

ESE 531: Digital Signal Processing

Lec 17: March 26, 2019

Design of FIR Filters



Linear Filter Design

- ❑ Used to be an art
 - Now, lots of tools to design optimal filters
- ❑ For DSP there are two common classes
 - Infinite impulse response IIR
 - Finite impulse response FIR
- ❑ Both classes use finite order of parameters for design
- ❑ Today we will focus on FIR designs



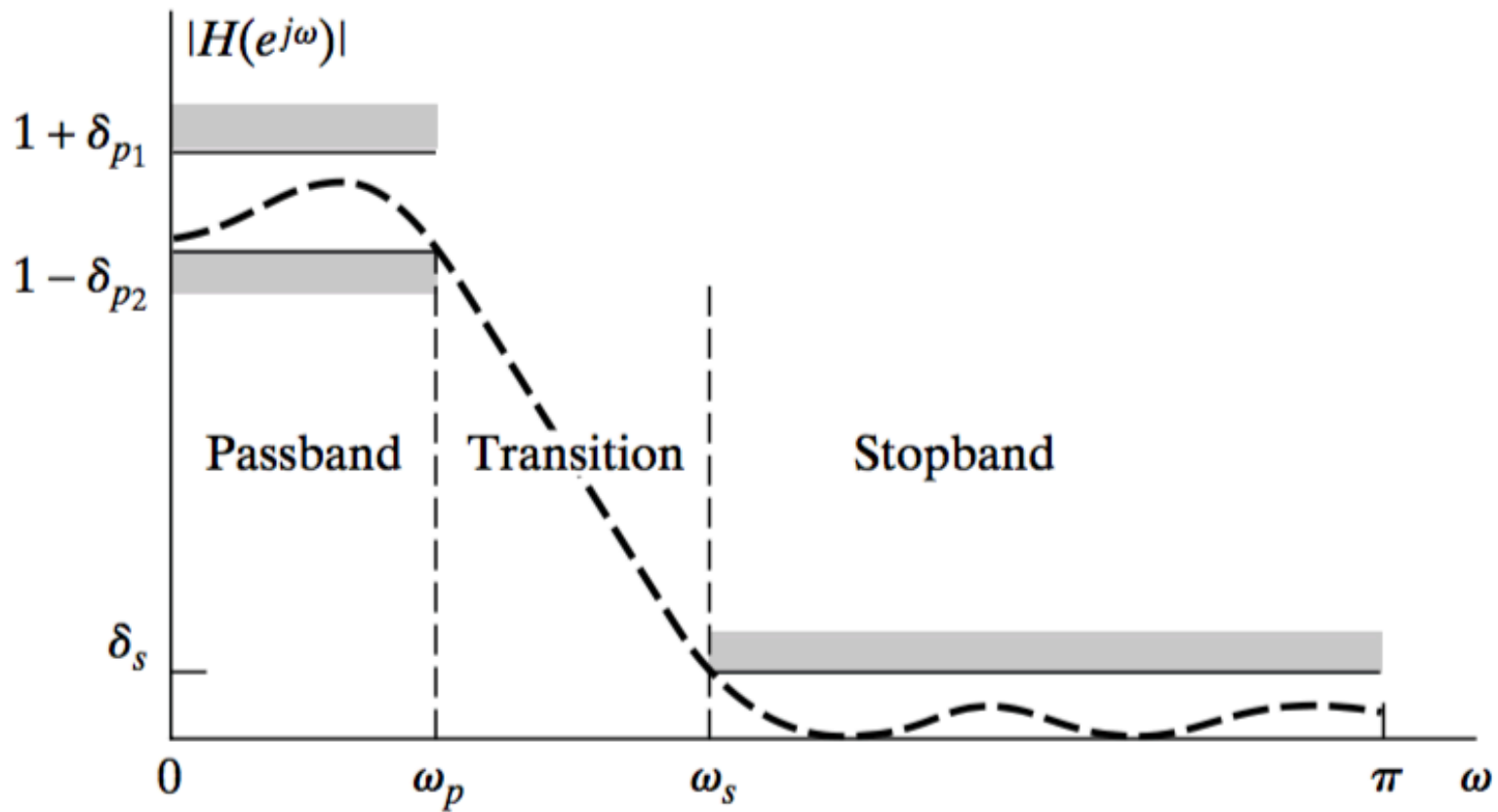
What is a Linear Filter?

- ❑ Attenuates certain frequencies
- ❑ Passes certain frequencies
- ❑ Affects both phase and magnitude

- ❑ IIR
 - Mostly non-linear phase response
 - Could be linear over a range of frequencies

- ❑ FIR
 - Much easier to control the phase
 - Both non-linear and linear phase

Filter Specifications



Impulse Invariance

□ Let,

$$h[n] = T h_c(nT)$$

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

□ If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c \left[j \left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

$$H(e^{j\omega}) = H_c \left(j \frac{\omega}{T} \right), \quad |\omega| < \pi$$



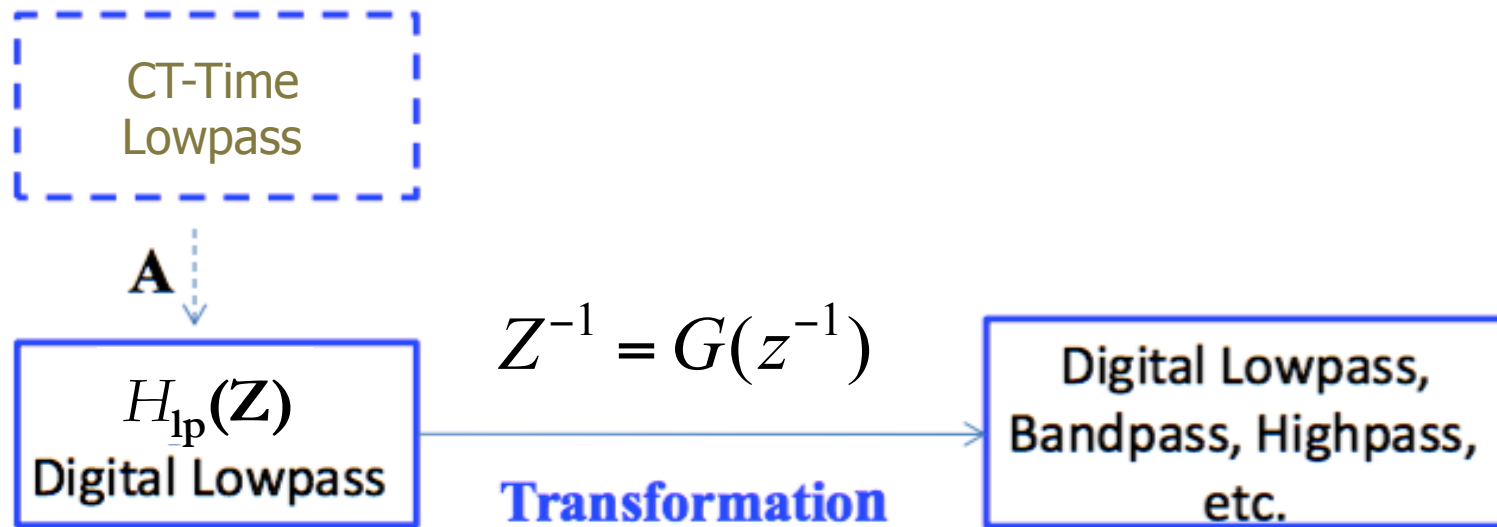
Bilinear Transformation

- The technique uses an algebraic transformation between the variables s and z that maps the entire $j\Omega$ -axis in the s -plane to one revolution of the unit circle in the z -plane.

$$s = \frac{2}{T_d} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right);$$

$$H(z) = H_c \left(\frac{2}{T_d} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) \right).$$

Transformation of DT Filters



- ❑ Z – complex variable for the LP filter
- ❑ z – complex variable for the transformed filter
- ❑ Map Z -plane $\rightarrow$ z -plane with transformation G



CT Filters

- ❑ Butterworth
 - Monotonic in pass and stop bands
- ❑ Chebyshev, Type I
 - Equiripple in pass band and monotonic in stop band
- ❑ Chebyshev, Type II
 - Monotonic in pass band and equiripple in stop band
- ❑ Elliptic
 - Equiripple in pass and stop bands

- ❑ Appendix B in textbook



Design Comparison

❑ Design specifications

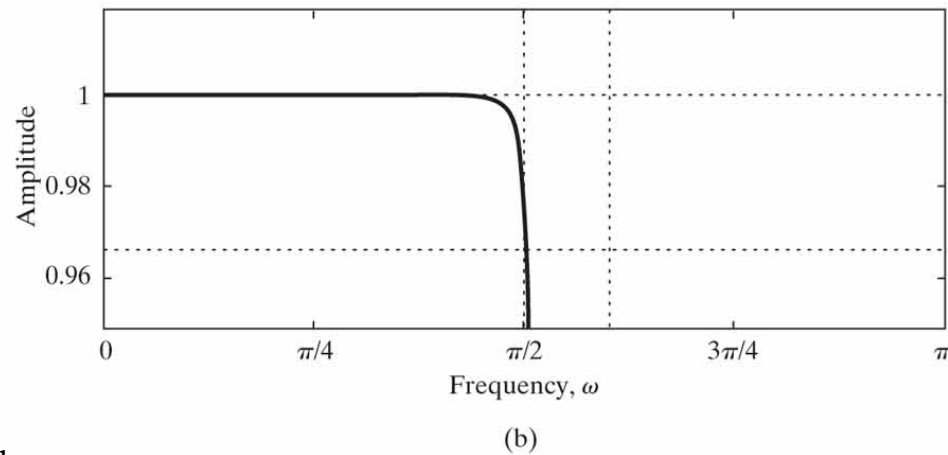
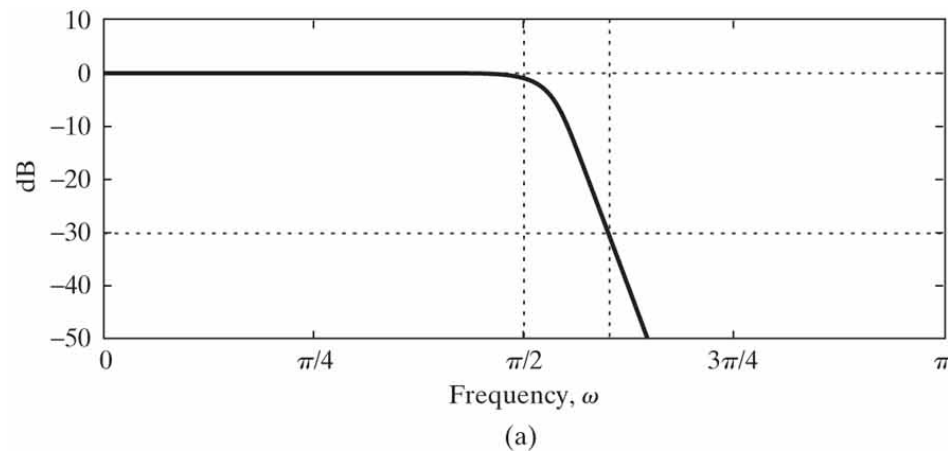
- passband edge frequency $\omega_p = 0.5 \pi$
- stopband edge frequency $\omega_s = 0.6 \pi$
- maximum passband gain = 0 dB
- minimum passband gain = -0.3dB
- maximum stopband gain = -30dB

❑ Use bilinear transformation to design DT low pass filter for each type

Butterworth

□ Butterworth

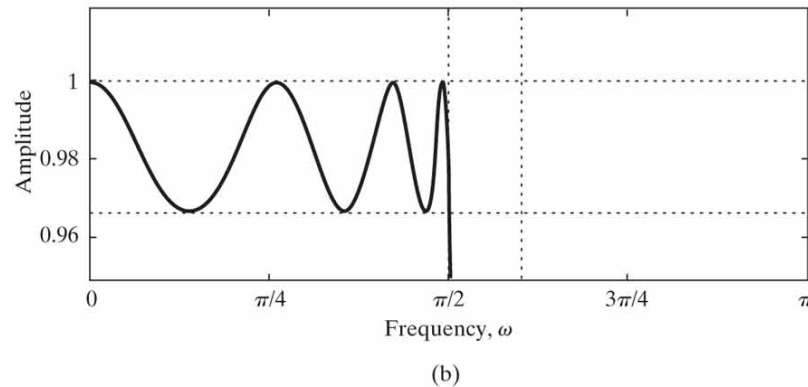
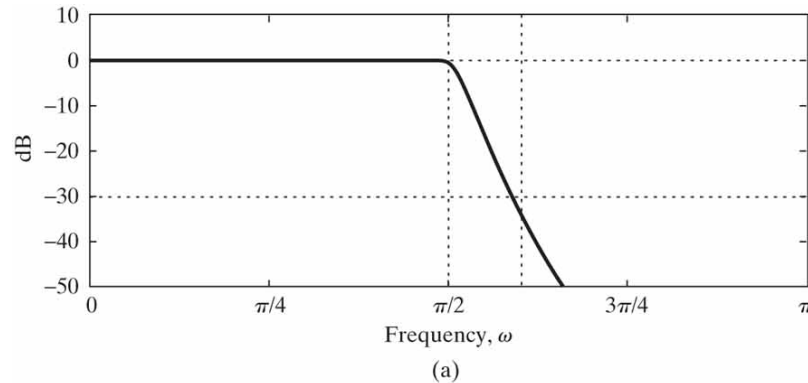
- Monotonic in pass and stop bands



Chebyshev

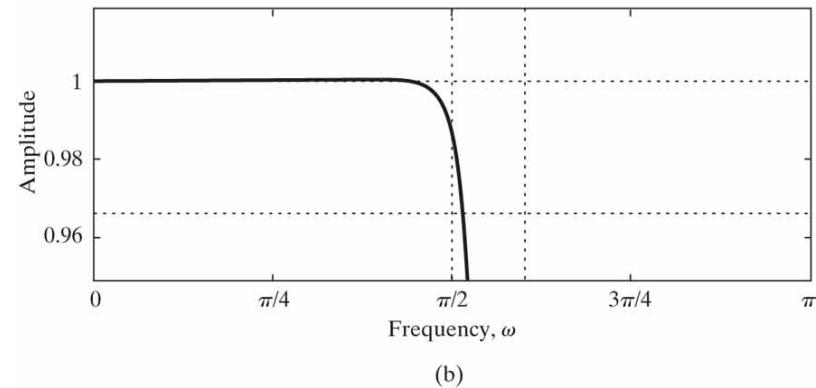
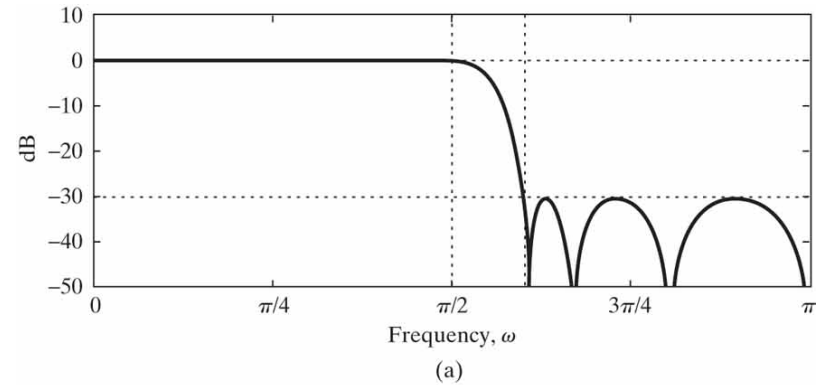
□ Type I

- Equiripple in pass band and monotonic in stop band



□ Type II

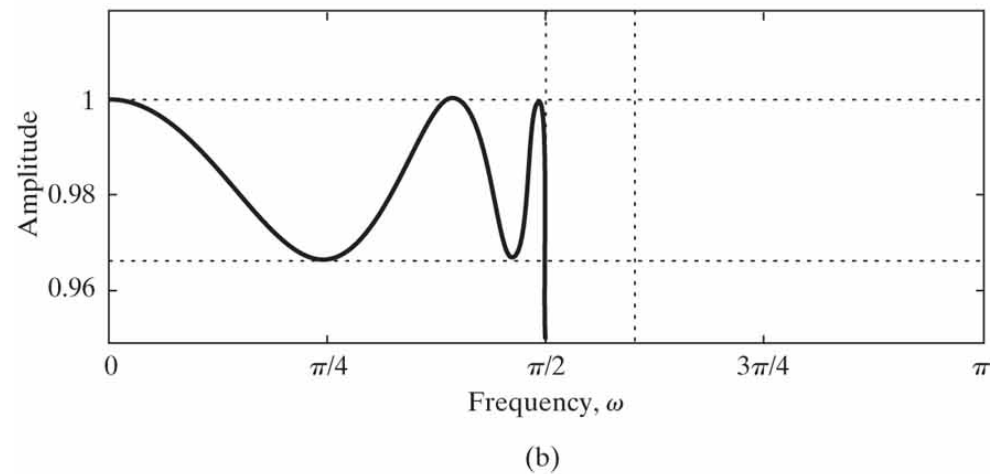
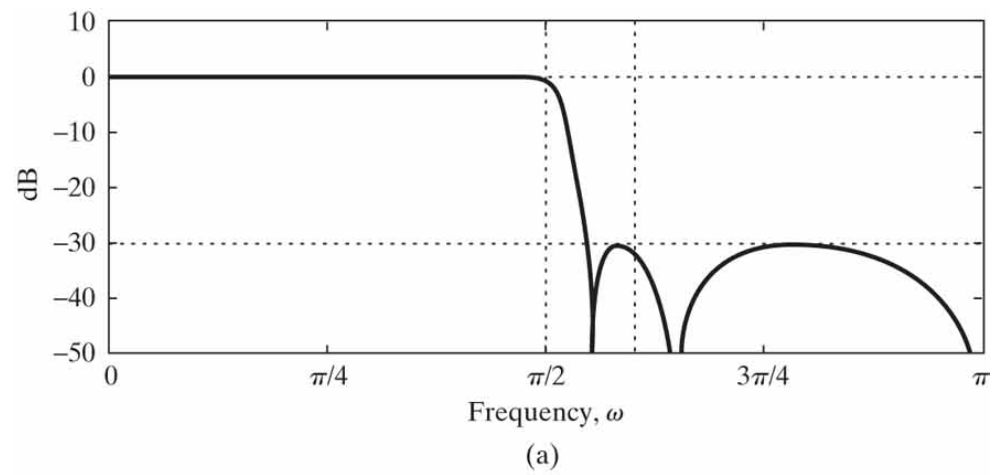
- Monotonic in pass band and equiripple in stop band



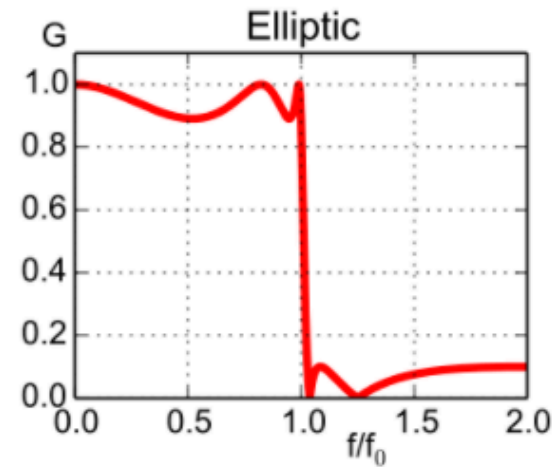
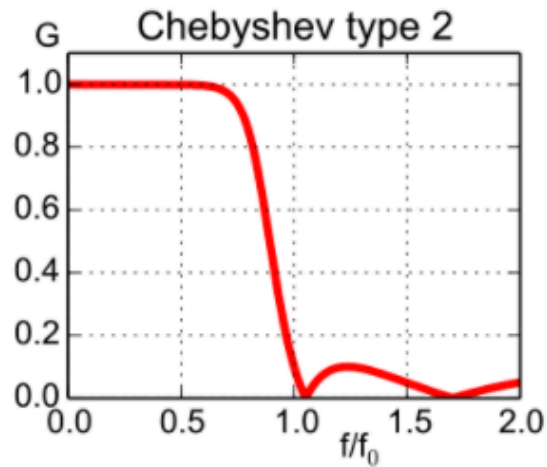
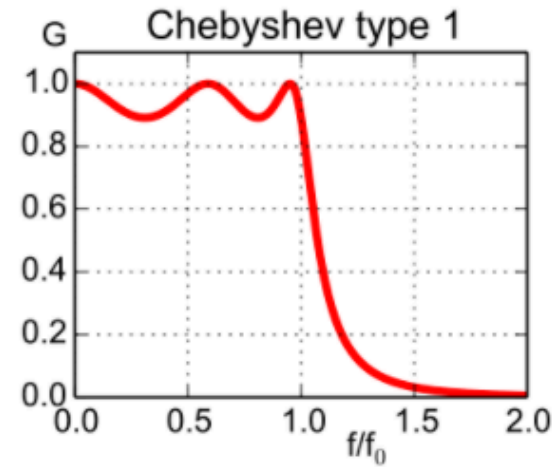
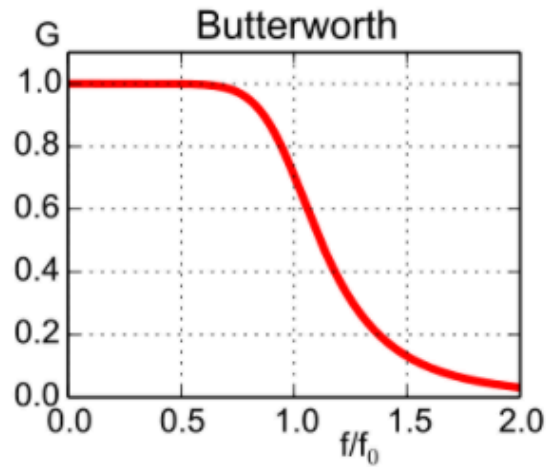
Elliptic

□ Elliptic

- Equiripple in pass and stop bands



Comparisons



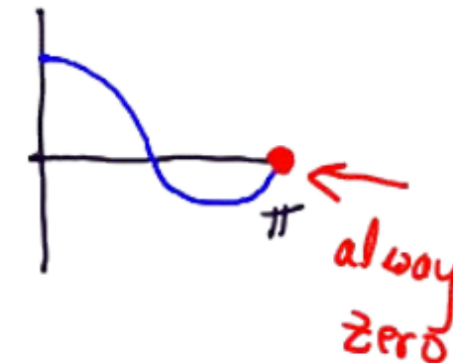
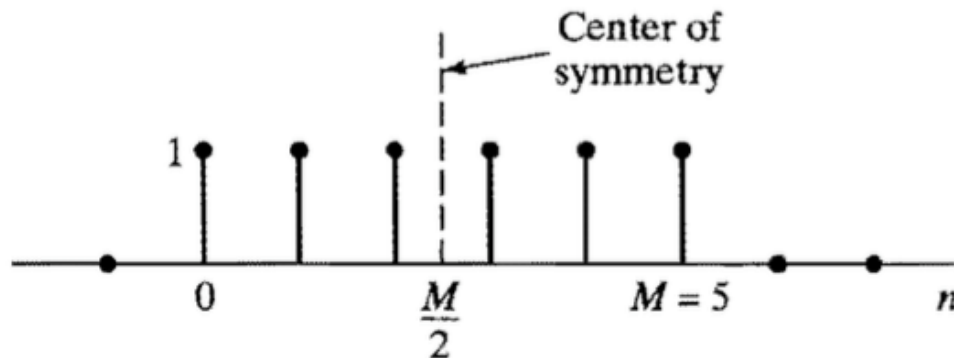
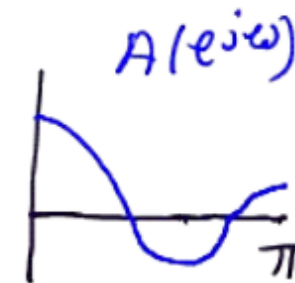
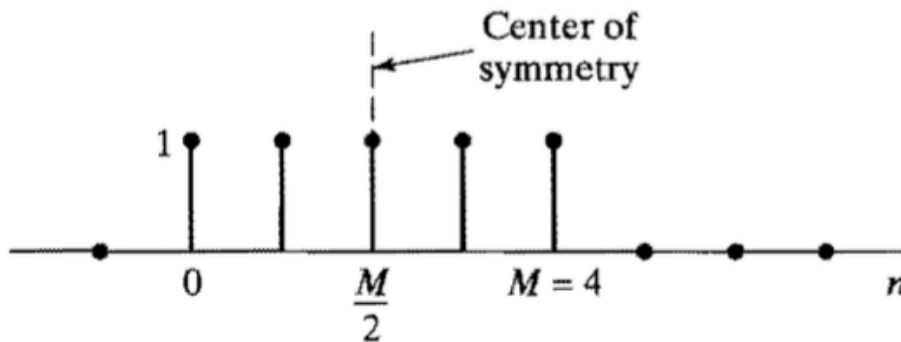


What is a Linear Filter?

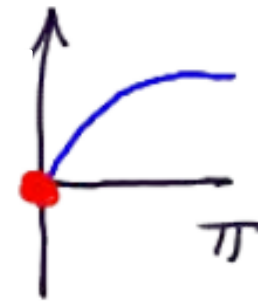
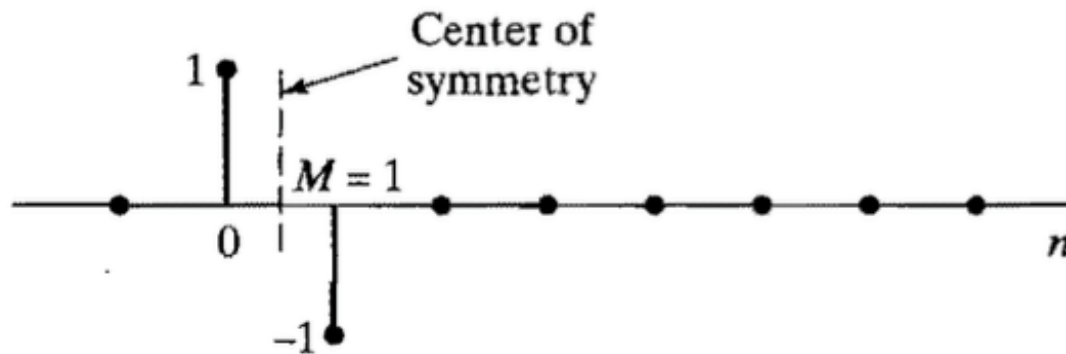
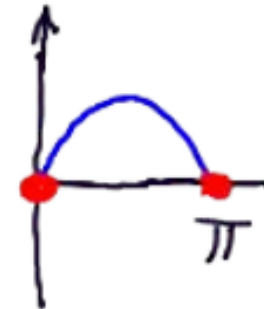
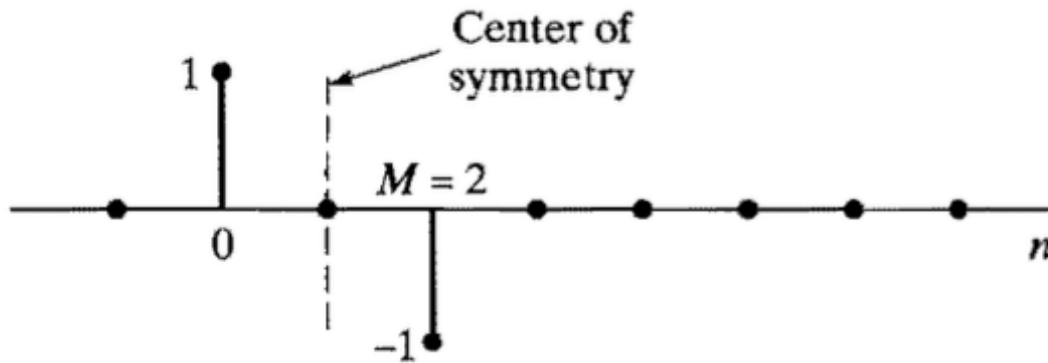
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- ❑ FIR
 - Much easier to control the phase
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FIR GLP: Type I and II



FIR GLP: Type III and IV



FIR Design by Windowing

- Given desired frequency response, $H_d(e^{j\omega})$, find an impulse response

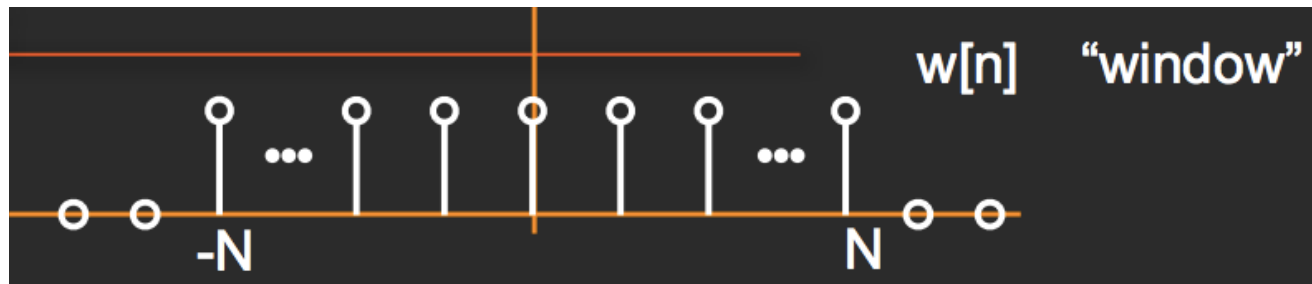
$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

ideal

- Obtain the M^{th} order causal FIR filter by truncating/windowing it

$$h[n] = \left\{ \begin{array}{ll} h_d[n]w[n] & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{array} \right\}$$

Example: Moving Average

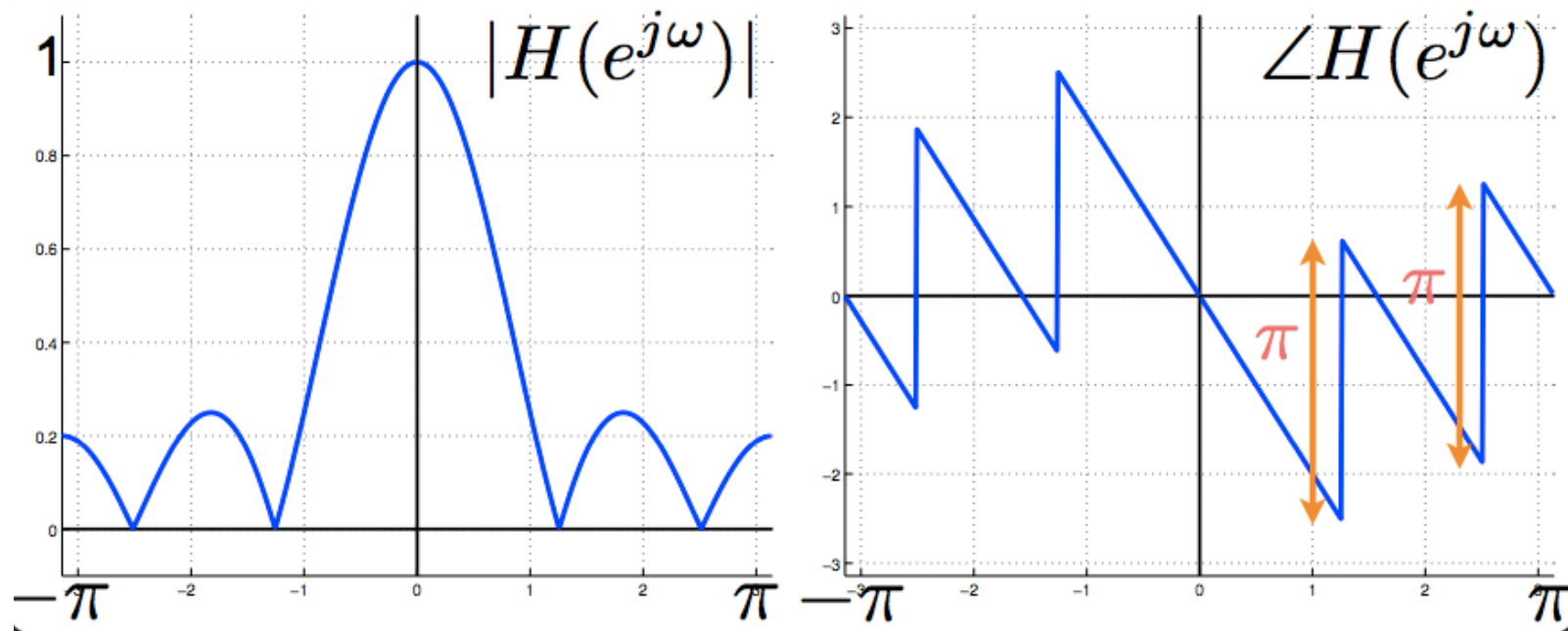


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$\frac{1}{M+1} w[n - M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Moving Average



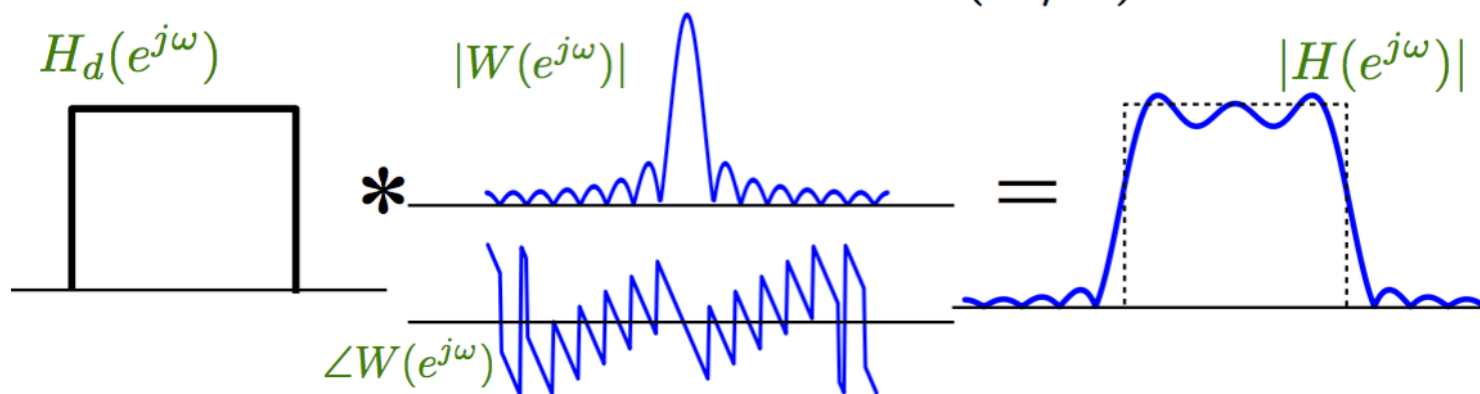
FIR Design by Windowing

- We already saw this before,

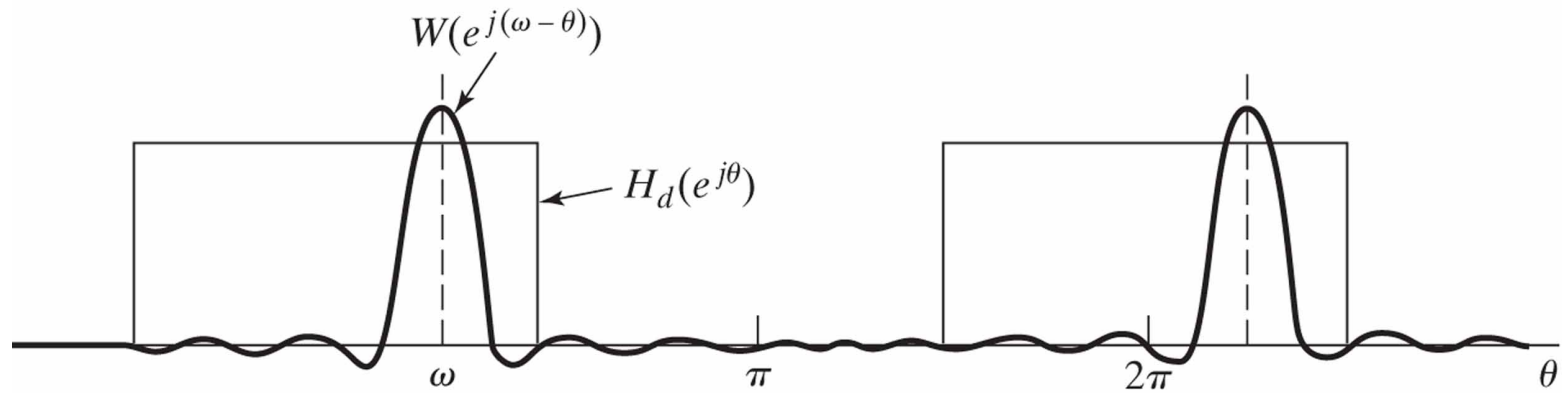
$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

- For Boxcar (rectangular) window

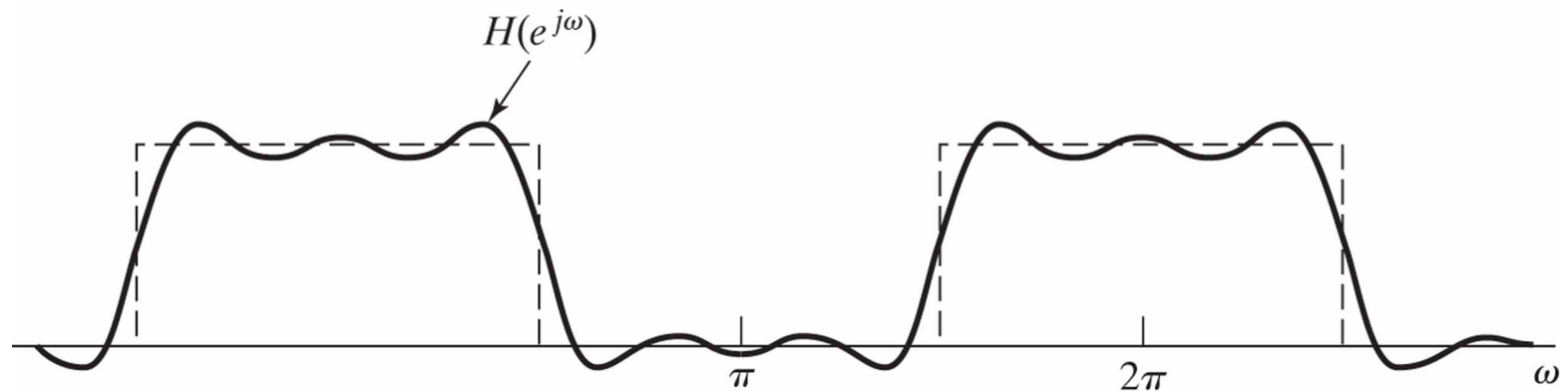
$$W(e^{j\omega}) = e^{-j\omega \frac{M}{2}} \frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)}$$



FIR Design by Windowing

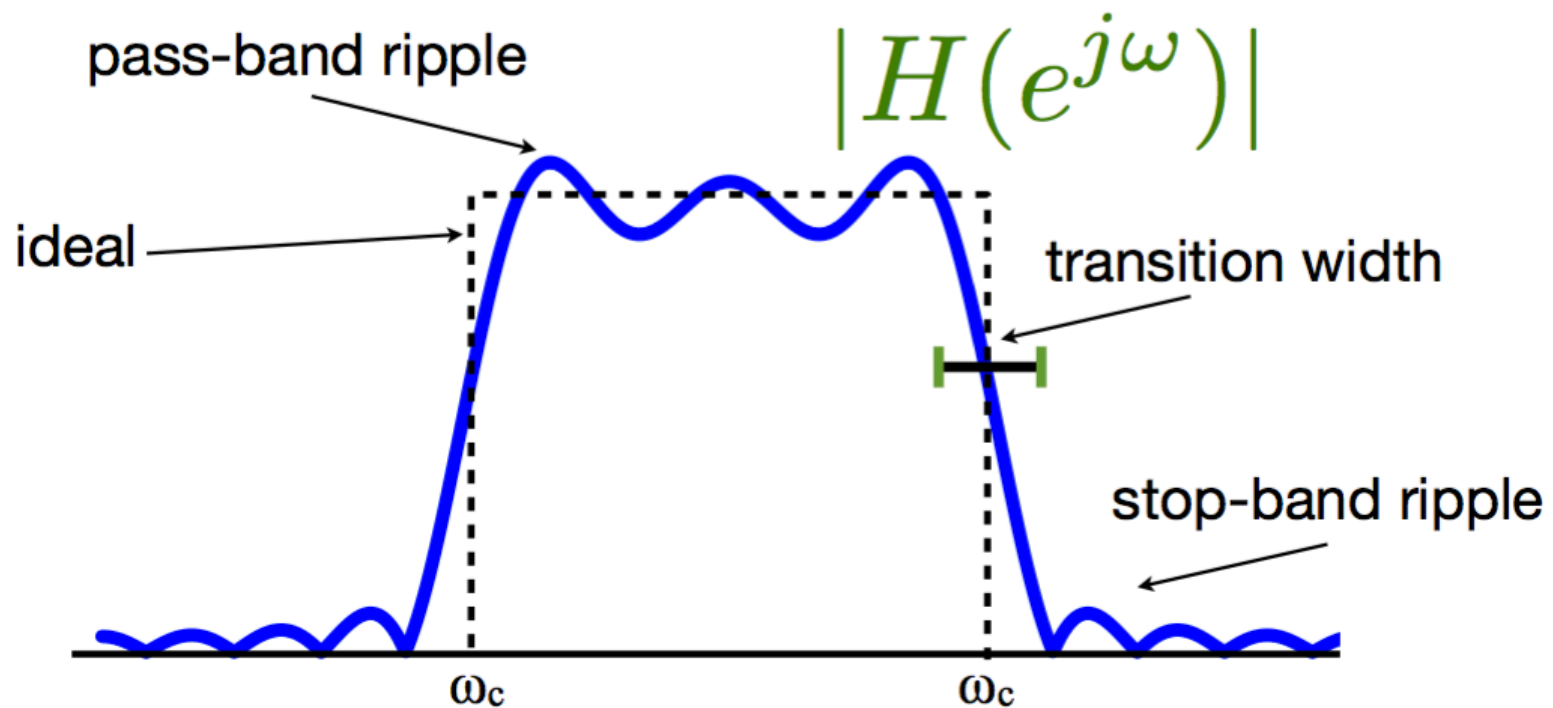


(a)

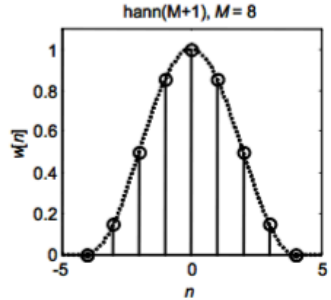
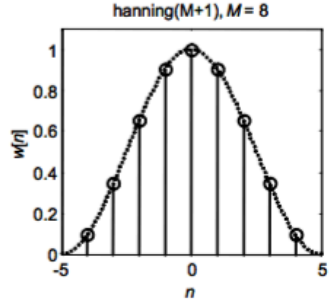
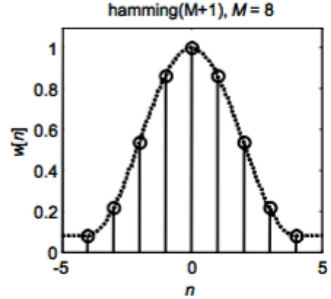


(b)

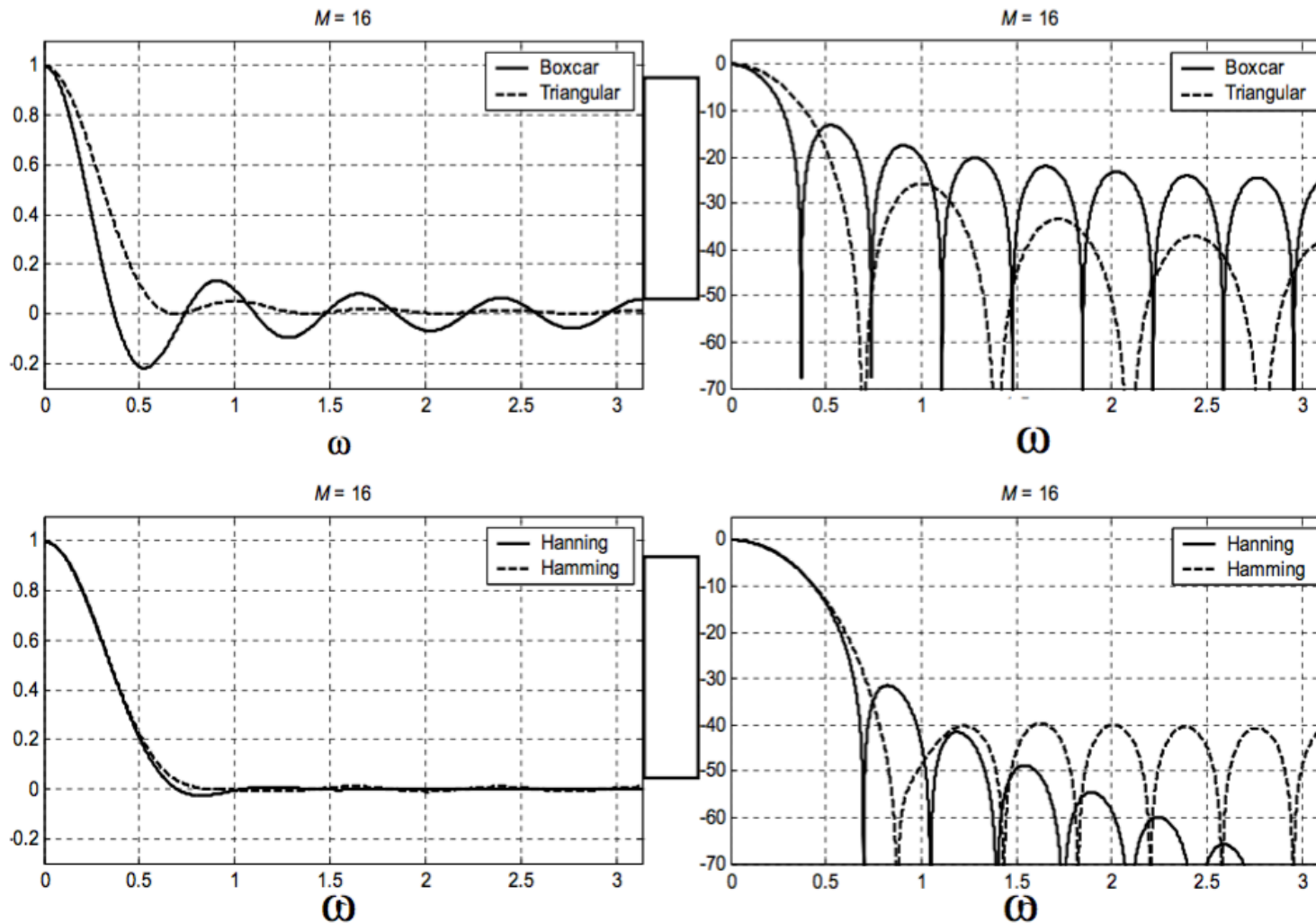
FIR Design by Windowing



Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2} \left[1 + \cos\left(\frac{\pi n}{M/2}\right) \right] & n \leq M/2 \\ 0 & n > M/2 \end{cases}$	<code>hann(M+1)</code>	
Hanning	$w[n] = \begin{cases} \frac{1}{2} \left[1 + \cos\left(\frac{\pi n}{M/2 + 1}\right) \right] & n \leq M/2 \\ 0 & n > M/2 \end{cases}$	<code>hanning(M+1)</code>	
Hamming	$w[n] = \begin{cases} 0.54 + 0.46 \cos\left(\frac{\pi n}{M/2}\right) & n \leq M/2 \\ 0 & n > M/2 \end{cases}$	<code>hamming(M+1)</code>	

Tradeoff – Ripple vs. Transition Width





Kaiser Window

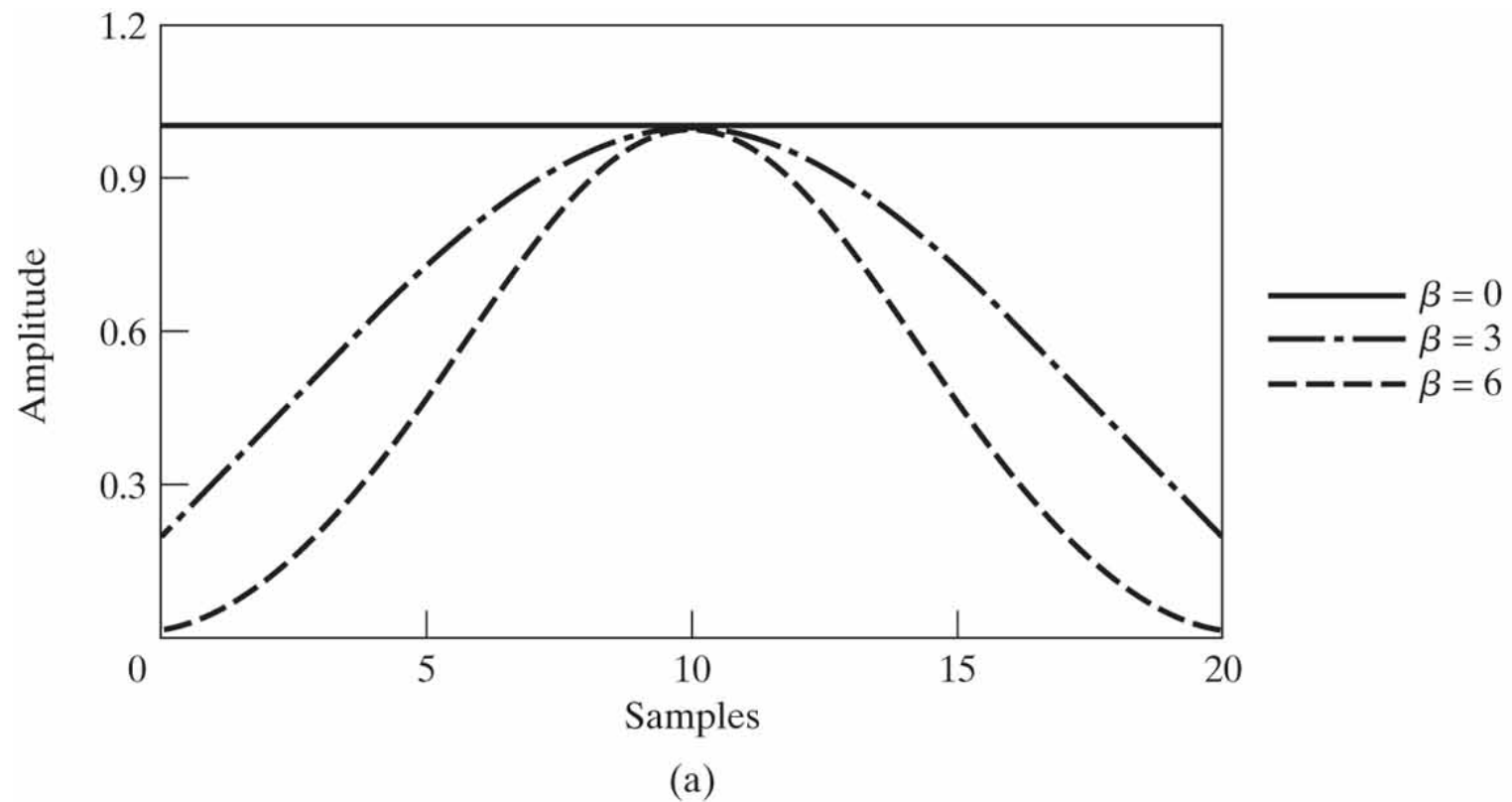
- ❑ Near optimal window quantified as the window maximally concentrated around $\omega=0$

$$w[n] = \begin{cases} \frac{I_0[\beta(1 - [(n - \alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \leq n \leq M, \\ 0, & \text{otherwise,} \end{cases}$$

- ❑ Two parameters – M and β
- ❑ $\alpha = M/2$
- ❑ $I_0(x)$ – zeroth order Bessel function of the first kind

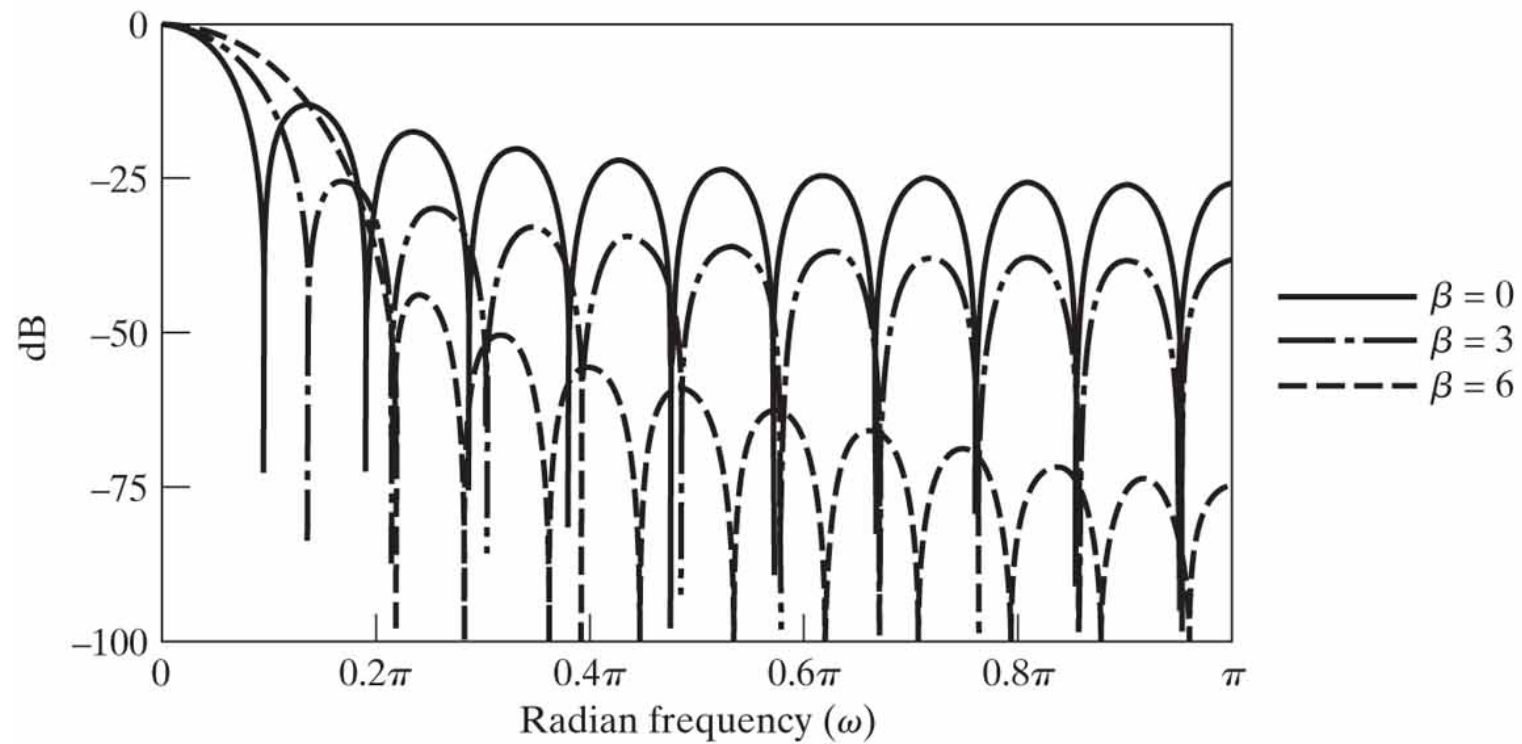
Kaiser Window

□ $M=20$



Kaiser Window

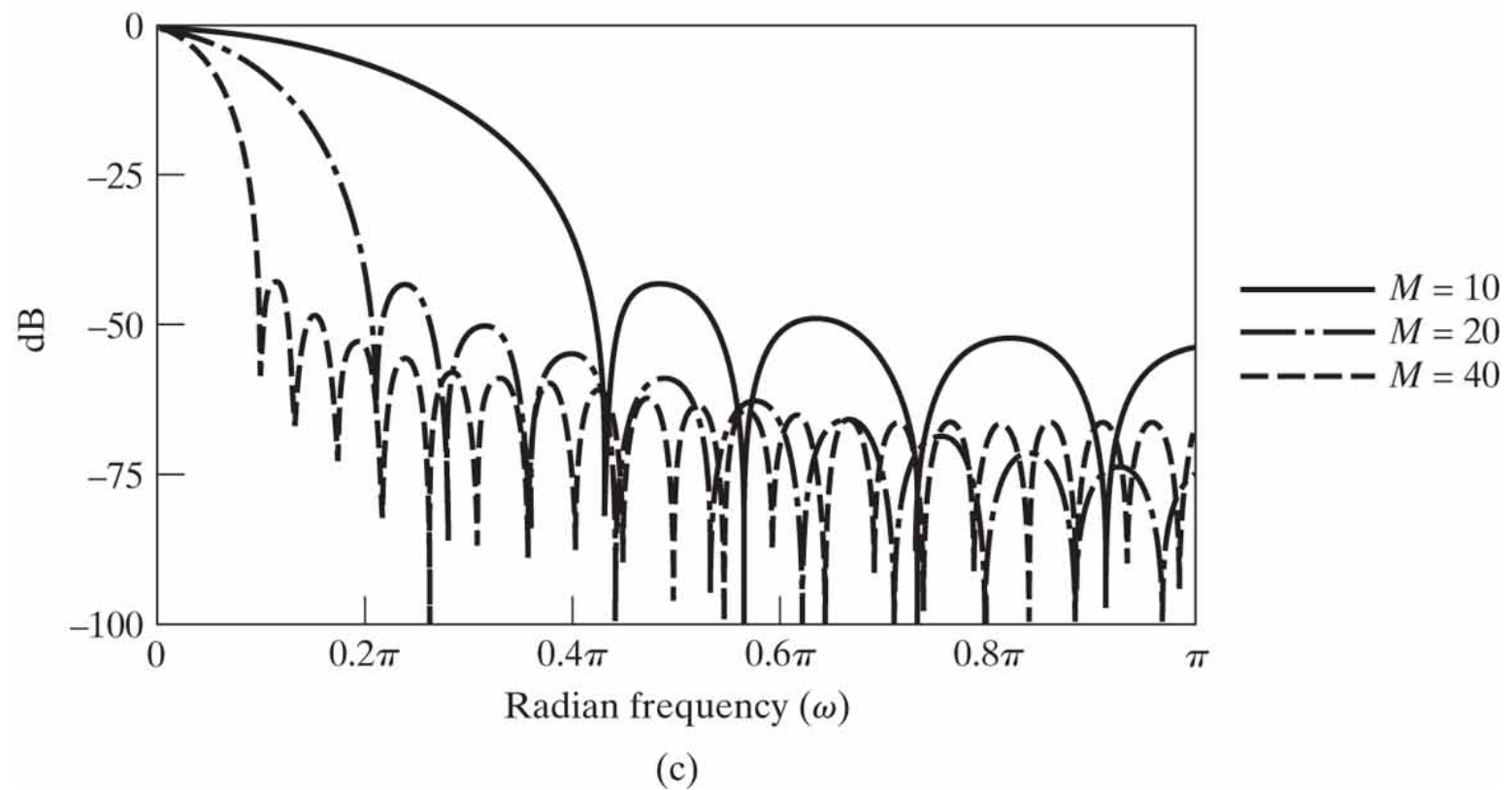
□ $M=20$



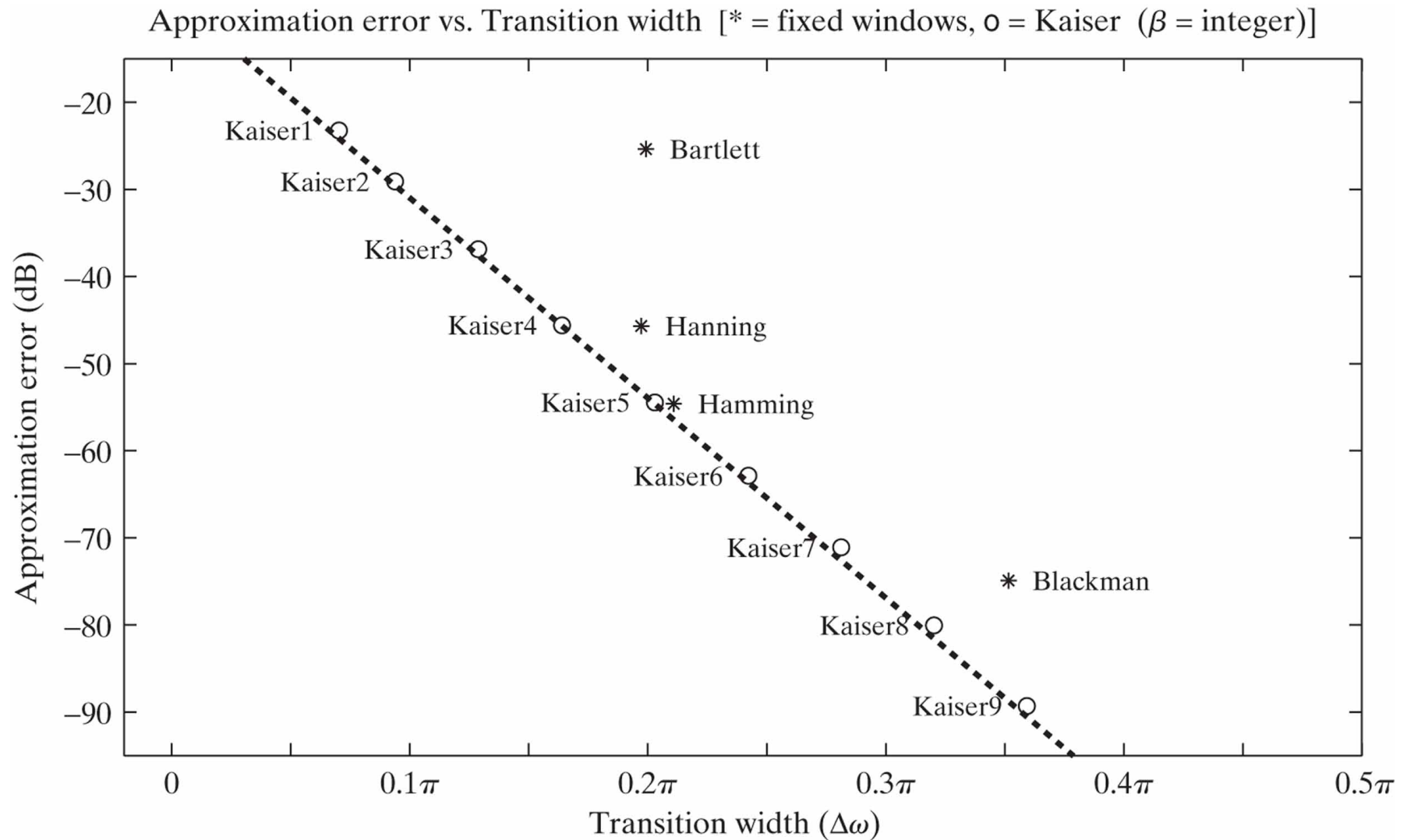
(b)

Kaiser Window

□ $\beta = 6$



Approximation Error





FIR Filter Design

- ❑ Choose a desired frequency response $H_d(e^{j\omega})$
 - non causal (zero-delay), and infinite imp. response
 - If derived from C.T, choose T and use:

$$H_d(e^{j\omega}) = H_c(j\frac{\Omega}{T})$$

- ❑ Window:
 - Length $M+1 \Leftrightarrow$ affects transition width
 - Type of window $\Leftrightarrow$ transition-width/ ripple

FIR Filter Design

- Choose a desired frequency response $H_d(e^{j\omega})$
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$$H_d(e^{j\omega}) = H_c(j\frac{\Omega}{T})$$

- Window:
 - Length M+1 $\Leftrightarrow$ affects transition width
 - Type of window $\Leftrightarrow$ transition-width/ ripple
 - Modulate to shift impulse response
 - Force causality

$$H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$



FIR Filter Design

- ❑ Determine truncated impulse response $h_1[n]$

$$h_1[n] = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{-j\omega \frac{M}{2}} e^{j\omega n} & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

- ❑ Apply window

$$h_w[n] = w[n]h_1[n]$$

- ❑ Check:

- Compute $H_w(e^{j\omega})$, if does not meet specs increase M or change window

Example: FIR Low-Pass Filter Design

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Choose $M \Rightarrow$ Window length and set

$$H_1(e^{j\omega}) = H_d(e^{j\omega})e^{-j\omega \frac{M}{2}}$$

$$h_1[n] = \begin{cases} \frac{\sin(\omega_c(n-M/2))}{\pi(n-M/2)} & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$


$$\frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c}{\pi}(n - M/2)\right)$$

Example: FIR Low-Pass Filter Design

- The result is a windowed sinc function

$$h_w[n] = w[n]h_1[n]$$


$$\frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c}{\pi}(n - M/2)\right)$$

- High Pass Design:
 - Design low pass
 - Transform to $h_w[n](-1)^n$
- General bandpass
 - Transform to $2h_w[n]\cos(\omega_0 n)$ or $2h_w[n]\sin(\omega_0 n)$



Lowpass Filter w/ Kaiser Window

□ Filter specs:

- passband edge frequency $\omega_p = 0.4 \pi$
- stopband edge frequency $\omega_s = 0.6 \pi$
- Max ripple (δ) = .001

Lowpass Filter w/ Kaiser Window

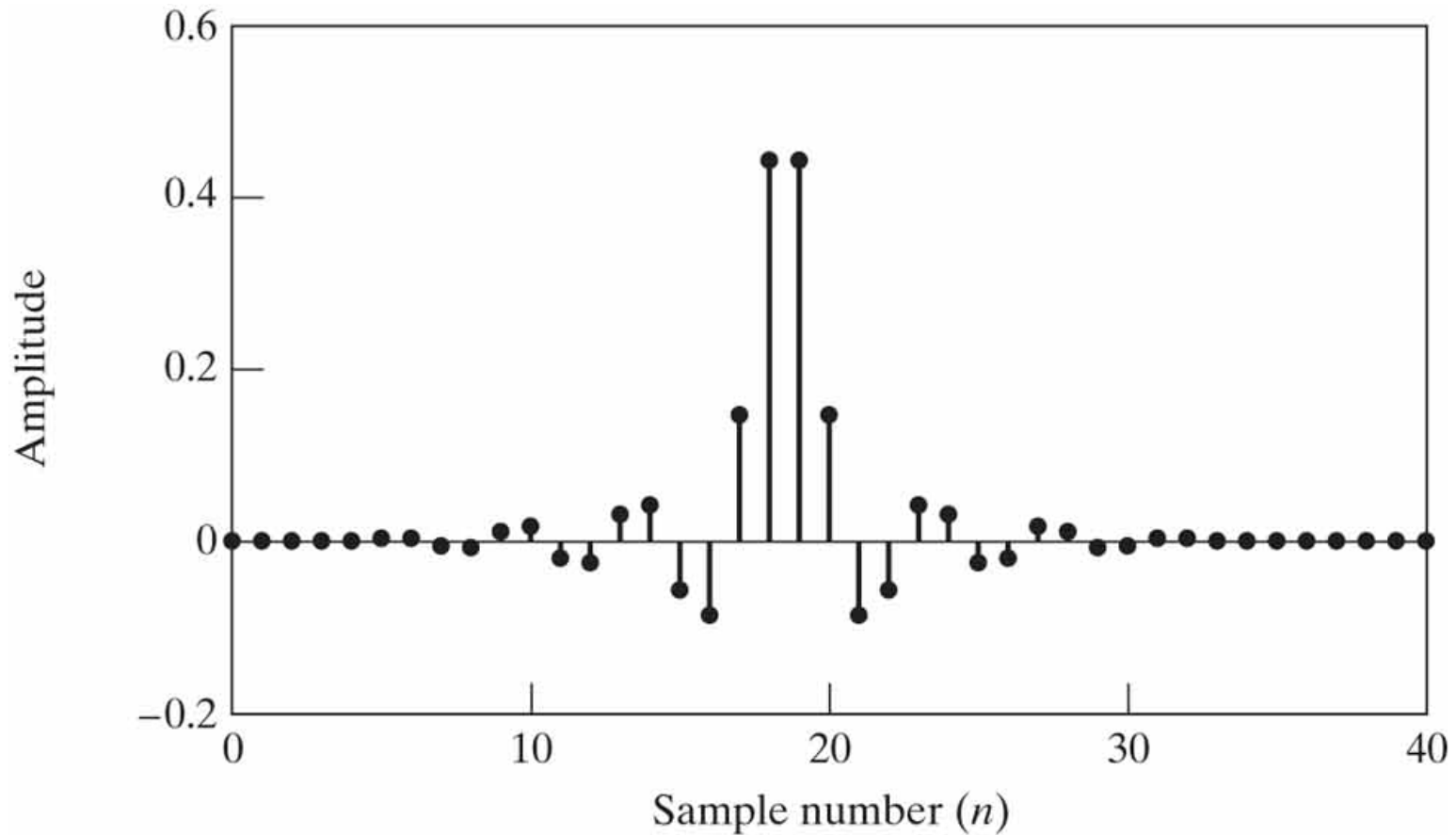
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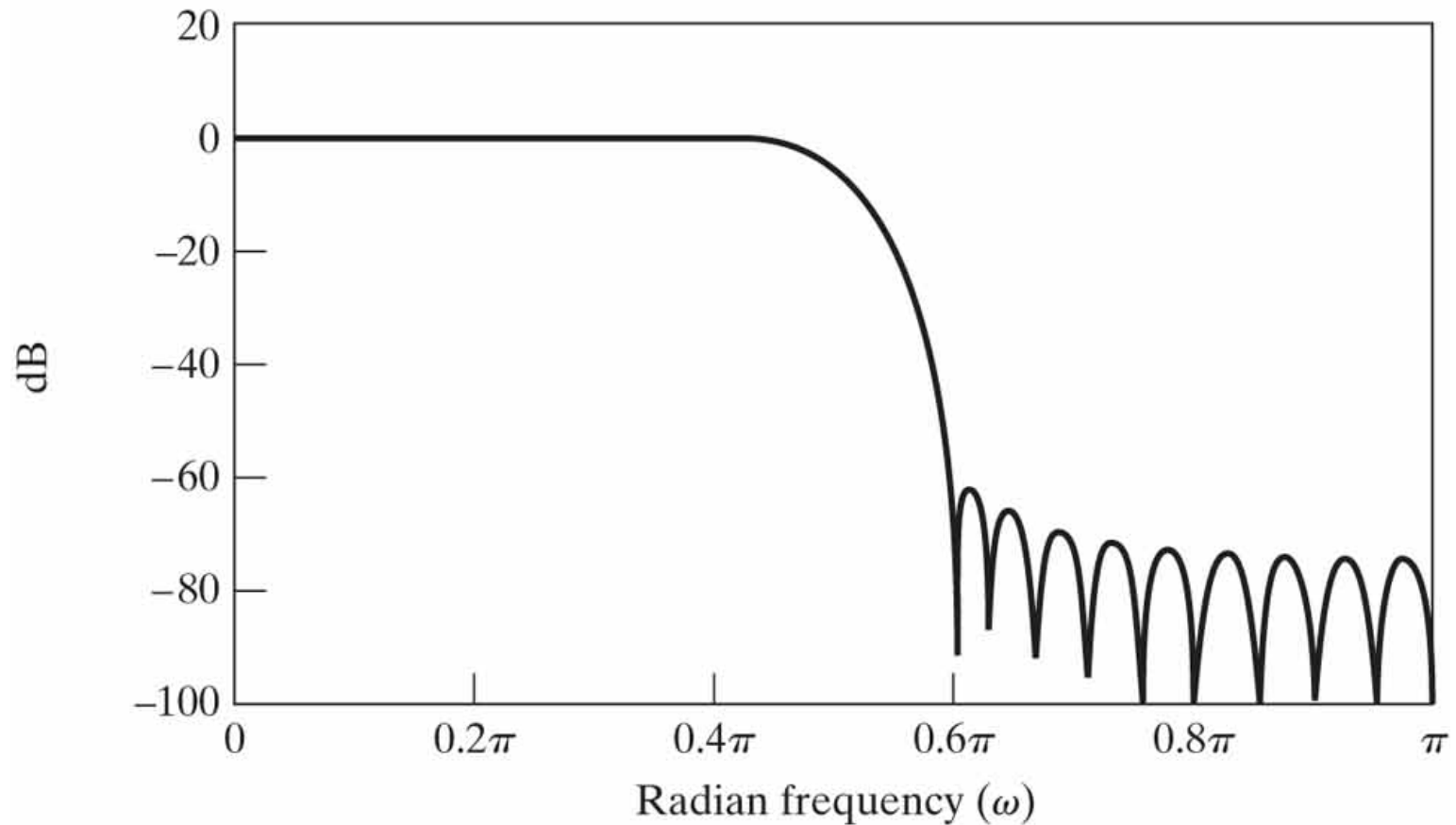
$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50, \\ 0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \leq A \leq 50, \\ 0.0, & A < 21. \end{cases}$$

$$M = \frac{A - 8}{2.285 \Delta \omega}.$$

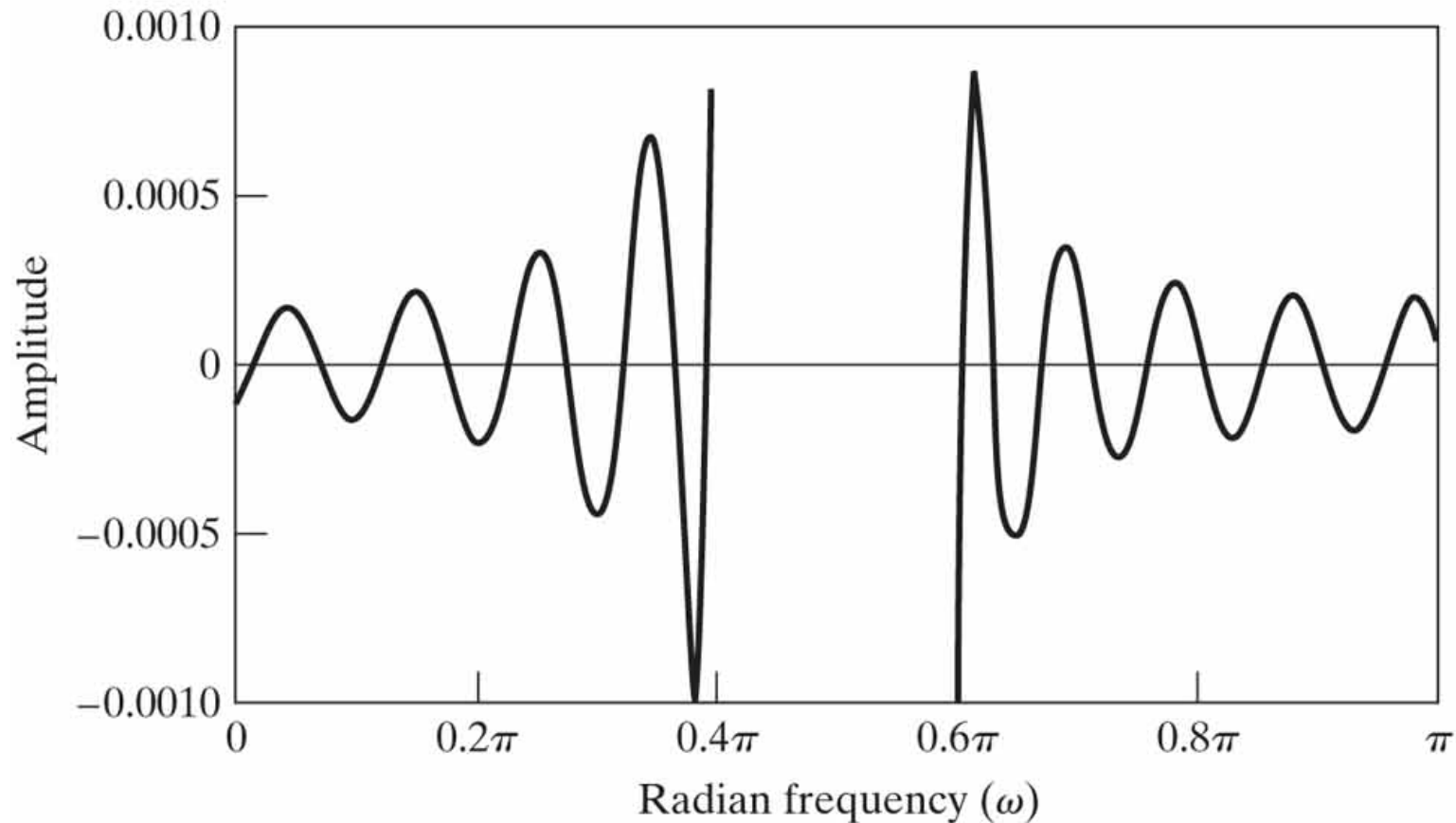
Lowpass Filter w/ Kaiser Window



Lowpass Filter w/ Kaiser Window



Lowpass Filter w/ Kaiser Window

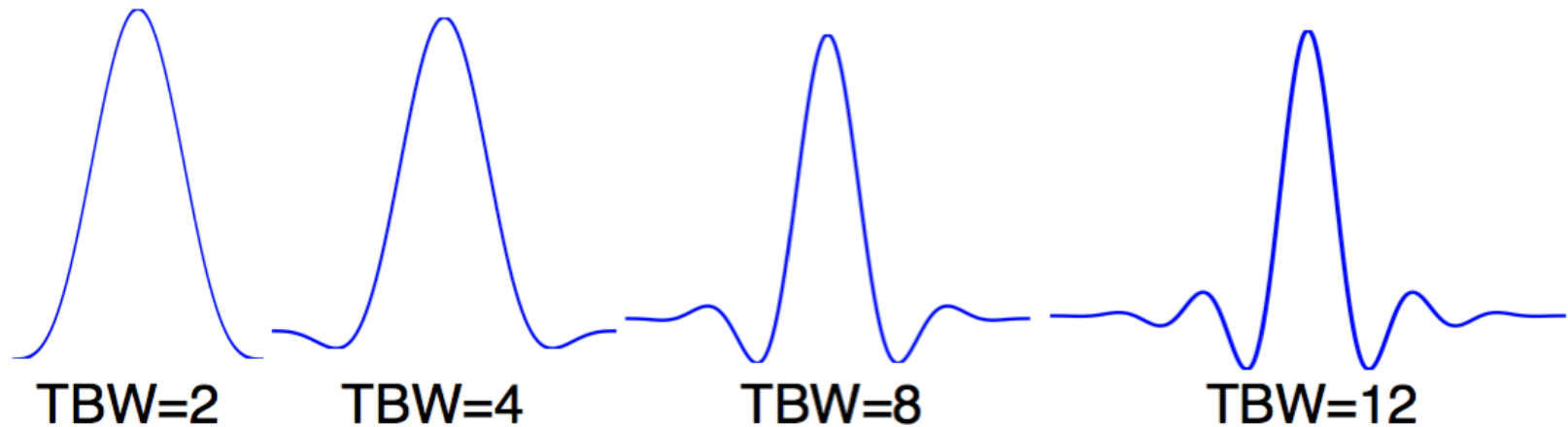


$$E_A(\omega) = \begin{cases} 1 - A_e(e^{j\omega}), & 0 \leq \omega \leq \omega_p, \\ 0 - A_e(e^{j\omega}), & \omega_s \leq \omega \leq \pi. \end{cases}$$

Characterization of Filter Shape

Time-Bandwidth Product, a unitless measure

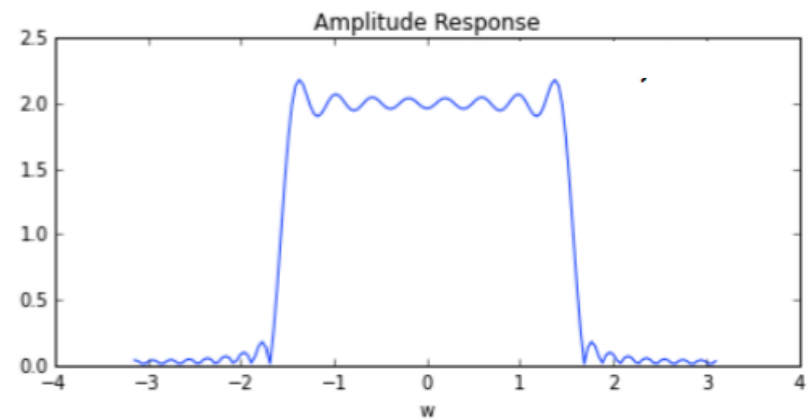
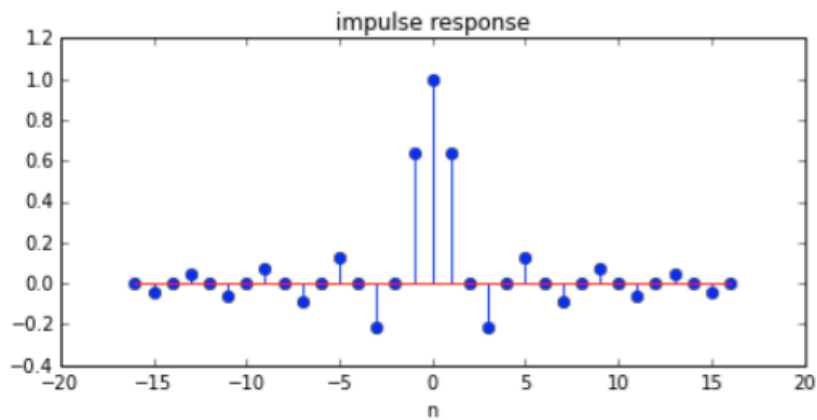
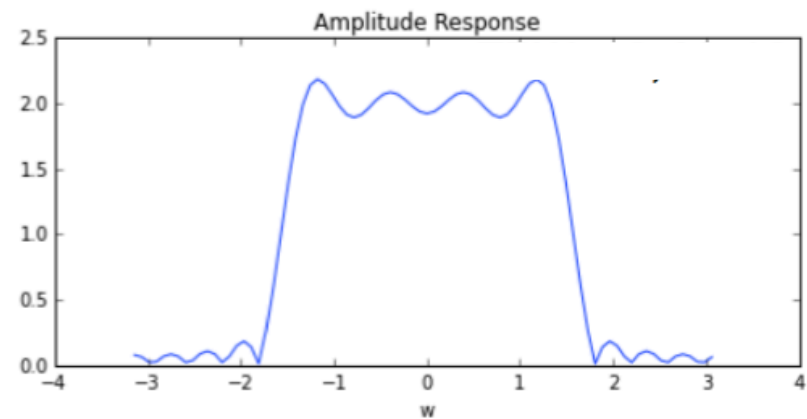
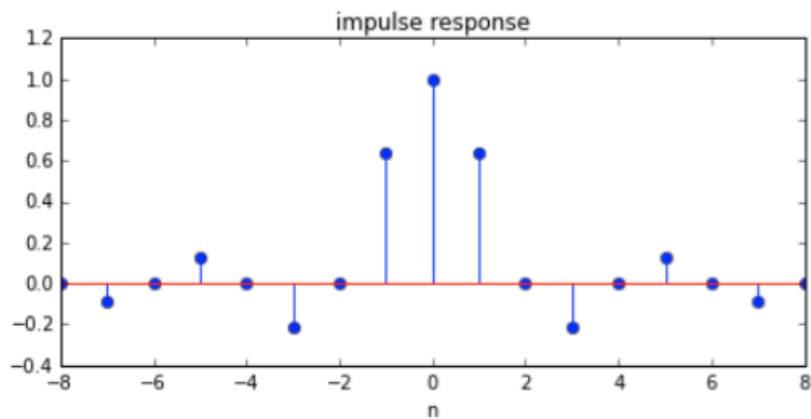
$$T(BW) = (M+1)\omega/2\pi \Rightarrow \text{also, total \# of zero crossings}$$



Larger TBW $\Rightarrow$ More of the “sinc” function

hence, frequency response looks more like a rect function

Time Bandwidth Product



Design through FFT

- ❑ To design order M filter:
- ❑ Over-Sample/discretize the frequency response at P points where $P \gg M$ ($P=15M$ is good)

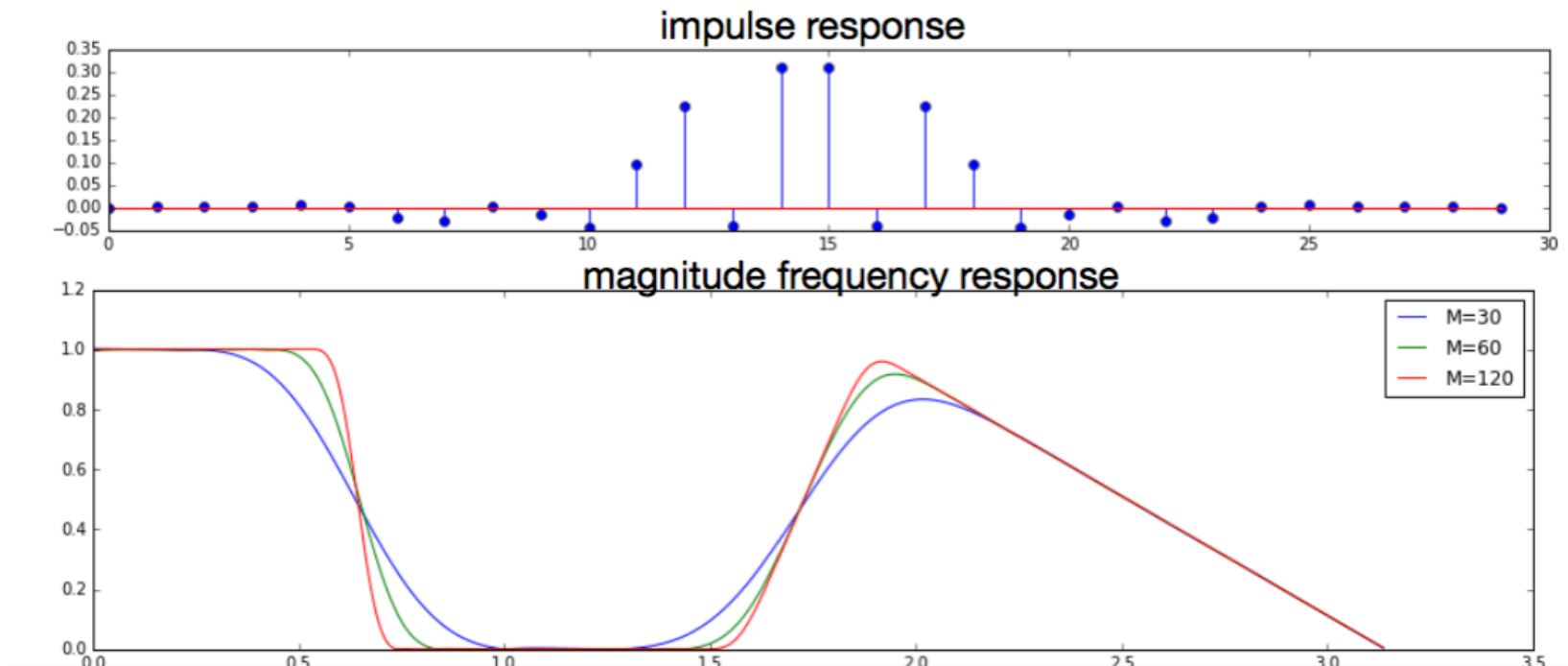
$$H_1(e^{j\omega_k}) = H_d(e^{j\omega_k})e^{-j\omega_k \frac{M}{2}}$$

- ❑ Sampled at: $\omega_k = k \frac{2\pi}{P} \quad |k = [0, \dots, P-1]$
- ❑ Compute $h_1[n] = \text{IDFT}_P(H_1[k])$
- ❑ Apply M+1 length window:

$$h_w[n] = w[n]h_1[n]$$

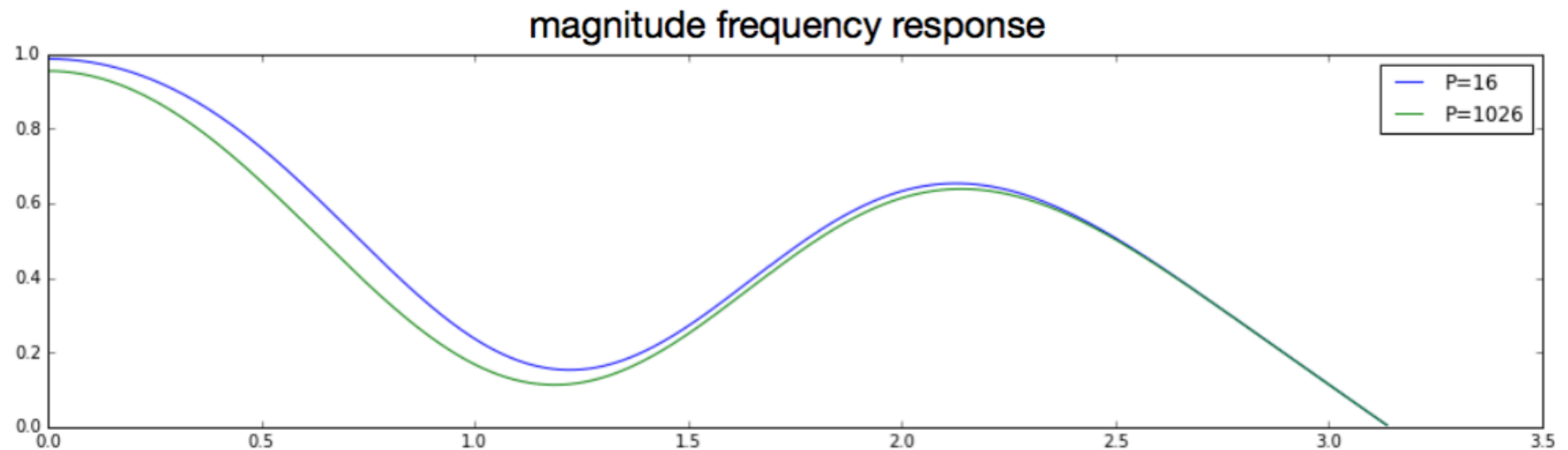
Example

- `signal.firwin2(M+1,omega_vec/pi, amp_vec)`
- `taps1 = signal.firwin2(30, [0.0,0.2,0.21,0.5, 0.6, 1.0], [1.0, 1.0, 0.0,0.0,1.0,0.0])`



Example

- For $M+1=14$
 - $P = 16$ and $P = 1026$

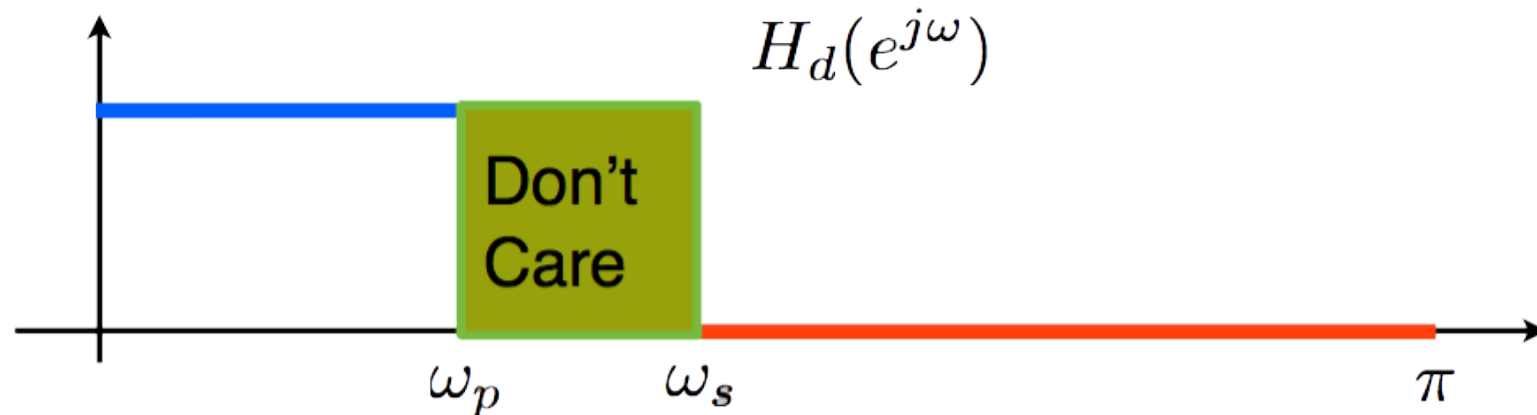




Optimal Filter Design

- ❑ Window method
 - Design Filters heuristically using windowed sinc functions
- ❑ Optimal design
 - Design a filter $h[n]$ with $H(e^{j\omega})$
 - Approximate $H_d(e^{j\omega})$ with some optimality criteria - or satisfies specs.

Optimality



- ❑ Least Squares:

$$\text{minimize} \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- ❑ Variation: Weighted Least Squares:

$$\text{minimize} \int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$



Optimality

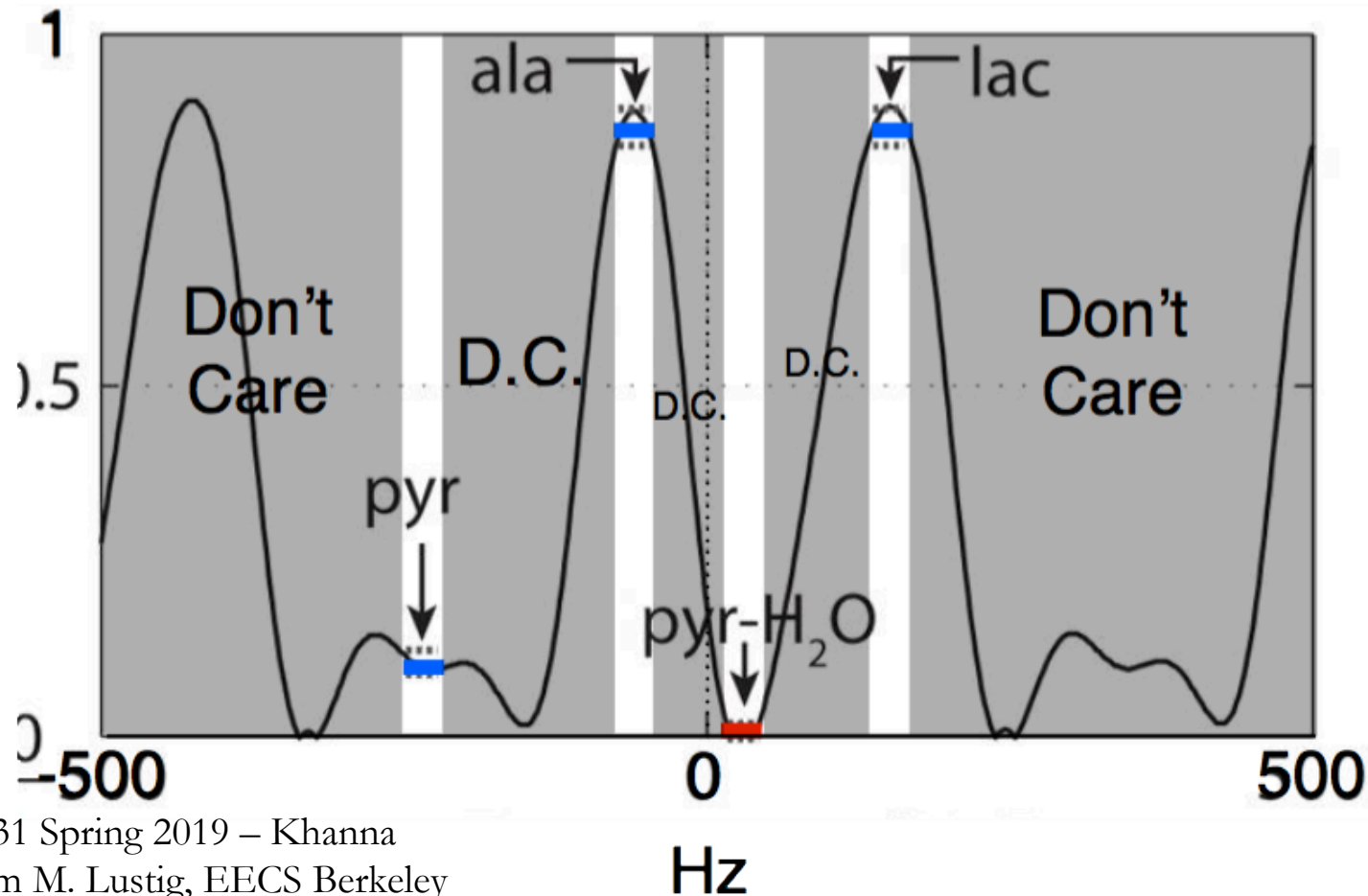
❑ Chebychev Design (min-max)

$$\text{minimize}_{\omega \in \text{care}} \quad \max |H(e^{j\omega}) - H_d(e^{j\omega})|$$

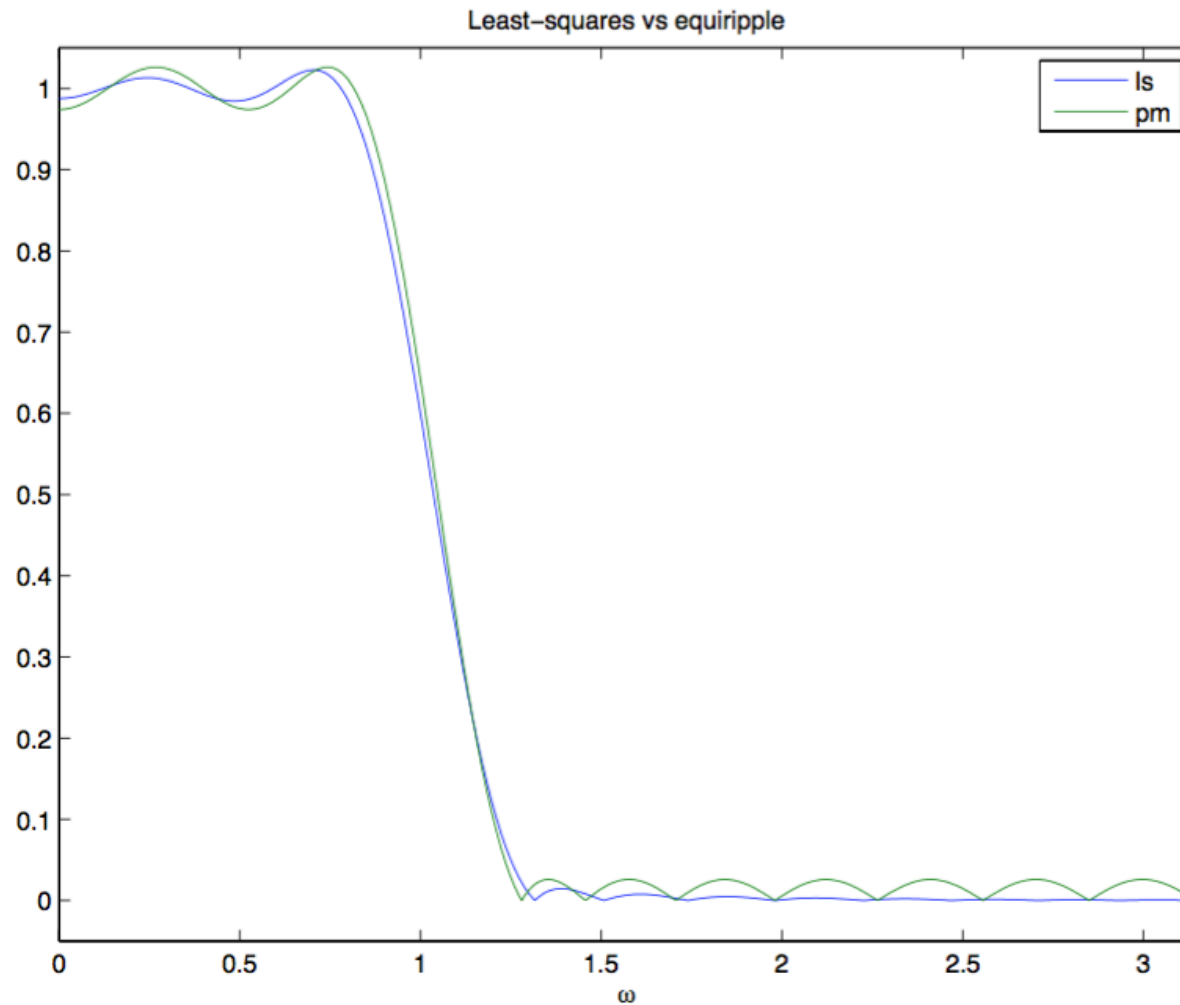
- Parks-McClellan algorithm - equiripple
- Also known as Remez exchange algorithms (signal.remez)
- Can also use convex optimization

Example of Complex Filter

- ❑ Larson et. al, “Multiband Excitation Pulses for Hyperpolarized ^{13}C Dynamic Chemical Shift Imaging” JMR 2008;194(1):121-127
- ❑ Need to design 11 taps filter with following frequency response:



Least-Squares v.s. Min-Max





Admin

- HW 7
 - Out now
 - Due Sunday 3/31
 - More Matlab
 - Less book