

## ESE 531: Digital Signal Processing

Lec 13: February 28, 2018  
Frequency Response of LTI Systems

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## Lecture Outline

- Noise Shaping
- Frequency Response of LTI Systems
  - Magnitude Response
    - Simple Filters
  - Phase Response
    - Group Delay
  - Example: Zero on Real Axis

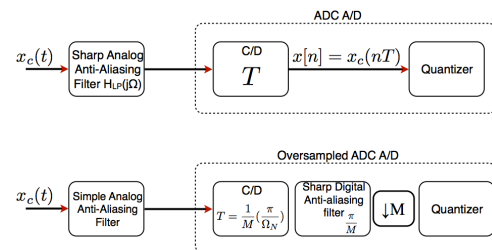
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## Noise Shaping



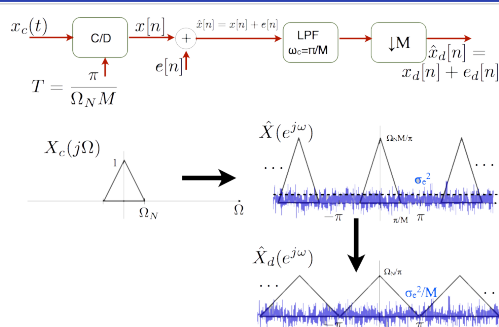
## Oversampled ADC



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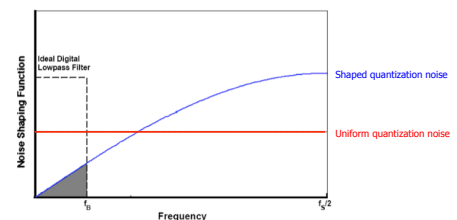
## Quantization Noise with Oversampling



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## Noise Shaping

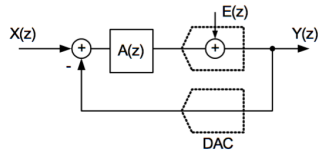


- Idea: "Somehow" build an ADC that has most of its quantization noise at high frequencies
- Key: Feedback

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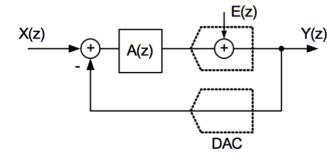
## Noise Shaping Using Feedback



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## Noise Shaping Using Feedback



$$\begin{aligned} Y(z) &= E(z) + A(z)X(z) - A(z)Y(z) \\ &= E(z) \frac{1}{1+A(z)} + X(z) \frac{A(z)}{1+A(z)} \\ &= E(z) \underbrace{\frac{1}{1+A(z)}}_{\text{Noise Transfer Function}} + X(z) \underbrace{\frac{A(z)}{1+A(z)}}_{\text{Signal Transfer Function}} \end{aligned}$$

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## Noise Shaping Using Feedback

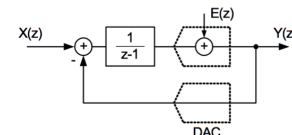
$$Y(z) = E(z) \underbrace{\frac{1}{1+A(z)}}_{\text{Noise Transfer Function}} + X(z) \underbrace{\frac{A(z)}{1+A(z)}}_{\text{Signal Transfer Function}}$$

- Objective
  - Want to make STF unity in the signal frequency band
  - Want to make NTF "small" in the signal frequency band
- If the frequency band of interest is around DC ( $0 \dots f_B$ ) we achieve this by making  $|A(z)| \gg 1$  at low frequencies
  - Means that  $NTF \ll 1$
  - Means that  $STF \approx 1$

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## First Order Sigma-Delta Modulator



$$\begin{aligned} Y(z) &= E(z) \frac{1}{1 + \frac{1}{z-1}} + X(z) \frac{\frac{1}{z-1}}{1 + \frac{1}{z-1}} \\ &= E(z) (1 - z^{-1}) + X(z) z^{-1} \end{aligned}$$

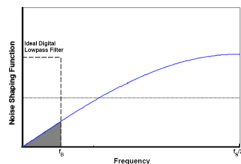
- Output is equal to delayed input plus filtered quantization noise

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## NTF Frequency Domain Analysis

$$\begin{aligned} H_e(z) &= 1 - z^{-1} \\ H_e(j\omega) &= (1 - e^{-j\omega T}) = 2e^{-j\omega T/2} \left( \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2} \right) \\ &= 2e^{-j\omega T/2} \left( j \sin\left(\frac{\omega T}{2}\right) \right) = 2 \sin\left(\frac{\omega T}{2}\right) e^{-j\omega T/2 - \pi/2} \\ |H_e(f)| &= 2 \left| \sin\left(\frac{\omega T}{2}\right) \right| = 2 \left| \sin\left(\pi \frac{f}{f_s}\right) \right| \end{aligned}$$



- "First order noise Shaping"
  - Quantization noise is attenuated at low frequencies, amplified at high frequencies

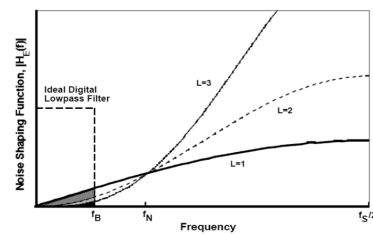
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## Higher Order Noise Shaping

- $L^{\text{th}}$  order noise transfer function

$$H_E(z) = (1 - z^{-1})^L$$



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## Frequency Response of LTI Systems



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## Frequency Response of LTI System

- LTI Systems are uniquely determined by their impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[k] * h[k]$$

- We can write the input-output relation also in the z-domain

$$Y(z) = H(z)X(z)$$

- Or we can define an LTI system with its frequency response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- $H(e^{j\omega})$  defines magnitude and phase change at each frequency

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## Frequency Response of LTI System

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- We can define a magnitude response

$$|Y(e^{j\omega})| = |H(e^{j\omega})| |X(e^{j\omega})|$$

- And a phase response

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

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## Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$

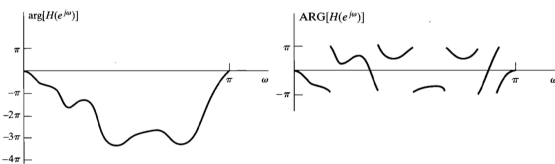
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## Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

- More later...

Unwrapped phase

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## Linear Difference Equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

**Example:**  $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

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## Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

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Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$

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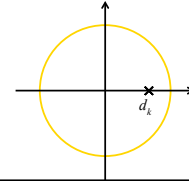
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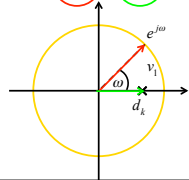
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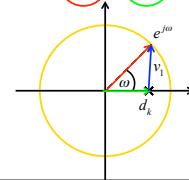
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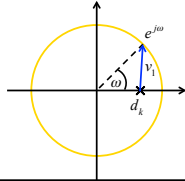
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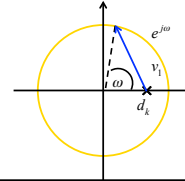
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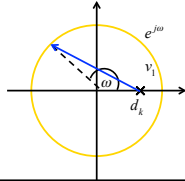
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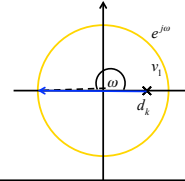
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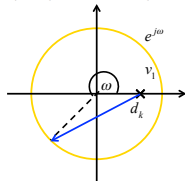
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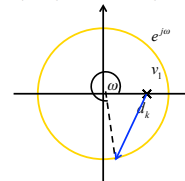
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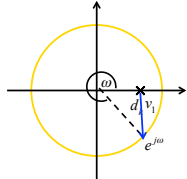
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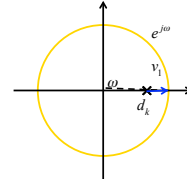
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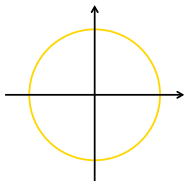
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## Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



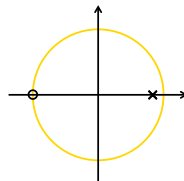
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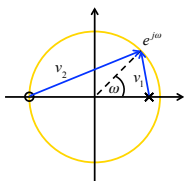
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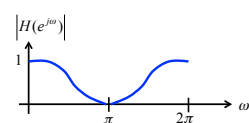
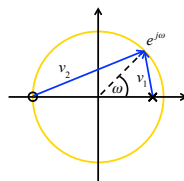
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### Simple Low Pass Filter

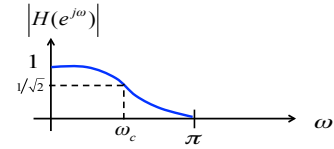
$$H_{LP}(z) = \frac{1-\alpha}{2} \frac{1+z^{-1}}{1-\alpha z^{-1}} \quad |\alpha| < 1$$

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### Simple Low Pass Filter

$$H_{LP}(z) = \frac{1-\alpha}{2} \frac{1+z^{-1}}{1-\alpha z^{-1}} \quad |\alpha| < 1$$



$\omega_c$  is the 3dB cutoff frequency  $\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$

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### Simple High Pass Filter

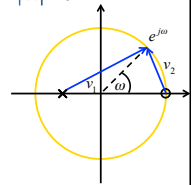
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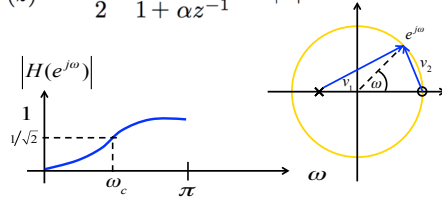


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### Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$

**Note:**  $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$   
 $\cos(\omega_0) = \beta$

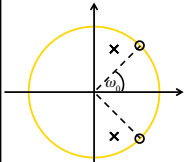
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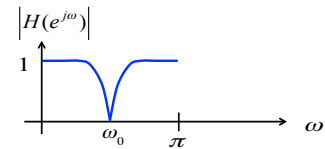
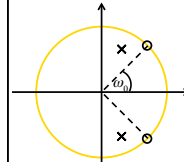
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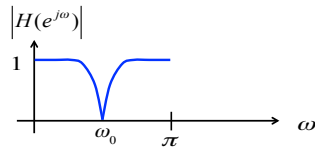
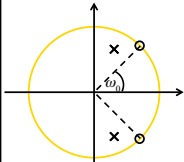
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### Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1-2\beta z^{-1} + z^{-2}}{1-\beta(1+\alpha)z^{-1} + \alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$

**Note:**

$$H_{BS}(\mp 1) = \frac{1+\alpha}{2} \frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$$



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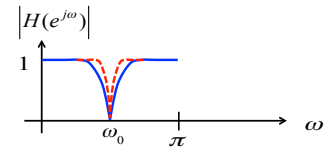
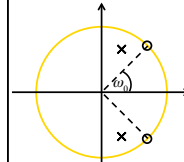
45

### Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1-2\beta z^{-1} + z^{-2}}{1-\beta(1+\alpha)z^{-1} + \alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$

**Note:** As  $\alpha \rightarrow 1$  poles approach zeros

$$H_{BS}(\mp 1) = \frac{1+\alpha}{2} \frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$$



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### Simple Band-Pass Filter

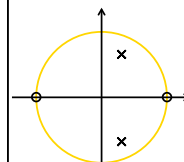
$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1} + \alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$

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### Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1} + \alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$

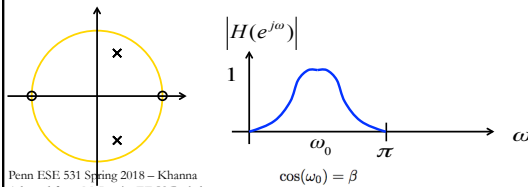


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## Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$

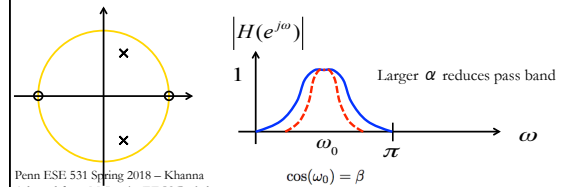


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## Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$



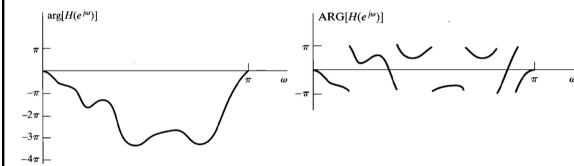
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## Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$



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## Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

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## Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

ARG is the wrapped phase  
arg is the unwrapped phase

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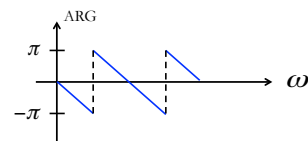
## Phase Response Example

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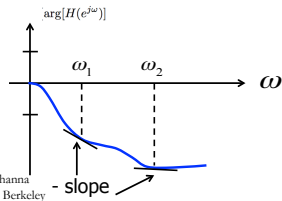
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## Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



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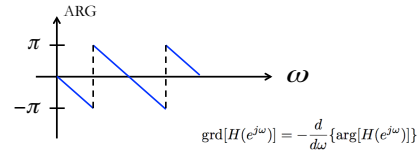
## Phase Response Example

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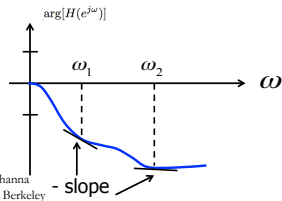
For linear phase system, group delay is  $n_d$

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## Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

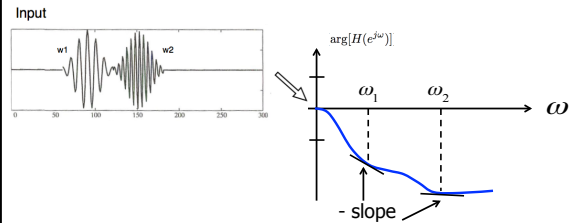


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## Group Delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

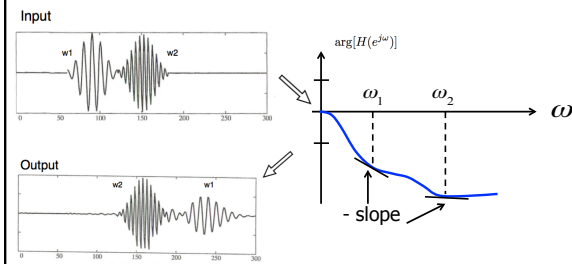


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## Group Delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



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## Group Delay Math

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})} \quad H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^M (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^N (1 - d_k e^{-j\omega})}$$

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## Group Delay Math

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})} \quad H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^M (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^N (1 - d_k e^{-j\omega})}$$

arg of products is sum of args

$$\arg[H(e^{j\omega})] = \sum_{k=1}^M \arg[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \arg[1 - d_k e^{-j\omega}]$$

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

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## Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

Look at each factor:

$$\arg[1 - r e^{j\theta} e^{-j\omega}] = \tan^{-1} \left( \frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right)$$

$$\text{grd}[1 - r e^{j\theta} e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - r e^{j\theta} e^{-j\omega}|^2}$$

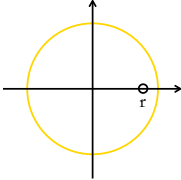
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## Example: Zero on Real Axis

Geometric Interpretation for ( $\theta = 0$ )

$$\arg[1 - r e^{-j\omega}]$$



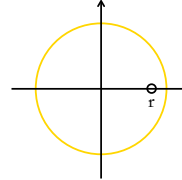
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## Example: Zero on Real Axis

Geometric Interpretation for ( $\theta = 0$ )

$$\arg[1 - r e^{-j\omega}] = \arg[(e^{j\omega} - r) e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



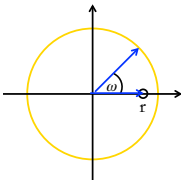
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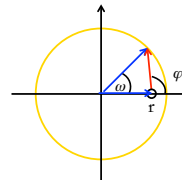
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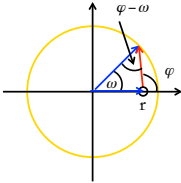
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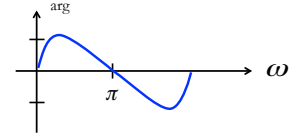
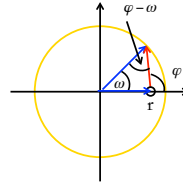
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$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{-j\omega}]}_{\omega}$$



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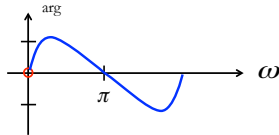
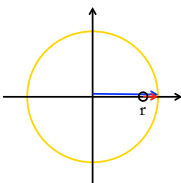
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### Example: Zero on Real Axis

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$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{-j\omega}]}_{\omega}$$

$\omega = 0$



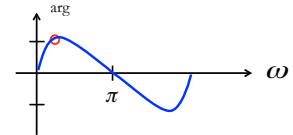
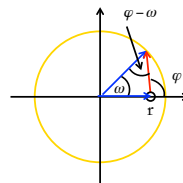
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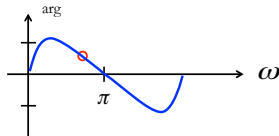
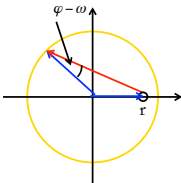
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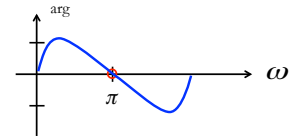
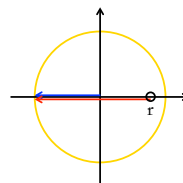
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### Example: Zero on Real Axis

- Geometric Interpretation for ( $\theta = 0$ )

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{-j\omega}]}_{\omega}$$

$\omega = \pi$



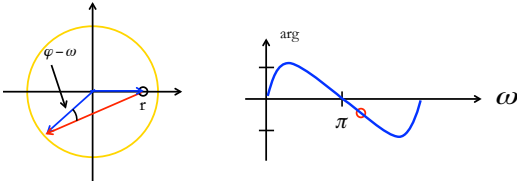
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### Example: Zero on Real Axis

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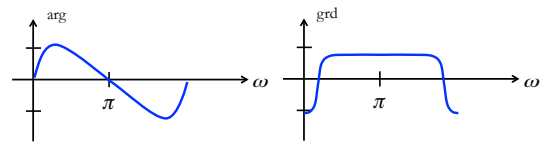
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### Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

- Look at each factor:

$\theta \neq 0?$

$$\arg[1 - re^{j\theta} e^{-j\omega}] = \tan^{-1} \left( \frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right)$$

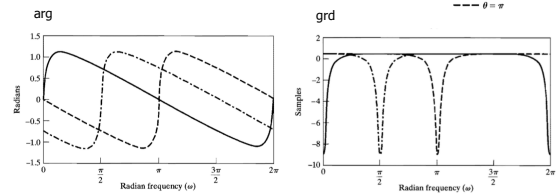
$$\text{grd}[1 - re^{j\theta} e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta} e^{-j\omega}|^2}$$

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### Example: Zero on Real Axis

- For  $\theta \neq 0$



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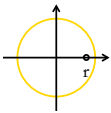
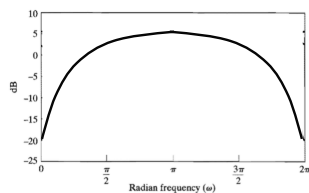
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### Example: Zero on Real Axis

- Magnitude Response

$\theta = 0$

$$1 - re^{j\theta} e^{-j\omega} = 1 - re^{-j\omega}$$



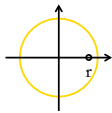
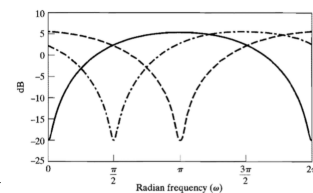
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### Example: Zero on Real Axis

- Magnitude Response

$$1 - re^{j\theta} e^{-j\omega}$$

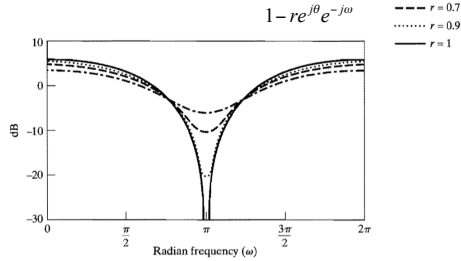


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### Example: Zero on Real Axis

- For  $\theta = \pi$ , how does zero location effect magnitude, phase and group delay?

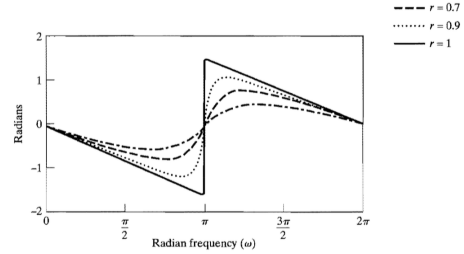


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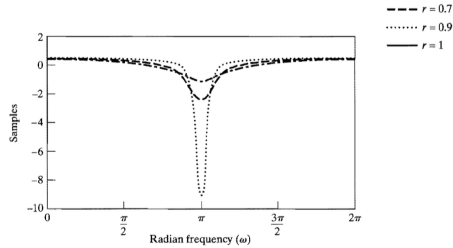


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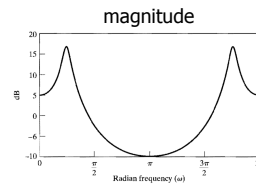


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### 2<sup>nd</sup> Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} \quad r=0.9, \theta = \pi/4$$

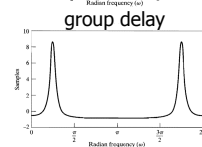
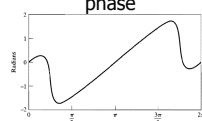
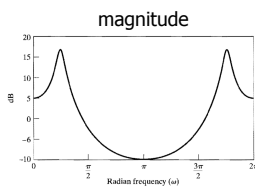


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### 2<sup>nd</sup> Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} \quad r=0.9, \theta = \pi/4$$

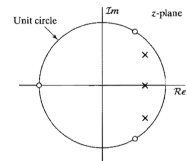


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### 3<sup>rd</sup> Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$

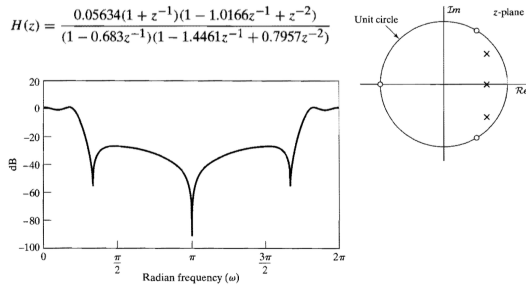


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### 3<sup>rd</sup> Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$

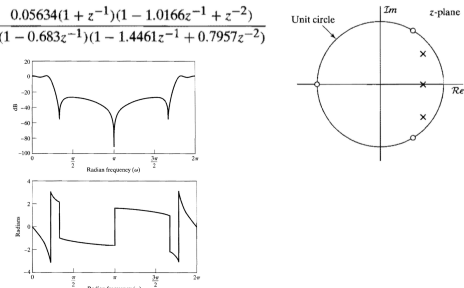


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### 3<sup>rd</sup> Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$

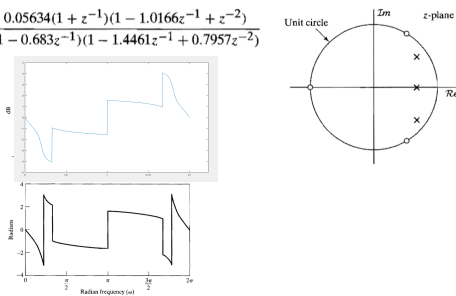


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### 3<sup>rd</sup> Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$

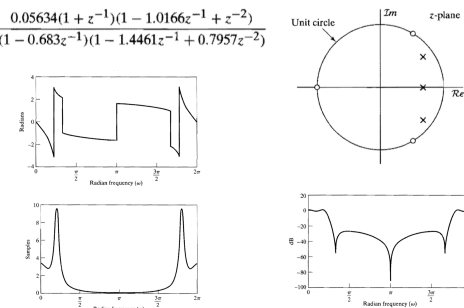


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### 3<sup>rd</sup> Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



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### Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response
    - Simple Filters
  - Phase Response
    - Group Delay
  - Example: Zero on Real Axis

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Adapted from M. Lustig, EECS Berkeley

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### Admin

- HW 5 due Sunday
- Midterm after spring break 3/12
  - During class
    - Starts at exactly 4:30pm, ends at exactly 5:50pm (80 minutes)
  - Location DRLB A2
  - Old exams posted on previous years' website
    - Disclaimer: old exams covered more material
  - Covers Lec 1- 11 ←changed from last lecture
  - Closed book, one page cheat sheet allowed
  - Calculators allowed, no smart phones
  - Review session (likely Sunday before exam)
    - Keep an eye on Piazza

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