ESE 531: Digital Signal Processing

Lec 14: March 14th, 2019

All-Pass Systems and Min Phase Decomposition

Lecture Outline

- Review: Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
  - Examples:
    - Zero on Real Axis
    - 2^{nd} order IIR
    - 3^{rd} order Low Pass
- Stability and Causality
- All Pass Systems
- Minimum Phase Systems

Review: Frequency Response of LTI System

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- We can define a magnitude response…

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right|\left|X\left(e^{j\omega}\right)\right|$$

- a phase response…

$$\angle Y\left(e^{j\omega}\right) = \angle H\left(e^{j\omega}\right) + \angle X\left(e^{j\omega}\right)$$

- and group delay

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Review: Group Delay Math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} \qquad H(e^{j\omega}) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}$$

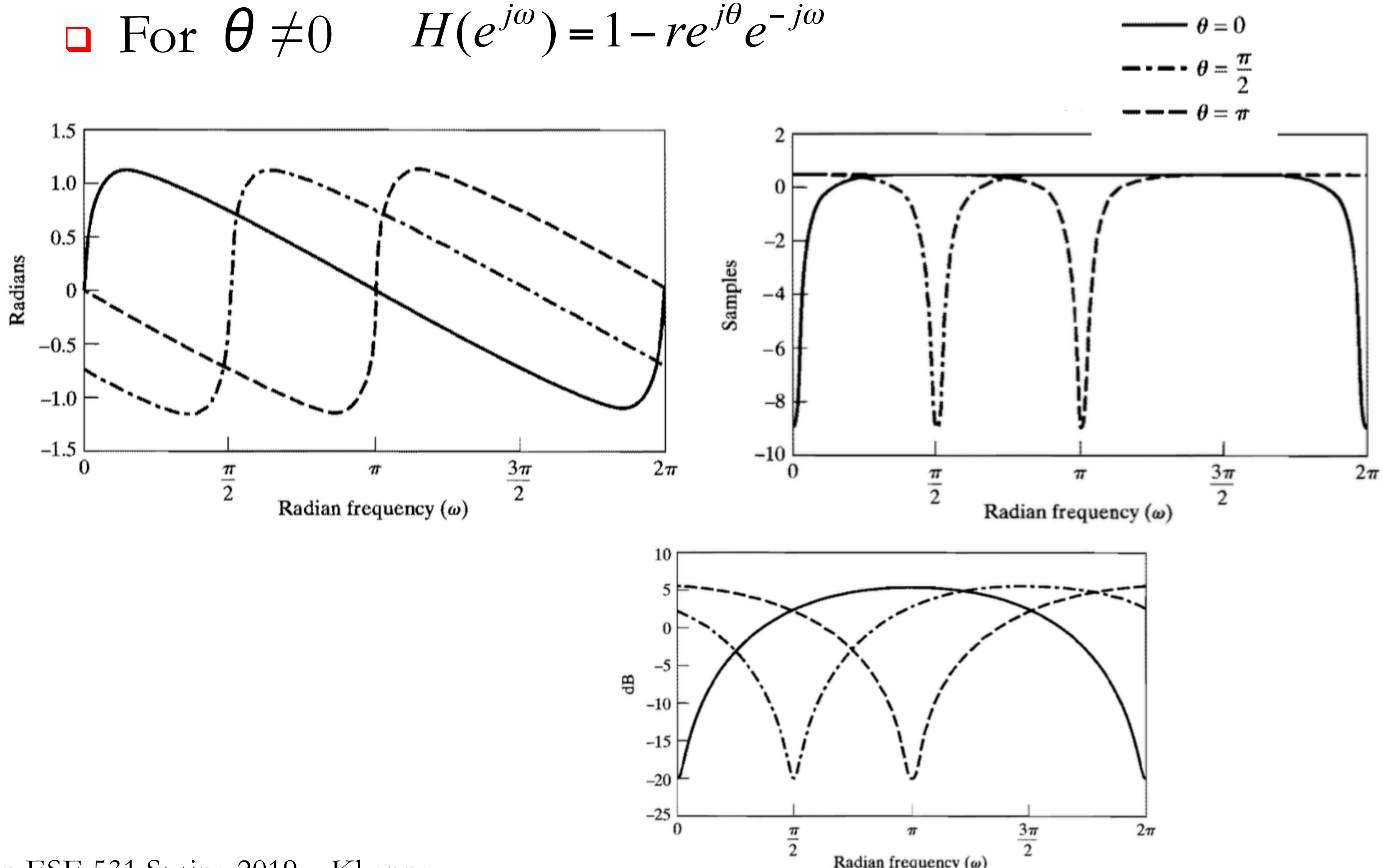
- Look at each factor:

$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

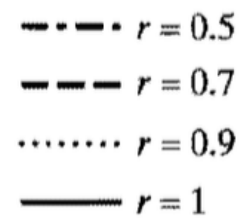
Example: Zero on Real Axis

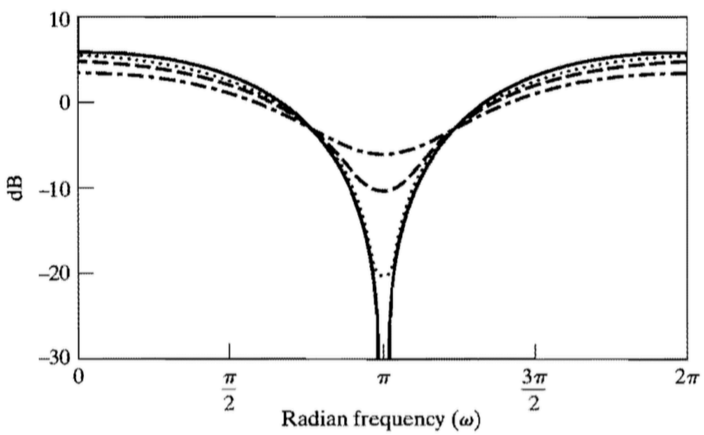
- For $\theta \neq 0$ $\quad H(e^{j\omega}) = 1 - re^{j\theta}e^{-j\omega}$

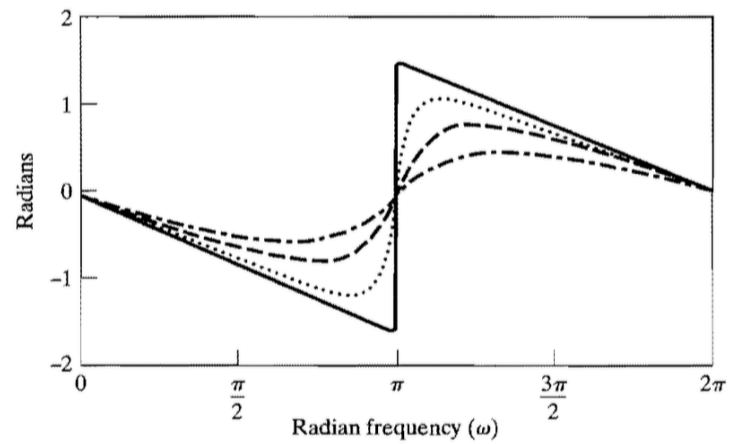


Example: Zero on Real Axis

- For $\theta = \pi$, how zero location affects magnitude, phase and group delay

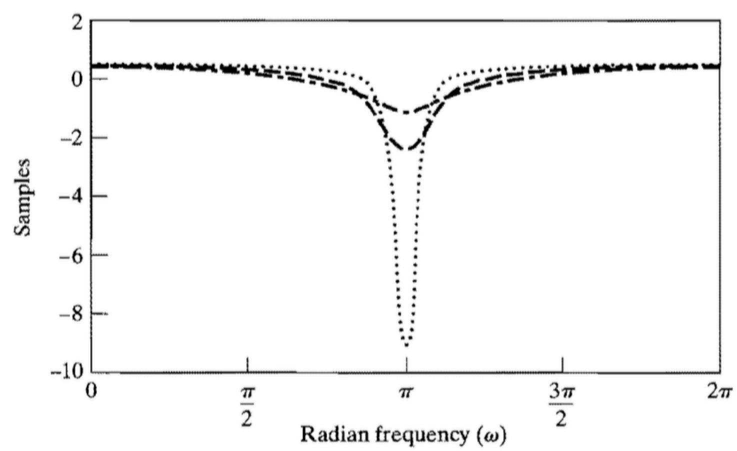






$$H(e^{j\omega}) = 1 - re^{j\theta}e^{-j\omega}$$

$$H(e^{j\omega}) = 1 + re^{-j\omega}$$

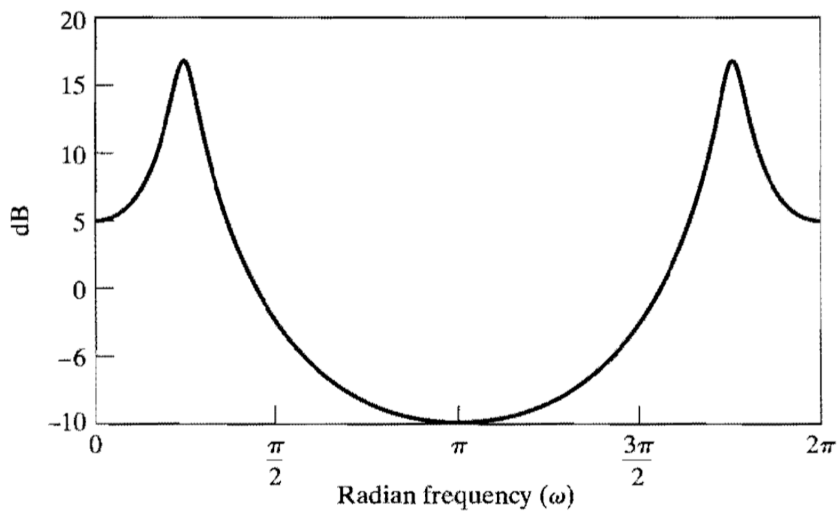


2^{nd} Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

r=0.9, $\theta = \pi/4$

magnitude



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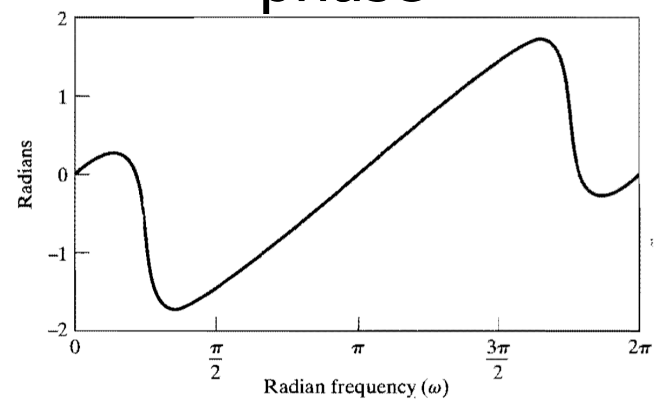
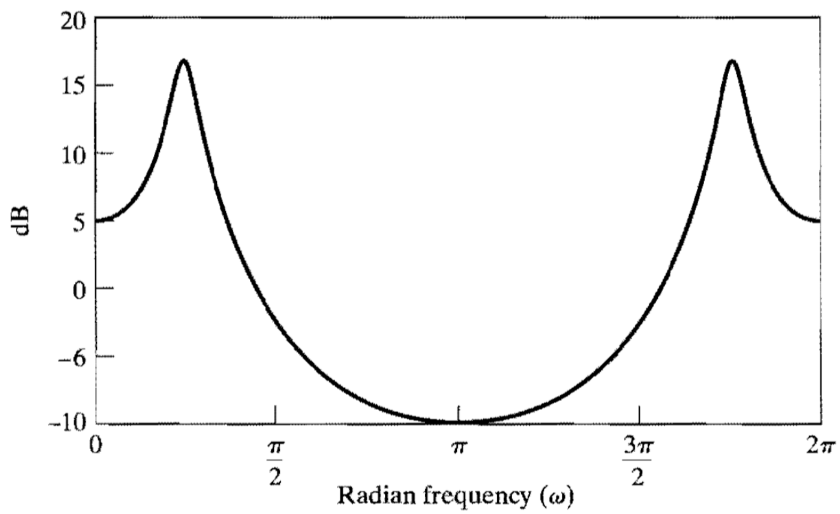
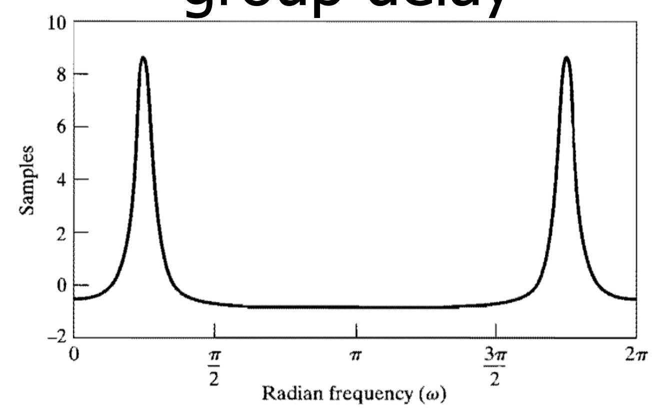
phase



magnitude

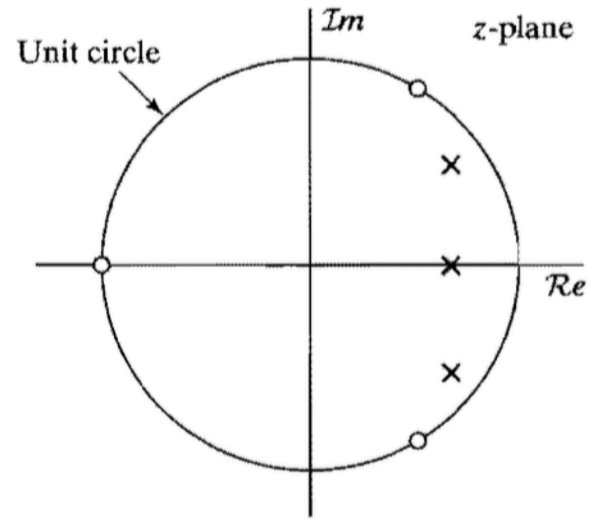


group delay



3rd Order IIR Example

$$H(z) = \frac{0.05634(1+z^{-1})(1-1.0166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$$



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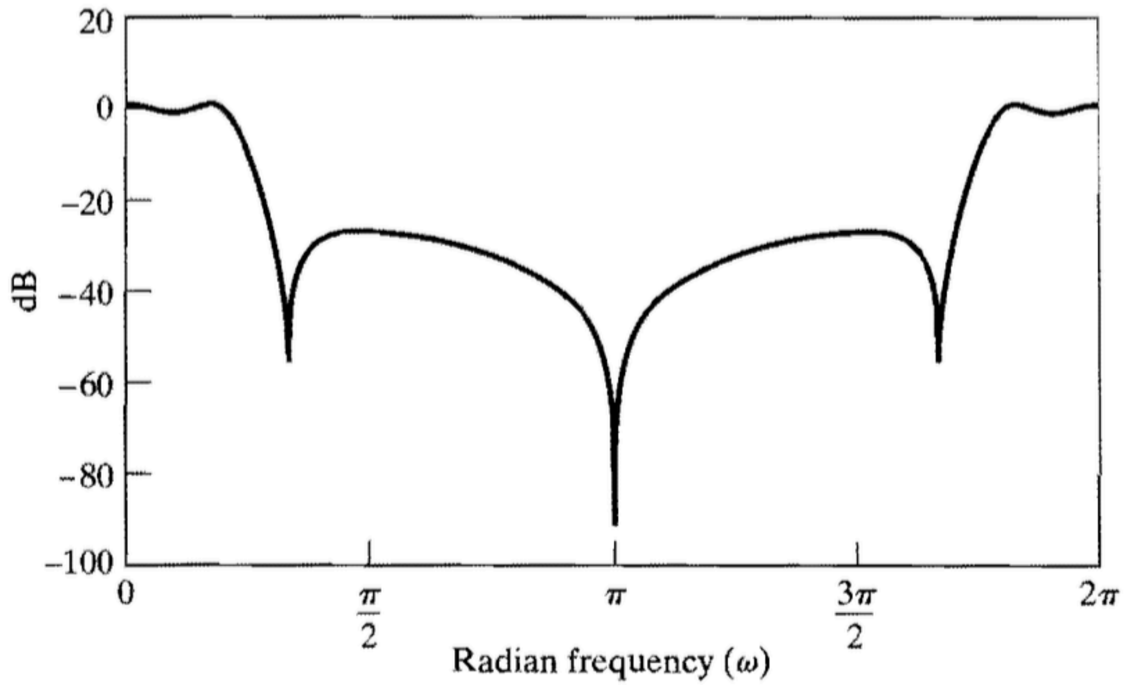

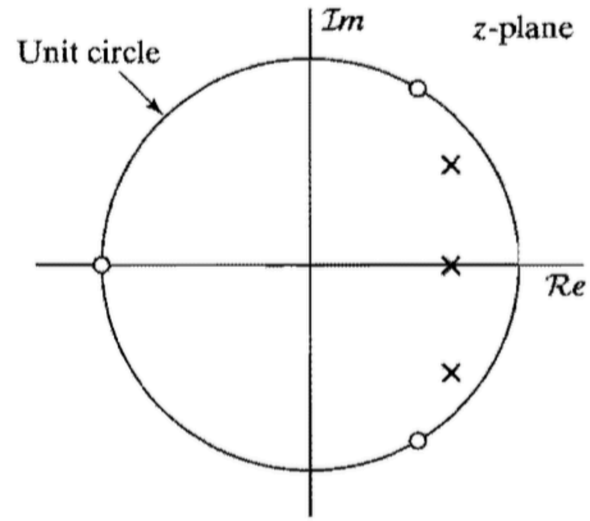

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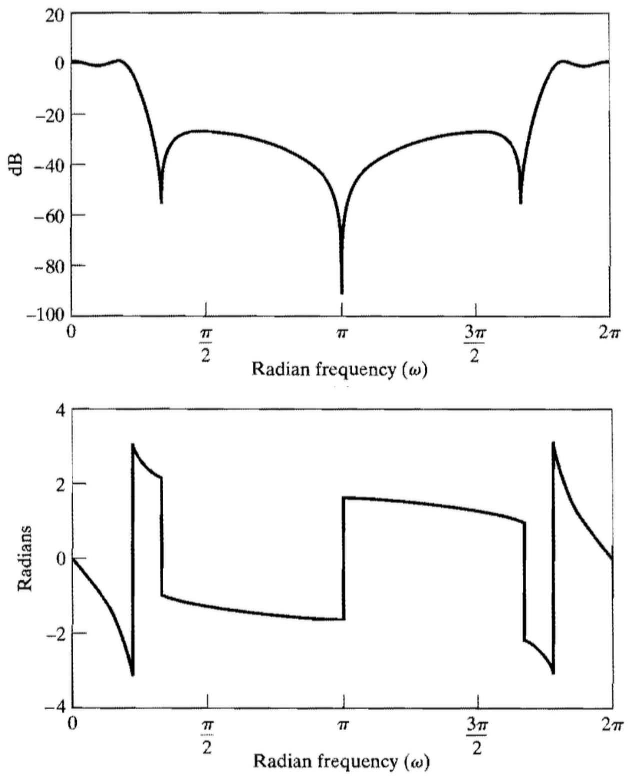

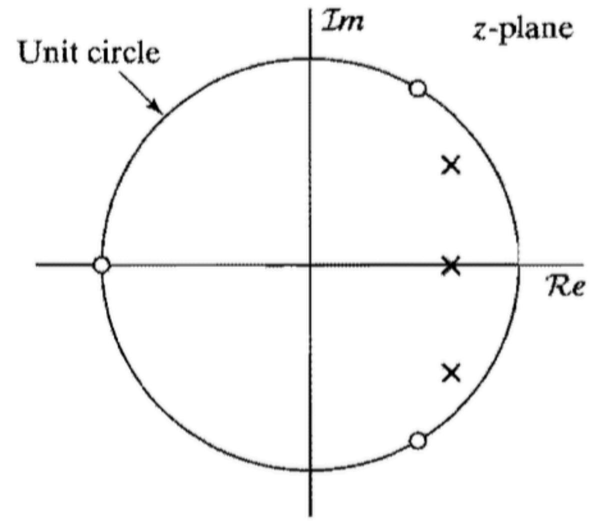

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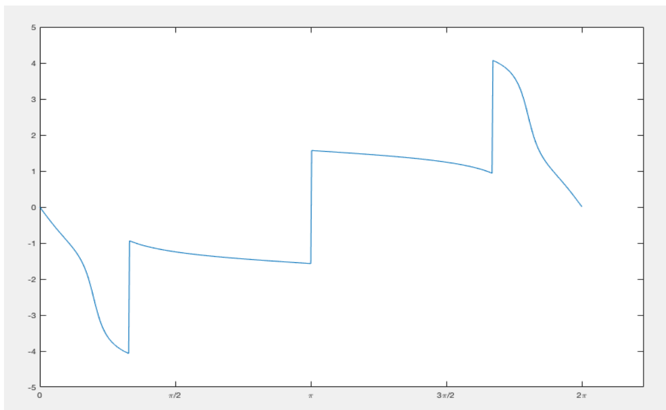

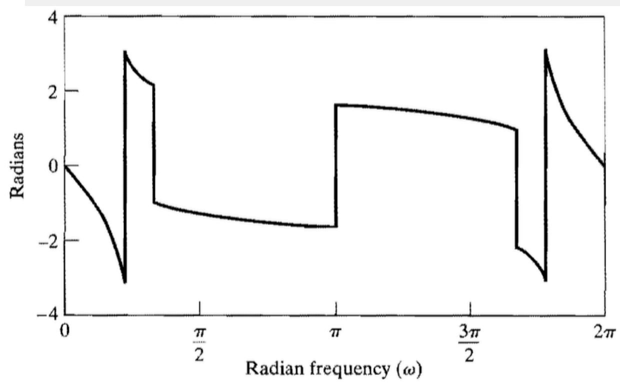

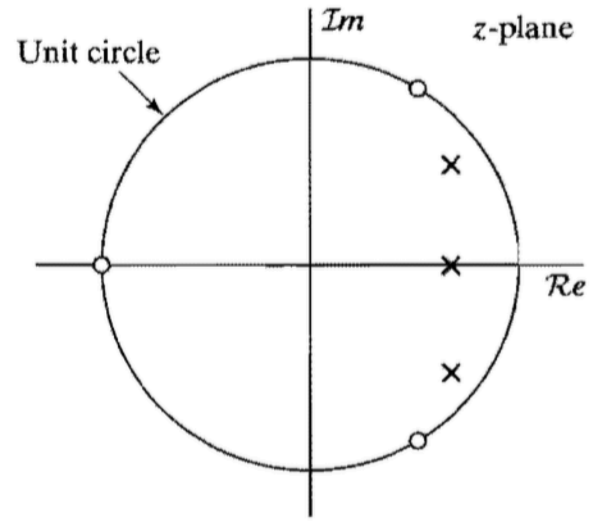

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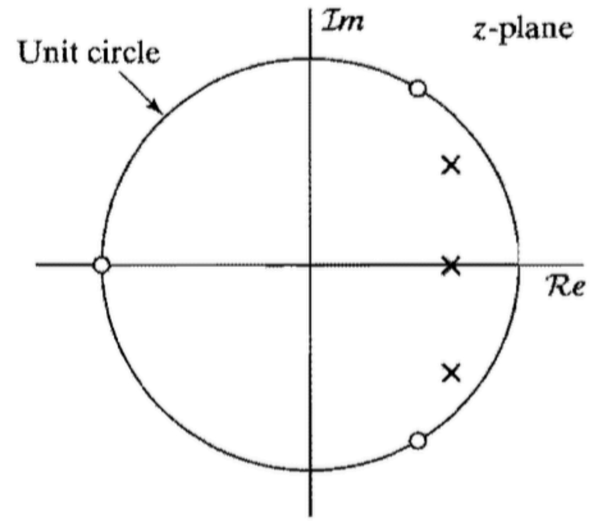
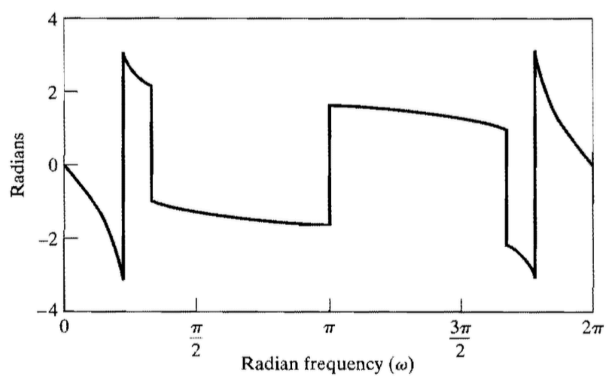
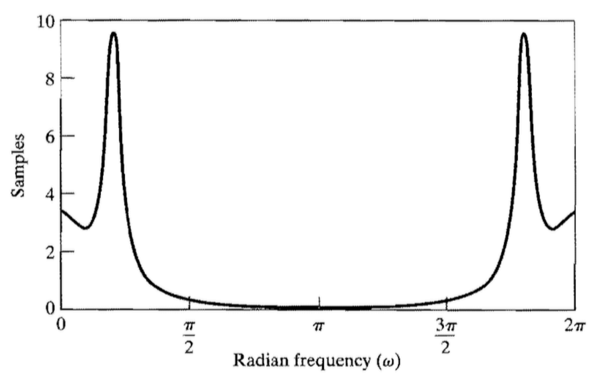
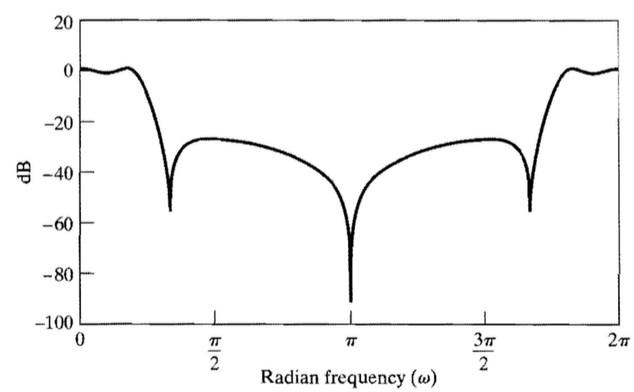








Stability and Causality

LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

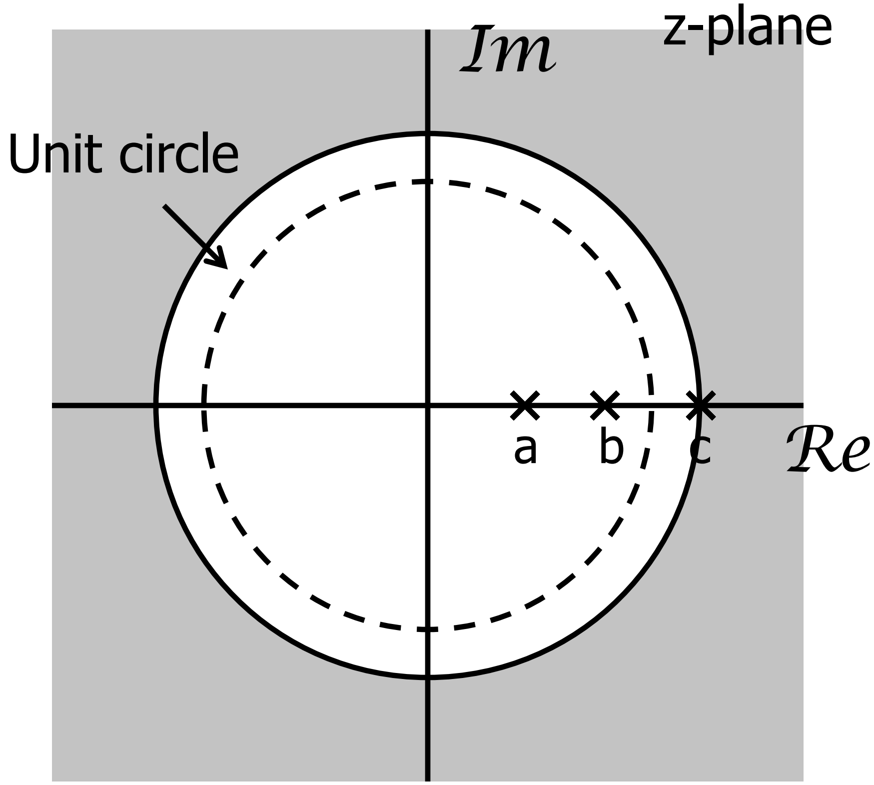
Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

Example: ROC from Pole-Zero Plot

ROC 1: right-sided



LTI System

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Example: $y[n] = x[n] + 0.1y[n-1]$

Stable and causal if all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
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All-Pass Systems

All-Pass Filters

- A system is an all-pass system if

$$\left|H(e^{j\omega})\right| = 1, \text{ all } \omega$$

- Its phase response $\theta(\omega)$ may be non-trivial

First Order All-Pass Filter (a real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

First Order All-Pass Filter (*a* real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$\left|H(e^{j\omega})\right| = \frac{\left|e^{-j\omega} - a^*\right|}{\left|1 - ae^{-j\omega}\right|}$$

$$= \frac{\left|e^{-j\omega}(1 - a^* e^{j\omega})\right|}{\left|1 - ae^{-j\omega}\right|}$$

$$= \frac{\left|1 - a^* e^{j\omega}\right|}{\left|1 - ae^{-j\omega}\right|} = 1$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\mathrm{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

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- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

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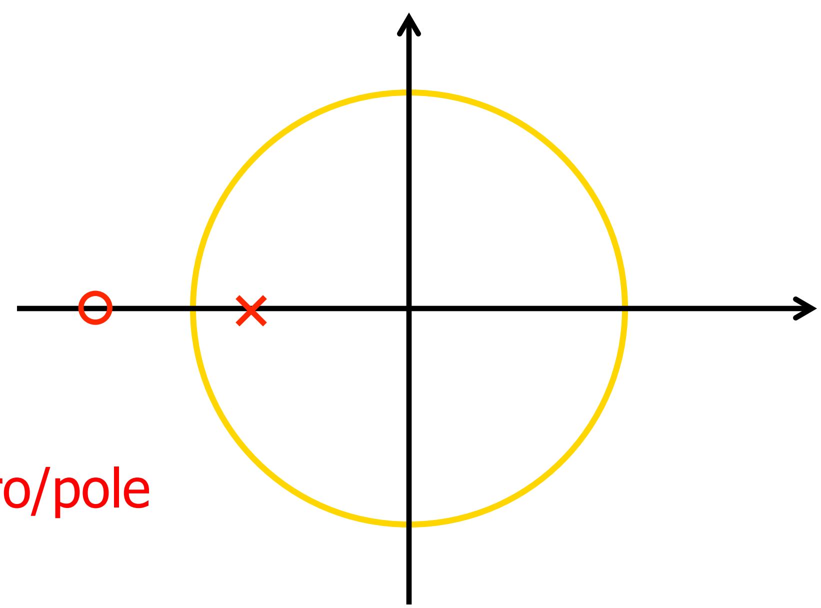

Real zero/pole

General All-Pass Filter

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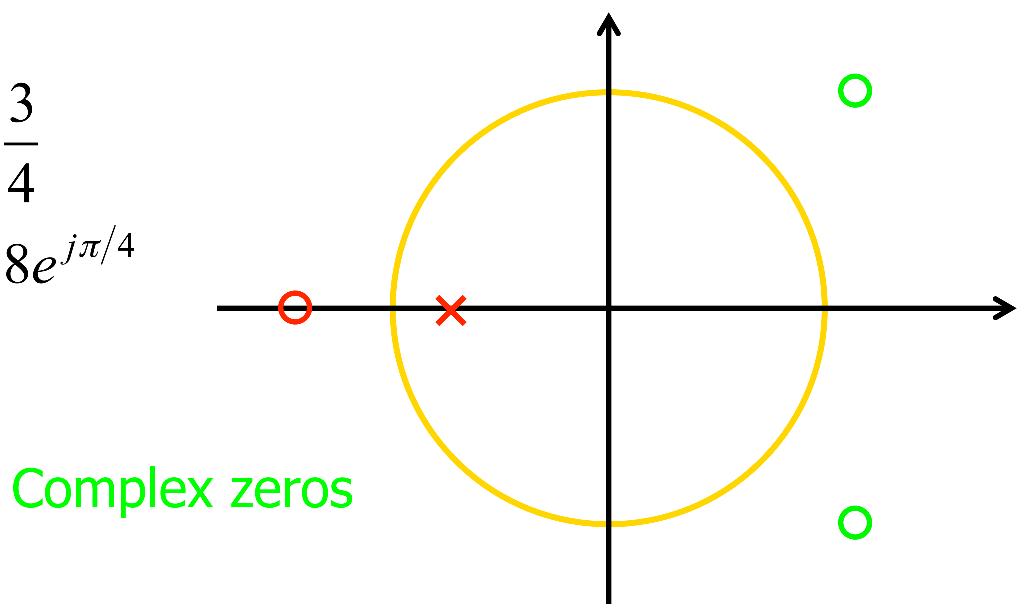

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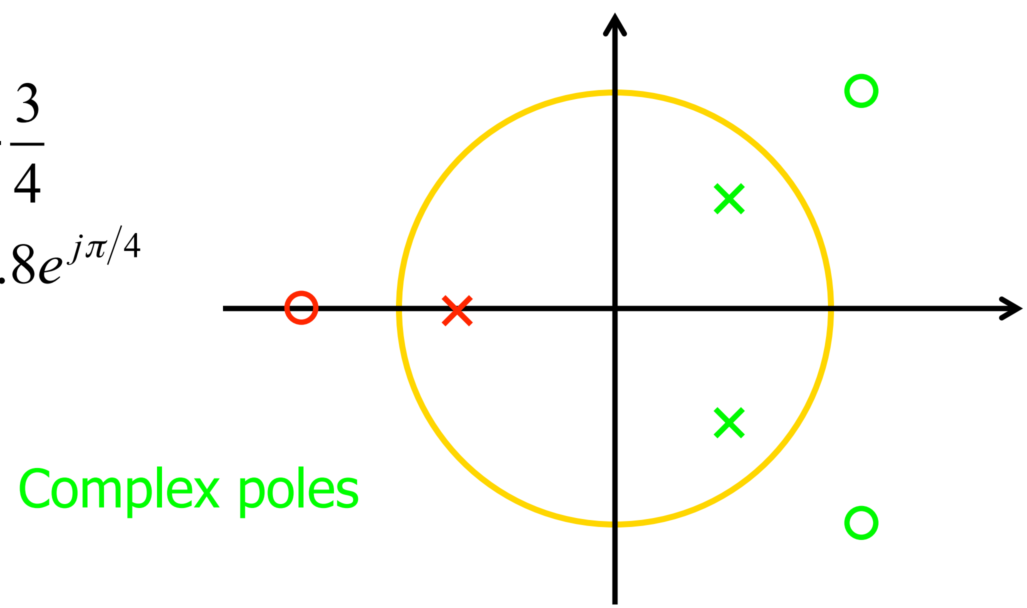
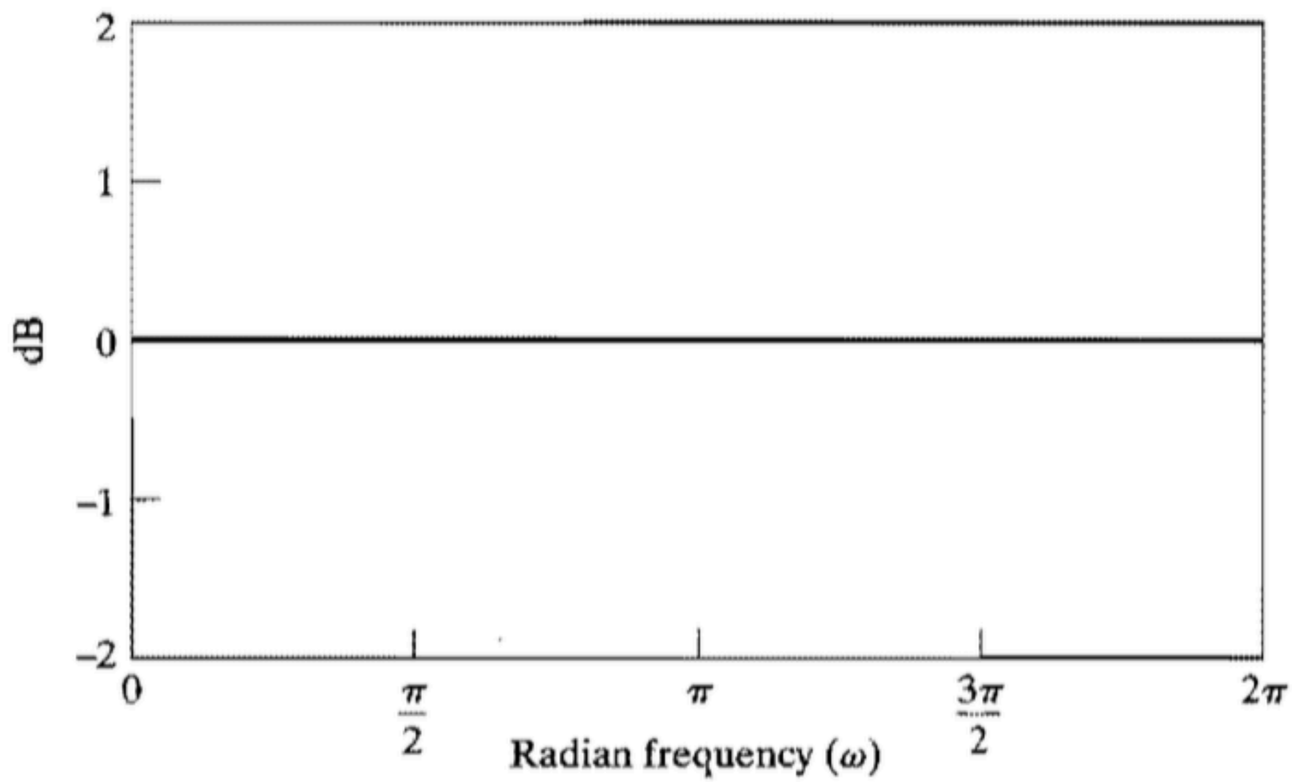


All Pass Filter Phase Response

- First order system $$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$ $$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$
- phase

First Order Example

- Magnitude:



All Pass Filter Phase Response

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- phase
$$\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$$

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$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$
$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

- phase
$$\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$$
$$= \arg\left(\frac{e^{-j\omega}(1 - re^{-j\theta}e^{j\omega})}{1 - re^{j\theta}e^{-j\omega}}\right)$$
$$= \arg(e^{-j\omega}) + \arg(1 - re^{-j\theta}e^{j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$
$$= -\omega - \arg(1 - re^{j\theta}e^{-j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$
$$= -\omega - 2\arg(1 - re^{j\theta}e^{-j\omega})$$

Group Delay Math

$$\mathrm{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \mathrm{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \mathrm{grd}[1 - d_k e^{-j\omega}]$$

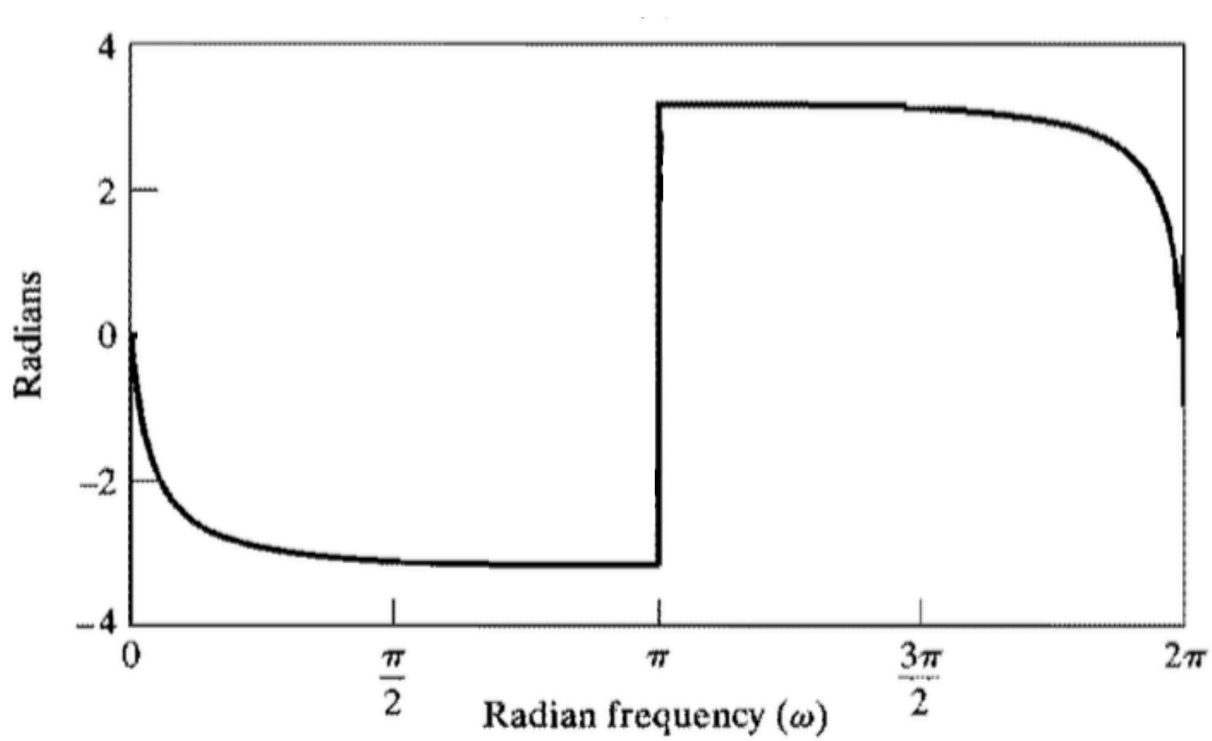
- Look at each factor:

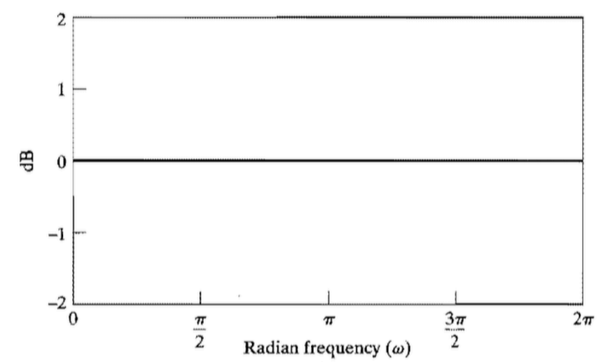
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

$$\mathrm{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

First Order Example

- Phase:



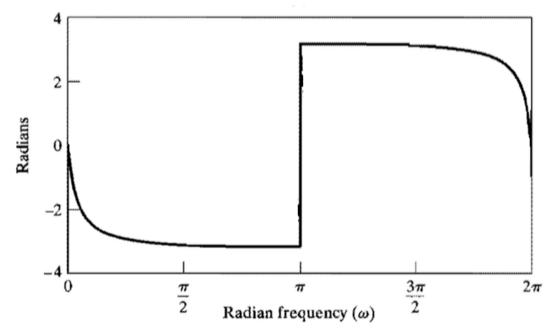


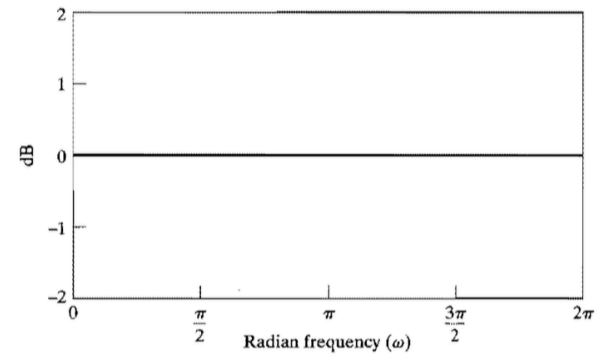
$z = 0.9$

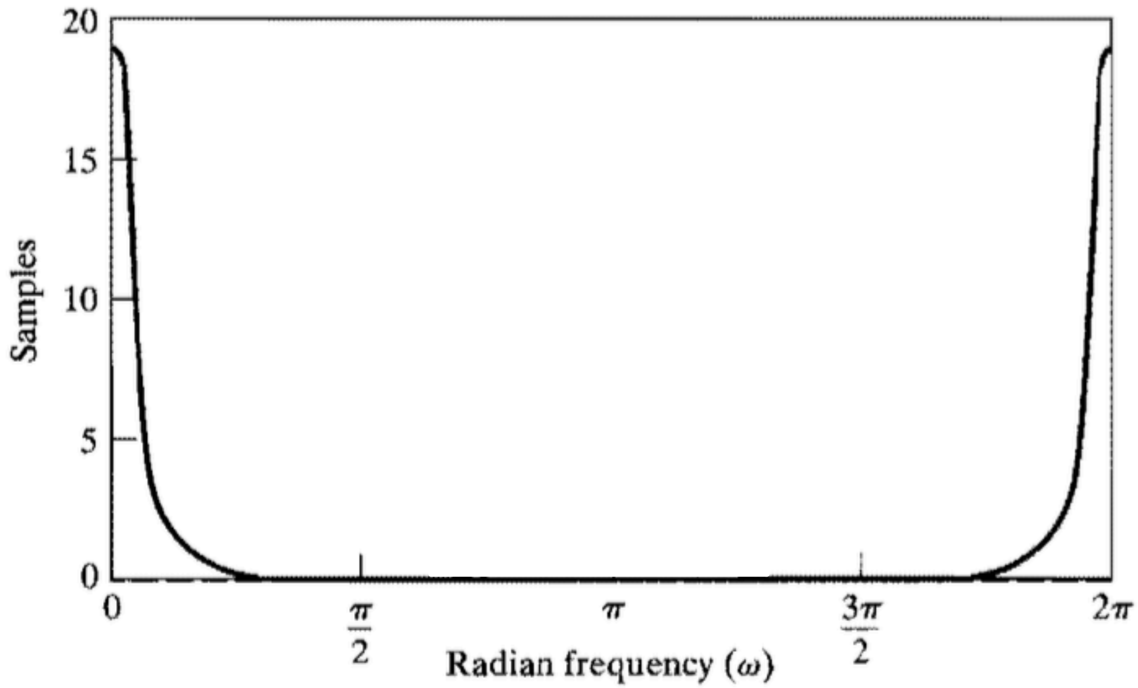
$\theta = 0$

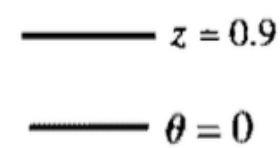
First Order Example

- Group Delay:









All Pass Filter Phase Response

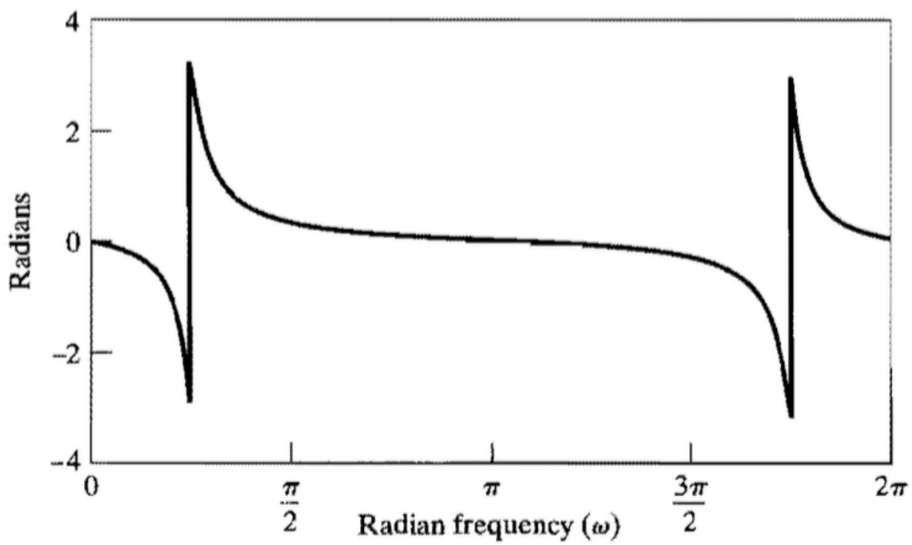
- Second order system with poles at $z = re^{j\theta}, re^{-j\theta}$

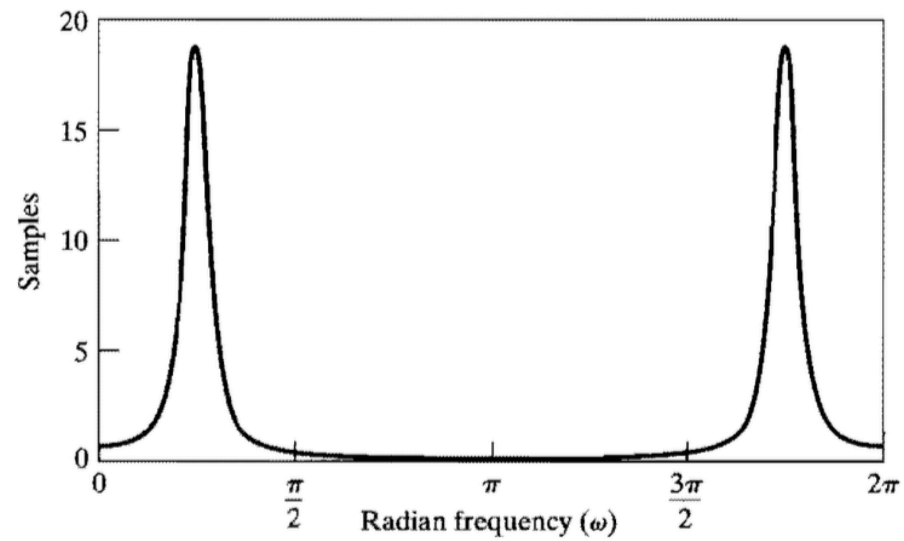
$$\angle\left[\frac{(e^{-j\omega} - re^{-j\theta})(e^{-j\omega} - re^{j\theta})}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}\right] = -2\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right]$$

$$-2\arctan\left[\frac{r\sin(\omega+\theta)}{1 - r\cos(\omega+\theta)}\right].$$

Second Order Example

- Poles at $z = 0.9e^{\pm j\pi/4}$ (zeros at conjugates)





All-Pass Properties

- Claim: For a stable, causal ($r < 1$) all-pass system:
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
    - Unwrapped phase always non-positive and decreasing
  - $\mathrm{grd}[H_{ap}(e^{j\omega})] > 0$
    - Group delay always positive

Pg 309 in text book

  - Intuition
    - delay is positive → system is causal
    - Phase negative → phase lag

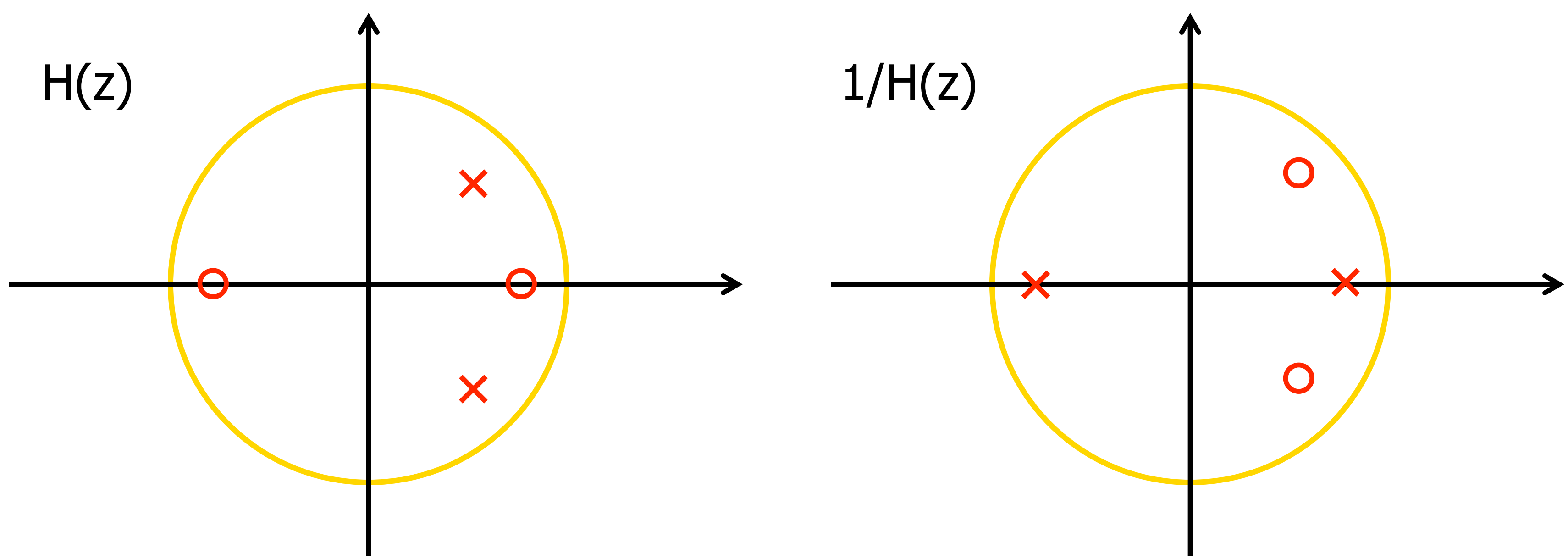
Minimum-Phase Systems

Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)

Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle



All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

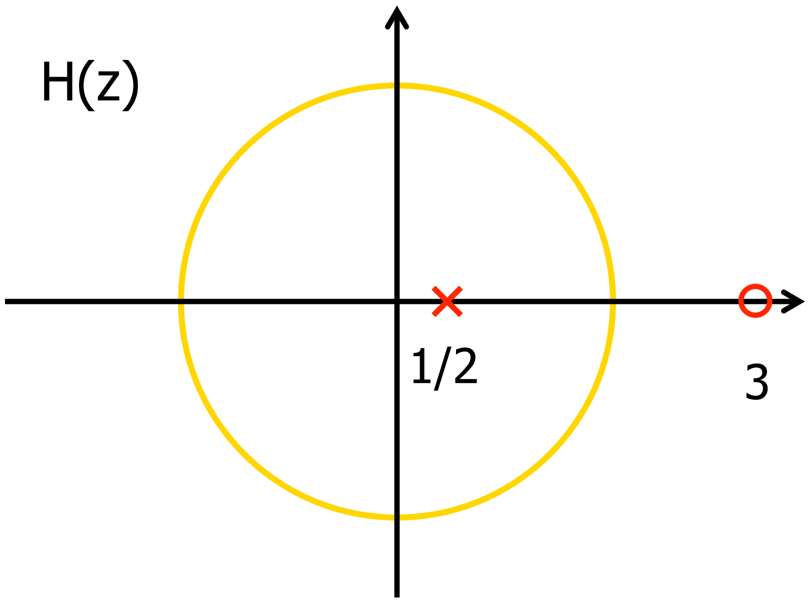
$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

- Approach:
  - (1) First construct H_{ap} with all zeros outside unit circle
  - (2) Compute

$$H_{\min}(z) = \frac{H(z)}{H_{ap}(z)}$$

Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$



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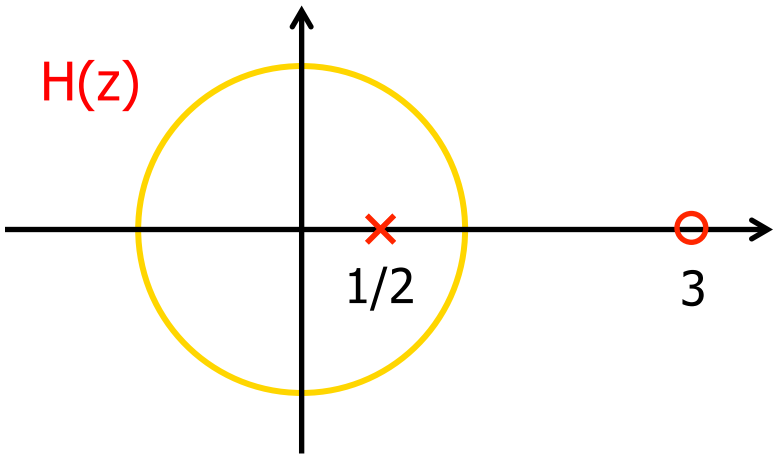

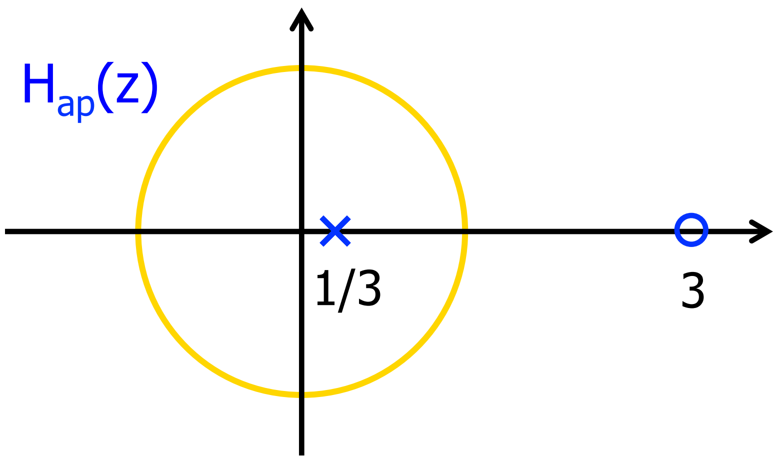

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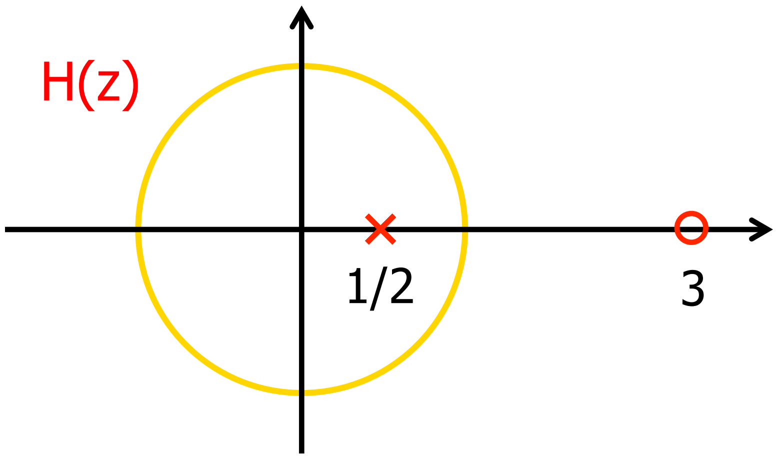
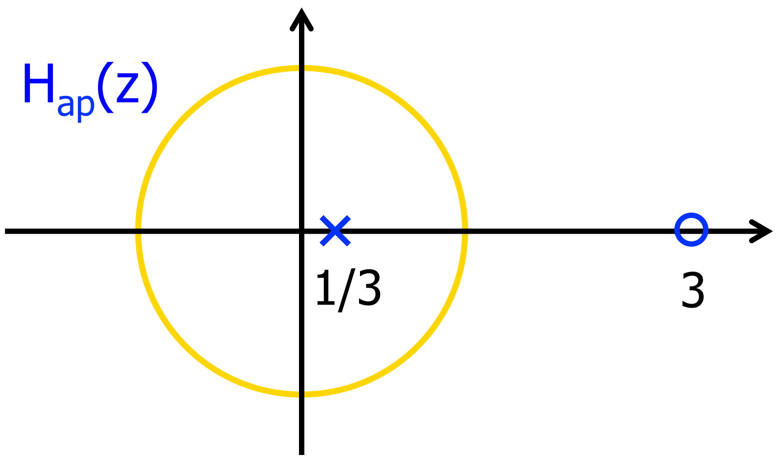
$$H_{min}(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}\left(\frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}\right)^{-1}$$





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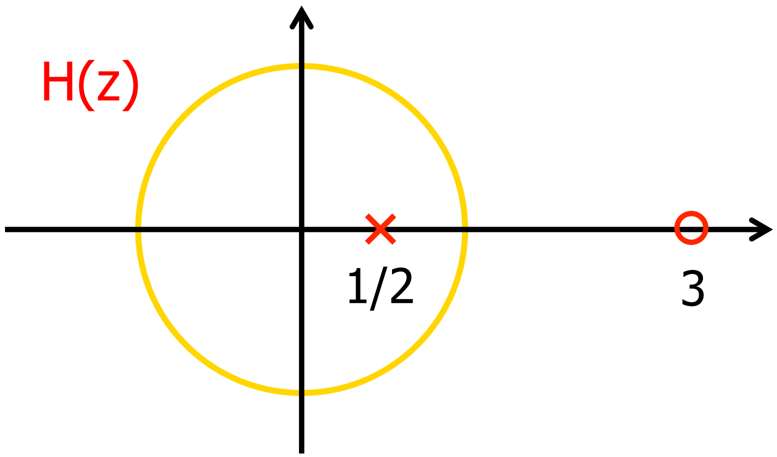
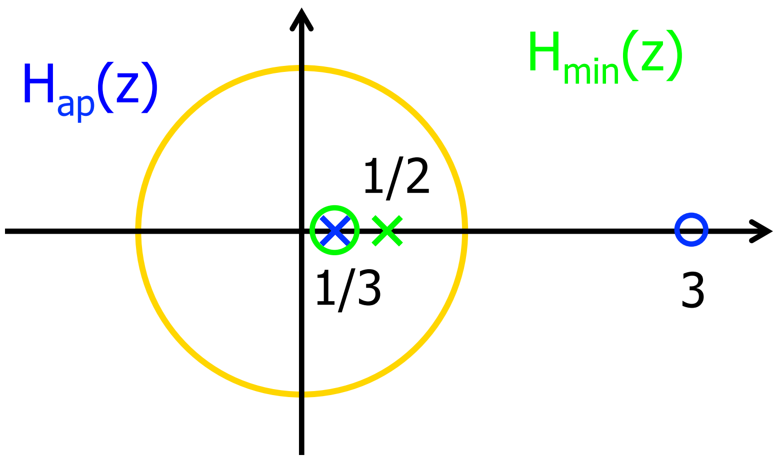
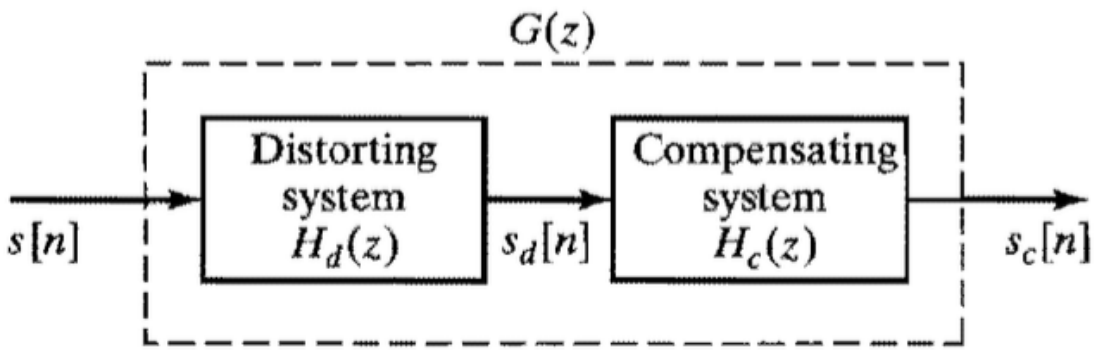
$$H_{\min}(z) = -3\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$





Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



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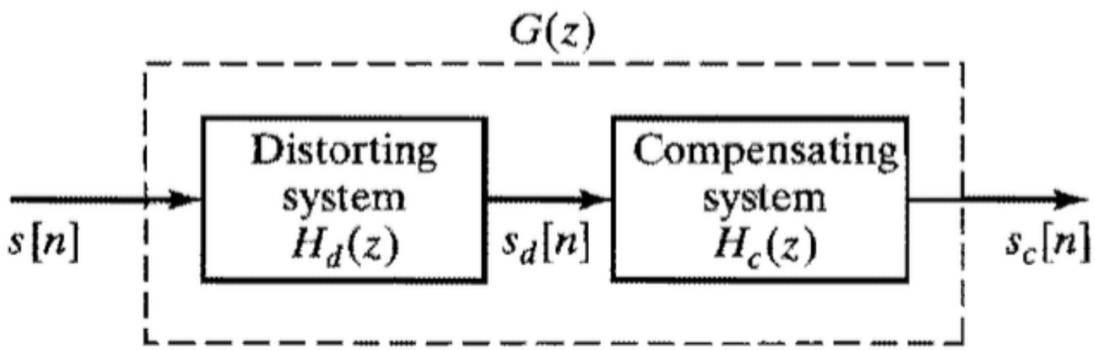

- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal

Min-Phase Decomposition Purpose

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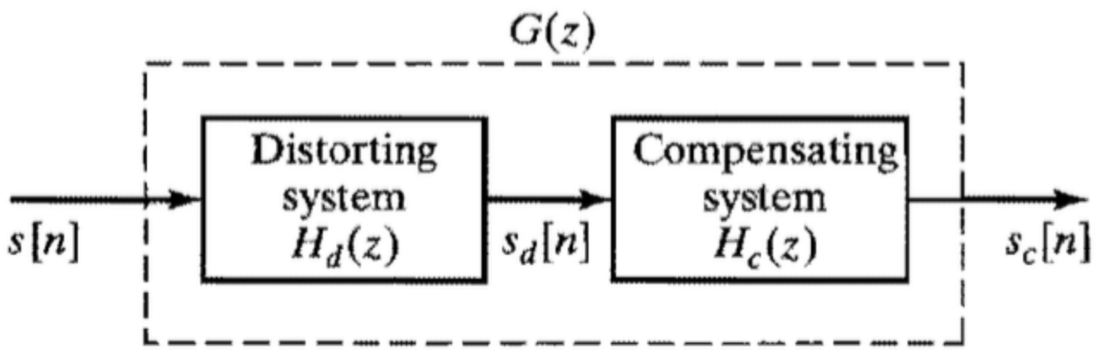
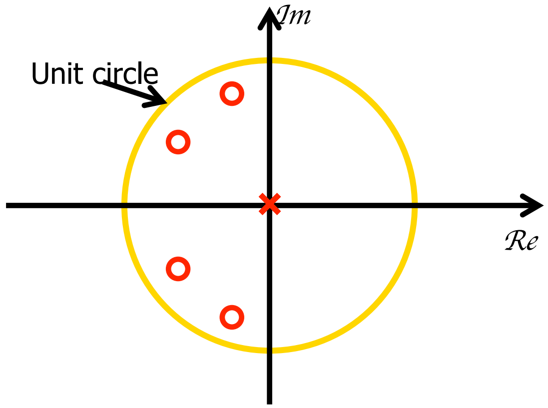


- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z) \rightarrow H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

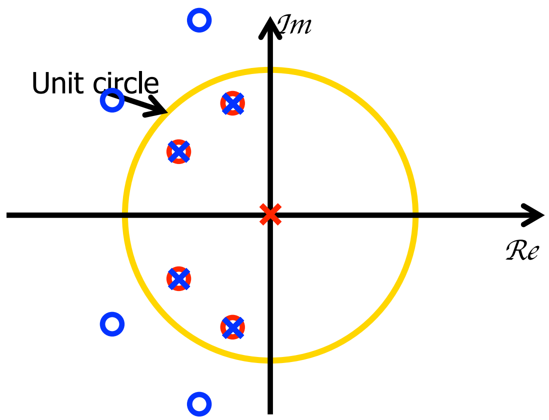
Minimum Phase to Max Phase

Min phase



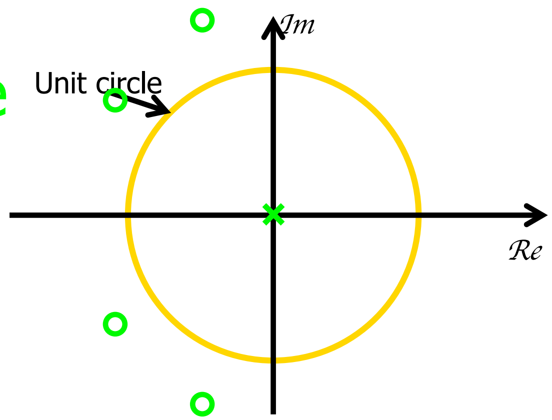
Minimum Phase to Max Phase

Min phase
x All Pass



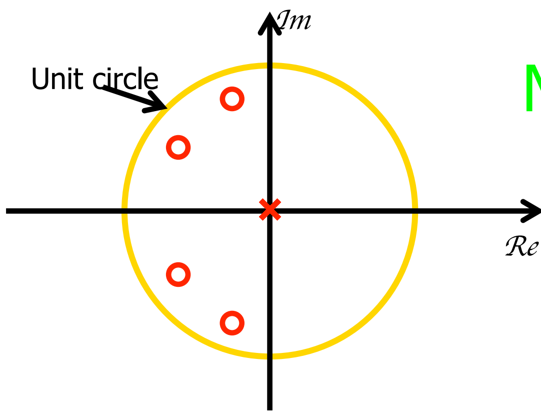
Minimum Phase to Max Phase

Max phase

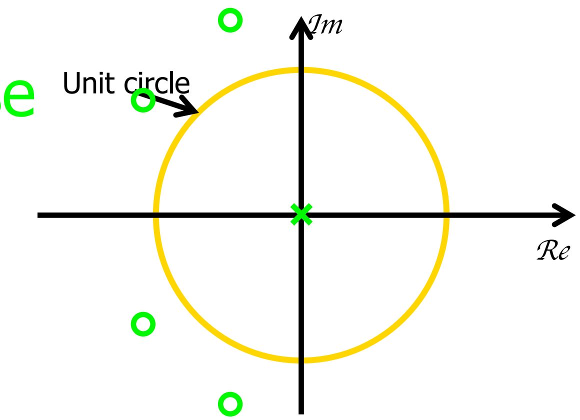


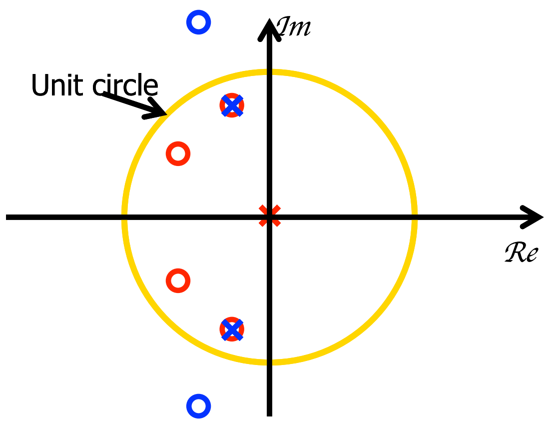
Minimum Phase

Min phase



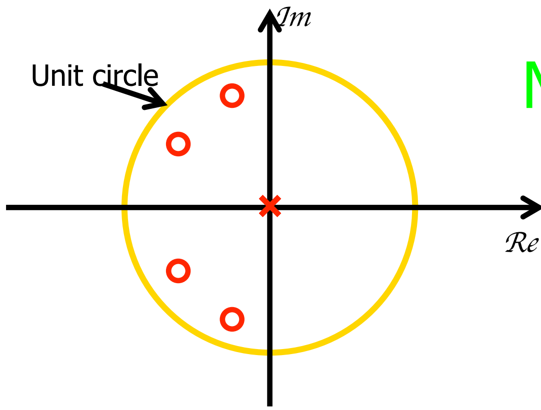
Max phase



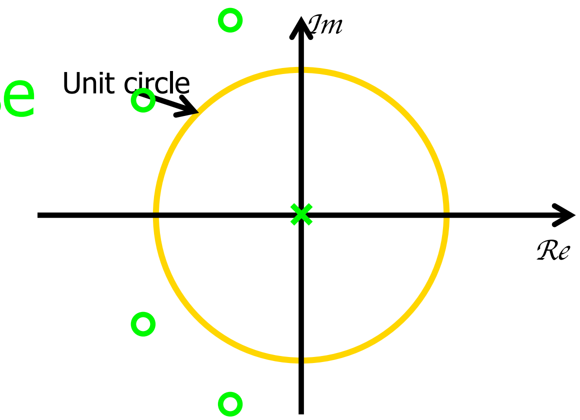


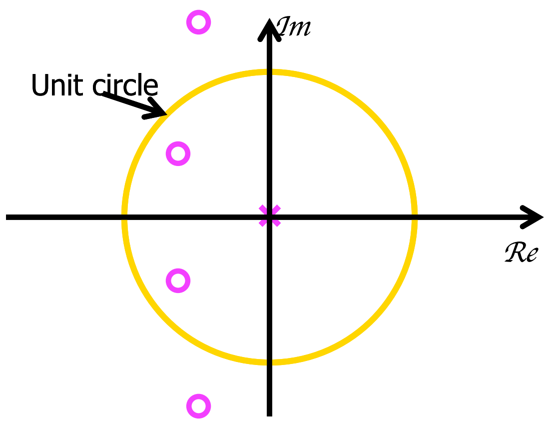
Minimum Phase

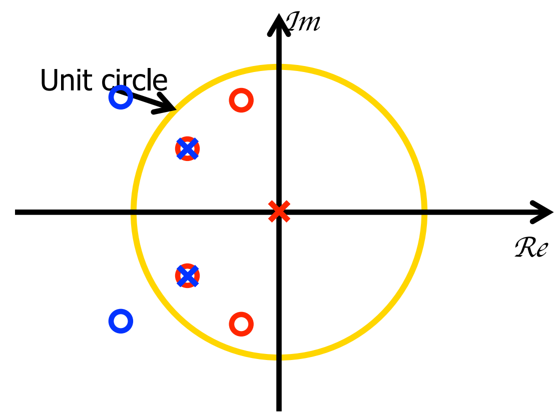
Min phase



Max phase



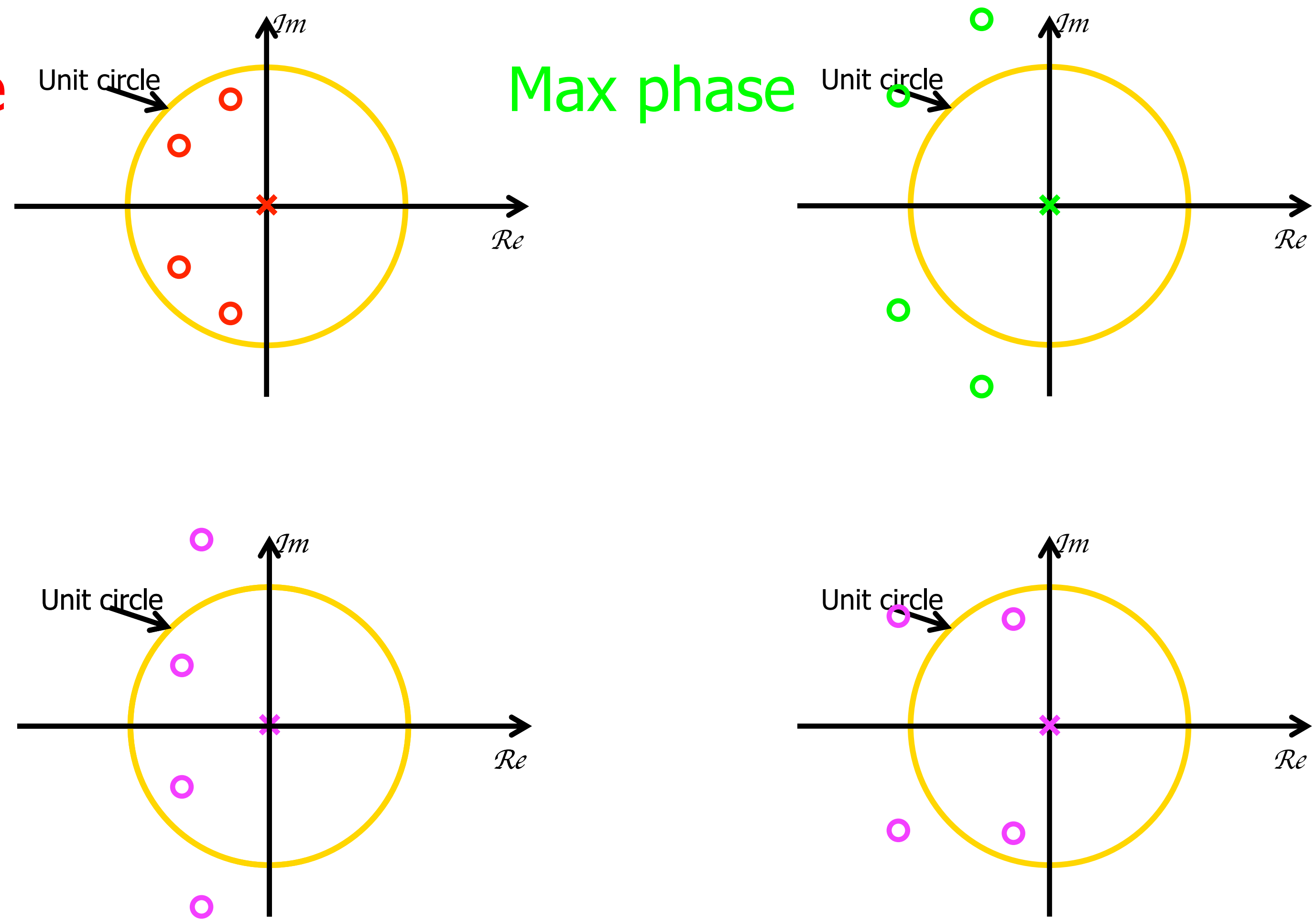




Minimum Phase

Min phase

Max phase



Minimum Phase Lag Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\mathrm{grd}[H_{ap}(e^{j\omega})] > 0$

- $$\begin{aligned} \arg[H_{max}(e^{j\omega})] &= \arg[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})] \\ &= \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})] \\ &= \qquad \leq 0 \qquad + \qquad \leq 0 \end{aligned}$$

Minimum Group Delay Property

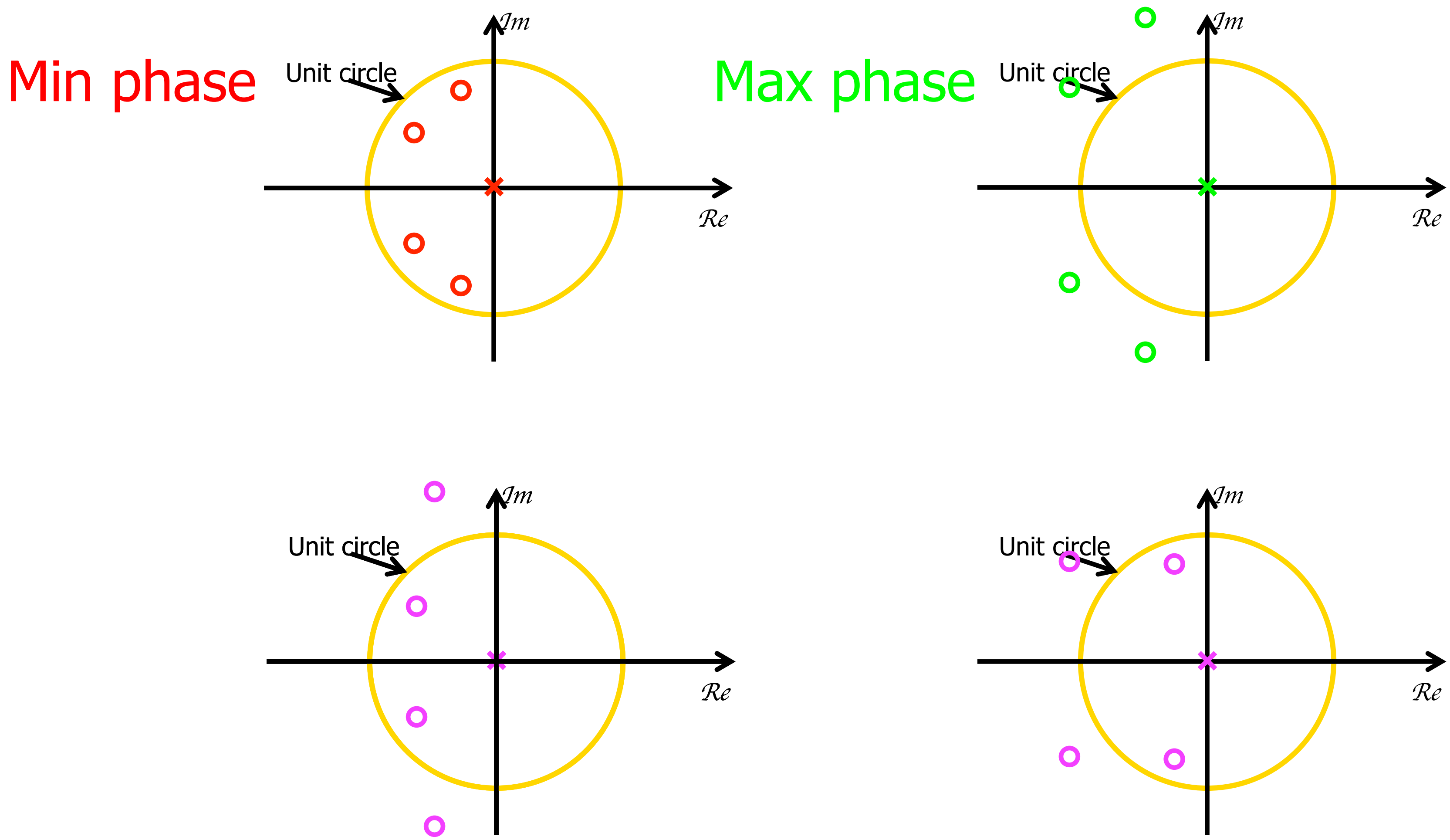
- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\mathrm{grd}[H_{ap}(e^{j\omega})] > 0$

- $\mathrm{grd}[H_{max}(e^{j\omega})] = \mathrm{grd}[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

$$= \mathrm{grd}[H_{min}(e^{j\omega})] + \mathrm{grd}[H_{ap}(e^{j\omega})]$$

$$= \quad \geq 0 \quad + \quad \geq 0$$

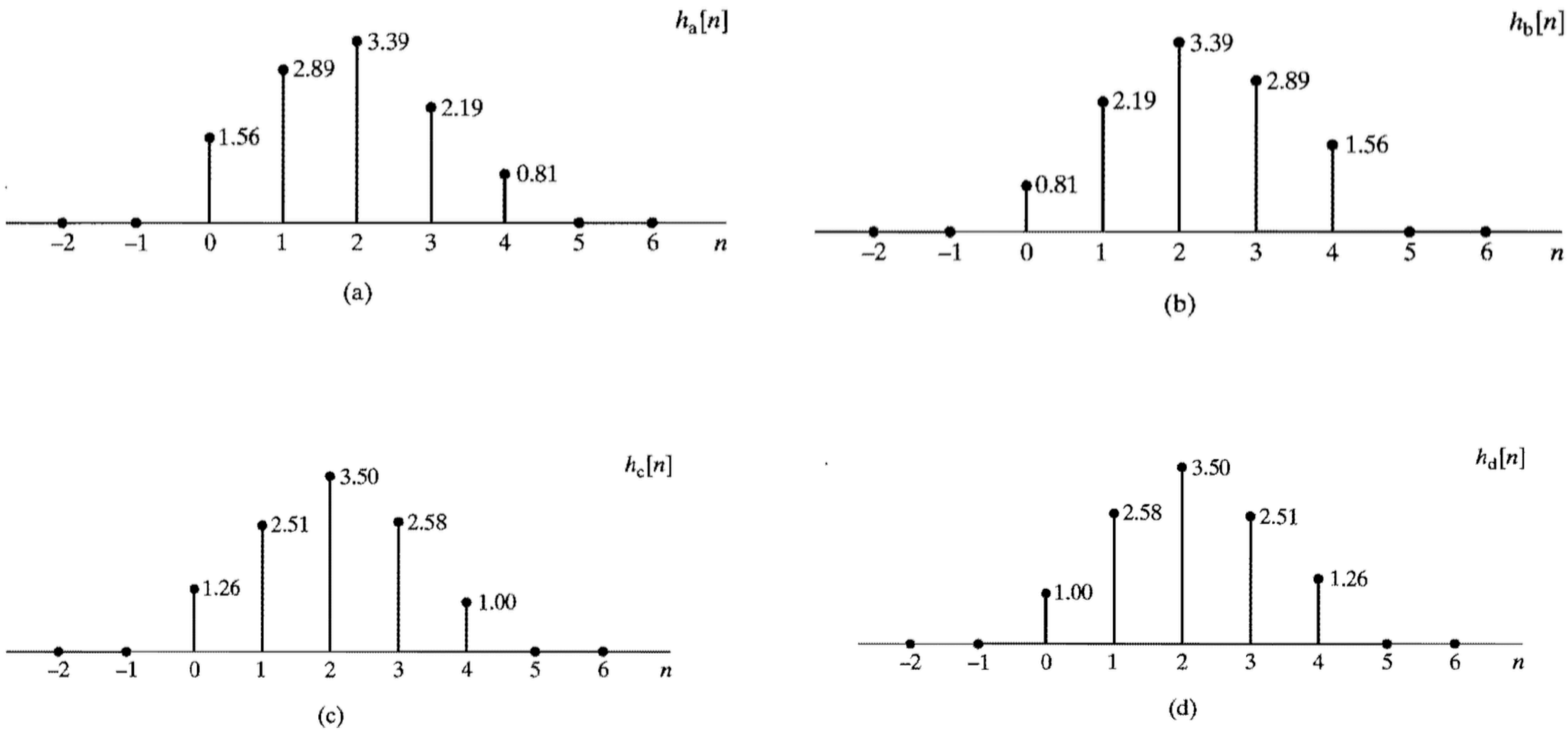
Minimum Phase



Minimum Energy-Delay Property

Min phase

Max phase



Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
- LTI Stability and Causality
  - If all poles inside unit circle
- All Pass Systems
  - Used for delay compensation
- Minimum Phase Systems
  - Can compensate for magnitude distortion
  - Minimum energy-delay property

Admin

- HW 6
  - Out now
  - Due Sunday 3/24