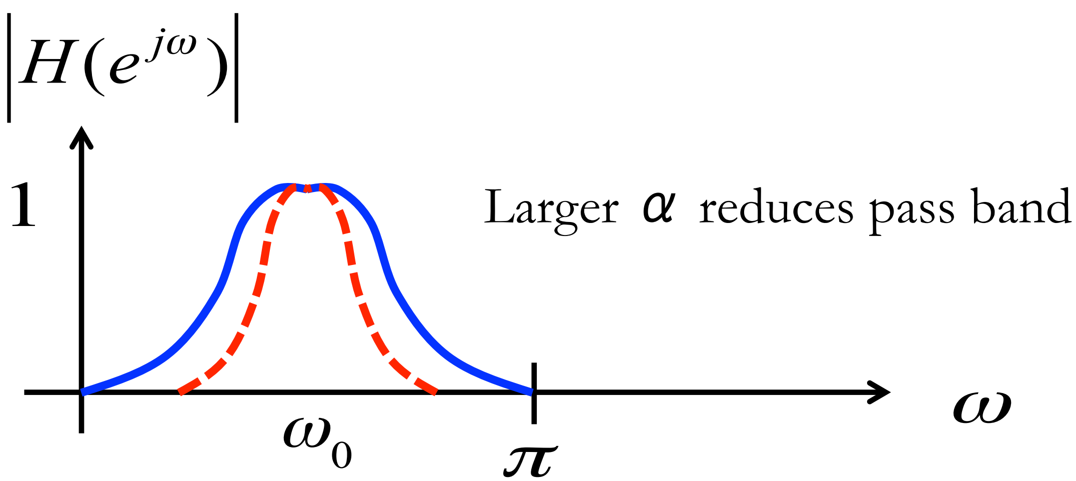


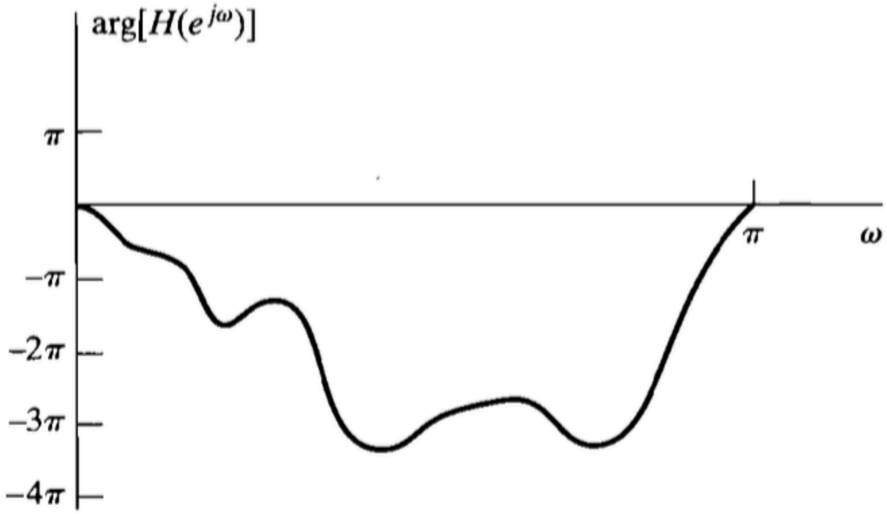


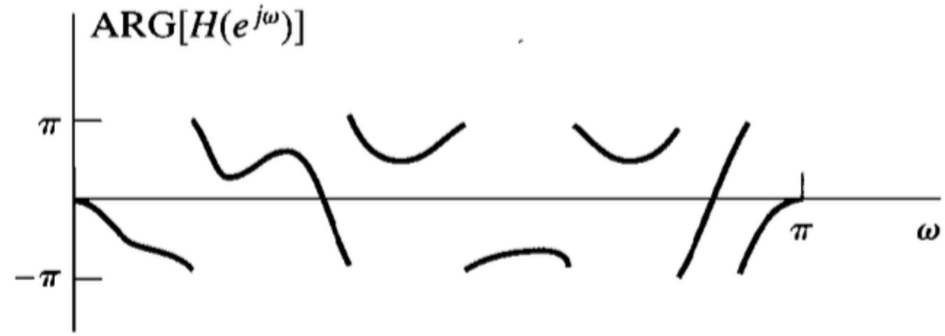
$$\cos(\omega_0) = \beta$$

Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$





Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \quad \leftrightarrow \quad h[n] = \delta[n - n_d]$$

Phase Response Example

$$H(e^{j\omega}) = e^{-j\omega n_d} \quad \leftrightarrow \quad h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

ARG is the wrapped phase
arg is the unwrapped phase

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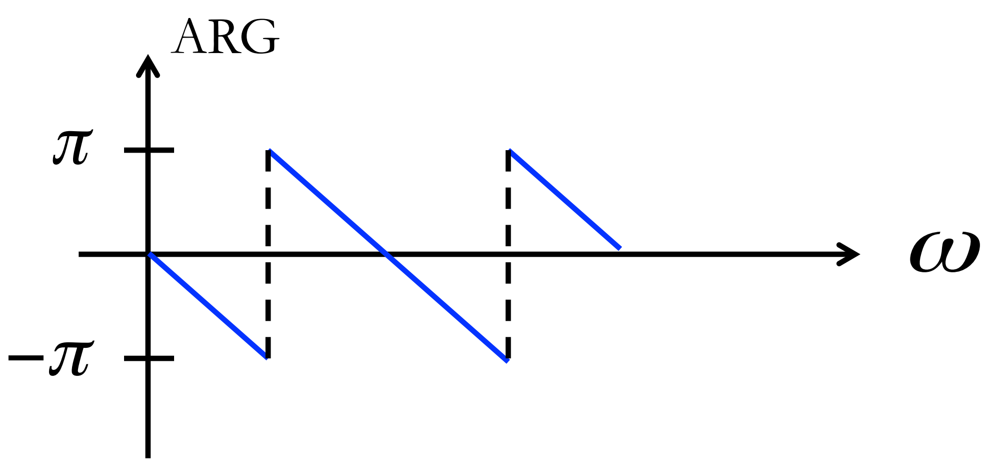
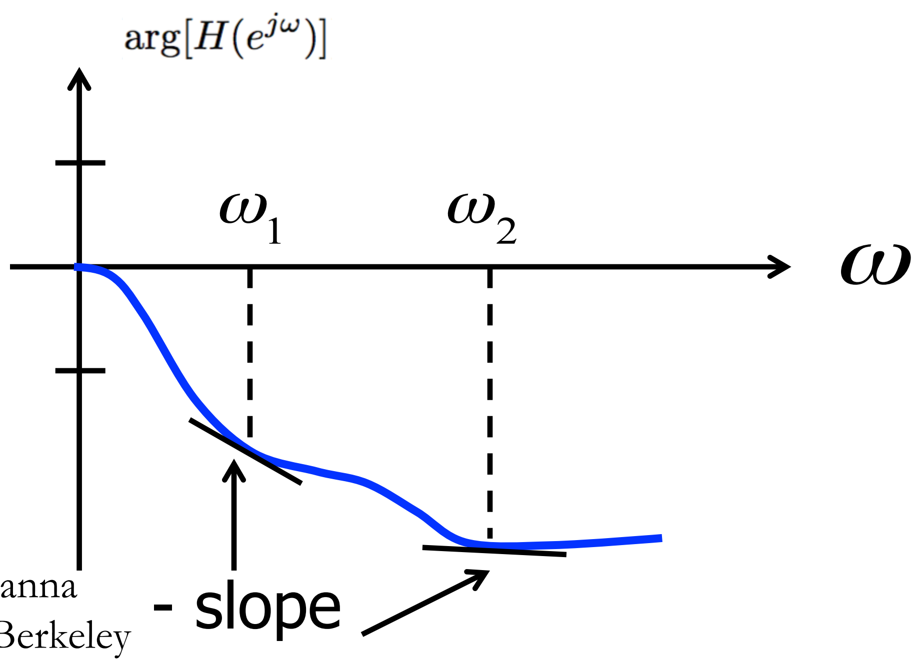


Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



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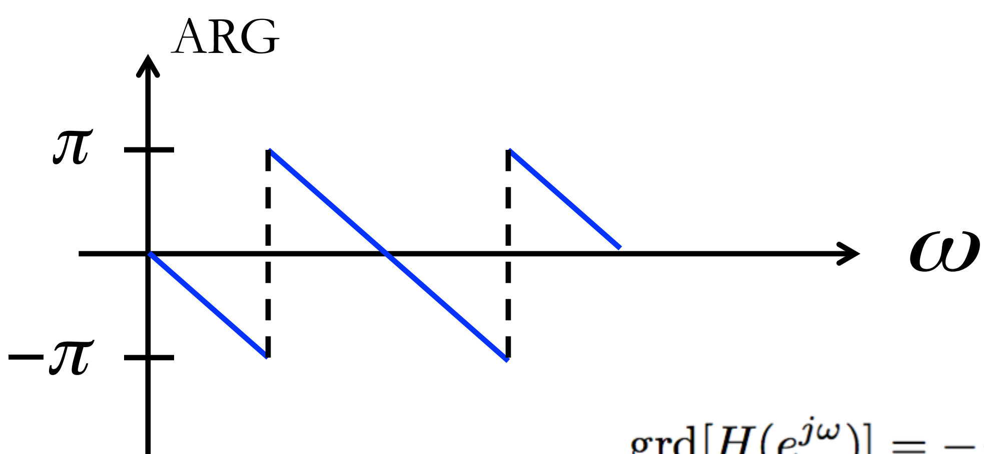

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For linear phase system, group delay is n_d

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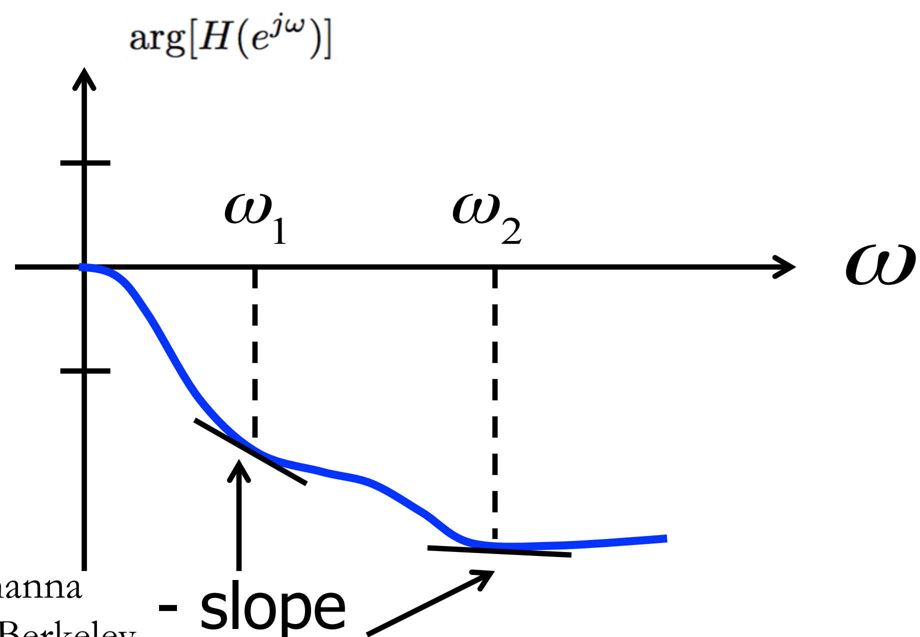

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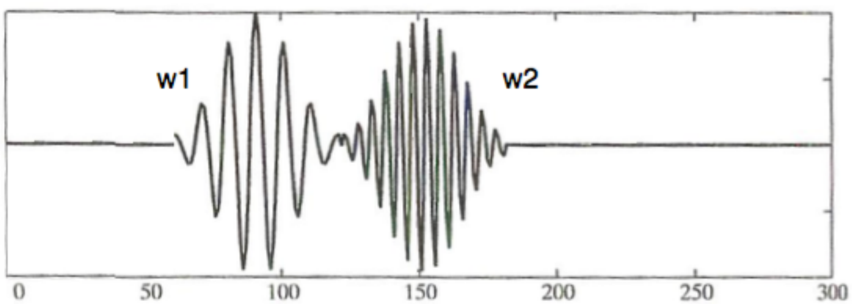
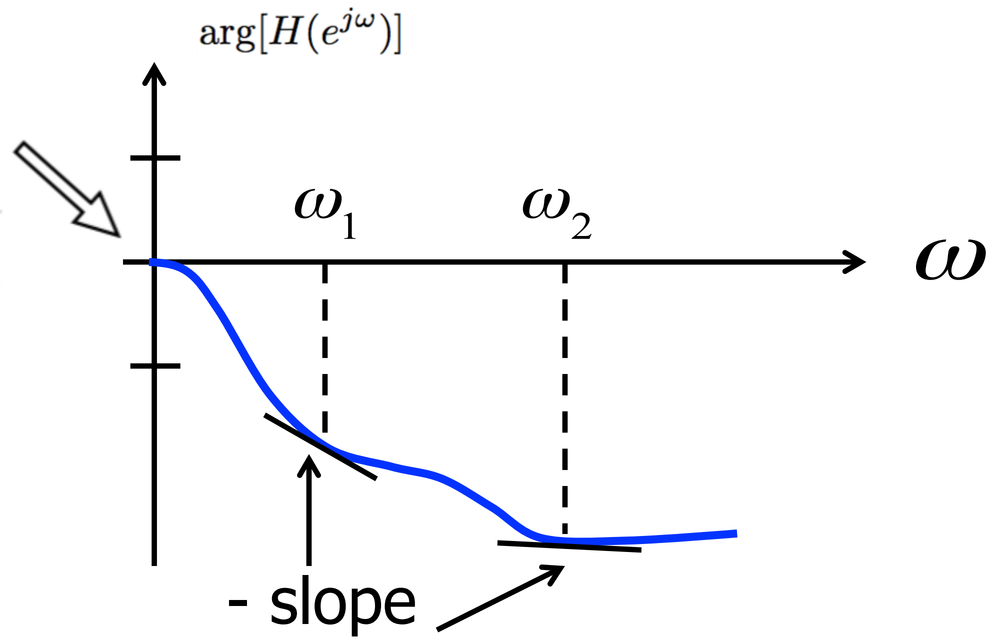
Input





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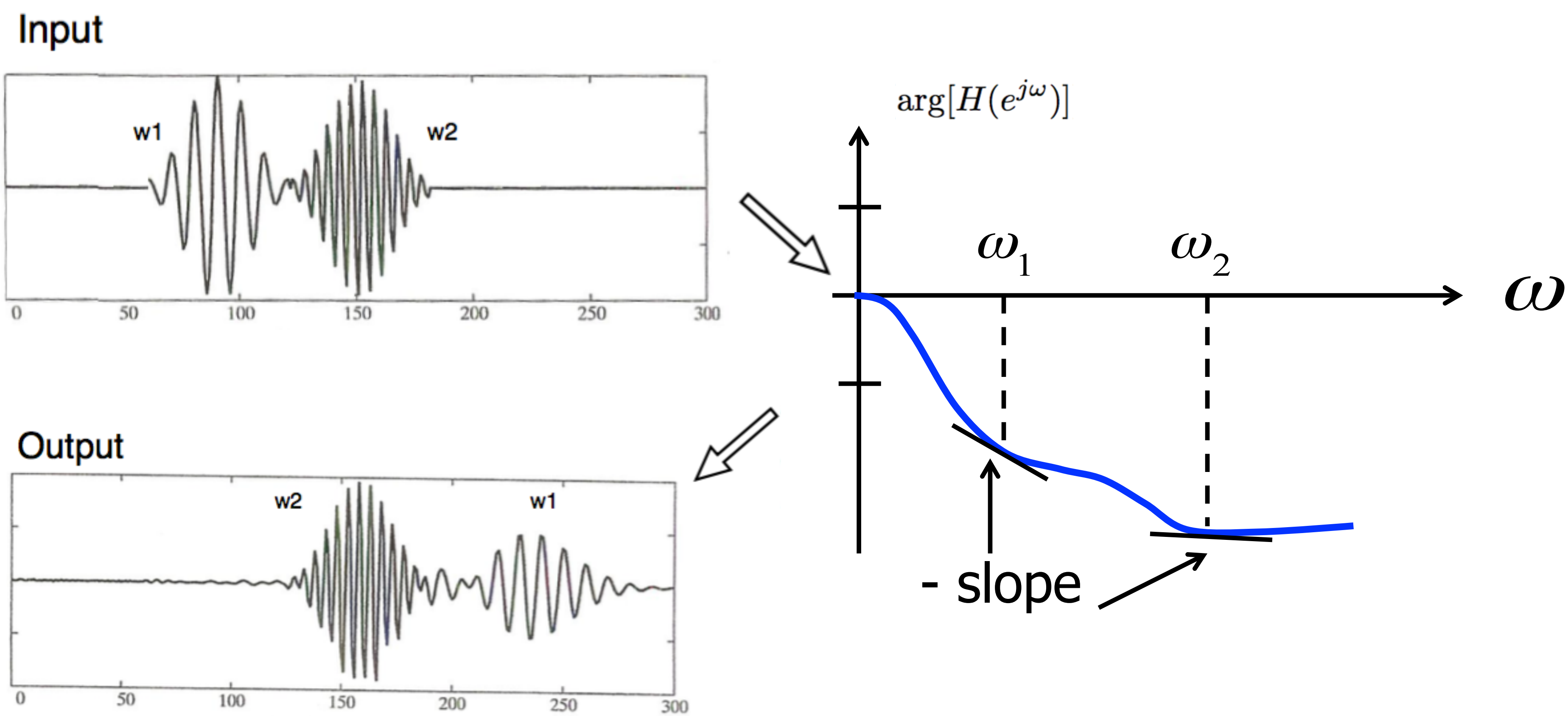

Group Delay Math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \qquad H(e^{j\omega}) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

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arg of products is sum of args

$$\arg[H(e^{j\omega})] = \sum_{k=1}^{M} \arg[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \arg[1 - d_k e^{-j\omega}]$$

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- Look at each factor:

$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

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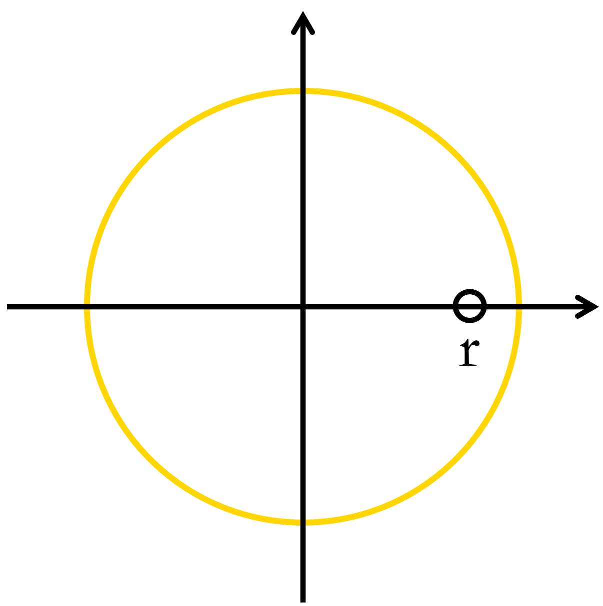
Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

$$\arg[1 - re^{-j\omega}]$$



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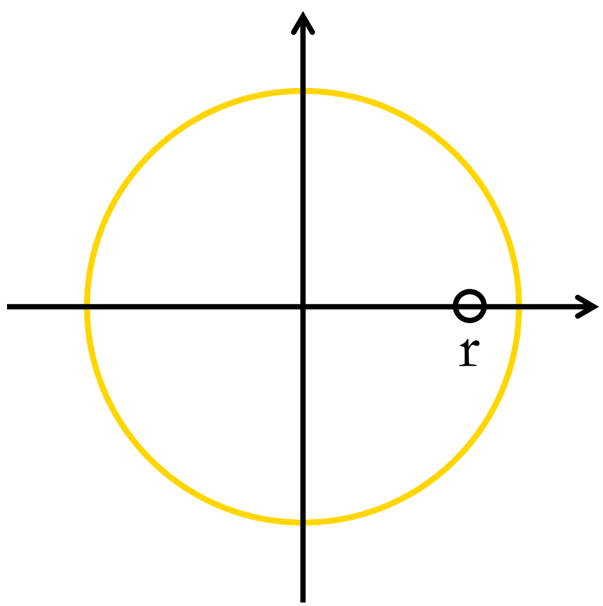

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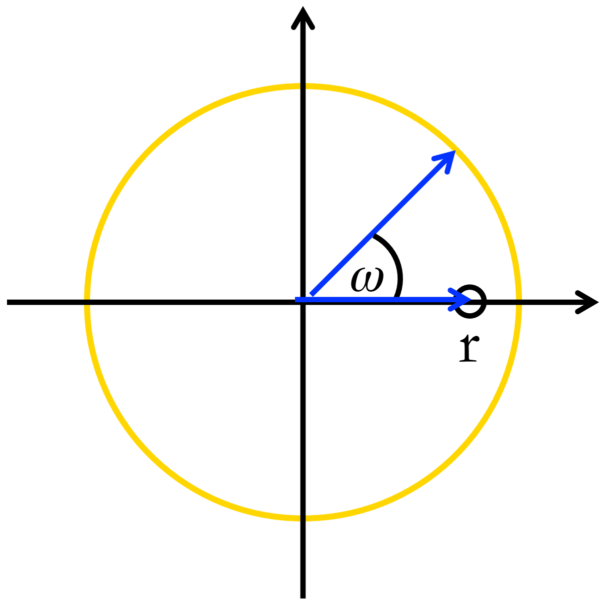

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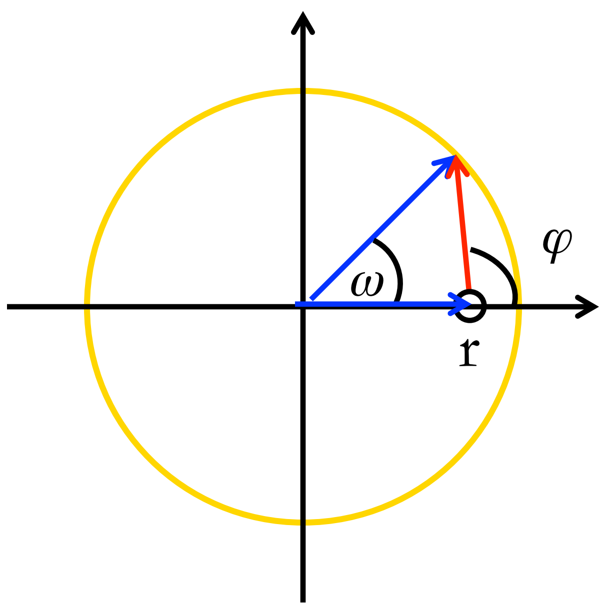

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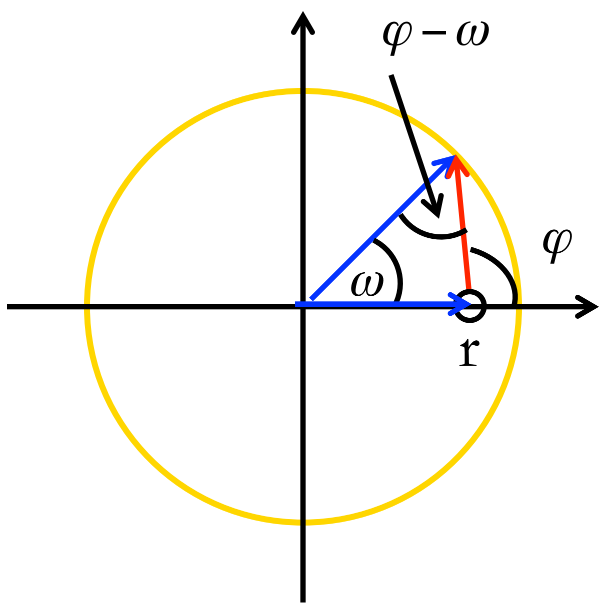

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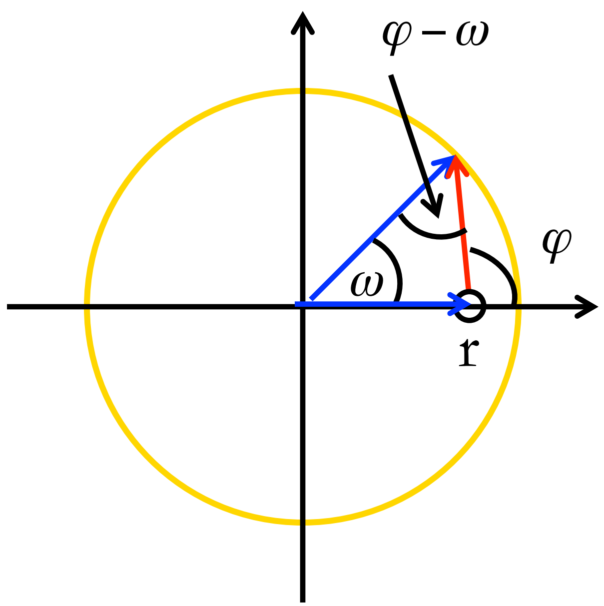

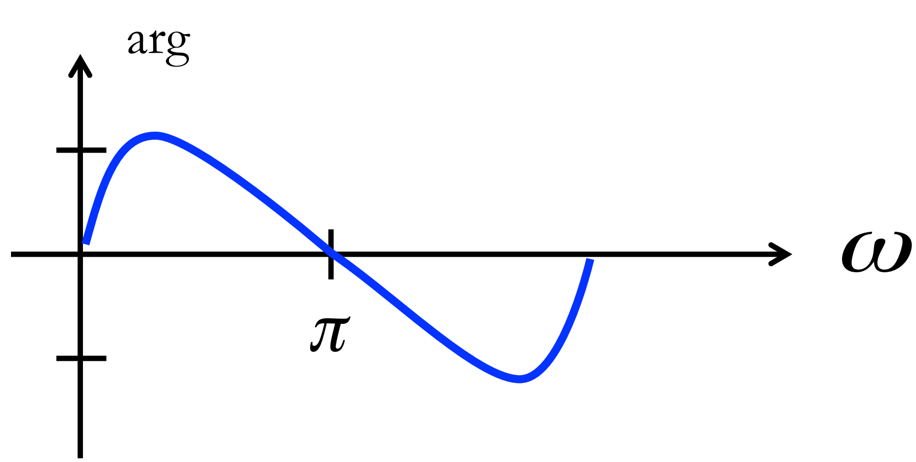

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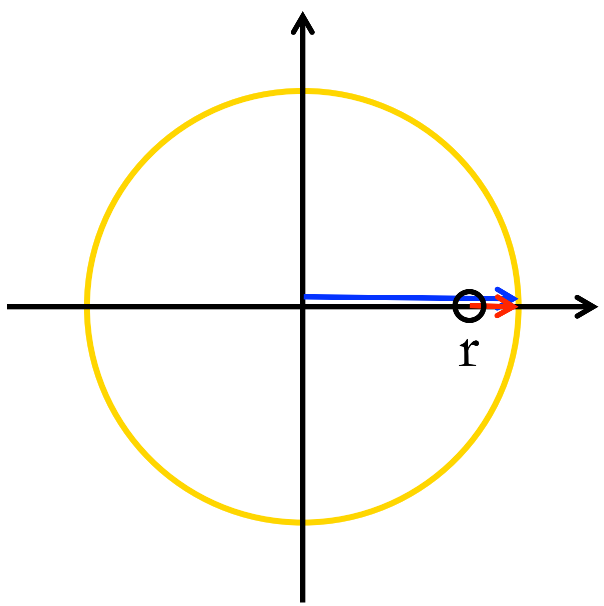
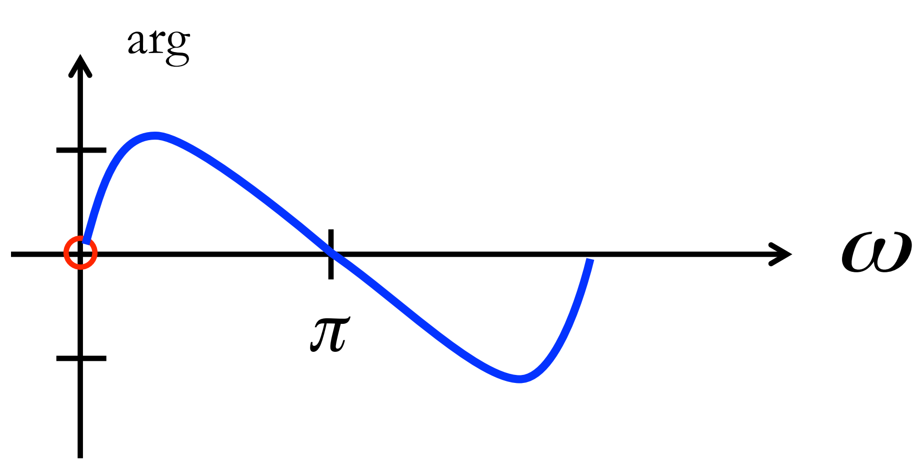
$\omega = 0$





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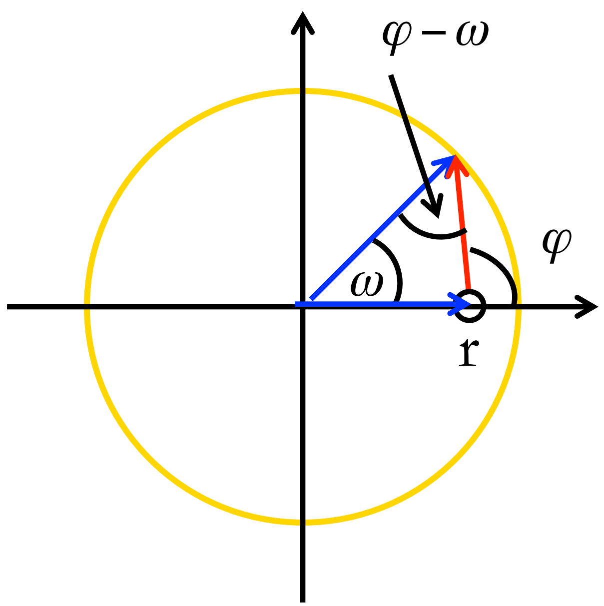

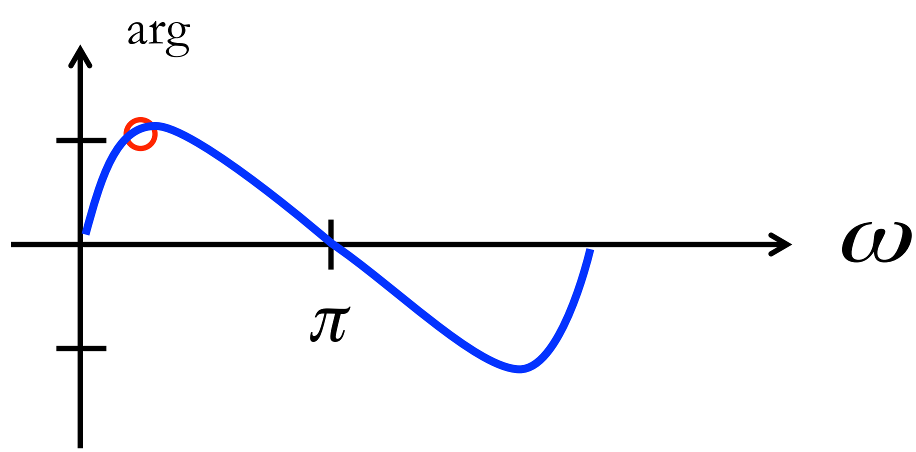

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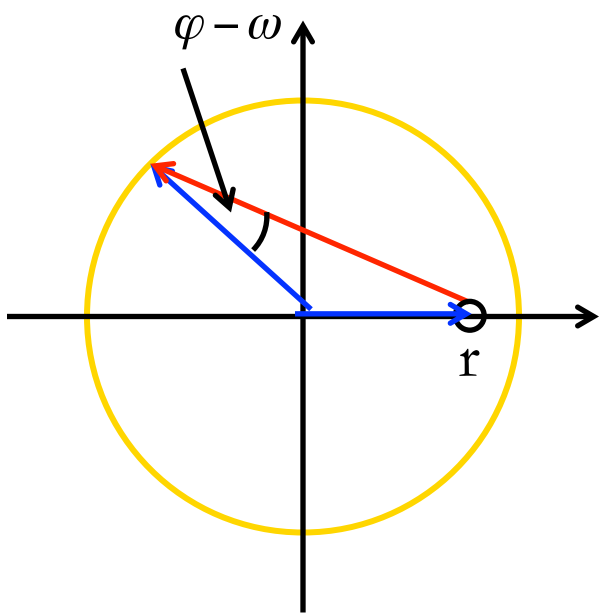

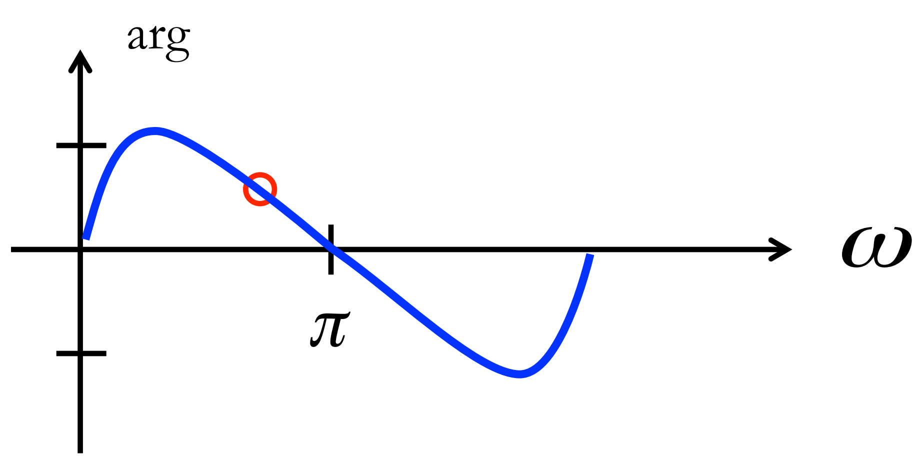

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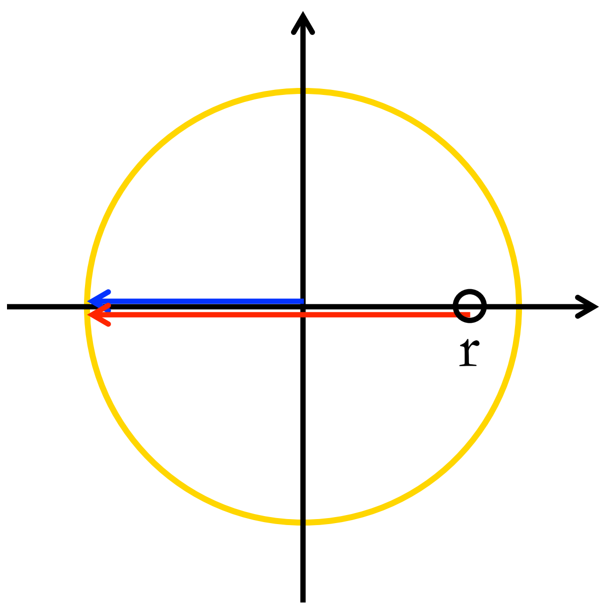
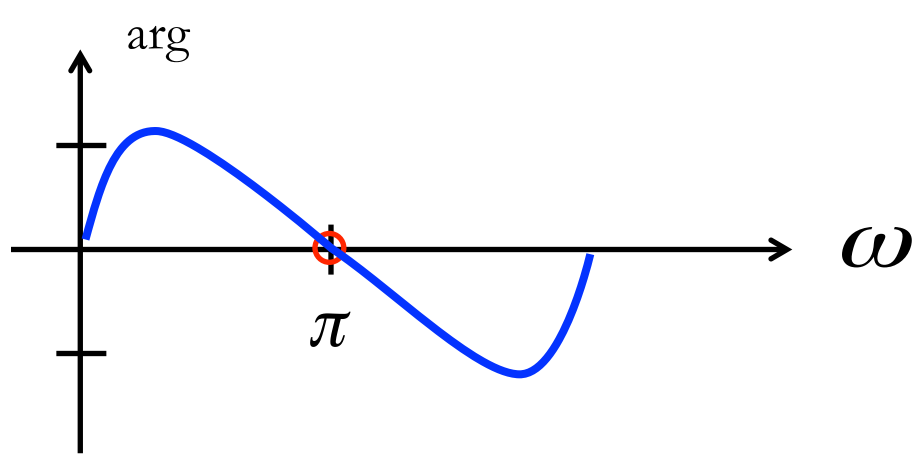
$\omega = \pi$





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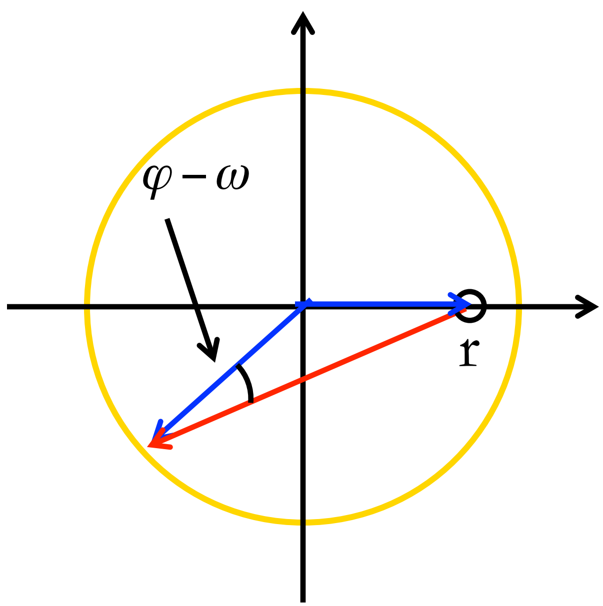

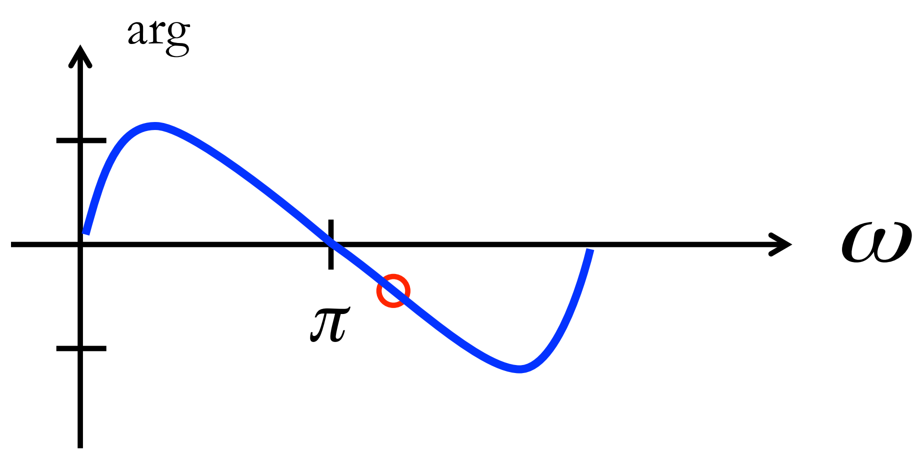

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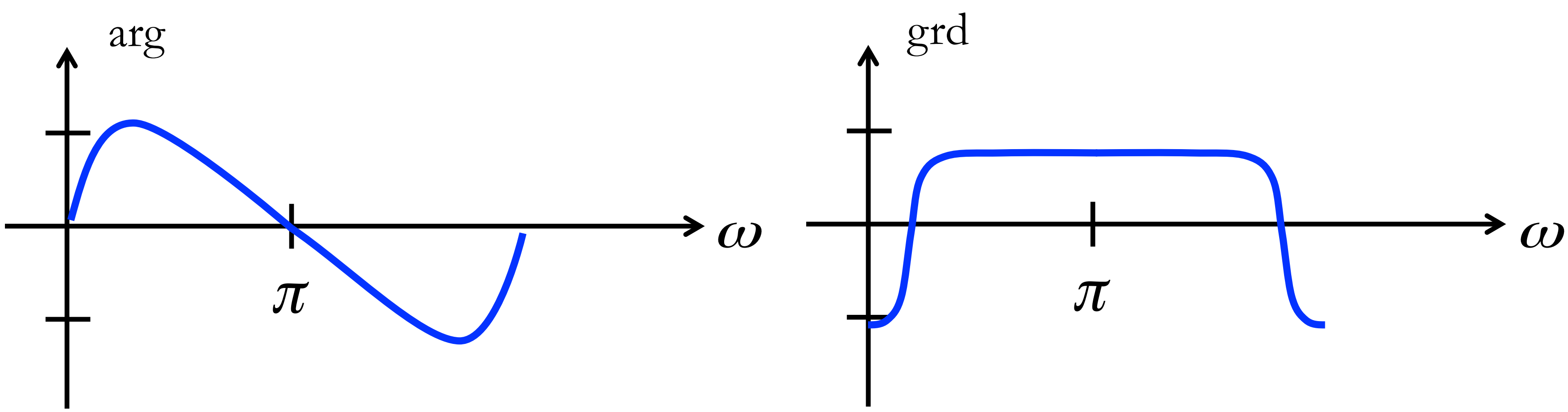

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- Look at each factor:

θ≠0?

$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right)$$

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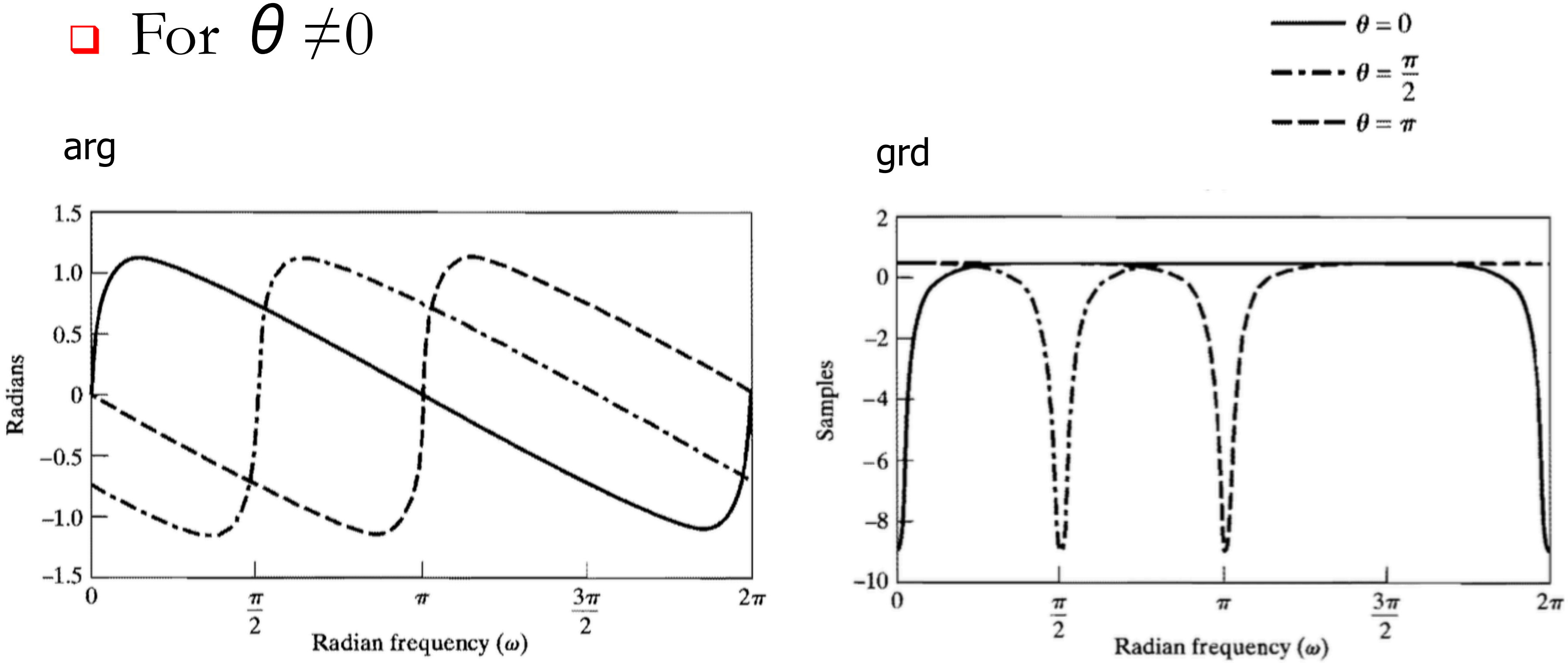
Example: Zero on Real Axis

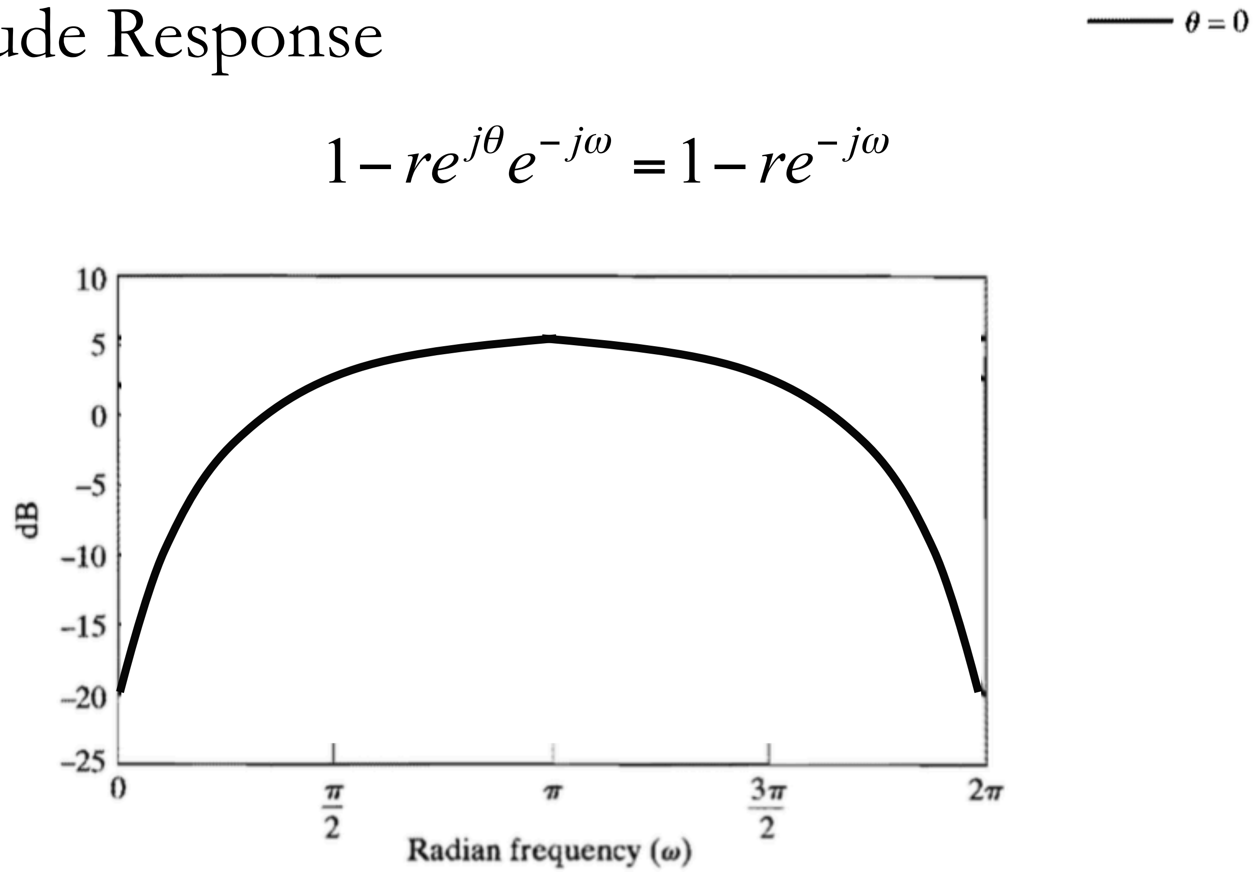
- For $\theta \neq 0$

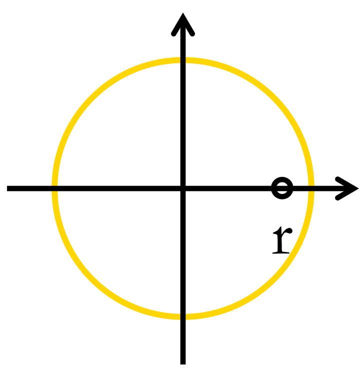


Example: Zero on Real Axis

- Magnitude Response

$$1 - re^{j\theta}e^{-j\omega} = 1 - re^{-j\omega}$$





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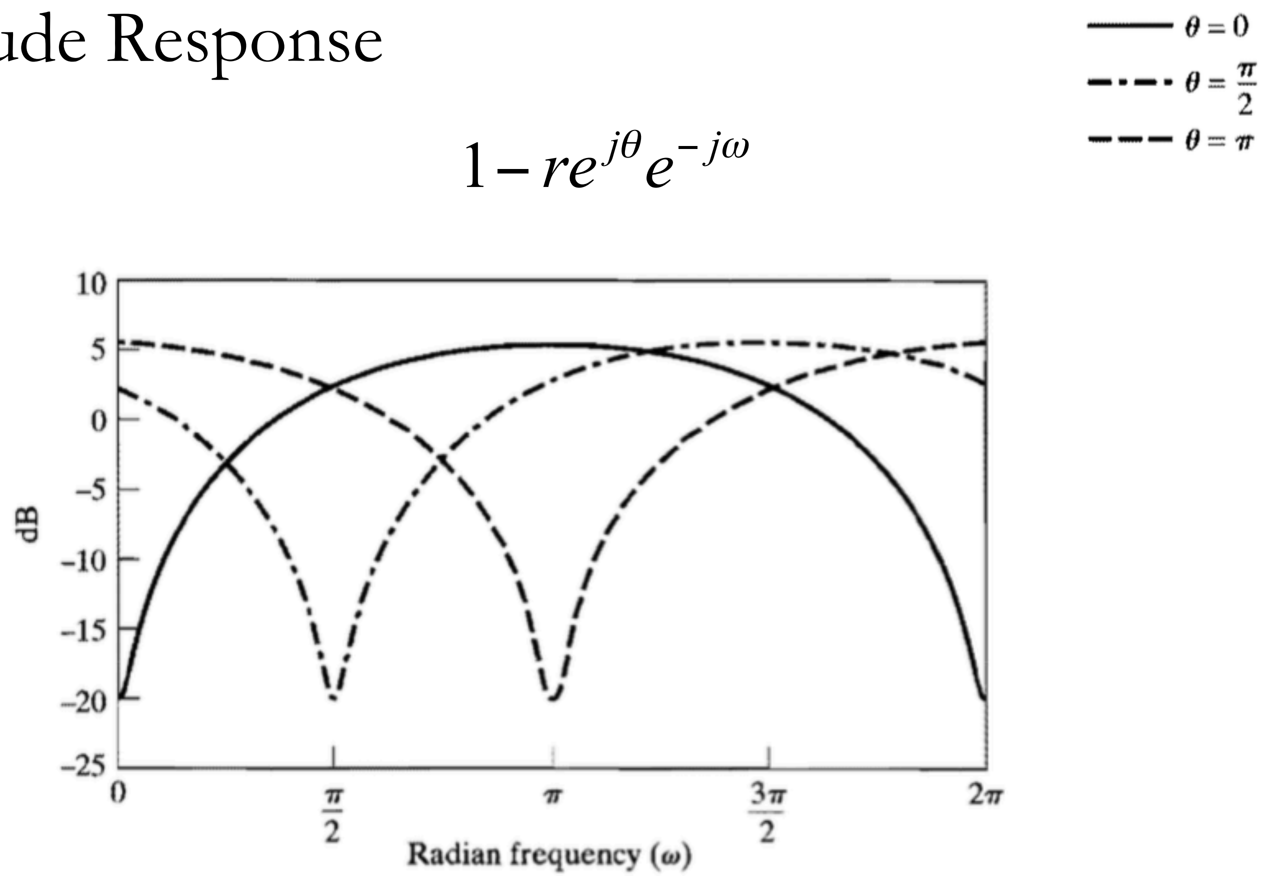
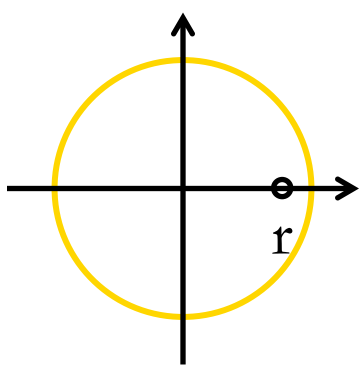
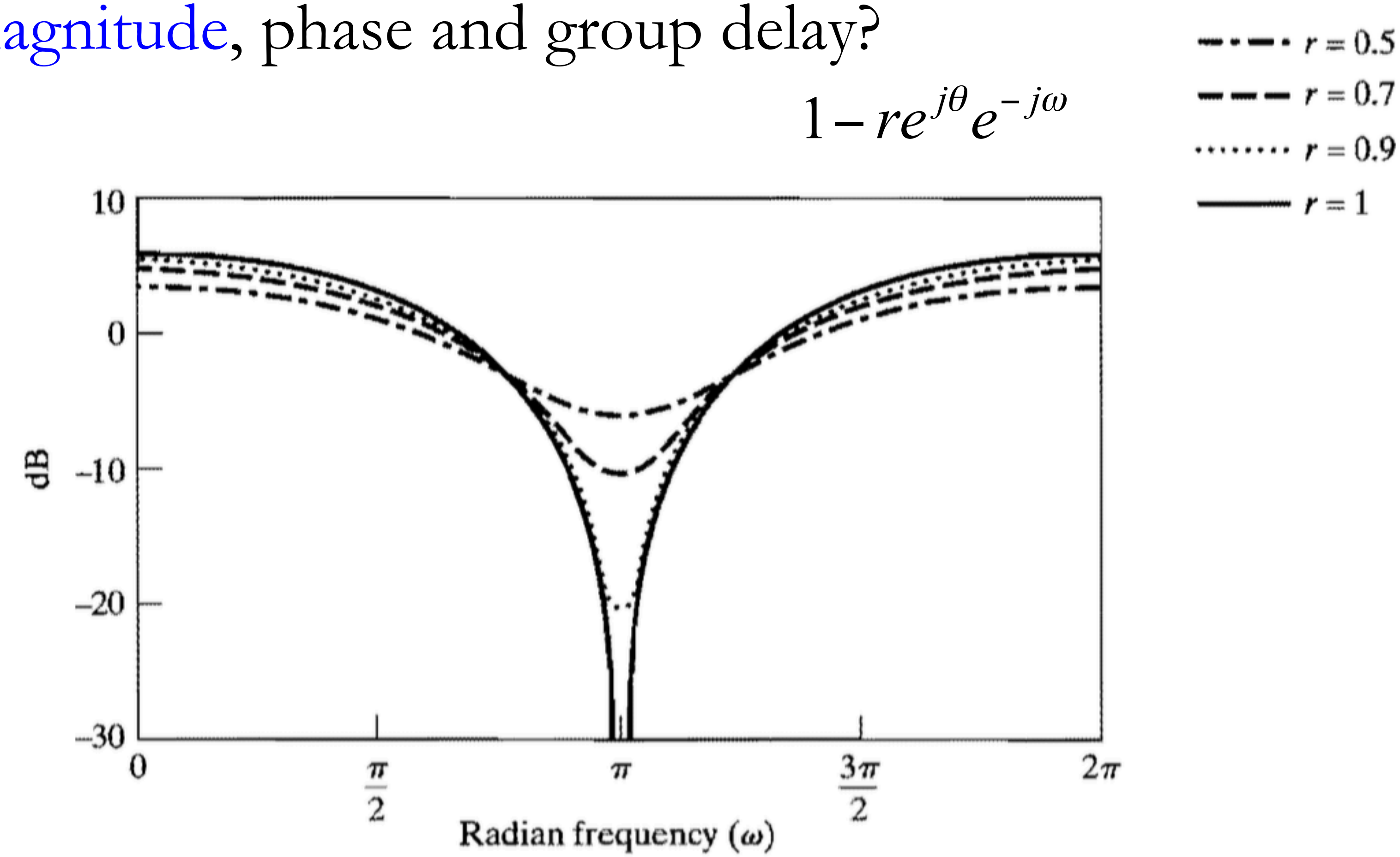




Example: Zero on Real Axis

- For $\theta = \pi$, how does zero location effect magnitude, phase and group delay?

$$1 - re^{j\theta}e^{-j\omega}$$



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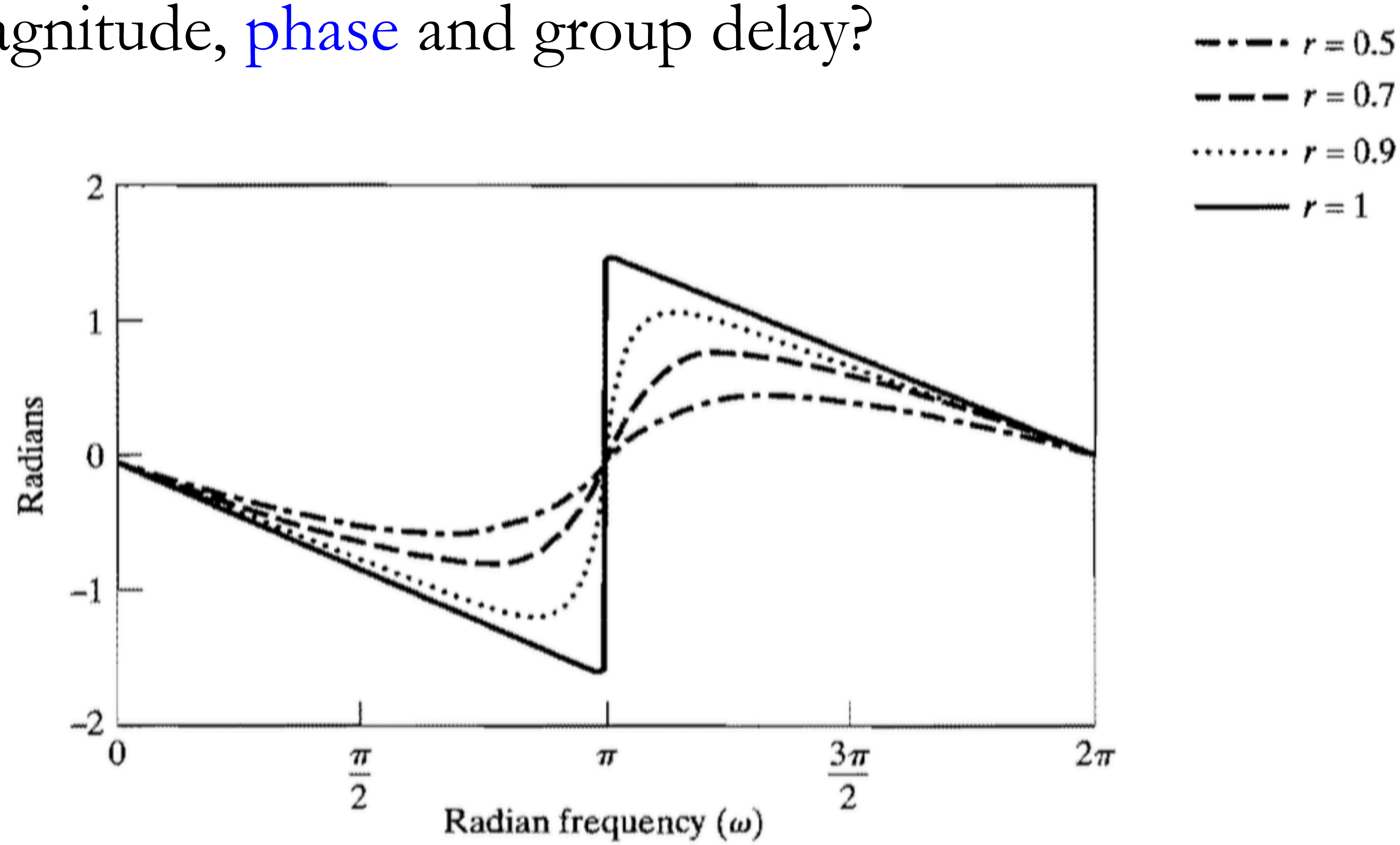

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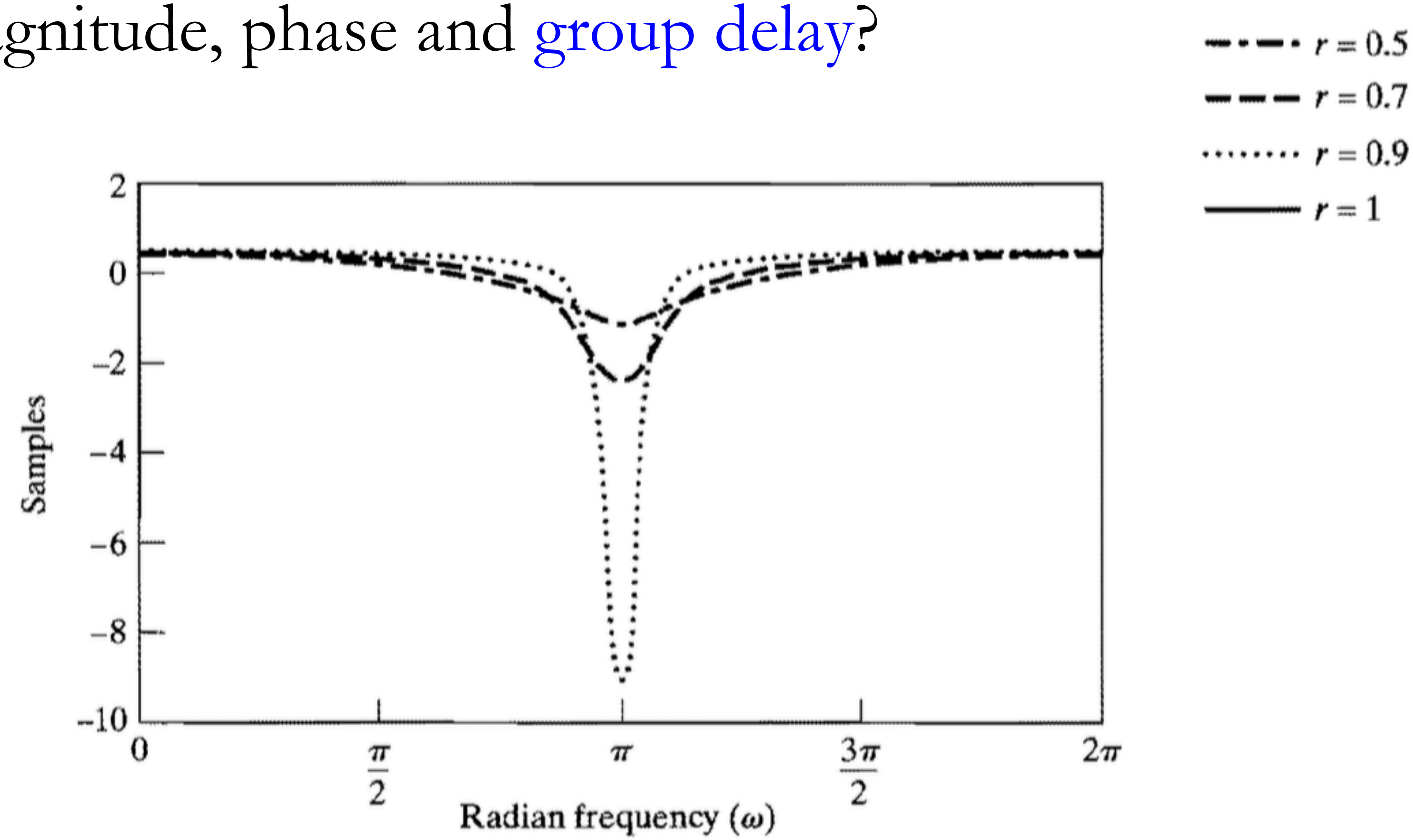
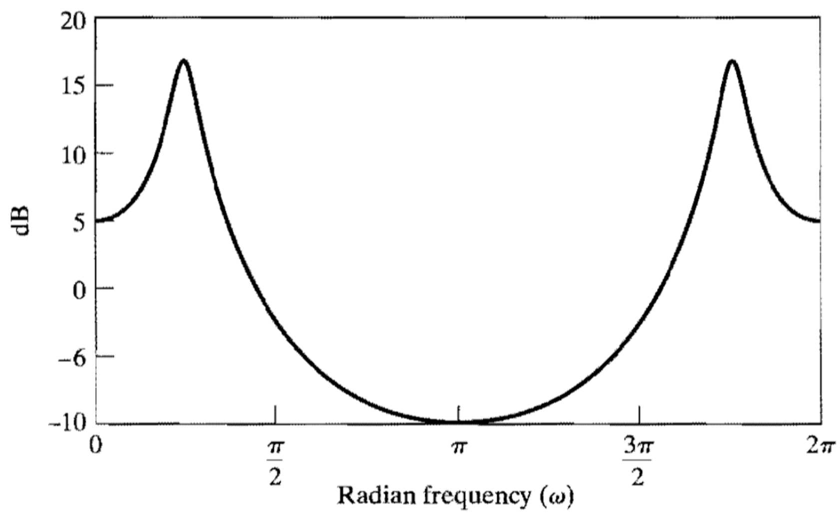


2^{nd} Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

r=0.9, $\theta = \pi / 4$

magnitude



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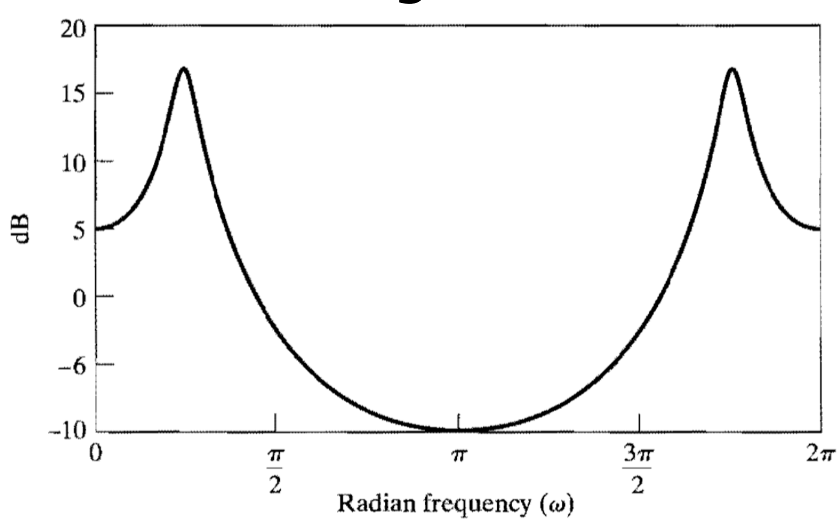
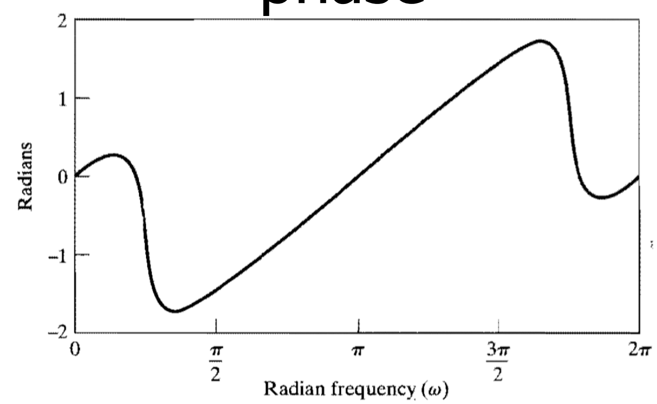
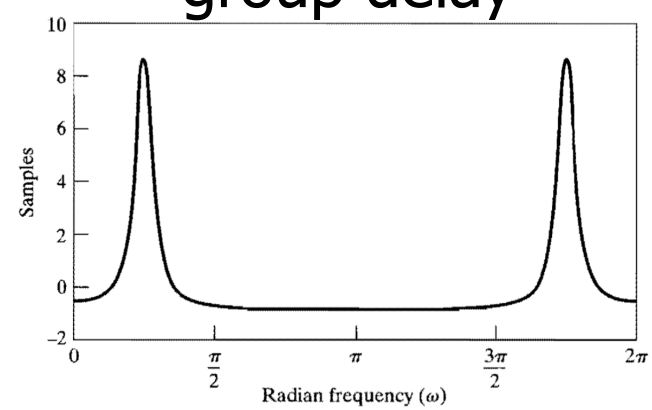


phase

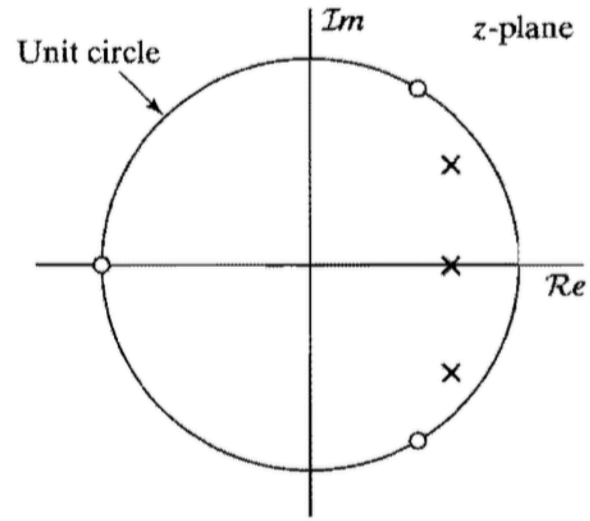


group delay



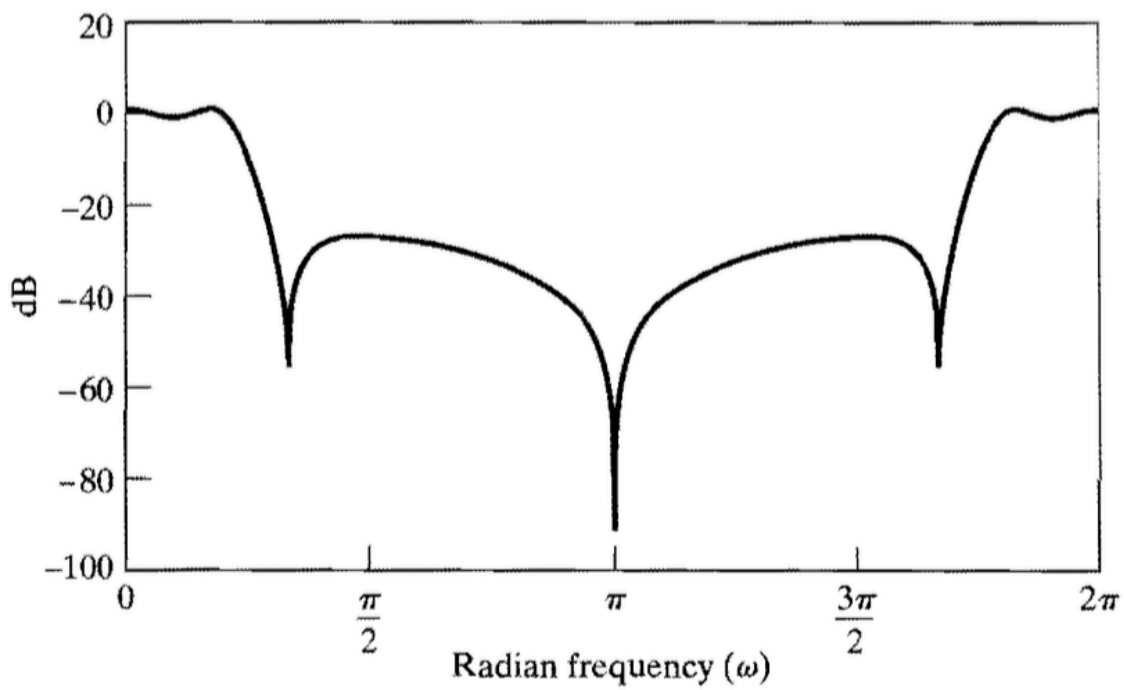
3^{rd} Order IIR Example

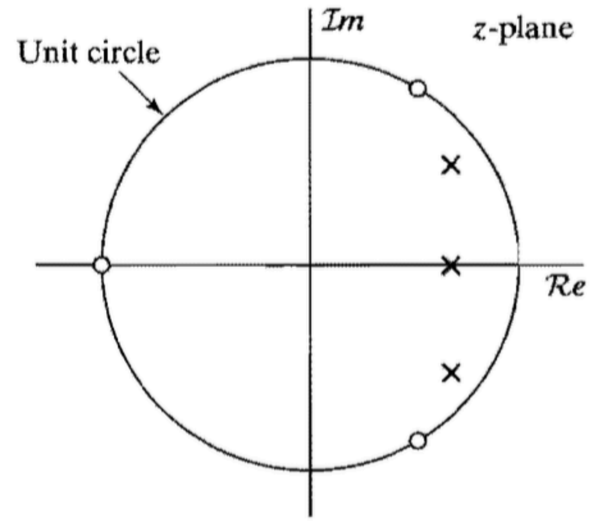
$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$





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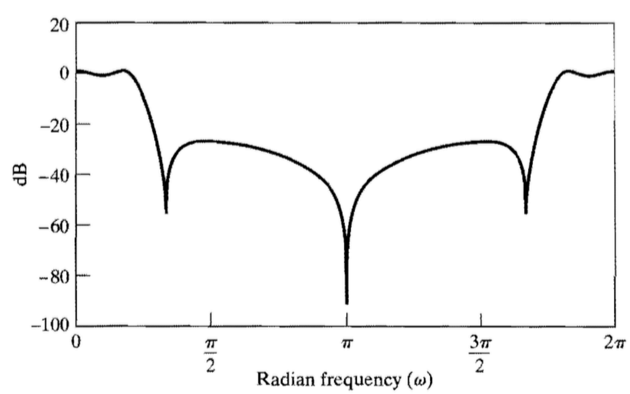

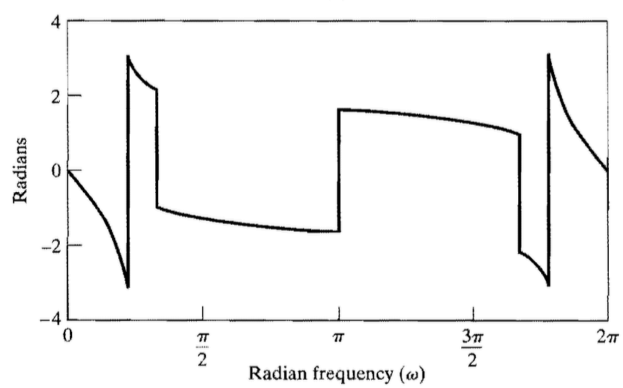

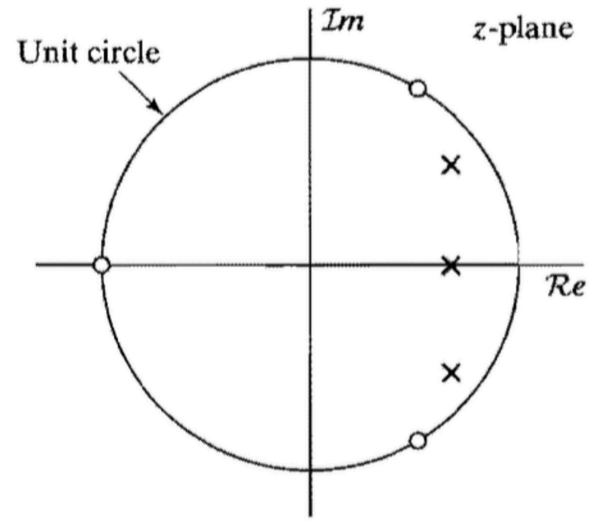

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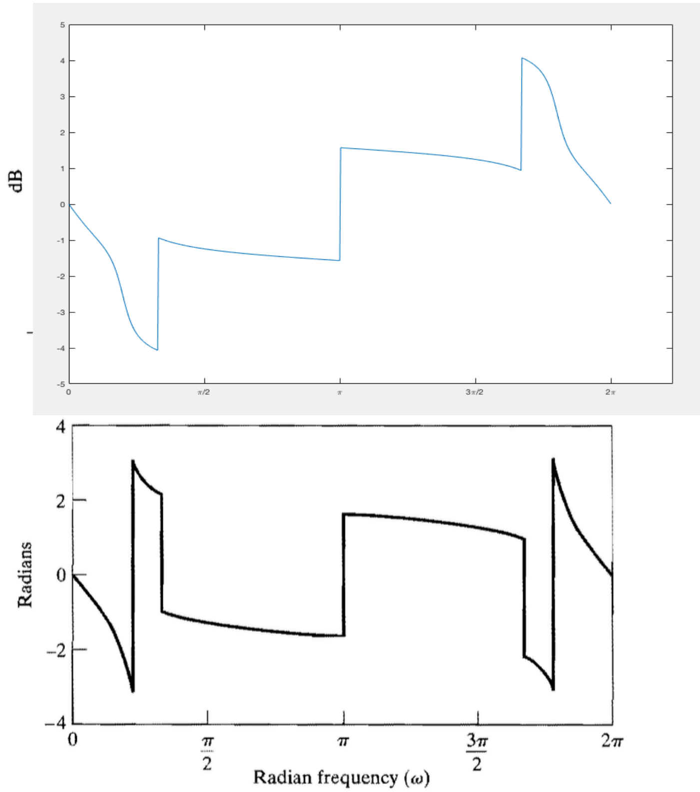

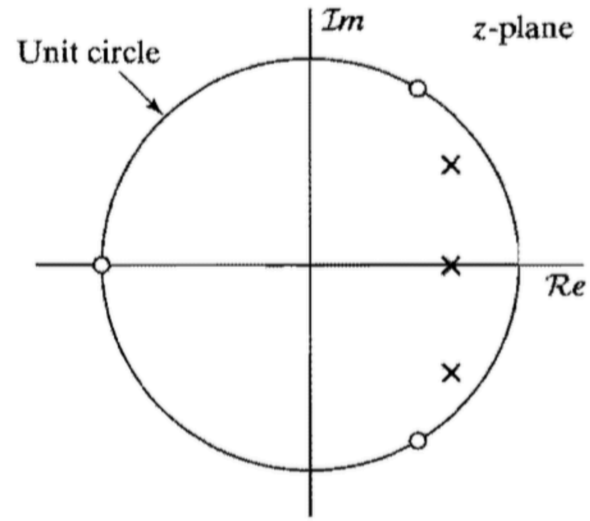

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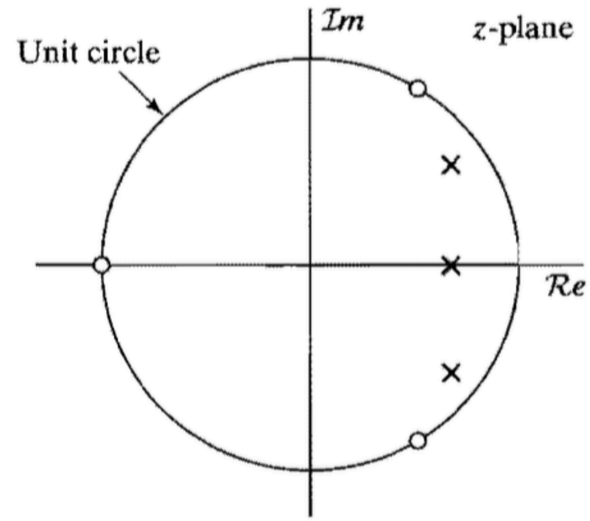

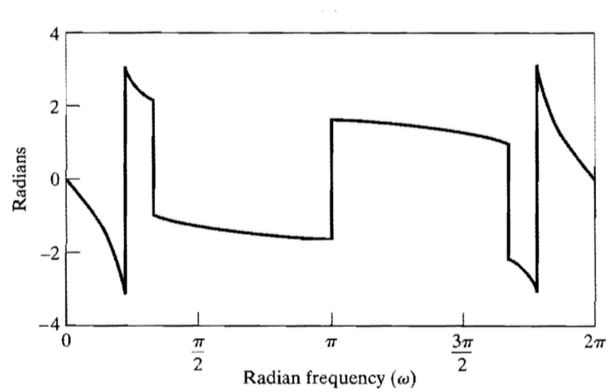

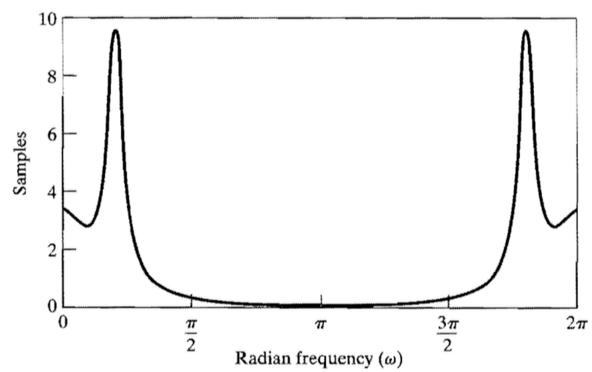

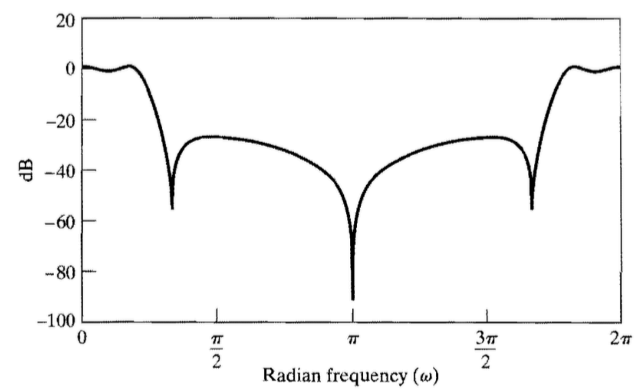

Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response
    - Simple Filters
  - Phase Response
    - Group Delay
  - Example: Zero on Real Axis

Admin

- HW 5 due Sunday
- Midterm after spring break 3/12
  - During class
    - Starts at exactly 4:30pm, ends at exactly 5:50pm (80 minutes)
  - Location DRLB A2
  - Old exams posted on previous years' website
    - Disclaimer: old exams covered more material
  - Covers Lec 1- 11 ←changed from last lecture
  - Closed book, one page cheat sheet allowed
  - Calculators allowed, no smart phones
  - Review session (likely Sunday before exam)
    - Keep an eye on Piazza