ESE 531: Digital Signal Processing

Lec 14: March 14th, 2019

All-Pass Systems and Min Phase Decomposition

Lecture Outline

- Review: Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
  - Examples:
    - Zero on Real Axis
    - 2^{nd} order IIR
    - 3^{rd} order Low Pass
- Stability and Causality
- All Pass Systems
- Minimum Phase Systems

Review: Frequency Response of LTI System

$$Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- We can define a magnitude response…

$$\left|Y\left(e^{j\omega}\right)\right|=\left|H\left(e^{j\omega}\right)\right|\left|X\left(e^{j\omega}\right)\right|$$

- a phase response…

$$\angle Y\left(e^{j\omega}\right)=\angle H\left(e^{j\omega}\right)+\angle X\left(e^{j\omega}\right)$$

- and group delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Review: Group Delay Math

$$H(z)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad H(e^{j\omega})=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})}$$

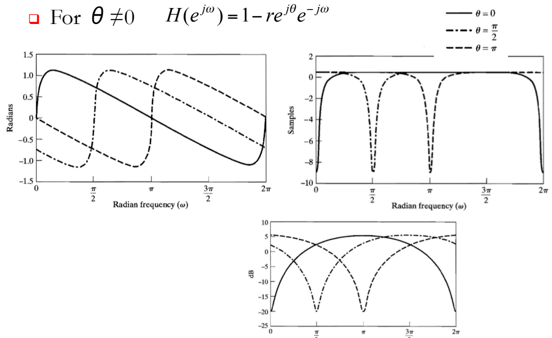
- Look at each factor:

$$\arg[1-re^{j\theta}e^{-j\omega}]=\tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)$$

$$\text{grd}[1-re^{j\theta}e^{-j\omega}]=\frac{r^2-r\cos(\omega-\theta)}{\left|1-re^{j\theta}e^{-j\omega}\right|^2}$$

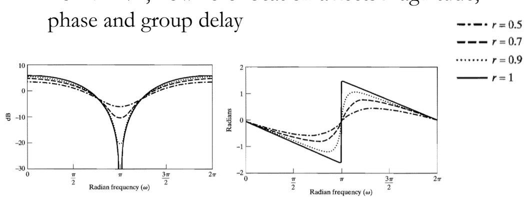
Example: Zero on Real Axis

- For $\theta \neq 0$ $\quad H(e^{j\omega})=1-re^{j\theta}e^{-j\omega}$

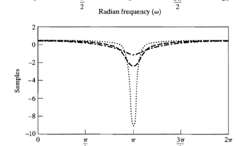


Example: Zero on Real Axis

- For $\theta = \pi$, how zero location affects magnitude, phase and group delay



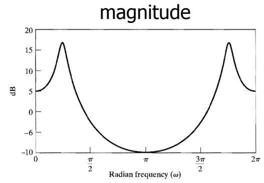
$$H(e^{j\omega})=1-re^{j\theta}e^{-j\omega}$$
$$H(e^{j\omega})=1+re^{-j\omega}$$



2^{nd} Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$ r=0.9, $\theta = \pi/4$

magnitude



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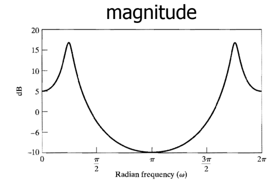
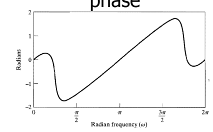
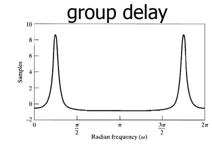


phase

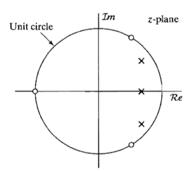


group delay



3^{rd} Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



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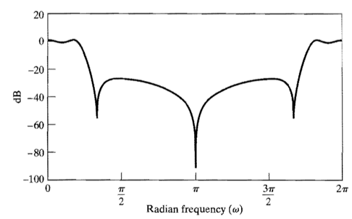

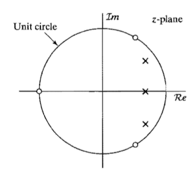

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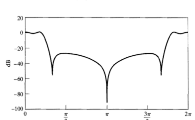

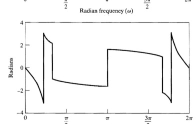

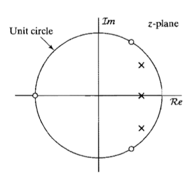

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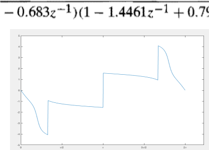

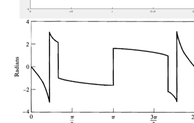

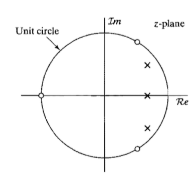

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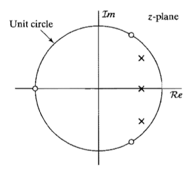
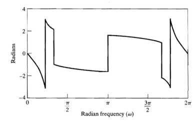
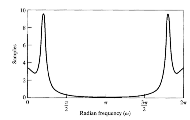
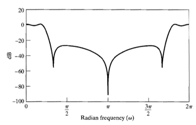








Stability and Causality

LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

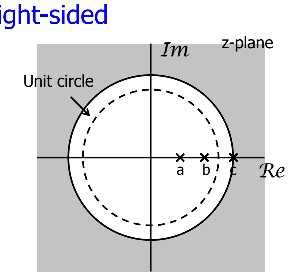
Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0 \prod_{k=1}^{M}(1-c_k z^{-1})}{a_0 \prod_{k=1}^{N}(1-d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

Example: ROC from Pole-Zero Plot

ROC 1: right-sided



LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

Stable and causal if all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0 \prod_{k=1}^{M}(1-c_k z^{-1})}{a_0 \prod_{k=1}^{N}(1-d_k z^{-1})}$$

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All-Pass Systems

All-Pass Filters

- A system is an all-pass system if

$$\left|H(e^{j\omega})\right| = 1, \quad \text{all } \omega$$

- Its phase response $\theta(\omega)$ may be non-trivial

First Order All-Pass Filter (*a* real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

First Order All-Pass Filter (*a* real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$\begin{aligned} \left|H(e^{j\omega})\right| &= \frac{\left|e^{-j\omega} - a^*\right|}{\left|1 - ae^{-j\omega}\right|} \\ &= \frac{\left|e^{-j\omega}(1 - a^* e^{j\omega})\right|}{\left|1 - ae^{-j\omega}\right|} \\ &= \frac{\left|1 - a^* e^{j\omega}\right|}{\left|1 - ae^{-j\omega}\right|} = 1 \end{aligned}$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\mathrm{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

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- Example:

$$d_k = -\frac{3}{4}$$
$$e_k = 0.8e^{j\pi/4}$$

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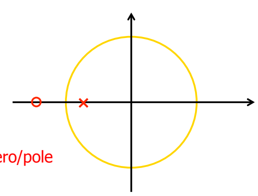

Real zero/pole

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

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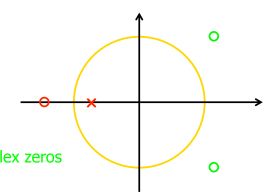

Complex zeros

General All-Pass Filter

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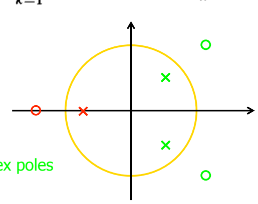


Complex poles

All Pass Filter Phase Response

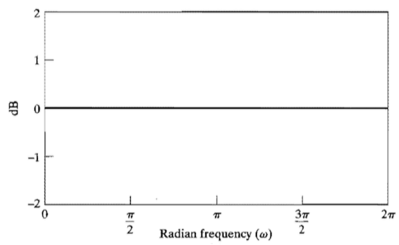
- First order system

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$
$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

- phase

First Order Example

- Magnitude:



All Pass Filter Phase Response

- First order system $H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$

- phase $\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$

All Pass Filter Phase Response

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- phase

$$
\begin{aligned}
&\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right) \\
&= \arg\left(\frac{e^{-j\omega}(1 - re^{-j\theta}e^{j\omega})}{1 - re^{j\theta}e^{-j\omega}}\right) \\
&= \arg(e^{-j\omega}) + \arg(1 - re^{-j\theta}e^{j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega}) \\
&= -\omega - \arg(1 - re^{j\theta}e^{-j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega}) \\
&= -\omega - 2\arg(1 - re^{j\theta}e^{-j\omega})
\end{aligned}
$$

Group Delay Math

$$\mathrm{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \mathrm{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \mathrm{grd}[1 - d_k e^{-j\omega}]$$

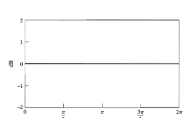
- Look at each factor:

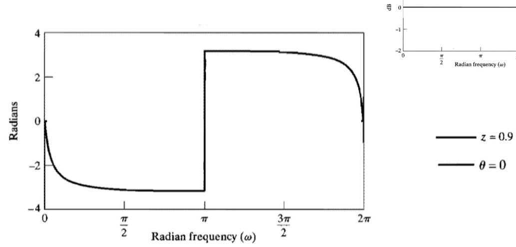
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right)$$

$$\mathrm{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega - \theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

First Order Example

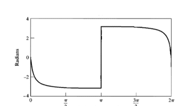
- Phase:

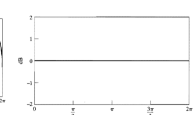


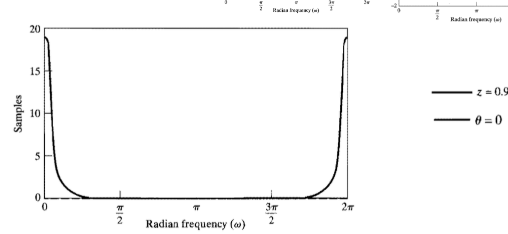


First Order Example

- Group Delay:







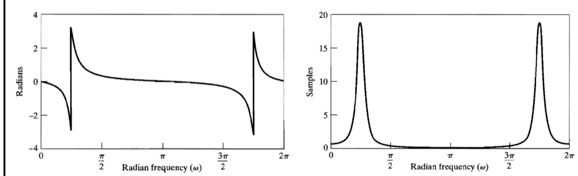
All Pass Filter Phase Response

- Second order system with poles at $z = re^{j\theta}, re^{-j\theta}$

$$\angle\left[\frac{(e^{-j\omega}-re^{-j\theta})(e^{-j\omega}-re^{j\theta})}{(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{-j\omega})}\right] = -2\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$
$$-2\arctan\left[\frac{r\sin(\omega+\theta)}{1-r\cos(\omega+\theta)}\right].$$

Second Order Example

- Poles at $z = 0.9e^{\pm j\pi/4}$ (zeros at conjugates)



All-Pass Properties

- Claim: For a stable, causal ($r < 1$) all-pass system:
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
    - Unwrapped phase always non-positive and decreasing
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$
    - Group delay always positive

Pg 309 in text book

  - Intuition
    - delay is positive → system is causal
    - Phase negative → phase lag

Minimum-Phase Systems

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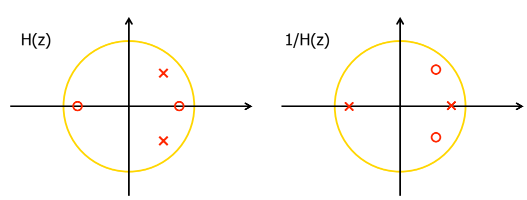
Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)

Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle



All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

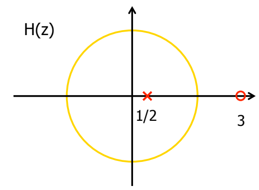
$$H(z)=H_{\min}(z)\cdot H_{ap}(z)$$

- Approach:
  - (1) First construct H_{ap} with all zeros outside unit circle
  - (2) Compute

$$H_{\min}(z)=\frac{H(z)}{H_{ap}(z)}$$

Min-Phase Decomposition Example

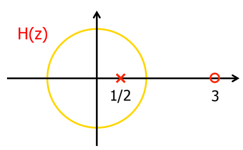
$$H(z)=\frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

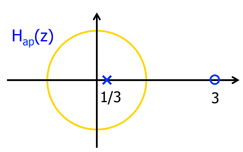


Min-Phase Decomposition Example

$$H(z)=\frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z)=\dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$



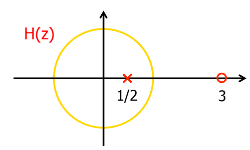


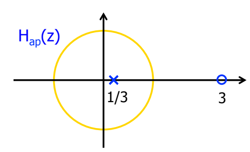
Min-Phase Decomposition Example

$$H(z)=\frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z)=\dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$

$$H_{\min}(z)=\frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}\left(\frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}\right)^{-1}$$



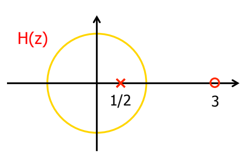


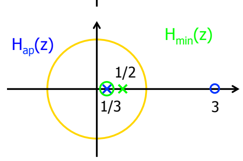
Min-Phase Decomposition Example

$$H(z)=\frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z)=\dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$

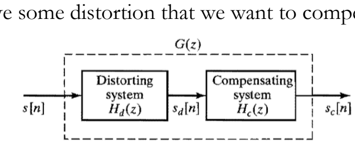
$$H_{\min}(z)=-3\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$





Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



Min-Phase Decomposition Purpose

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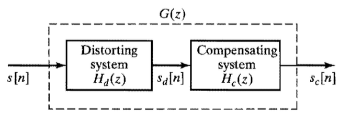
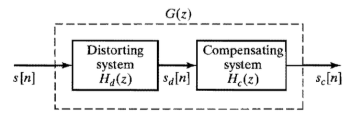


- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal

Min-Phase Decomposition Purpose

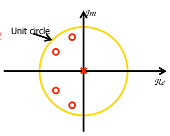
- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z) \rightarrow H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

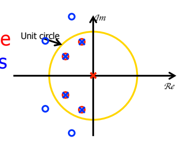
Minimum Phase to Max Phase

Min phase



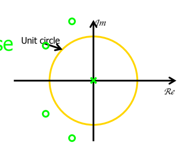
Minimum Phase to Max Phase

Min phase
x All Pass



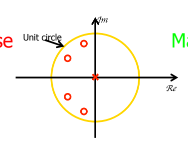
Minimum Phase to Max Phase

Max phase

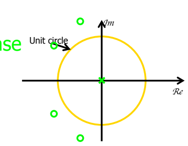


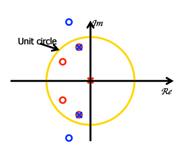
Minimum Phase

Min phase

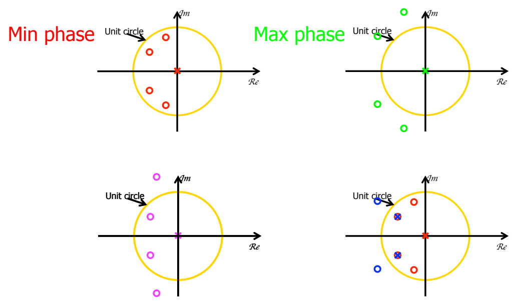


Max phase

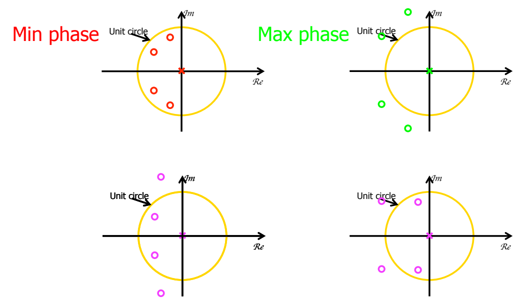




Minimum Phase



Minimum Phase



Minimum Phase Lag Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

- $\arg[H_{max}(e^{j\omega})] = \arg[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

$$= \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

$$= \quad \leq 0 \quad + \quad \leq 0$$

Minimum Group Delay Property

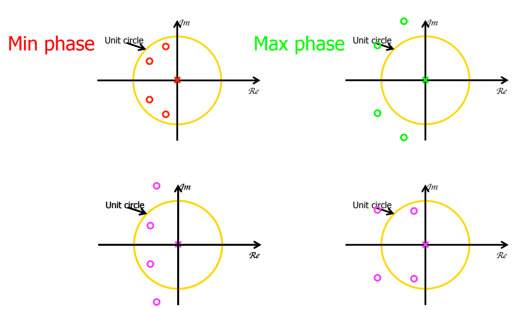
- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

- $\text{grd}[H_{max}(e^{j\omega})] = \text{grd}[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

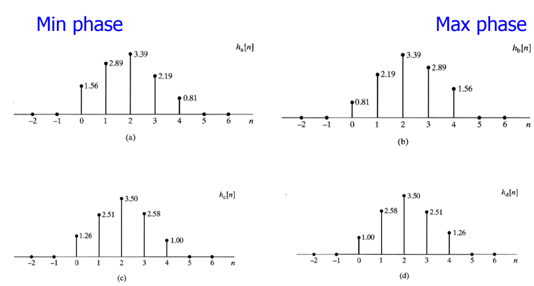
$$= \text{grd}[H_{min}(e^{j\omega})] + \text{grd}[H_{ap}(e^{j\omega})]$$

$$= \quad \geq 0 \quad + \quad \geq 0$$

Minimum Phase



Minimum Energy-Delay Property



Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
- LTI Stability and Causality
  - If all poles inside unit circle
- All Pass Systems
  - Used for delay compensation
- Minimum Phase Systems
  - Can compensate for magnitude distortion
  - Minimum energy-delay property

Admin

- HW 6
  - Out now
  - Due Sunday 3/24