ESE 531: Digital Signal Processing

Lec 2: January 22, 2019

Discrete Time Signals and Systems

Lecture Outline

- Discrete Time Signals
- Signal Properties
- Discrete Time Systems

Discrete Time Signals

Signals

DEFINITION

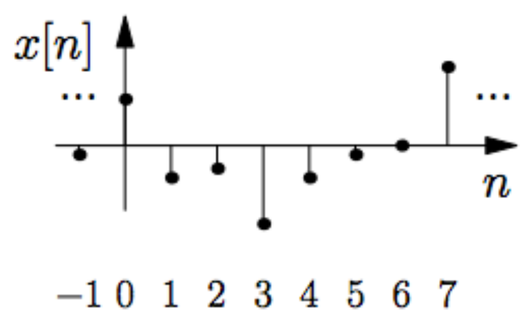
Signal (n): A detectable physical quantity . . . by which messages or information can be transmitted (Merriam-Webster)

- Signals carry information
- Examples:
  - Speech signals transmit language via acoustic waves
  - Radar signals transmit the position and velocity of targets via electromagnetic waves
  - Electrophysiology signals transmit information about processes inside the body
  - Financial signals transmit information about events in the economy
- Signal processing systems manipulate the information carried by signals

Signals are Functions

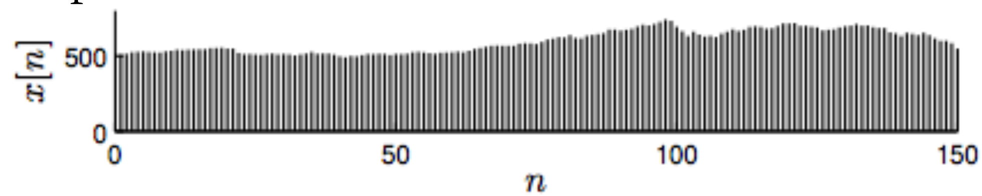
> **DEFINITION**
>
> A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course we will focus on **discrete-time** signals:
  - Independent variable is an **integer**: $n \in Z$ (will refer to n as time)
  - Dependent variable is a real or complex number: $x[n] \in R$

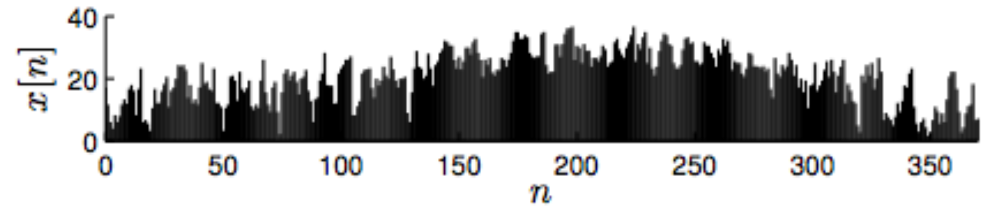


A Menagerie of Signals

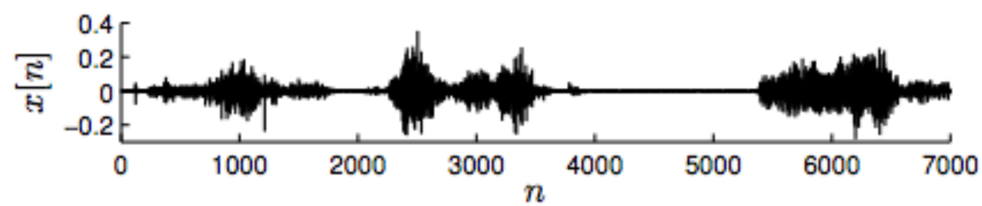
- Google Share daily share price for 5 months



- Temperature at Houston International Airport in 2013



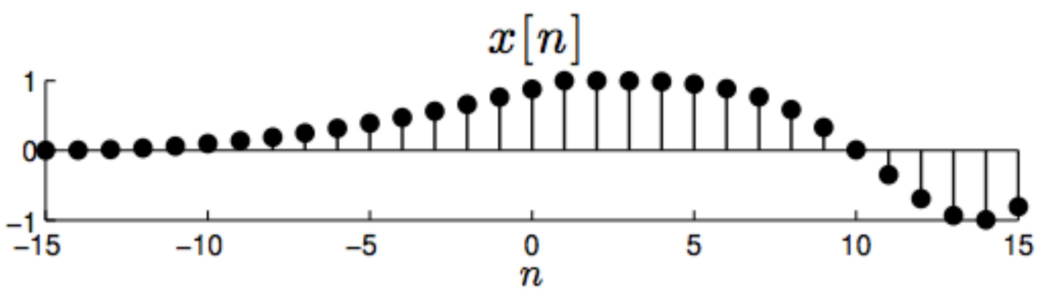
- Excerpt from a reading of Shakespeare's *Hamlet*



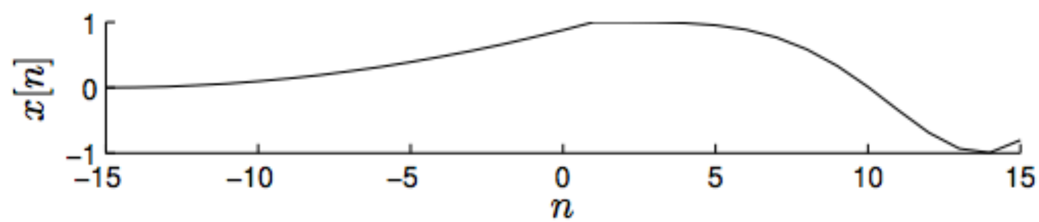
Plotting Signals Correctly

- In a discrete-time signal $x[n]$, the independent variable n is discrete
- To plot a discrete-time signal in a program like Matlab, you should use the **stem** or similar command and not the **plot** command

- Correct:



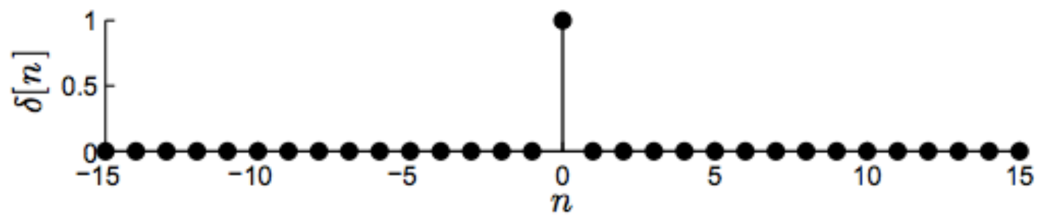
- Incorrect:



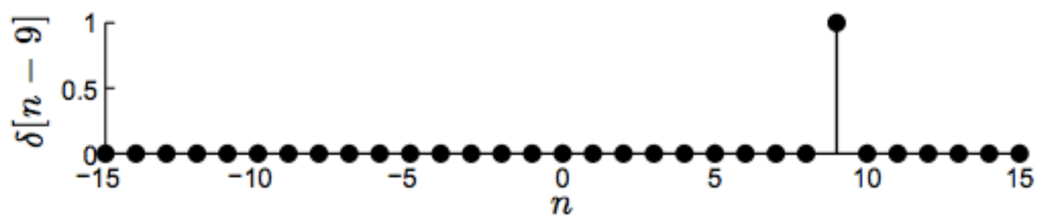
Unit Sample

DEFINITION

The **delta function** (aka unit impulse) $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$



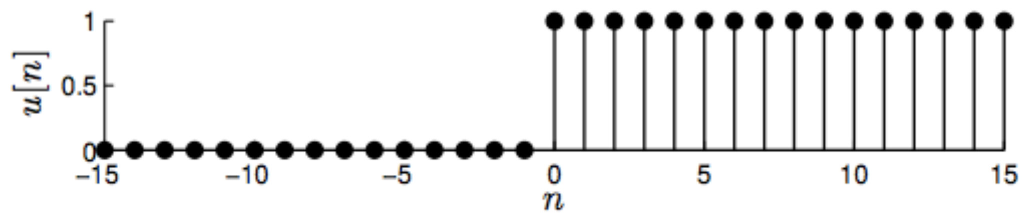
- The shifted delta function $\delta[n-m]$ peaks up at $n = m$; here $m = 9$



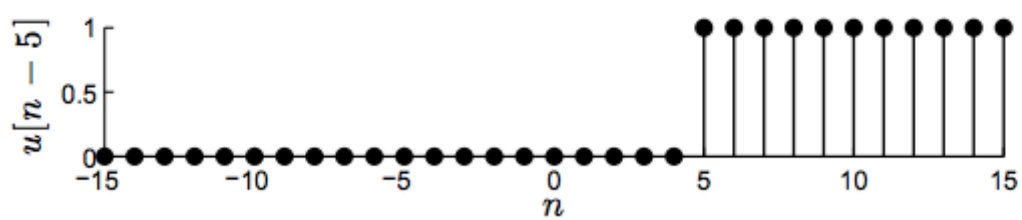
Unit Step

DEFINITION

The **unit step** $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



- The shifted unit step $u[n-m]$ jumps from 0 to 1 at $n = m$; here $m = 9$

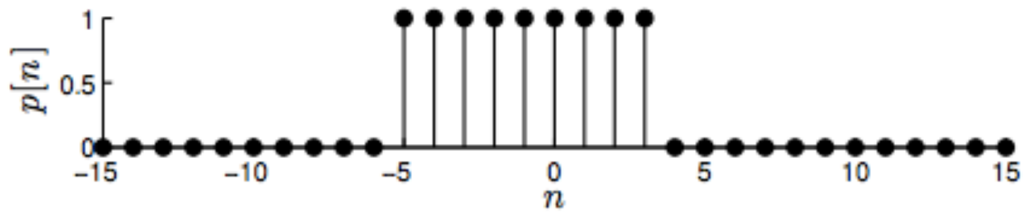


Unit Pulse

DEFINITION

The **unit pulse** (aka boxcar) $p[n] = \begin{cases} 0 & n < N_1 \\ 1 & N_1 \le n \le N_2 \\ 0 & n > N_2 \end{cases}$

- Ex: $p[n]$ for $N_1 = -5$ and $N_2 = 3$



- One of many different formulas for the unit pulse

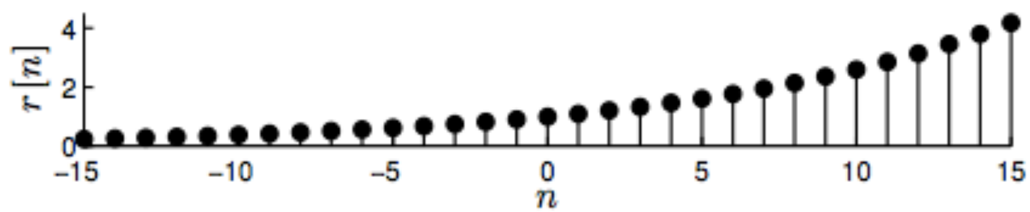
$$p[n] = u[n - N_1] - u[n - (N_2 + 1)]$$

Real Exponential

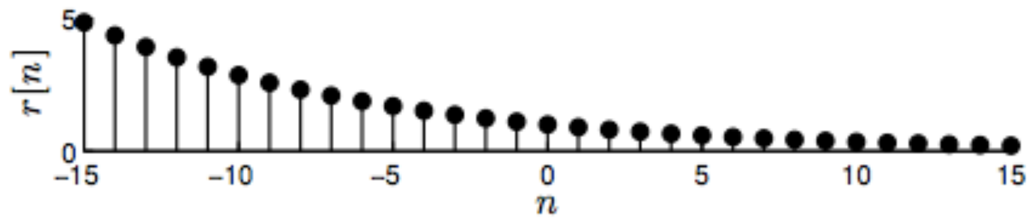
DEFINITION

The **real exponential** $r[n] = a^n,\ a \in \mathbb{R},\ a \geq 0$

- For $a > 1$, $r[n]$ shrinks to the left and grows to the right; here $a = 1.1$

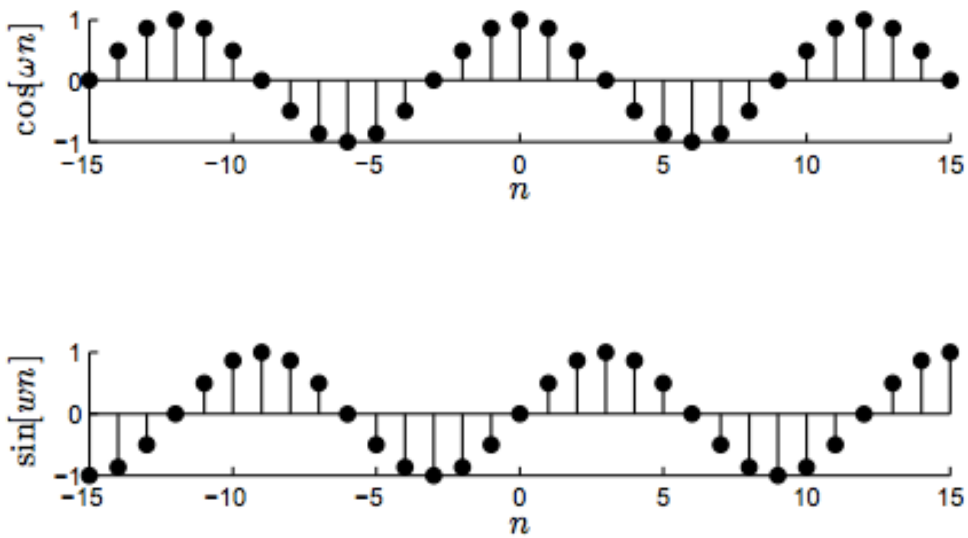


- For $0 < a < 1$, $r[n]$ grows to the left and shrinks to the right; here $a = 0.9$



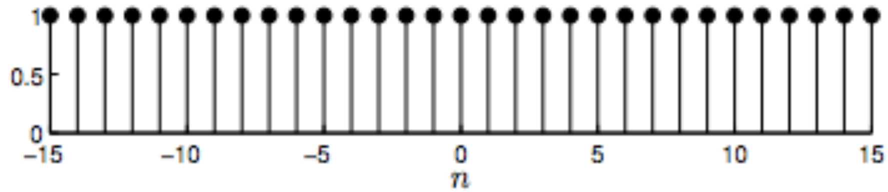
Sinusoids

- There are two natural real-value sinusoids: $\cos(\omega n+\phi)$ and $\sin(\omega n+\phi)$
- **Frequency:** ω (units: radians/sample)
- **Phase:** ϕ (units: radians)

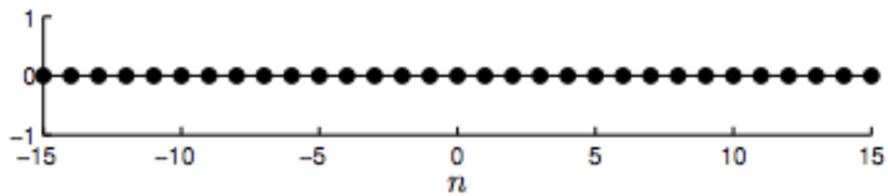


Sinusoid Examples

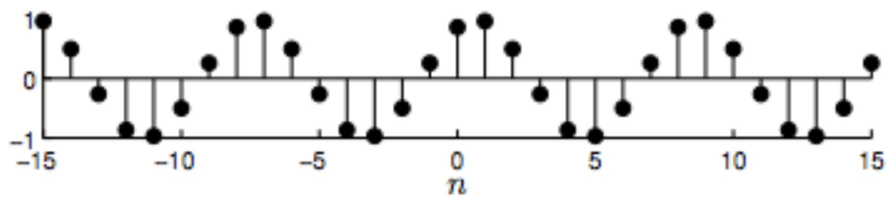
- $\cos(0n)$



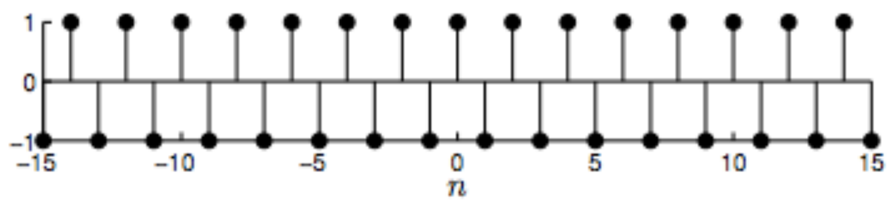
- $\sin(0n)$



- $\sin\left(\frac{\pi}{4}n+\frac{2\pi}{6}\right)$



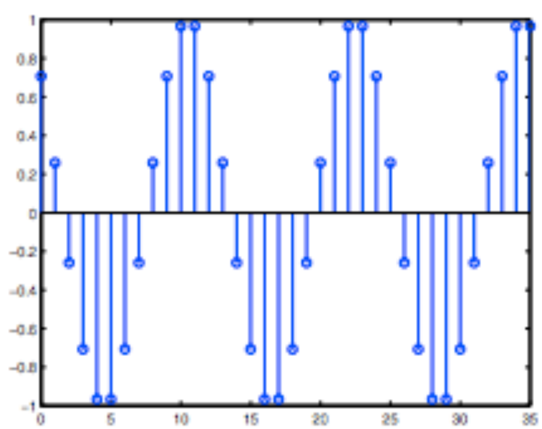
- $\cos(\pi n)$



Sinusoid in Matlab

- It's easy to play around in Matlab to get comfortable with the properties of sinusoids

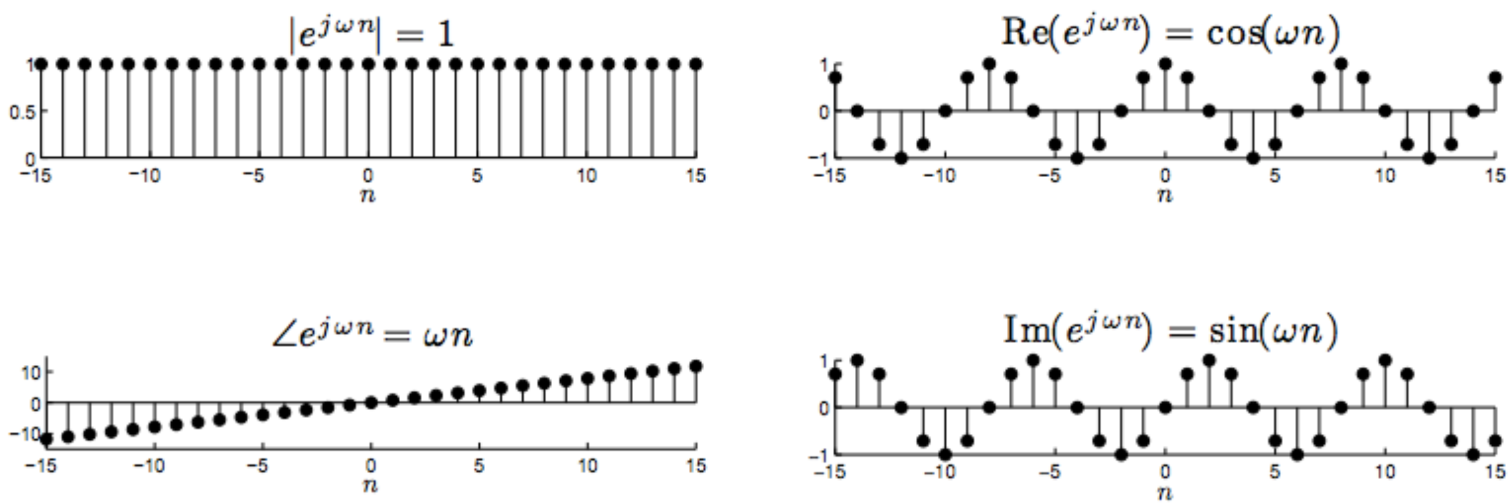
```
N=36;
n=0:N-1;
omega=pi/6;
phi=pi/4;
x=cos(omega*n+phi);
stem(n,x)
```



Complex Sinusoid

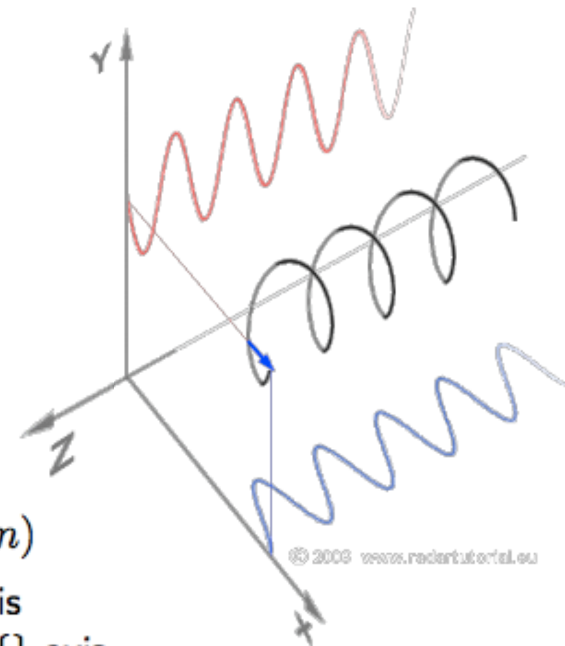
- The complex-value sinusoid combines both the cos and sin terms using Euler's identity:

$$e^{j(\omega n+\phi)} = \cos(\omega n+\phi) + j\sin(\omega n+\phi)$$



Complex Sinusoid as Helix

$$e^{j(\omega n+\phi)} = \cos(\omega n + \phi) + j\sin(\omega n + \phi)$$



- A complex sinusoid is a **helix** in 3D space $(\mathrm{Re}\{\}, \mathrm{Im}\{\}, n)$
  - Real part $(\cos$ term$)$ is the projection onto the $\mathrm{Re}\{\}$ axis
  - Imaginary part $(\sin$ term$)$ is the projection onto the $\mathrm{Im}\{\}$ axis
- Frequency ω determines rotation speed and direction of helix
  - $\omega > 0 \Rightarrow$ anticlockwise rotation
  - $\omega < 0 \Rightarrow$ clockwise rotation

Animation: https://upload.wikimedia.org/wikipedia/commons/4/41/Rising_circular.gif

Negative Frequency?

- Negative frequency is nothing to be afraid of!

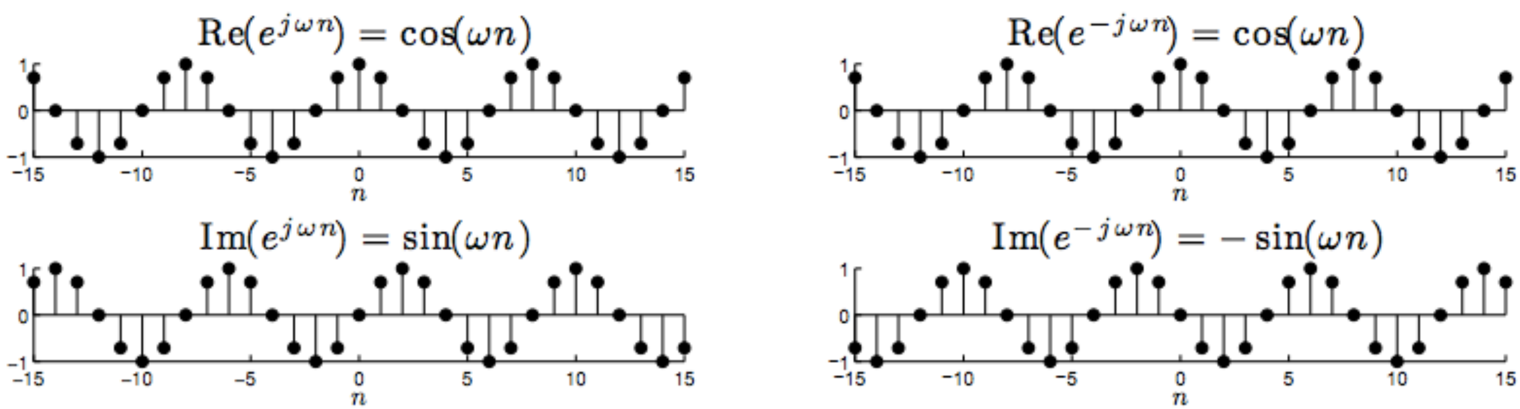
Negative Frequency

- Negative frequency is nothing to be afraid of! Consider a sinusoid with a negative frequency:

$$e^{j(-\omega)n} = e^{-j\omega n} = \cos(-\omega n) + j\sin(-\omega n) = \cos(\omega n) - j\sin(\omega n)$$

- Also note: $e^{j(-\omega)n} = e^{-j\omega n} = \left(e^{j\omega n}\right)^*$

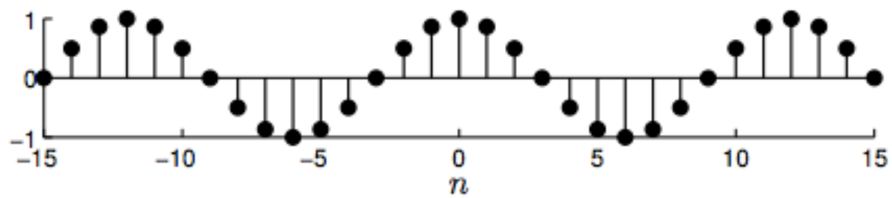
- **Takeaway:** negating the frequency is equivalent to complex conjugating a complex sinusoid—flips the sign of the imaginary sin term



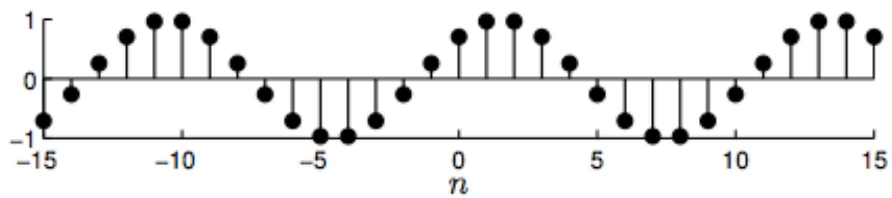
Phase of a Sinusoid

- ϕ is a (frequency independent) shift that is referenced to one period of oscillation

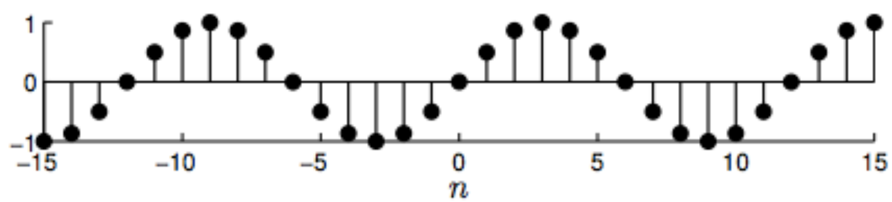
$\cos\left(\frac{\pi}{6}n - 0\right)$



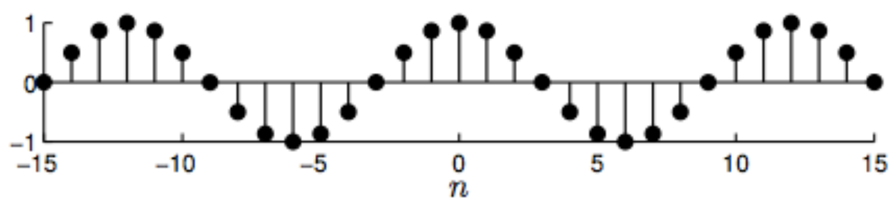
$\cos\left(\frac{\pi}{6}n - \frac{\pi}{4}\right)$



$\cos\left(\frac{\pi}{6}n - \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{6}n\right)$



$\cos\left(\frac{\pi}{6}n - 2\pi\right) = \cos\left(\frac{\pi}{6}n\right)$



Complex Exponentials

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- Consider the general complex number $z = |z|\, e^{j\omega},\ z \in \mathbb{C}$
  - $|z|$ = magnitude of z
  - $\omega = \angle(z)$, phase angle of z
  - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**

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  - $|z|$ = magnitude of z
  - $\omega = \angle(z)$, phase angle of z
  - Can visualize $z \in \mathbb{C}$ as a **point** in the **complex plane**
- Now we have

$$z^n = (|z|e^{j\omega})^n = |z|^n (e^{j\omega})^n = |z|^n e^{j\omega n}$$

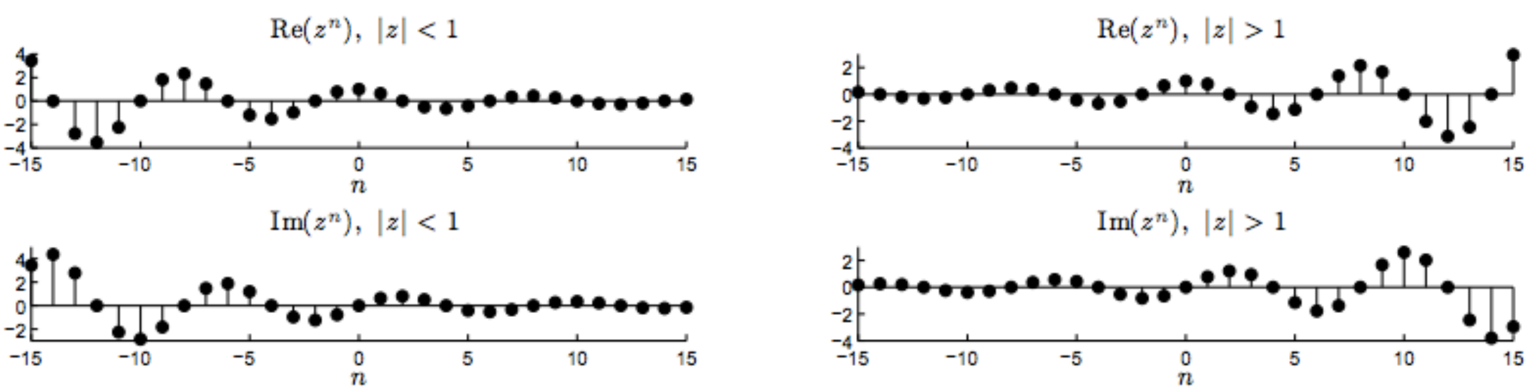
  - $|z|^n$ is a **real exponential** $(a^n$ with $a = |z|)$
  - $e^{j\omega n}$ is a **complex sinusoid**

Complex Exponentials

$$z^n = \left(|z|\, e^{j\omega n}\right)^n = |z|^n\, e^{j\omega n}$$

- $|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a **complex sinusoid**

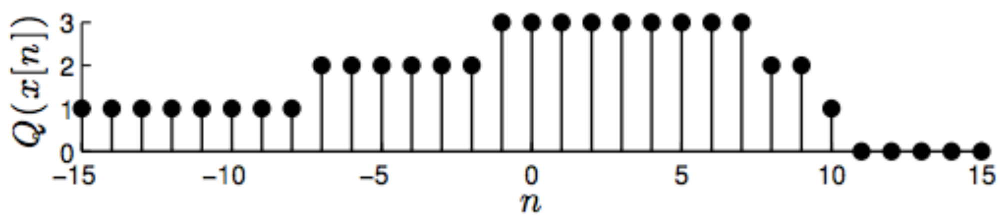
$|z| < 1$ $\qquad\qquad$ $|z| > 1$



Bounded $\qquad\qquad$ Unbounded

Digital Signals

- **Digital signals** are a special sub-class of discrete-time signals
  - Independent variable is still an integer: $n \in \mathbb{Z}$
  - Dependent variable is from a **finite set of integers**: $x[n] \in \{0, 1, \ldots, D-1\}$
  - Typically, choose $D = 2^q$ and represent each possible level of $x[n]$ as a digital code with q bits
  - Ex: Digital signal with $q = 2$ bits $\Rightarrow D = 2^2 = 4$ levels

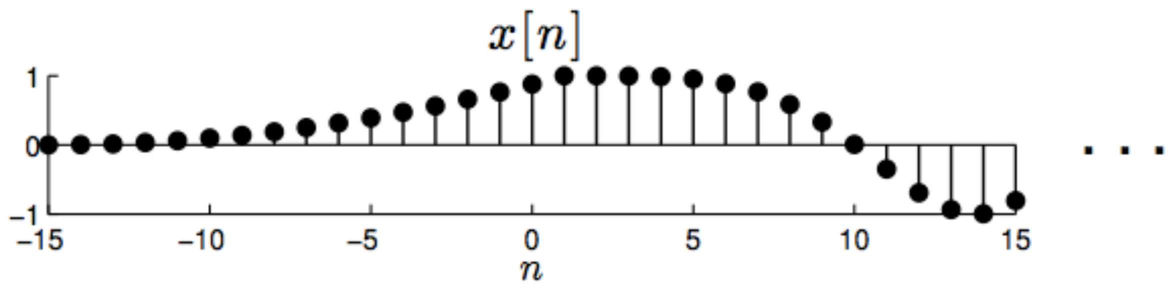


  - Ex: Compact discs use $q = 16$ bits $\Rightarrow D = 2^{16} = 65536$ levels

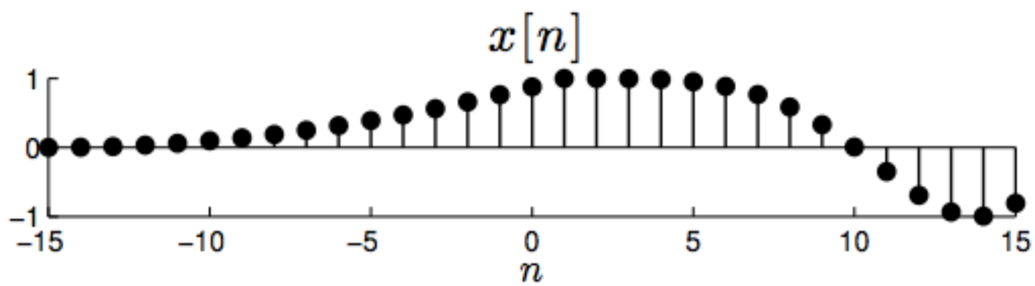
Signal Properties

Finite/Infinite Length Sequences

- An **infinite-length** discrete-time signal $x[n]$ is defined for all $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$



- A **finite-length** discrete-time signal $x[n]$ is defined only for a finite range of $N_1 \leq n \leq N_2$

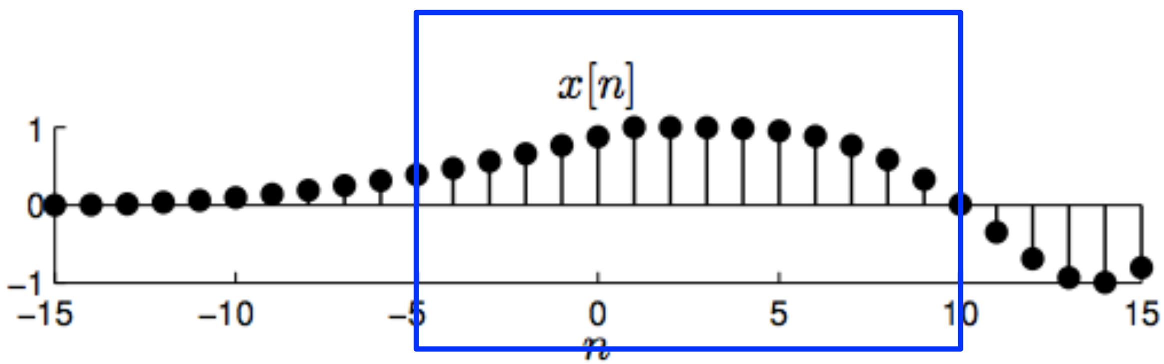


- Important: a finite-length signal is undefined for $n < N_1$ and $n > N_2$

Windowing

- Converts a longer signal into a shorter one

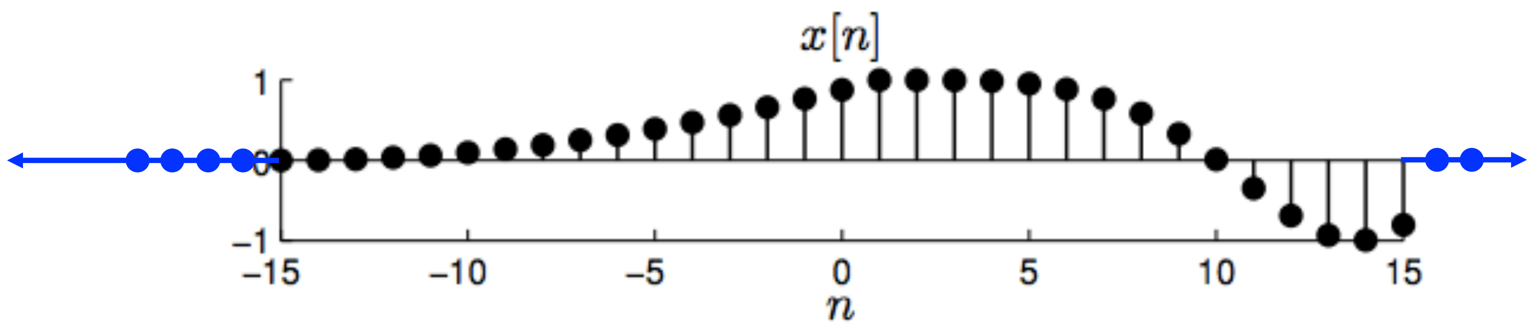
$$y[n] = \begin{cases} x[n] & N_1 \le n \le N_2 \\ 0 & \text{otherwise} \end{cases}$$



Zero Padding

- Converts a shorter signal into a longer one
- Say $x[n]$ is defined for $N_1 \leq n \leq N_2$
- Given $N_0 \leq N_1 \leq N_2 \leq N_3$

$$y[n] = \begin{cases} 0 & N_0 \leq n < N_1 \\ x[n] & N_1 \leq n \leq N_2 \\ 0 & N_2 < n \leq N_3 \end{cases}$$

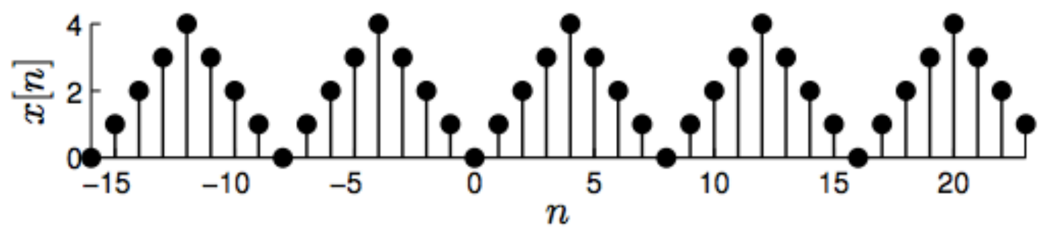


Periodic Signals

DEFINITION

A discrete-time signal is **periodic** if it repeats with period $N \in \mathbb{Z}$:

$$x[n + mN] = x[n] \quad \forall m \in \mathbb{Z}$$



Notes:

- The period N must be an integer
- A periodic signal is infinite in length

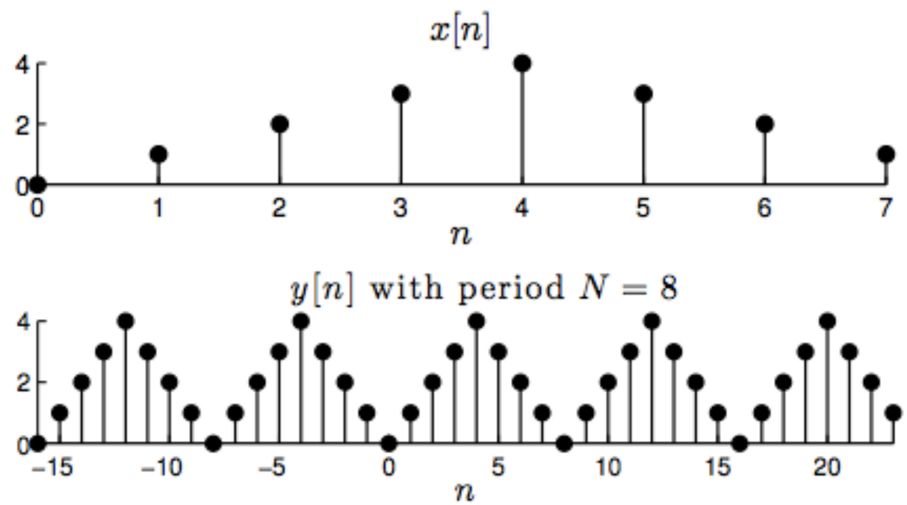
DEFINITION

A discrete-time signal is **aperiodic** if it is not periodic

Periodization

- Converts a finite-length signal into an infinite-length, periodic signal
- Given finite-length $x[n]$, replicate $x[n]$ periodically with period N

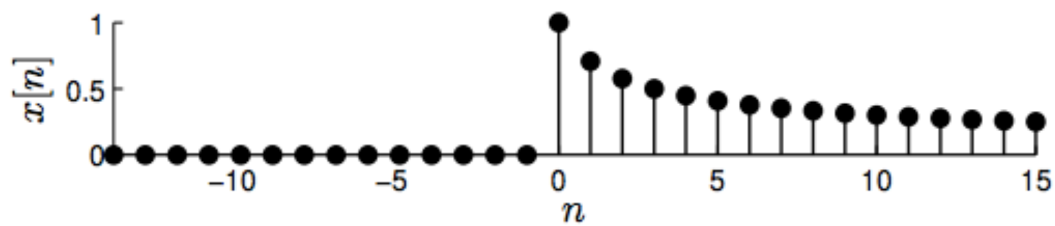
$$y[n] = \sum_{m=-\infty}^{\infty} x[n-mN], \quad n \in \mathbb{Z}$$
$$= \cdots + x[n+2N] + x[n+N] + x[n] + x[n-N] + x[n-2N] + \cdots$$



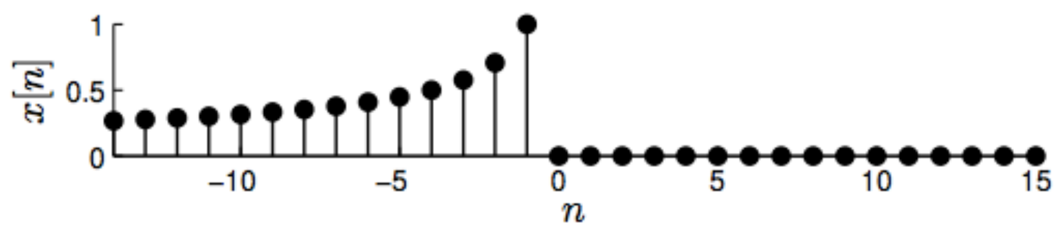
Causal Signals

DEFINITION

A signal $x[n]$ is **causal** if $x[n] = 0$ for all $n < 0$.



- A signal $x[n]$ is **anti-causal** if $x[n] = 0$ for all $n \geq 0$

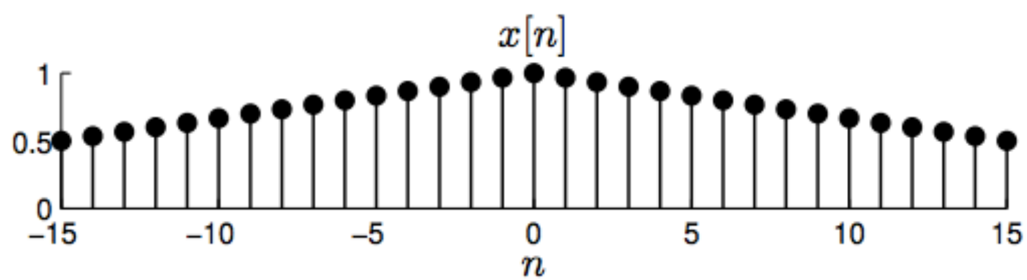


- A signal $x[n]$ is **acausal** if it is not causal

Even Signals

DEFINITION

A real signal $x[n]$ is **even** if $x[-n] = x[n]$

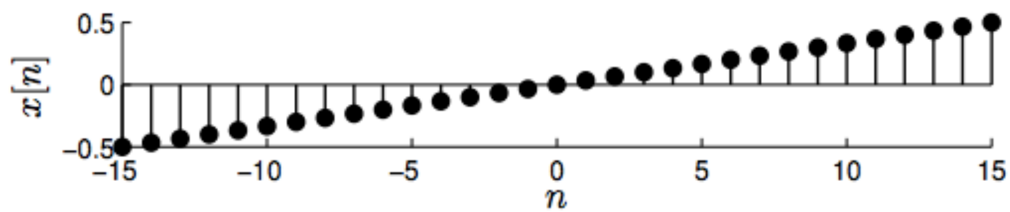


- Even signals are symmetrical around the point $n = 0$

Odd Signals

DEFINITION

A real signal $x[n]$ is **odd** if $x[-n] = -x[n]$



- Odd signals are anti-symmetrical around the point $n = 0$

Signal Decomposition

- **Useful fact:** Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part
- Even part: $e[n] = \frac{1}{2}(x[n] + x[-n])$ (easy to verify that $e[n]$ is even)
- Odd part: $o[n] = \frac{1}{2}(x[n] - x[-n])$ (easy to verify that $o[n]$ is odd)

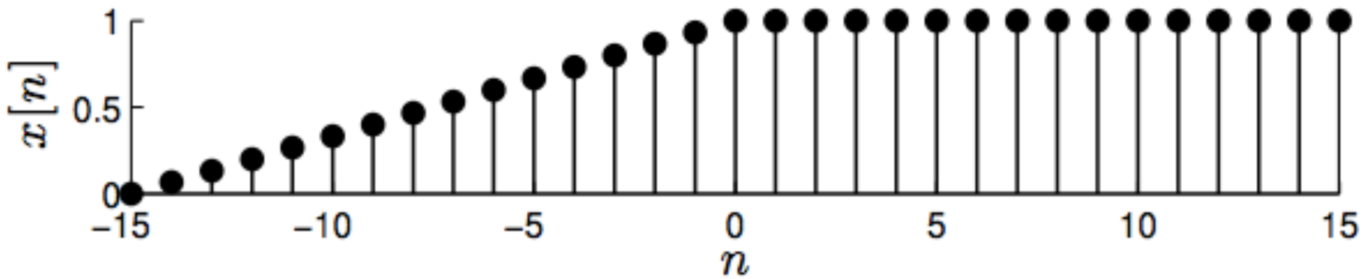
Signal Decomposition

- **Useful fact:** Every signal $x[n]$ can be decomposed into the sum of its even part + its odd part
- Even part: $e[n] = \frac{1}{2}(x[n] + x[-n])$ (easy to verify that $e[n]$ is even)
- Odd part: $o[n] = \frac{1}{2}(x[n] - x[-n])$ (easy to verify that $o[n]$ is odd)
- **Decomposition** $x[n] = e[n] + o[n]$
- Verify the decomposition:

$$\begin{aligned} e[n] + o[n] &= \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n]) \\ &= \frac{1}{2}(x[n] + x[-n] + x[n] - x[-n]) \\ &= \frac{1}{2}(2\,x[n]) = x[n] \quad \checkmark \end{aligned}$$

Decomposition Example

X[n]



Decomposition Example

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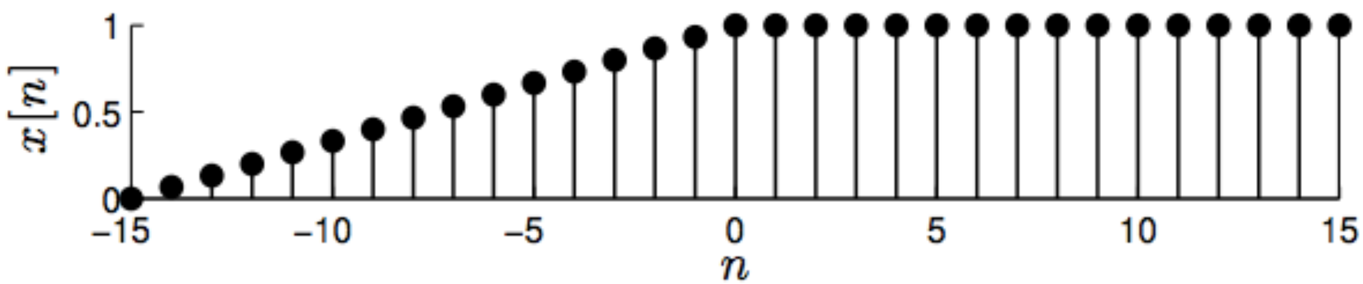
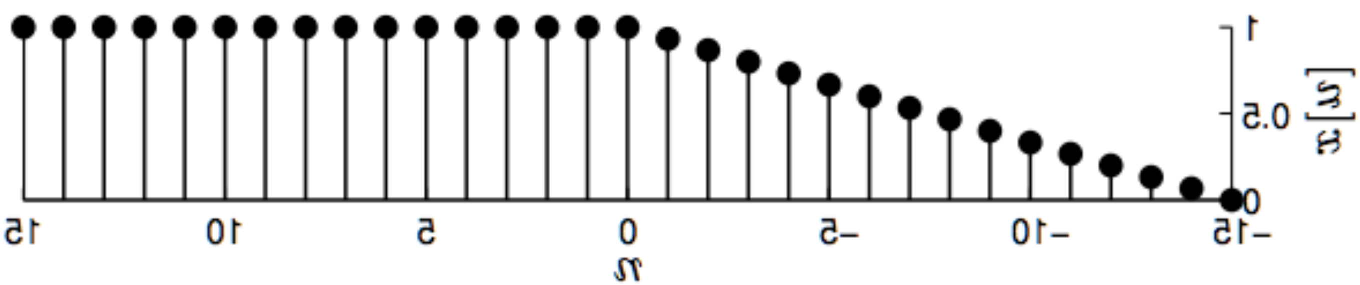
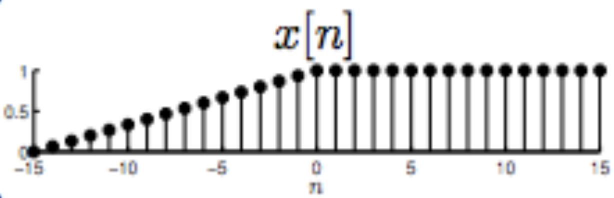


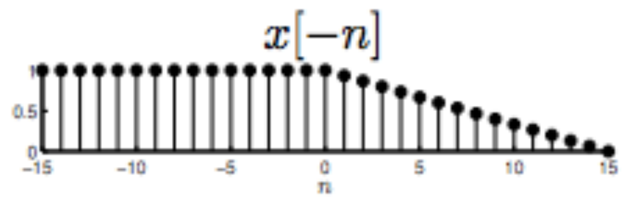
X[-n]



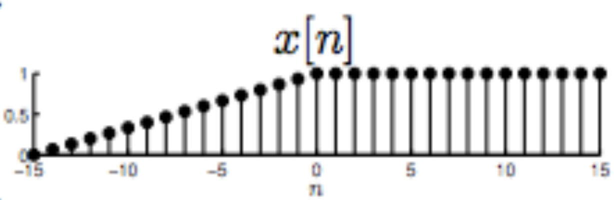
Decomposition Example

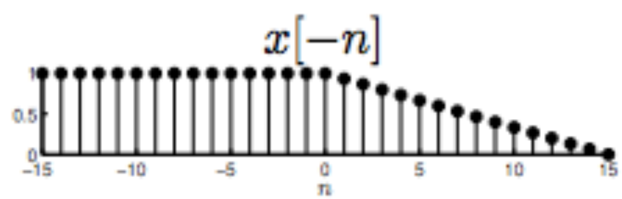
X[n] X[-n]

$\frac{1}{2}\Big($ 

$+$ 

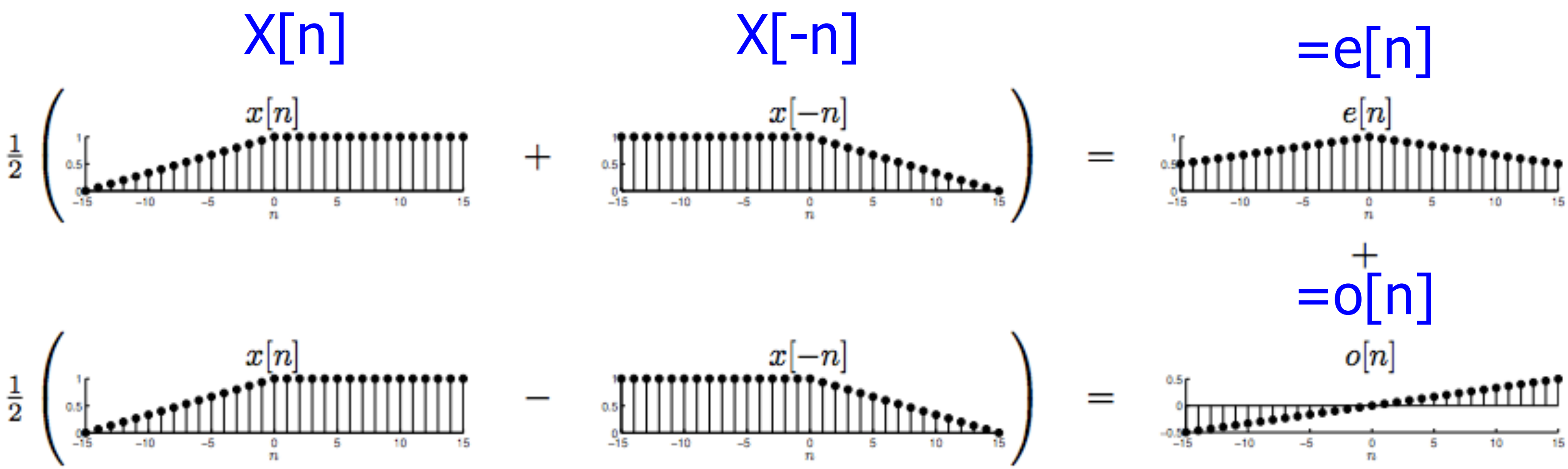
$\Big)$ =e[n]

$\frac{1}{2}\Big($ 

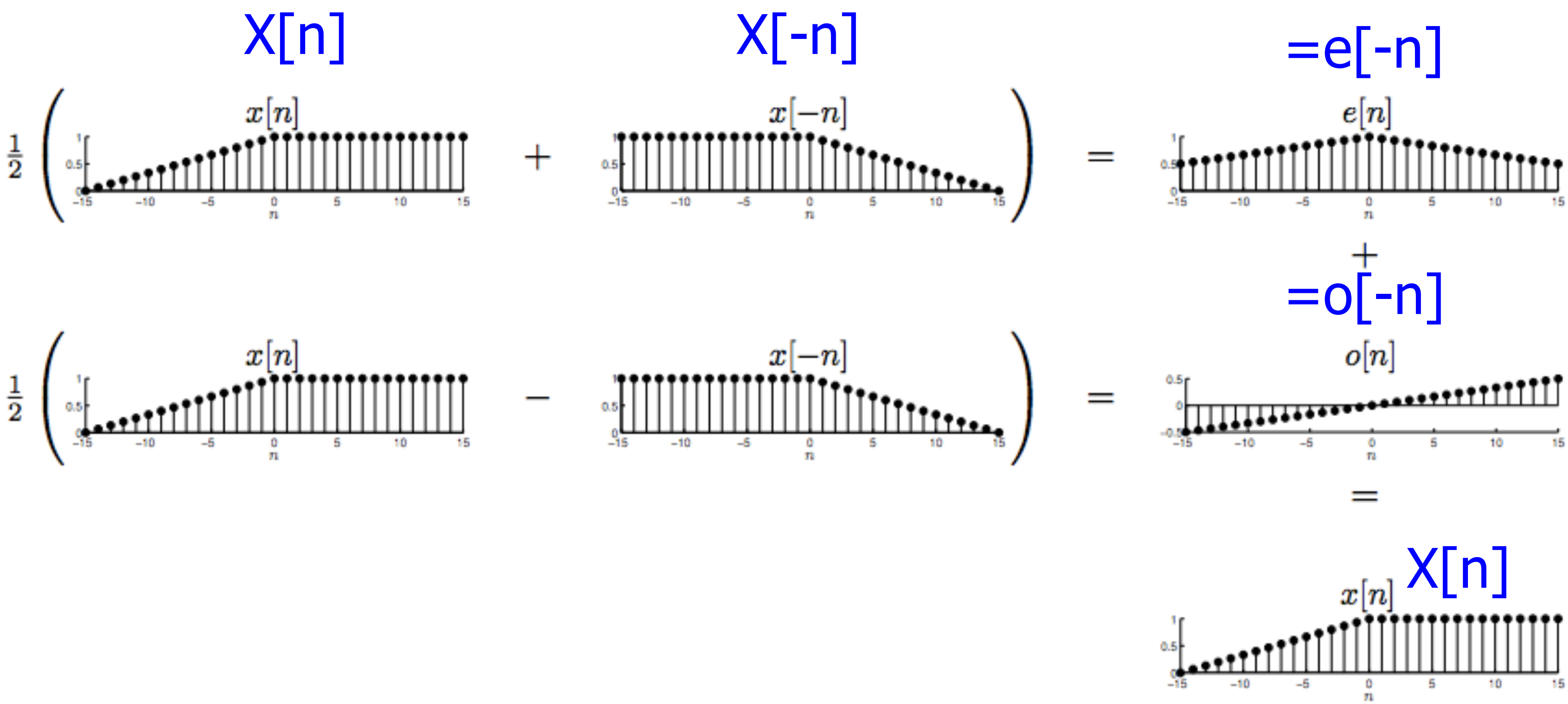
$-$ 

$\Big)$ =o[n]

Decomposition Example



Decomposition Example



Discrete-Time Sinusoids

- Discrete-time sinusoids $e^{j(\omega n+\phi)}$ have two counterintuitive properties
- Both involve the frequency ω
- Weird property #1: Aliasing
- Weird property #2: Aperiodicity

Property #1: Aliasing of Sinusoids

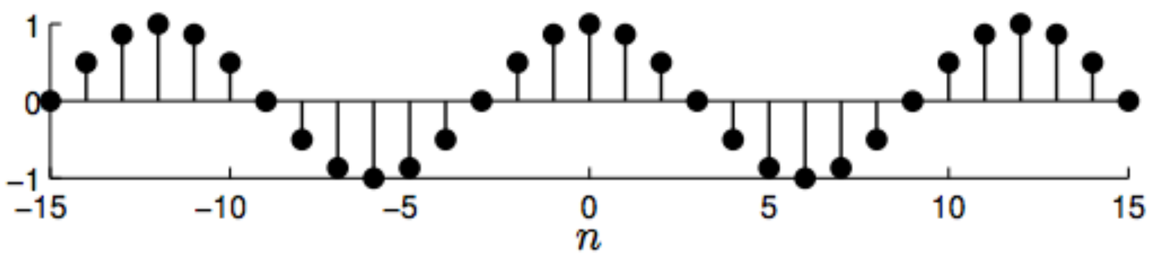
- Consider two sinusoids with two different frequencies
  - $\omega \quad \Rightarrow \quad x_1[n] = e^{j(\omega n+\phi)}$
  - $\omega + 2\pi \quad \Rightarrow \quad x_2[n] = e^{j((\omega+2\pi)n+\phi)}$
- But note that

$$x_2[n] = e^{j((\omega+2\pi)n+\phi)} = e^{j(\omega n+\phi)+j2\pi n} = e^{j(\omega n+\phi)}\, e^{j2\pi n} = e^{j(\omega n+\phi)} = x_1[n]$$

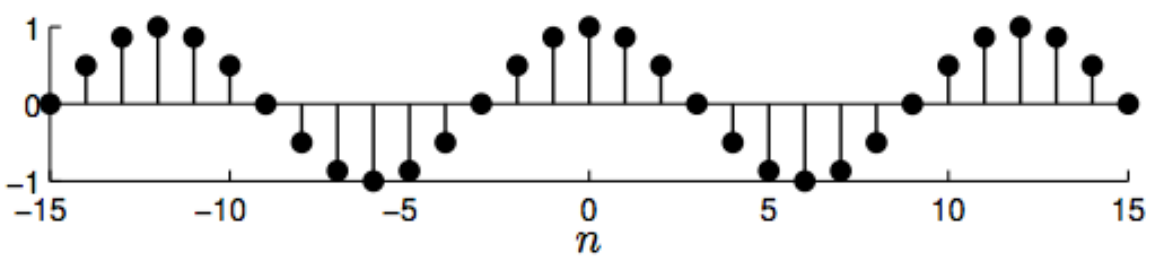
- The signals x_1 and x_2 have different frequencies but are **identical**!
- We say that x_1 and x_2 are aliases; this phenomenon is called **aliasing**
- Note: Any integer multiple of 2π will do; try with $x_3[n] = e^{j((\omega+2\pi m)n+\phi)}$, $m \in \mathbb{Z}$

Aliasing Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$



- $x_2[n] = \cos\left(\frac{13\pi}{6}n\right) = \cos\left(\left(\frac{\pi}{6} + 2\pi\right)n\right)$



Aliasing Example

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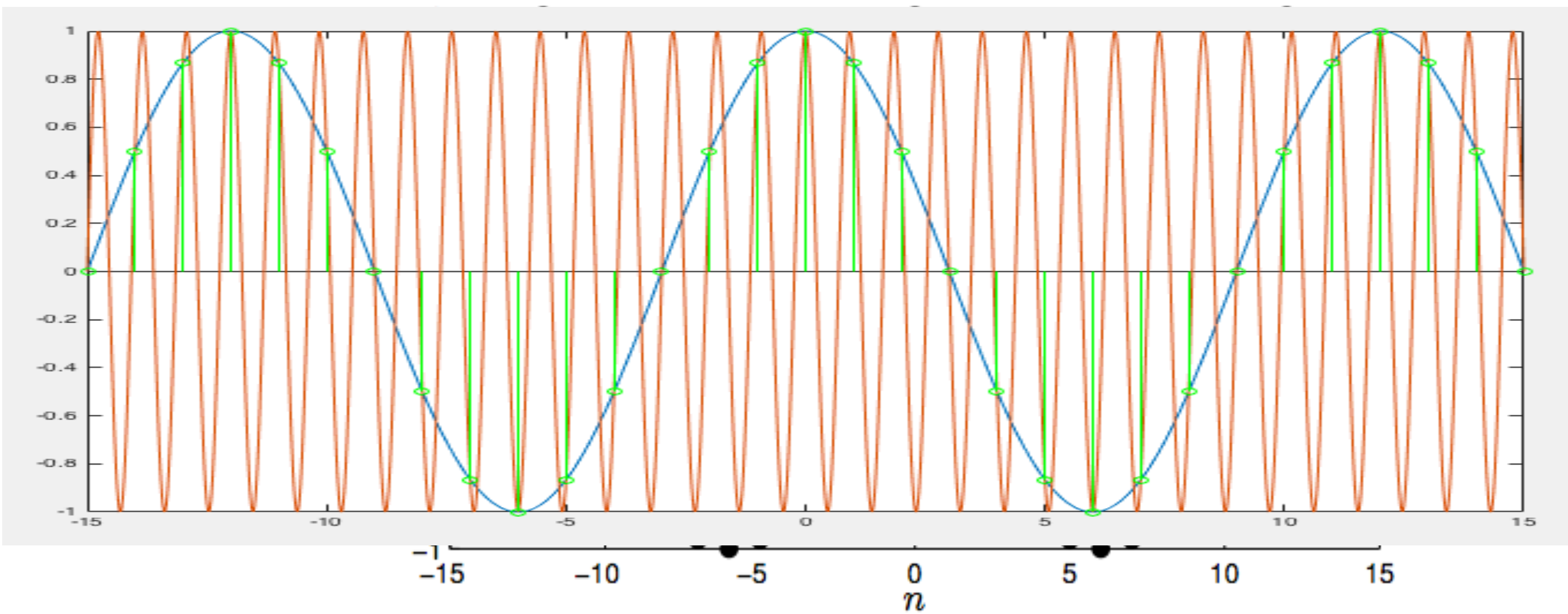


Alias-Free Frequencies

- Since

$$x_3[n] = e^{j(\omega+2\pi m)n+\phi)} = e^{j(\omega n+\phi)} = x_1[n] \quad \forall m \in \mathbb{Z}$$

the only frequencies that lead to unique (distinct) sinusoids lie in an interval of length 2π

- Two intervals are typically used in the signal processing literature (and in this course)
  - $0 \leq \omega < 2\pi$
  - $-\pi < \omega \leq \pi$

Which is higher in frequency?

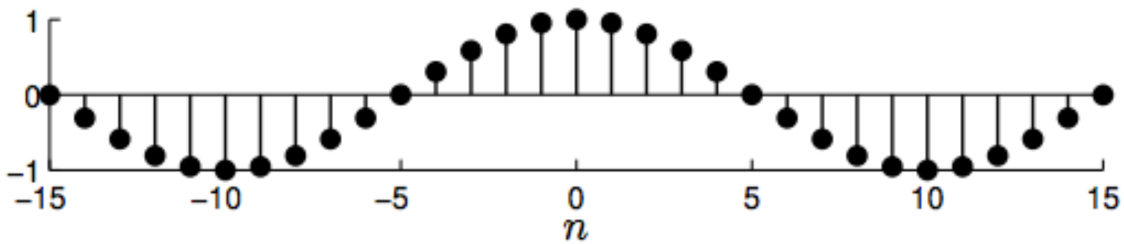
- $\cos(\pi n)$ or $\cos(3\pi/2n)$?

Low and High Frequencies

$$e^{j(\omega n+\phi)}$$

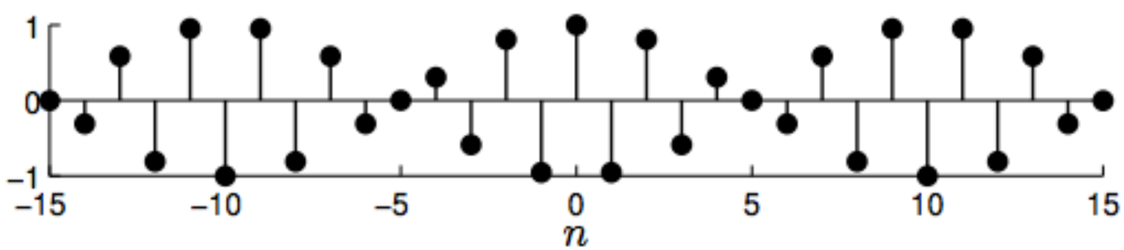
- **Low frequencies:** ω close to 0 or 2π rad

  Ex: $\cos\left(\frac{\pi}{10}n\right)$

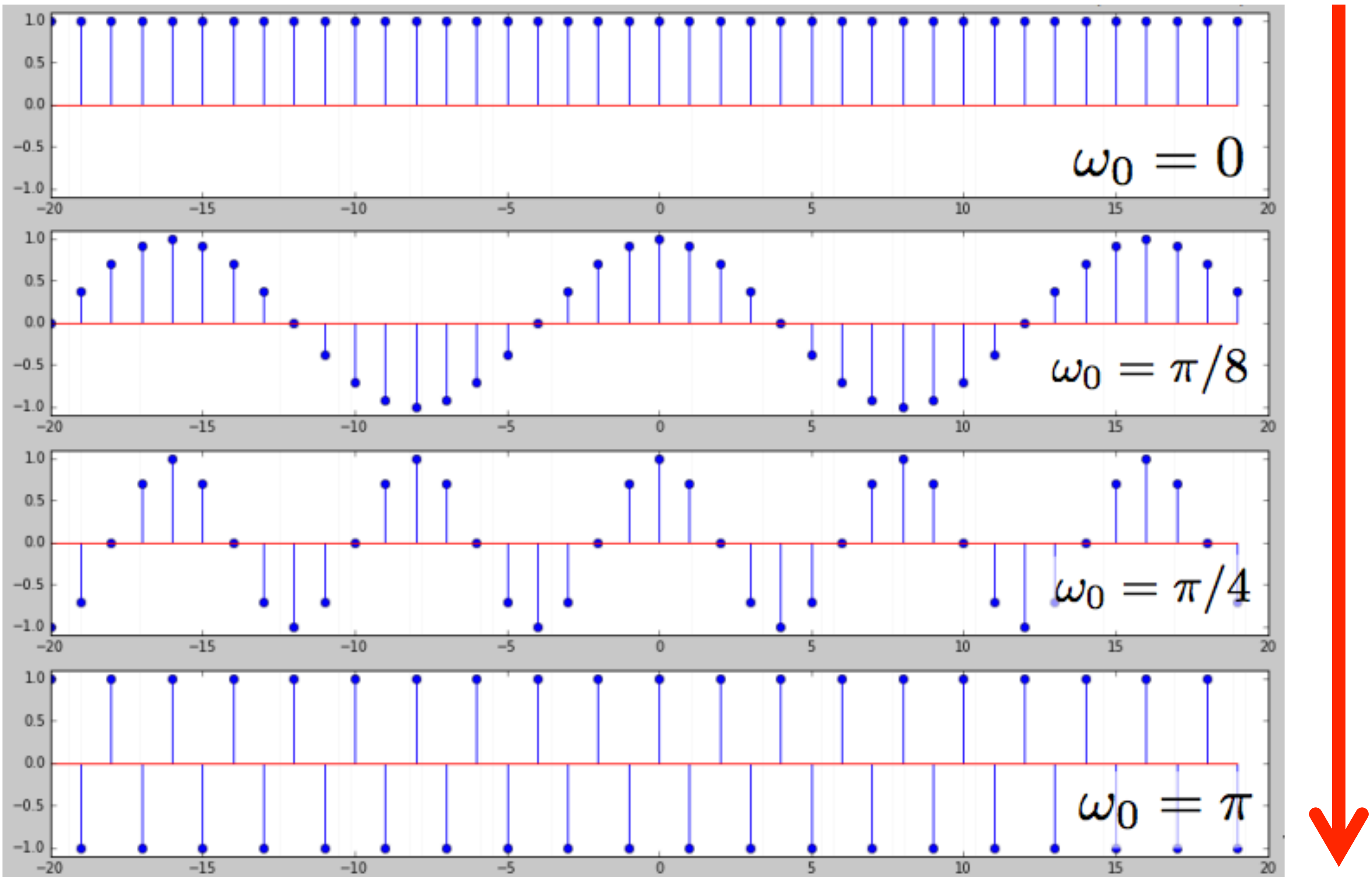


- **High frequencies:** ω close to π or $-\pi$ rad

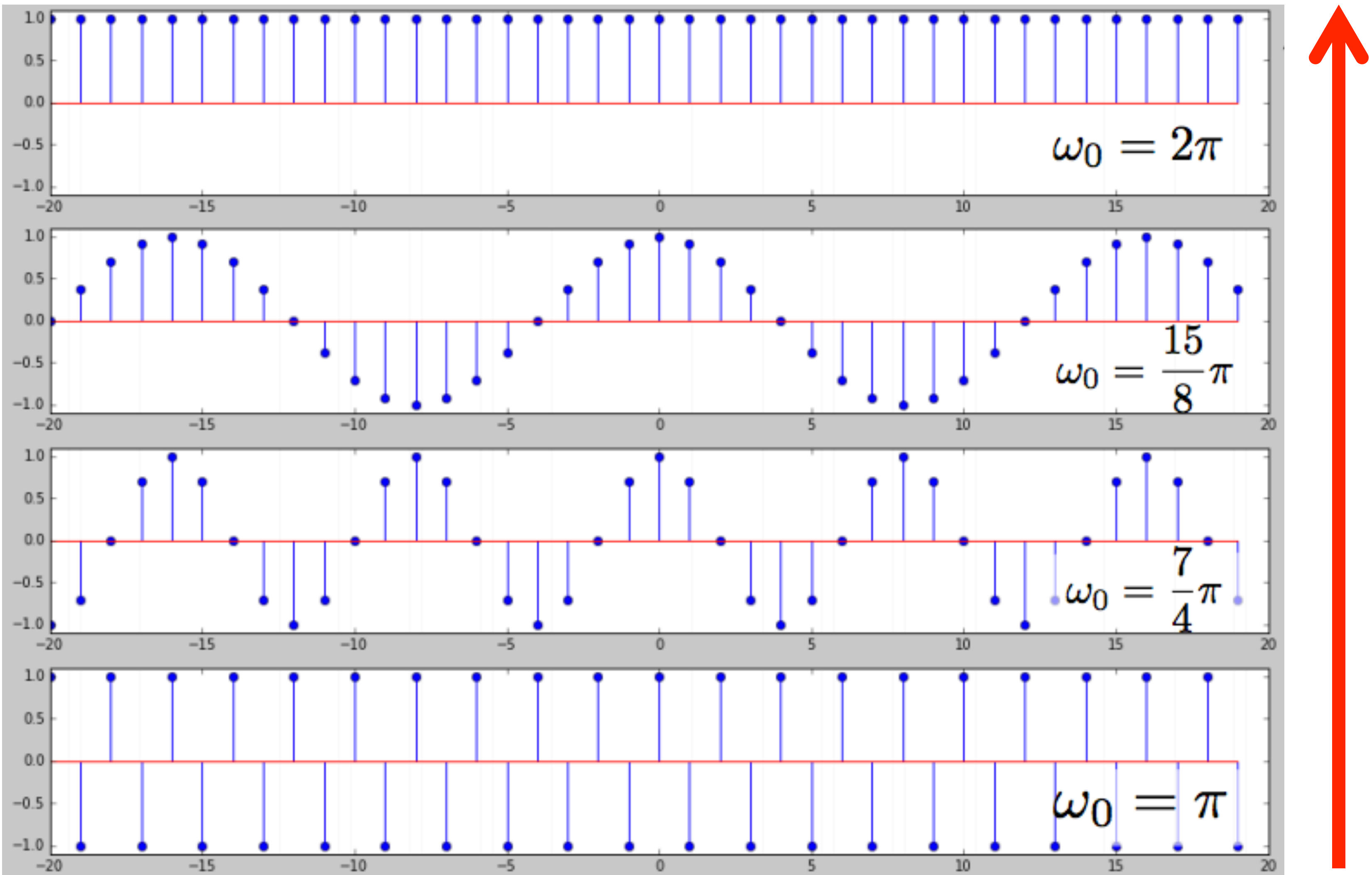
  Ex: $\cos\left(\frac{9\pi}{10}n\right)$



Increasing Frequency



Decreasing Frequency



Property #2: Periodicity of Sinusoids

- Consider $x_1[n] = e^{j(\omega n+\phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (harmonic frequency)

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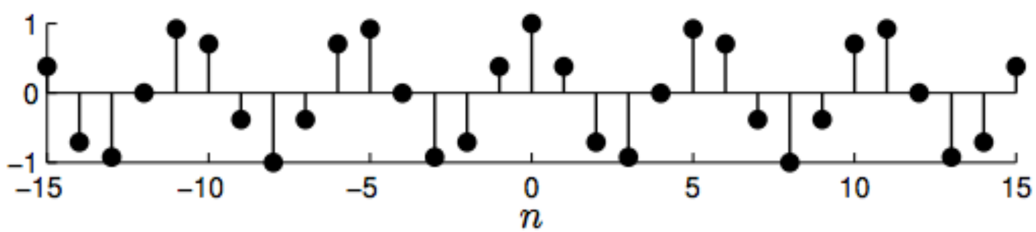
- Consider $x_1[n] = e^{j(\omega n+\phi)}$ with frequency $\omega = \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (harmonic frequency)

- It is easy to show that x_1 is periodic with period N, since

$$x_1[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n+\omega N+\phi)} = e^{j(\omega n+\phi)}\, e^{j(\omega N)} = e^{j(\omega n+\phi)}\, e^{j(\frac{2\pi k}{N} N)} = x_1[n] \checkmark$$

- Ex: $x_1[n] = \cos(\frac{2\pi 3}{16} n)$, $\quad N = 16$



- Note: x_1 is periodic with the (smaller) period of $\frac{N}{k}$ when $\frac{N}{k}$ is an integer

Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (not harmonic frequency)

Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (not harmonic frequency)

- Is x_2 periodic?

$$x_2[n+N] = e^{j(\omega(n+N)+\phi)} = e^{j(\omega n + \omega N + \phi)} = e^{j(\omega n + \phi)}\, e^{j(\omega N)} \neq x_1[n] \quad \text{NO!}$$

Aperiodicity of Sinusoids

- Consider $x_2[n] = e^{j(\omega n + \phi)}$ with frequency $\omega \neq \frac{2\pi k}{N}$, $\quad k, N \in \mathbb{Z}$ (not harmonic frequency)

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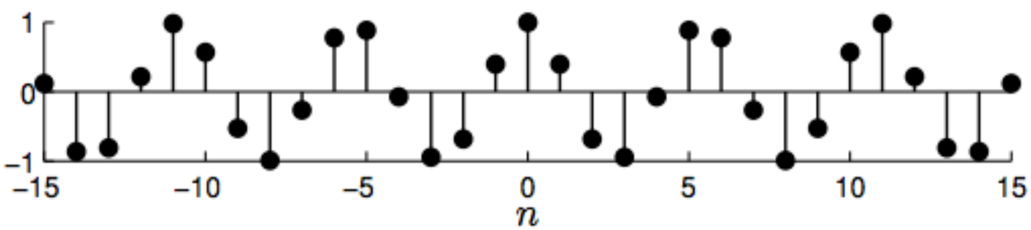
- Ex: $x_2[n] = \cos(1.16\, n)$



- If its frequency ω is not harmonic, then a sinusoid oscillates but is not periodic!

Harmonic Sinusoids

$$e^{j(\omega n+\phi)}$$

- Semi-amazing fact: The **only** periodic discrete-time sinusoids are those with **harmonic frequencies**

$$\omega = \frac{2\pi k}{N}, \quad k, N \in \mathbb{Z}$$

- Which means that
  - **Most** discrete-time sinusoids are **not** periodic!
  - The harmonic sinusoids are somehow magical (they play a starring role later in the DFT)

Periodic or not?

- $\cos(5/7\,\pi\,n)$

- $\cos(\pi/5n)$

- What are N and k? (I.e How many samples is one period?

Periodic or not?

- $\cos(5/7\,\pi n)$
  - N=14, k=5
  - $\cos(5/14*2\pi n)$
  - Repeats every N=14 samples
- $\cos(\pi/5n)$
  - N=10, k=1
  - $\cos(1/10*2\pi n)$
  - Repeats every N=10 samples

Periodic or not?

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  - Repeats every N=14 samples
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  - N=10, k=1
  - $\cos(1/10 * 2\pi n)$
  - Repeats every N=10 samples

- $\cos(5/7\,\pi n)+\cos(\pi/5n)$?

Periodic or not?

- $\cos(5/7\pi n)+\cos(\pi/5n)$?
  - $N=SCM\{10,14\}=70$
  - $\cos(5/7*\pi n)+\cos(\pi/5n)$
    - $n=N=70 \rightarrow \cos(5/7*70\pi)+\cos(\pi/5*70)=\cos(25*2\pi)+\cos(7*2\pi)$

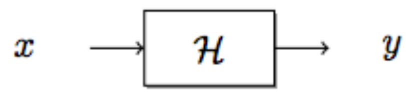
Discrete-Time Systems

Discrete Time Systems

DEFINITION

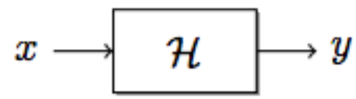
A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

Signal Length and Systems



- Recall that there are two kinds of signals: infinite-length and finite-length
- Accordingly, we will consider two kinds of systems:
  1. Systems that transform an infinite-length-signal x into an infinite-length signal y
  2. Systems that transform a length-N signal x into a length-N signal y

     (Such systems can also be used to process periodic signals with period N)
- For generality, we will assume that the input and output signals are complex valued

System Examples

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

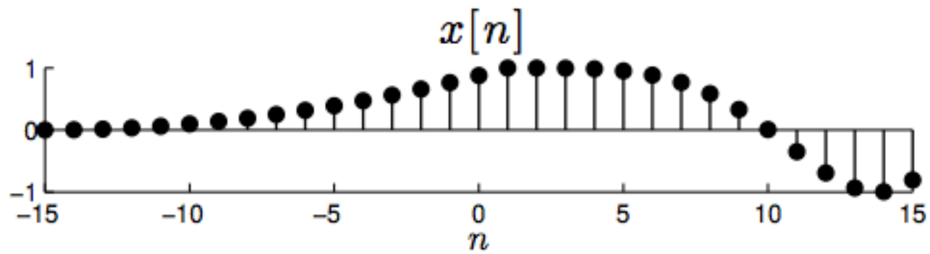
- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

System Examples



- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n-m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n-1]) \quad \forall n$$

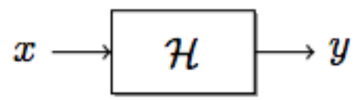
- Recursive average

$$y[n] = x[n] + \alpha\, y[n-1] \quad \forall n$$

System Properties

- Memoryless
- Linearity
- Time Invariance
- Causality
- BIBO Stability

Memoryless



- y[n] depends only on x[n]

- Examples:
- Ideal delay system (or shift system):
  - y[n]=x[n-m]   memoryless?

- Square system:
  - $y[n]=(x[n])^2$ memoryless?

Linear Systems

DEFINITION

A system $\mathcal{H}$ is (zero-state) **linear** if it satisfies the following two properties:

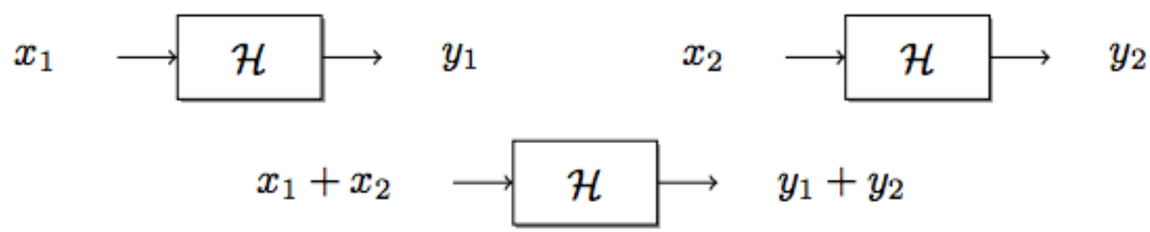
1. Scaling

$$\mathcal{H}\{\alpha\, x\} = \alpha\, \mathcal{H}\{x\} \quad \forall\, \alpha \in \mathbb{C}$$

2. Additivity

$$\text{If} \quad y_1 = \mathcal{H}\{x_1\} \quad \text{and} \quad y_2 = \mathcal{H}\{x_2\} \quad \text{then}$$

$$\mathcal{H}\{x_1 + x_2\} = y_1 + y_2$$



Proving Linearity

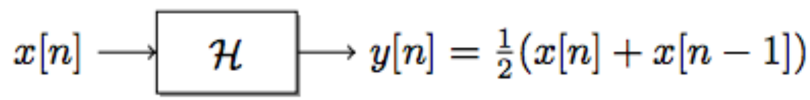
- A system that is not linear is called **nonlinear**
- To prove that a system is linear, you must prove rigorously that it has **both** the scaling and additivity properties for **arbitrary** input signals
- To prove that a system is nonlinear, it is sufficient to exhibit a **counterexample**

Linearity Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \tfrac{1}{2}(x[n] + x[n-1])$$

- **Scaling:** (Strategy to prove – Scale input x by $\alpha \in \mathbb{C}$, compute output y via the formula at top, and verify that it is scaled as well)
  - Let
  $$x'[n] = \alpha x[n], \quad \alpha \in \mathbb{C}$$
  - Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)
  - Then
  $$y'[n] = \frac{1}{2}(x'[n] + x'[n-1])$$

Linearity Example: Moving Average



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  - Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)
  - Then
  $$y'[n] \;=\; \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(\alpha x[n] + \alpha x[n-1]) \;=\; \alpha\left(\frac{1}{2}(x[n] + x[n-1])\right) \;=\; \alpha y[n] \;\checkmark$$

Linearity Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)
  - Let
    $$x'[n] \;=\; x_1[n] + x_2[n]$$
  - Let $y'/y_1/y_2$ denote the output when $x'/x_1/x_2$ is input

Linearity Example: Moving Average

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  - Then
  $$\begin{aligned} y'[n] &= \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(\{x_1[n] + x_2[n]\} + \{x_1[n-1] + x_2[n-1]\}) \\ &= \frac{1}{2}(x_1[n] + x_1[n-1]) + \frac{1}{2}(x_2[n] + x_2[n-1]) \;=\; y_1[n] + y_2[n] \quad \checkmark \end{aligned}$$

Example: Squaring is Nonlinear

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = (x[n])^2$$

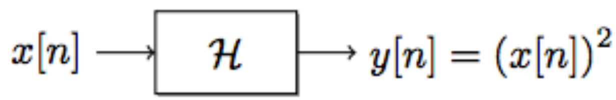
- **Additivity:** Input two signals into the system and see what happens
  - Let

$$y_1[n] = (x_1[n])^2, \qquad y_2[n] = (x_2[n])^2$$

  - Set

$$x'[n] \;=\; x_1[n] + x_2[n]$$

Example: Squaring is Nonlinear

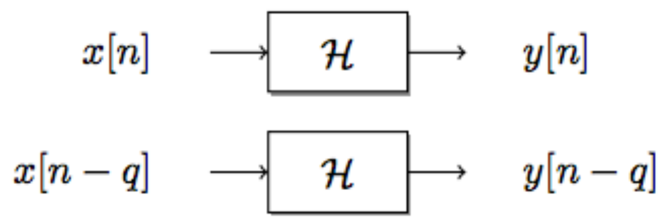


- **Additivity:** Input two signals into the system and see what happens
  - Let
  $$y_1[n] = (x_1[n])^2, \qquad y_2[n] = (x_2[n])^2$$
  - Set
  $$x'[n] = x_1[n] + x_2[n]$$
  - Then
  $$y'[n] = (x'[n])^2 = (x_1[n] + x_2[n])^2 = (x_1[n])^2 + 2x_1[n]x_2[n] + (x_2[n])^2 \neq y_1[n] + y_2[n]$$
  - Nonlinear!

Time-Invariant Systems

DEFINITION

A system $\mathcal{H}$ processing infinite-length signals is **time-invariant** (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal



- Intuition: A time-invariant system behaves the same no matter when the input is applied
- A system that is not time-invariant is called **time-varying**

Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- Let

$$x'[n] = x[n-q], \quad q \in \mathbb{Z}$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

Example: Moving Average

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- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] \;=\; \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(x[n-q] + x[n-q-1]) \;=\; y[n-q] \quad \checkmark$$

Example: Decimation

- This system is time-varying; demonstrate with a counter-example

- Let

$$x'[n] = x[n-1]$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] \;=\; x'[2n] \;=\; x[2n-1] \;\neq\; x[2(n-1)] \;=\; y[n-1]$$

Causal Systems

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- Forward difference system:
  - y[n]=x[n+1]-x[n] causal?

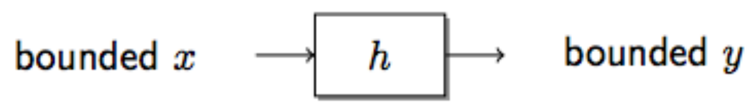
- Backward difference system:
  - y[n]=x[n]-x[n-1] causal?

Stability

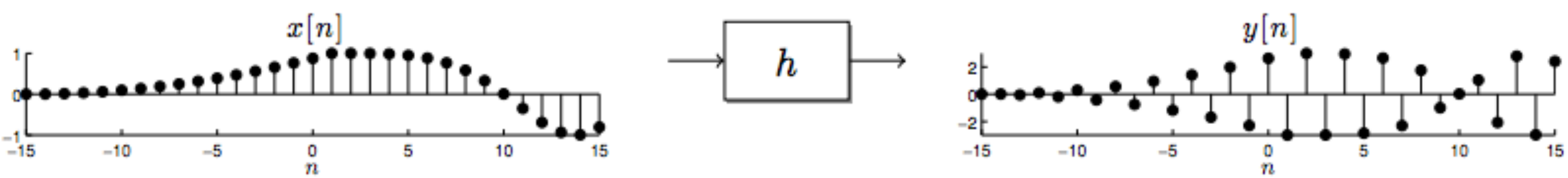
- BIBO Stability
  - Bounded-input bounded-output Stability

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$, or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all n



System Properties - Summary

- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max)

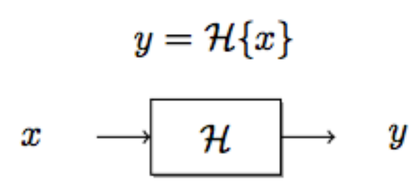
Examples

- Causal? Linear? Time-invariant? Memoryless? BIBO Stable?
- Time Shift:
  - $y[n] = x[n-m]$
- Accumulator:
  - $$y[n] = \sum_{k=-\infty}^{n} x[k]$$
- Compressor (M>1):

$$y[n] = x[Mn]$$

Big Ideas

- Discrete Time Signals
  - Unit impulse, unit step, exponential, sinusoids, complex sinusoids
  - Can be finite length, infinite length
  - Properties
    - Even, odd, causal
    - Periodicity and aliasing
      - Discrete frequency bounded!
- Discrete Time Systems
  - Transform one signal to another
  - Properties
    - Linear, Time-invariance, memoryless, causality, BIBO stability



Admin

- Behind the scenes programming note:
  - Additional grader: Zhefu Peng
- Enroll in Piazza site:
  - piazza.com/upenn/spring2019/ese531
- HW 0: Brush up on background and Matlab tutorial