# ESE 531: Digital Signal Processing

Lec 22: April 11, 2019

Fast Fourier Transform Pt 2

## Lecture Outline

- DFT vs. DTFT
- FFT practice
- Chirp Transform Algorithm
- Circular convolution as linear convolution with aliasing

## Discrete Fourier Transform

- The DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \qquad \text{Inverse DFT, synthesis}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \qquad \text{DFT, analysis}$$

- It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \le n \le N-1$$
$$X[k] = 0 \quad \text{outside } 0 \le k \le N-1$$

## DFT vs. DTFT

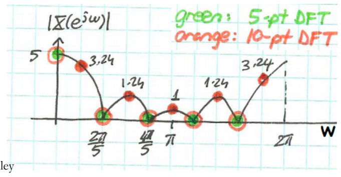
- The DFT are samples of the DTFT at N equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega = k\frac{2\pi}{N}} \qquad 0 \le k \le N-1$$

## DFT vs. DTFT

- Back to example

$$X[k] = \sum_{n=0}^{4} W_{10}^{nk}$$
$$= e^{-j\frac{4\pi}{10}k}\frac{\sin(\frac{\pi}{2}k)}{\sin(\frac{\pi}{10}k)}$$



## Fast Fourier Transform Algorithms

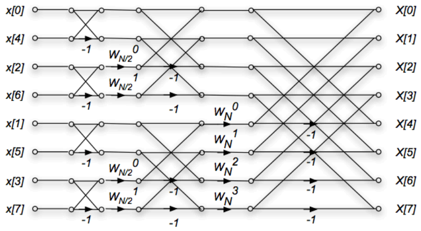
- We are interested in efficient computing methods for the DFT and inverse DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad k = 0, \dots, N-1$$
$$x[n] = \sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad n = 0, \dots, N-1$$

$$W_N = e^{-j(\frac{2\pi}{N})}.$$

## Decimation-in-Time FFT

Combining all these stages, the diagram for the 8 sample DFT is:



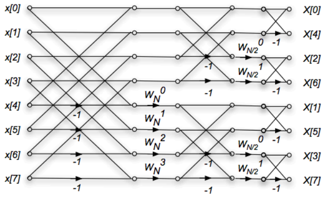
- 3=$\log_2$(N)=$\log_2$(8) stages
- 4=N/2=8/2 multiplications in each stage
  - 1st stage has trivial multiplication

## Decimation-in-Time FFT

- In general, there are $\log_2$N stages of decimation-in-time.
- Each stage requires N/2 complex multiplications, some of which are trivial.
- The total number of complex multiplications is (N/2) $\log_2$N, or O(N $\log_2$N)
- The order of the input to the decimation-in-time FFT algorithm must be permuted.
  - First stage: split into odd and even. Zero low-order bit (LSB) first
  - Next stage repeats with next zero-lower bit first.
  - Net effect is reversing the bit order of indexes

## Decimation-in-Frequency FFT

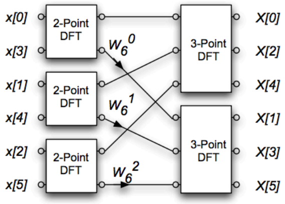
The diagram for and 8-point decimation-in-frequency DFT is as follows



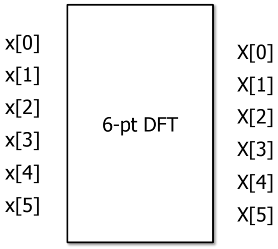
This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

## Non-Power-of-2 FFTs

- A similar argument applies for any length DFT, where the length *N* is a composite number
- For example, if N=6, with decimation-in-frequency you could compute three 2-point DFTs followed by two 3-point DFTs



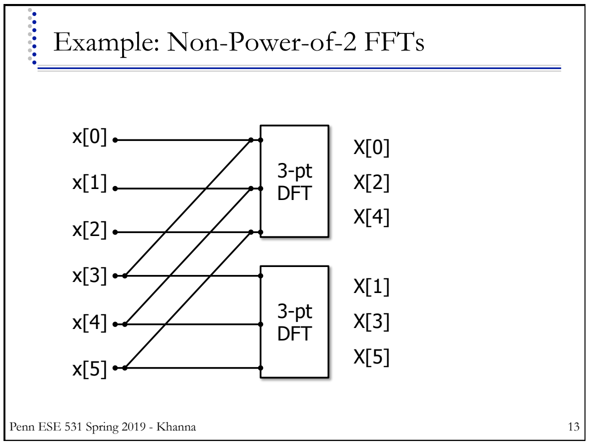
## Example: Non-Power-of-2 FFTs



## Decimation-in-Frequency FFT

$$\begin{aligned} X[2r] &= \mathrm{DFT}_{\frac{N}{2}}\{(x[n]+x[n+N/2])\} \\ X[2r+1] &= \mathrm{DFT}_{\frac{N}{2}}\{(x[n]-x[n+N/2])\,W_N^n\} \end{aligned}$$

(By a similar argument that gives the odd samples)



## Example: Non-Power-of-2 FFTs

## Decimation-in-Frequency FFT

$$X[2r] = \mathrm{DFT}_{\frac{N}{2}}\{(x[n] + x[n+N/2])\}$$
$$X[2r+1] = \mathrm{DFT}_{\frac{N}{2}}\{(x[n] - x[n+N/2])\, W_N^n\}$$

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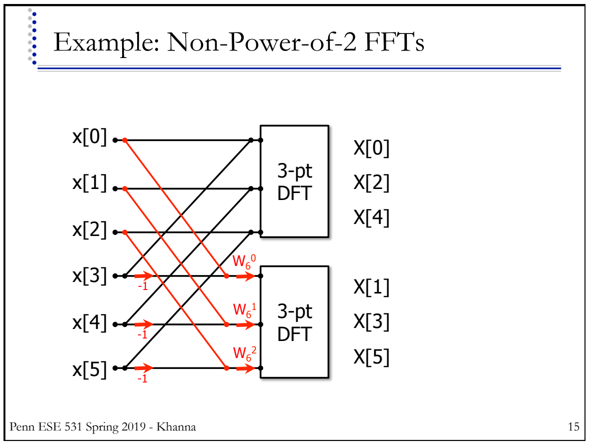
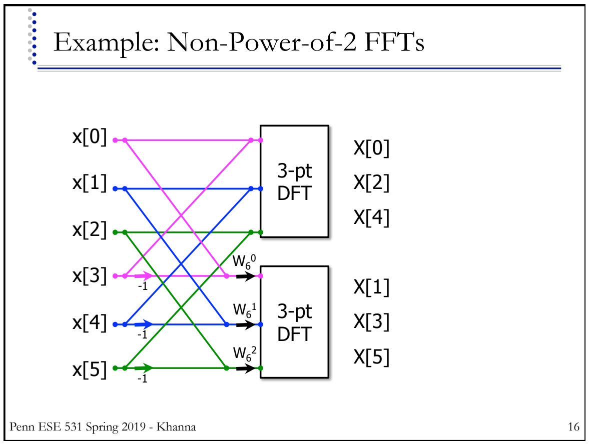


## Example: Non-Power-of-2 FFTs



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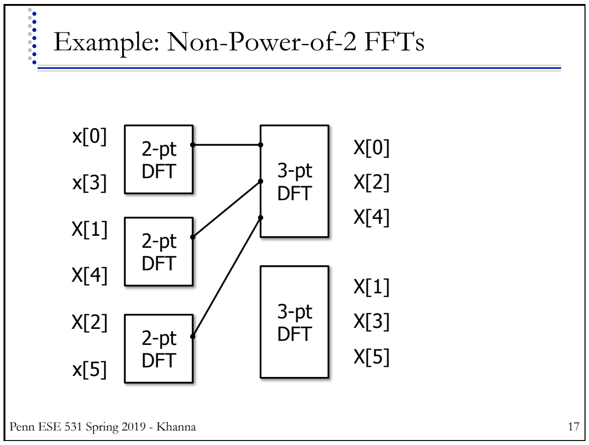

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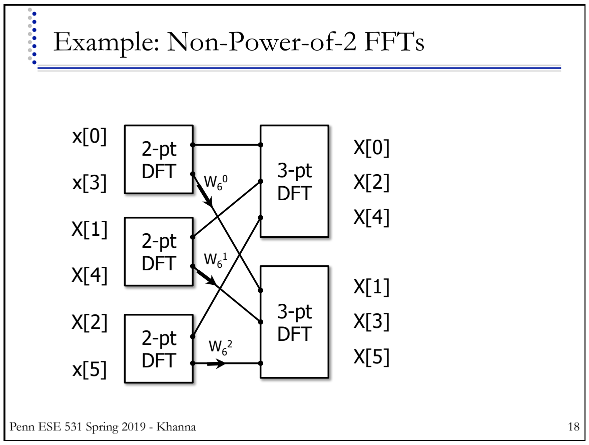

## Example: Non-Power-of-2 FFTs

## Example 1:

A long *periodic* sequence $x$ of period $N = 2^r$ ($r$ is an integer) is to be convolved with a finite-length sequence $h$ of length $K$.

(a) *Show* that the output $y$ of this convolution (filtering) is *periodic*; what is its period?

## Example 1:

A long *periodic* sequence $x$ of period $N = 2^r$ ($r$ is an integer) is to be convolved with a finite-length sequence $h$ of length $K$.

(a) *Show* that the output $y$ of this convolution (filtering) is *periodic*; what is its period?

(b) Let $K = mN$ where $m$ is an integer; $N$ is large. How would you implement this convolution *efficiently*? Explain your analysis clearly.
Compare the *total number of multiplications* required in your scheme to that in the direct implementation of FIR filtering. (Consider the case $r = 10$, $m = 10$).

## Example 2:

A sequence $x = \{x[n], n = 0,1,\ldots,N-1\}$ is given; let $X(e^{j\omega})$ be its DTFT.

(a) Suppose $N = 10$. You want to evaluate both $X(e^{j2\pi 7/12})$ and $X(e^{j2\pi 3/8})$. The only computation you can perform is one DFT, on any one input sequence of your choice. Can you find the desired DTFT values? *(Show your analysis and explain clearly.)*

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(b) Suppose $N$ is large. You want to obtain $X(e^{j\omega})$ at the following $2M$ frequencies:

$$\omega = \frac{2\pi}{M}m, \quad m = 0,1,\ldots,M-1 \quad \text{and} \quad \omega = \frac{2\pi}{M}m + \frac{2\pi}{N}, \quad m = 0,1,\ldots,M-1.$$

Here $M = 2^\mu \ll N = 2^\nu$

A standard radix-2 FFT algorithm is available. You may execute the FFT algorithm *once* or *more than once*, and *multiplications* and *additions* outside of the FFT are *allowed*, if necessary.

(i) You want to get the $2M$ DTFT values with as few *total multiplications* as possible (*including those in the FFT*). Give explicitly the best method you can find for this, with an estimate of the *total number of multiplications* needed in terms of $M$ and $N$.

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(ii) Does your result change if extra multiplications outside of FFTs are *not* allowed?

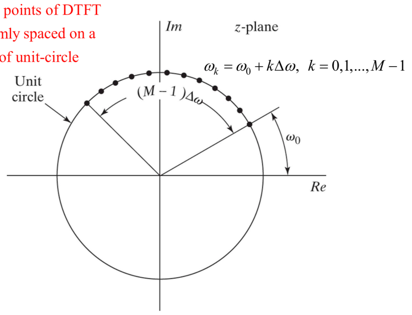
## Chirp Transfer Algorithm

## Chirp Transform Algorithm

- Uses convolution to evaluate the DFT
- This algorithm is not optimal in minimizing any measure of computational complexity, but it has been useful in a variety of applications, particularly when implemented in technologies that are well suited to doing convolution with a fixed, pre-specified impulse response.
- The CTA is also more flexible than the FFT, since it can be used to compute *any* set of equally spaced samples of the Fourier transform on the unit circle.

## Chirp Transform Algorithm

For $M$ points of DTFT uniformly spaced on a sector of unit-circle


$Im$
$z$-plane
$\omega_k = \omega_0 + k\Delta\omega,\ \ k = 0,1,\ldots,M-1$
Unit circle
$(M-1)\Delta\omega$
$\omega_0$
$Re$

## Chirp Transform Algorithm

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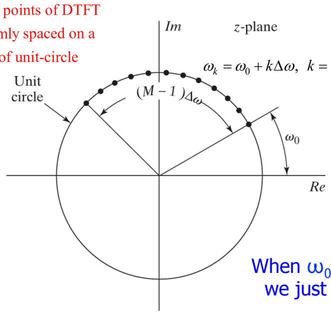
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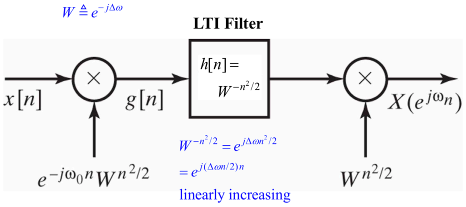
$\omega_k = \omega_0 + k\Delta\omega,\ \ k = 0,1,\ldots,M-1$
Unit circle
$(M-1)\Delta\omega$
$\omega_0$
$Re$

When $\omega_0$=0 and M=N, we just get the DFT

## Chirp Transform Algorithm

$W \triangleq e^{-j\Delta\omega}$



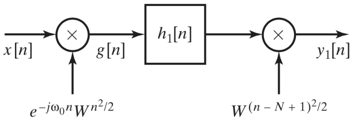
$W^{-n^2/2} = e^{j\Delta\omega n^2/2}$

$= e^{j(\Delta\omega n/2)n}$

linearly increasing frequency "chirp"

## Causal FIR CTA

$$h_1[n] = \begin{cases} W^{-(n-N+1)^2/2}, & n = 0, 1, \ldots, M+N-2, \\ 0, & \text{otherwise.} \end{cases}$$

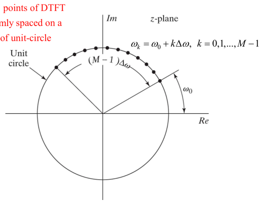


$$X(e^{j\omega_n}) = y_1[n+N-1], \qquad n = 0, 1, \ldots, M-1.$$

## Example: Chirp Transform Parameters

- We have a finite-length sequence x[n] that is nonzero only on the interval n = 0, ..., 25, (Length N=26) and we wish to compute 16 samples of the DTFT X(e$^{j\omega}$) at the frequencies $\omega_k = 2\pi/27 + 2\pi k/1024$ for k = 0, . . . , 15.

For $M$ points of DTFT uniformly spaced on a sector of unit-circle


$Im$
$z$-plane
$\omega_k = \omega_0 + k\Delta\omega,\ \ k = 0,1,\ldots,M-1$
Unit circle
$(M-1)\Delta\omega$
$\omega_0$
$Re$

## Circular Convolution

Linear Convolution with aliasing!

## Circular Convolution

- Circular Convolution:

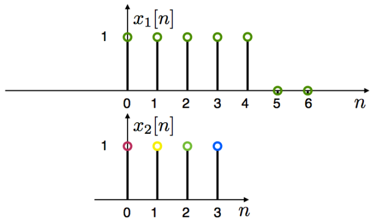
$$x_1[n] \textcircled{N} x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

For two signals of length N

Note: Circular convolution is commutative

$$x_2[n] \textcircled{N} x_1[n] = x_1[n] \textcircled{N} x_2[n]$$

## Compute Circular Convolution Sum



$$x_1[n] \textcircled{N} x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

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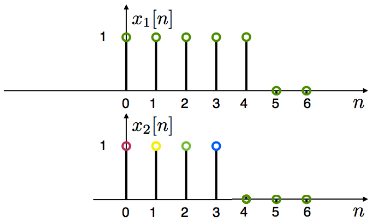

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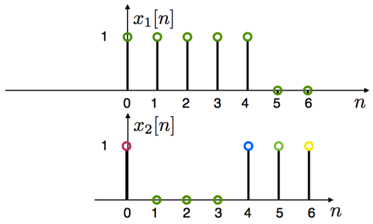

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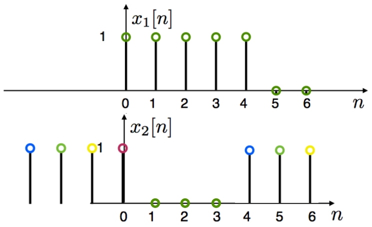
## Compute Circular Convolution Sum

y[0]=2



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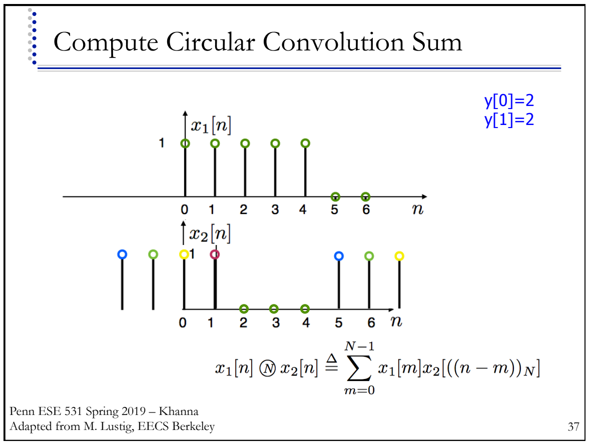
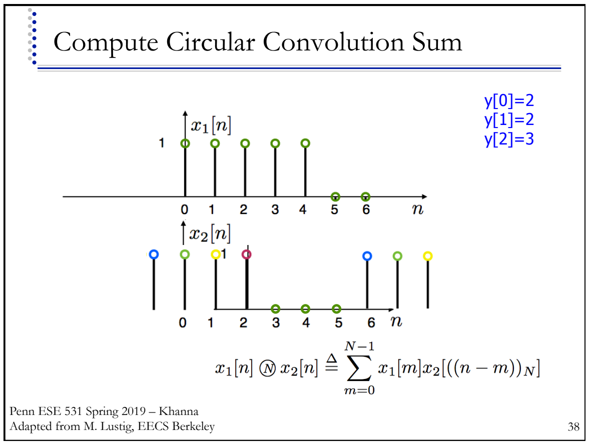


## Compute Circular Convolution Sum

y[0]=2
y[1]=2

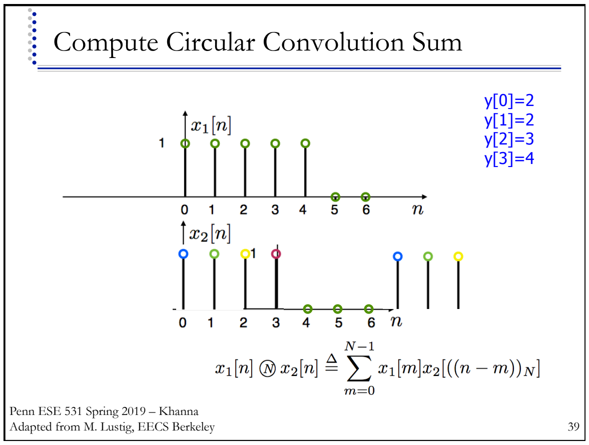
$$x_1[n] \textcircled{N} x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$



## Compute Circular Convolution Sum

y[0]=2
y[1]=2
y[2]=3

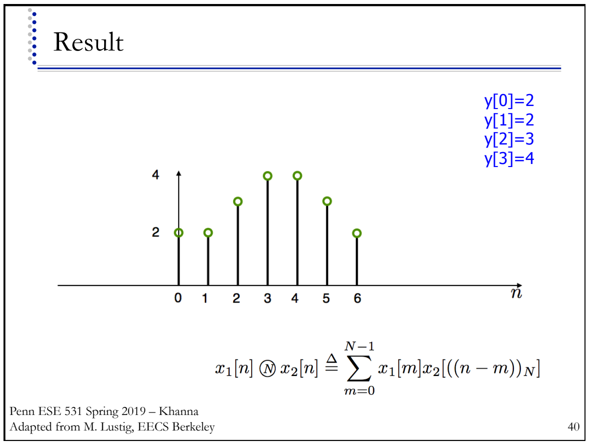
$$x_1[n] \textcircled{N} x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$



## Compute Circular Convolution Sum

y[0]=2
y[1]=2
y[2]=3
y[3]=4

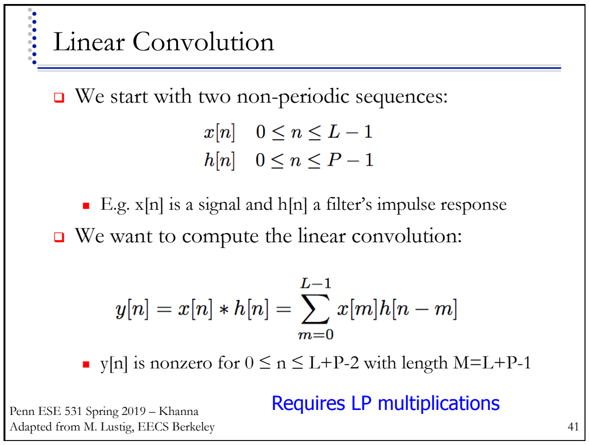
$$x_1[n] \textcircled{N} x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$



## Result

y[0]=2
y[1]=2
y[2]=3
y[3]=4

$$x_1[n] \textcircled{N} x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$



## Linear Convolution

- We start with two non-periodic sequences:

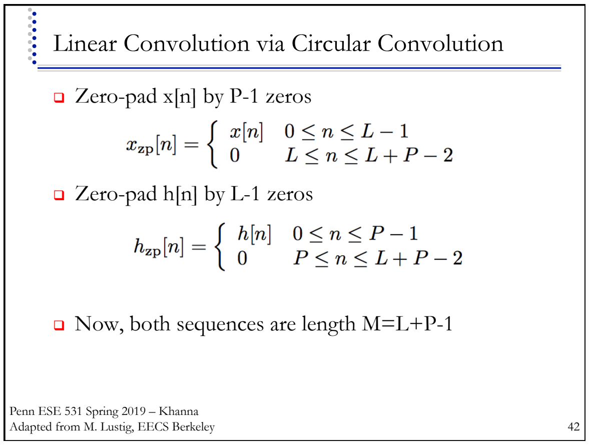
$$x[n] \quad 0 \le n \le L-1$$
$$h[n] \quad 0 \le n \le P-1$$

  - E.g. x[n] is a signal and h[n] a filter's impulse response
- We want to compute the linear convolution:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{L-1} x[m]h[n-m]$$

  - y[n] is nonzero for 0 ≤ n ≤ L+P-2 with length M=L+P-1

Requires LP multiplications



## Linear Convolution via Circular Convolution

- Zero-pad x[n] by P-1 zeros

$$x_{\text{zp}}[n] = \begin{cases} x[n] & 0 \le n \le L-1 \\ 0 & L \le n \le L+P-2 \end{cases}$$

- Zero-pad h[n] by L-1 zeros

$$h_{\text{zp}}[n] = \begin{cases} h[n] & 0 \le n \le P-1 \\ 0 & P \le n \le L+P-2 \end{cases}$$

- Now, both sequences are length M=L+P-1

### Circular Conv. via Linear Conv. w/ Aliasing

- If the DTFT X($e^{j\omega}$) of a sequence x[n] is sampled at N frequencies $\omega_k$=2$\pi$k/N, then the resulting sequence X[k] corresponds to the periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN].$$

- And $X[k] = \begin{cases} X(e^{j(2\pi k/N)}), & 0 \le k \le N-1, \\ 0, & \text{otherwise,} \end{cases}$ is the DFT of one period given as

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

### Circular Conv. via Linear Conv. w/ Aliasing

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

- If x[n] has length less than or equal to N, then $x_p$[n]=x[n]
- However if the length of x[n] is greater than N, this might not be true and we get aliasing in time
  - N-point convolution results in N-point sequence

### Circular Conv. via Linear Conv. w/ Aliasing

- Given two N-point sequences ($x_1$[n] and $x_2$[n]) and their N-point DFTs ($X_1$[k] and $X_2$[k])
- The N-point DFT of $x_3$[n]=$x_1$[n]*$x_2$[n] is defined as

$$X_3[k] = X_3(e^{j(2\pi k/N)})$$

### Circular Conv. via Linear Conv. w/ Aliasing

- Given two N-point sequences ($x_1$[n] and $x_2$[n]) and their N-point DFTs ($X_1$[k] and $X_2$[k])
- The N-point DFT of $x_3$[n]=$x_1$[n]*$x_2$[n] is defined as

$$X_3[k] = X_3(e^{j(2\pi k/N)})$$

- And $X_3$[k]=$X_1$[k]$X_2$[k], where the inverse DFT of $X_3$[k] is

$$x_{3p}[n] = \begin{cases} \displaystyle\sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise,} \end{cases}$$

### Circular Conv. as Linear Conv. w/ Aliasing

$$x_{3p}[n] = \begin{cases} \displaystyle\sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise,} \end{cases}$$

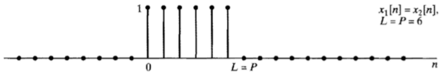
- Thus

$$x_{3p}[n] = x_1[n] \, \textcircled{N} \, x_2[n]$$

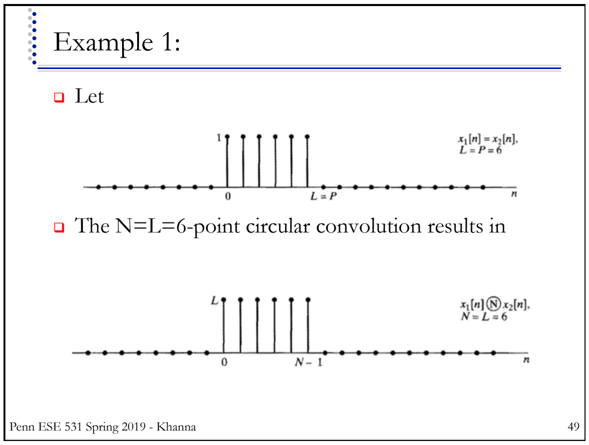
- The N-point circular convolution is the sum of linear convolutions shifted in time by N

### Example 1:

- Let



- The N=L=6-point circular convolution results in



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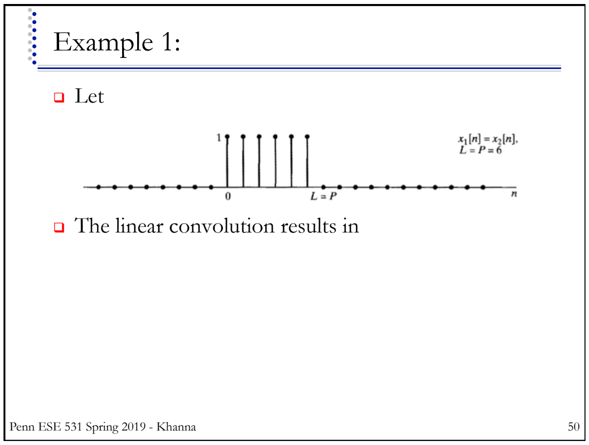

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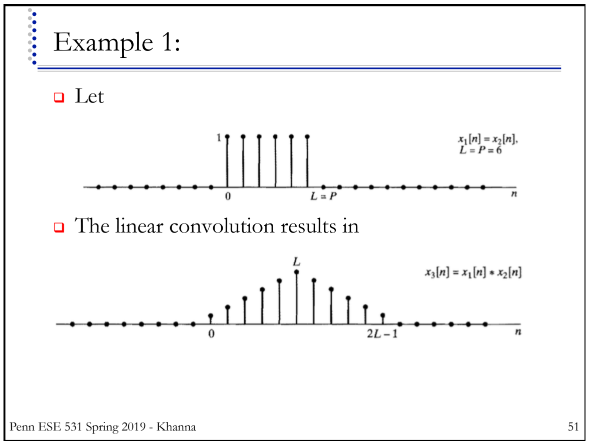

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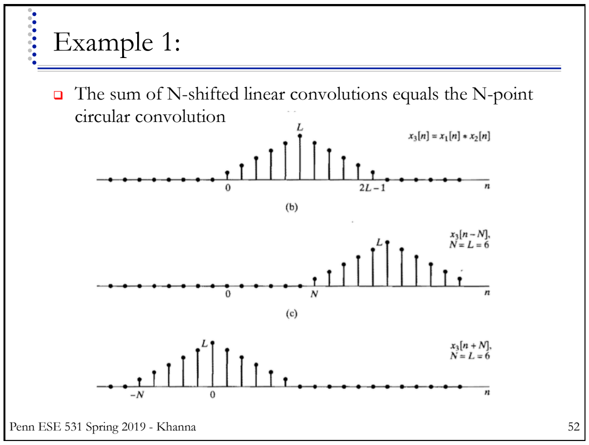
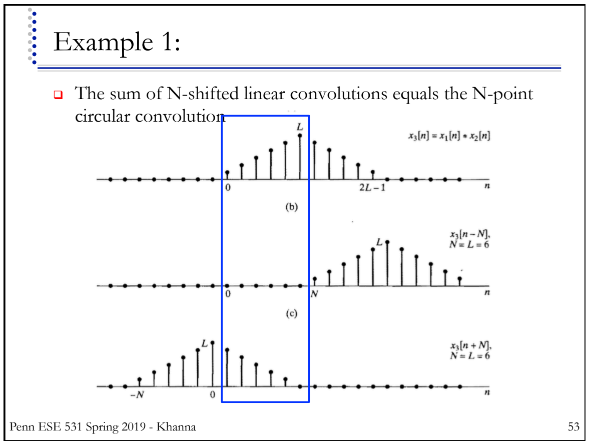


## Example 1:

- The sum of N-shifted linear convolutions equals the N-point circular convolution



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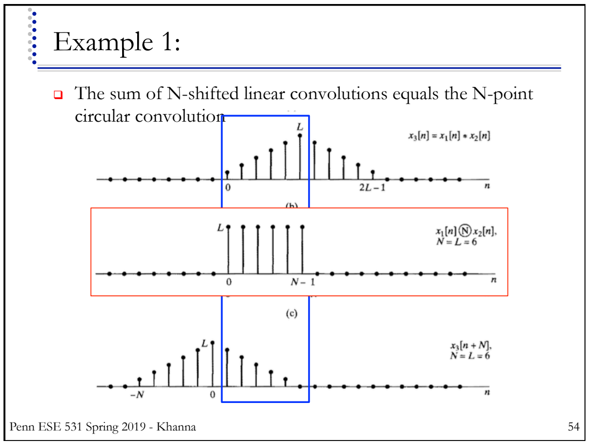
- The sum of N-shifted linear convolutions equals the N-point circular convolution



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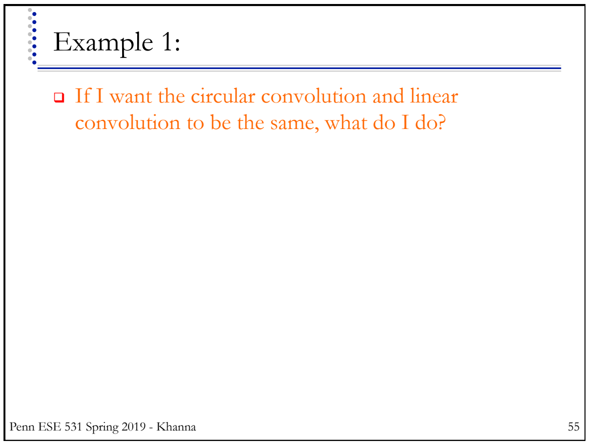
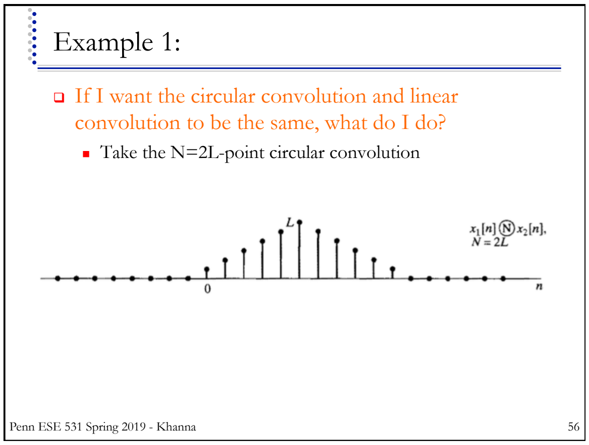
- The sum of N-shifted linear convolutions equals the N-point circular convolution



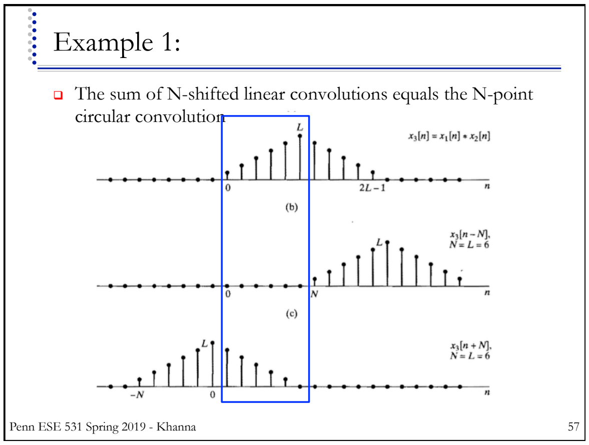
## Example 1:

- If I want the circular convolution and linear convolution to be the same, what do I do?



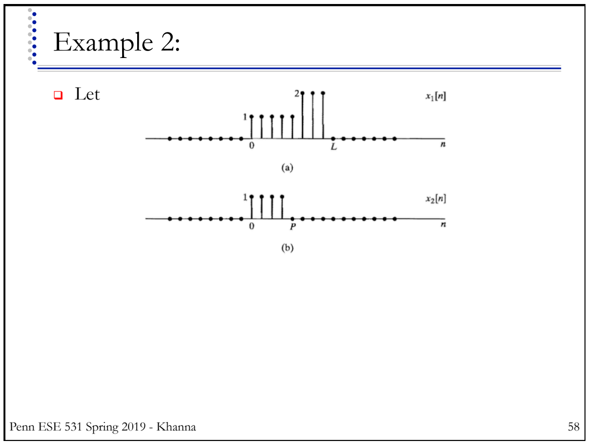
## Example 1:

- If I want the circular convolution and linear convolution to be the same, what do I do?
  - Take the N=2L-point circular convolution



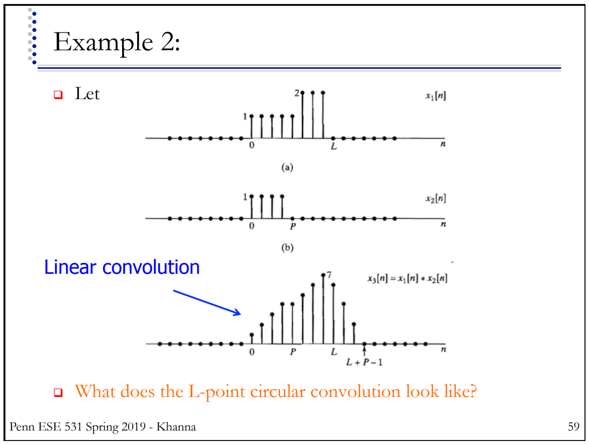
## Example 1:

- The sum of N-shifted linear convolutions equals the N-point circular convolution



## Example 2:

- Let

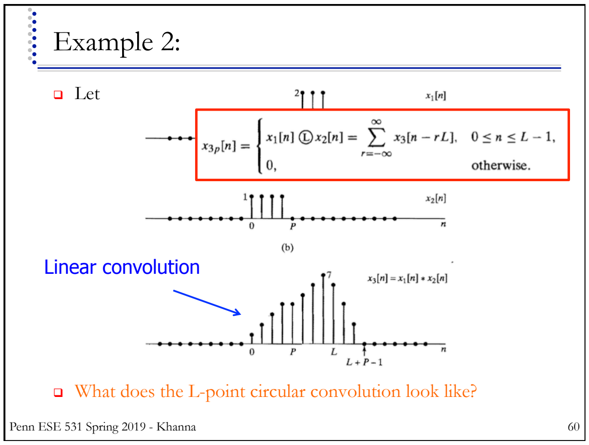


## Example 2:

- Let

Linear convolution

- What does the L-point circular convolution look like?



## Example 2:

- Let

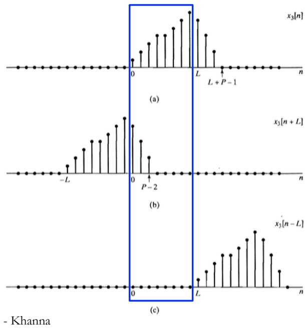
$$x_{3p}[n] = \begin{cases} x_1[n] \text{ Ⓛ } x_2[n] = \sum_{r=-\infty}^{\infty} x_3[n-rL], & 0 \le n \le L-1, \\ 0, & \text{otherwise.} \end{cases}$$

Linear convolution

- What does the L-point circular convolution look like?

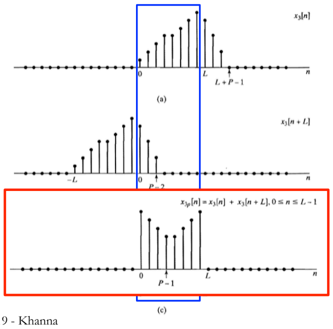
## Example 2:

- The L-shifted linear convolutions



## Example 2:

- The L-shifted linear convolutions



## Big Ideas

- Discrete Fourier Transform (DFT)
  - For finite signals assumed to be zero outside of defined length
  - N-point DFT is sampled DTFT at N points
  - Useful properties allow easier linear convolution
- Fast Fourier Transform
  - Enable computation of an N-point DFT (or DFT$^{-1}$) with the order of just N · log$_2$ N complex multiplications.
- Fast Convolution Methods
  - Use circular convolution (i.e DFT) to perform fast linear convolution
    - Overlap-Add, Overlap-Save
  - Circular convolution is linear convolution with aliasing
- Design DSP methods to minimize computations!

## Admin

- Read adaptive filter reference for next lecture

- Project: Adaptive Filtering
  - Handout posted now
  - Work in pairs
    - Use Piazza to find partners
  - Will discuss next lecture
    - Read handout linked on calendar before class!
    - Additional resources in Canvas Files
  - Due 4/30