

ESE 531: Digital Signal Processing

Lec 23: April 16, 2019
Adaptive Filters



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Lecture Outline

- Chirp Transfer Algorithm
- Circular convolution as linear convolution with aliasing
- Adaptive Filters

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Chirp Transfer Algorithm



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Chirp Transfer Algorithm

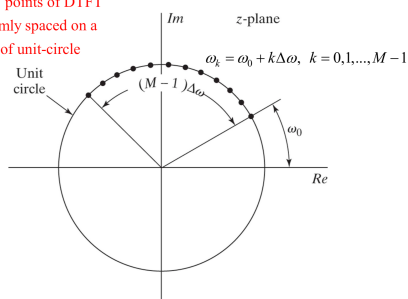
- Uses convolution to evaluate the DFT
- This algorithm is not optimal in minimizing any measure of computational complexity, but it has been useful in a variety of applications, particularly when implemented in technologies that are well suited to doing convolution with a fixed, pre-specified impulse response.
- The CTA is also more flexible than the FFT, since it can be used to compute *any* set of equally spaced samples of the Fourier transform on the unit circle.

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Chirp Transfer Algorithm

For M points of DTFT uniformly spaced on a sector of unit-circle

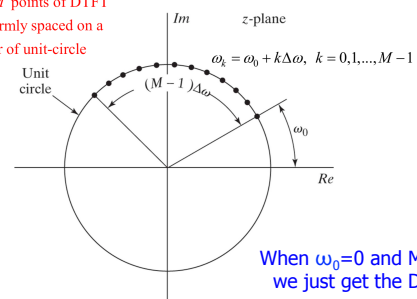


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Chirp Transfer Algorithm

For M points of DTFT uniformly spaced on a sector of unit-circle

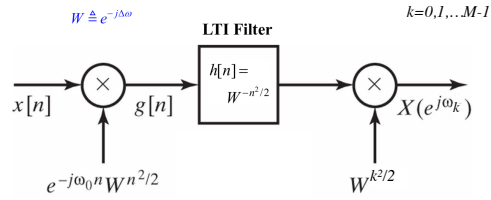


When $\omega_0=0$ and $M=N$, we just get the DFT

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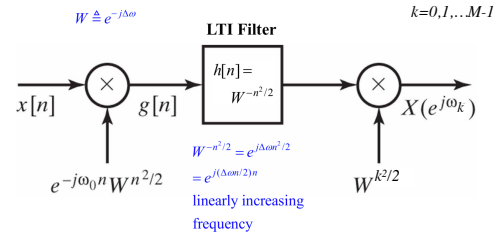
Chirp Transform Algorithm



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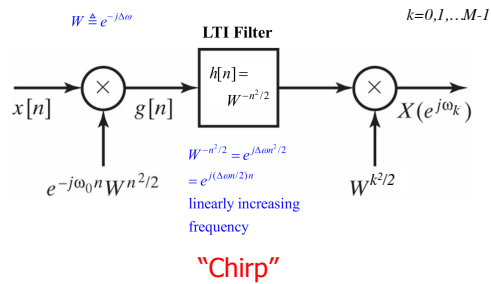
Chirp Transform Algorithm



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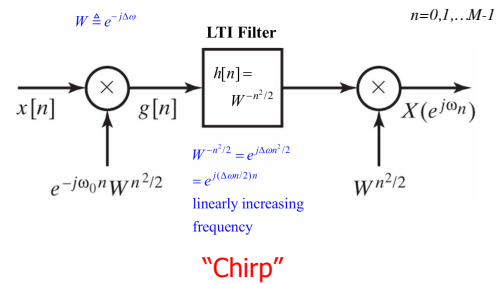
Chirp Transform Algorithm



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Chirp Transform Algorithm

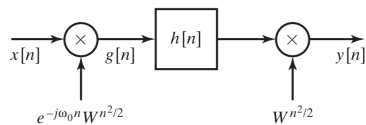


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FIR CTA

$$h[n] = \begin{cases} W^{-n^2/2}, & -(N-1) \leq n \leq M-1, \\ 0, & \text{otherwise,} \end{cases}$$



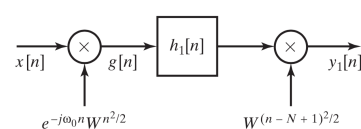
$$X(e^{j\omega_n}) = y[n], \quad n = 0, 1, \dots, M-1.$$

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Causal FIR CTA

$$h_1[n] = \begin{cases} W^{-(n-N+1)^2/2}, & n = 0, 1, \dots, M+N-2, \\ 0, & \text{otherwise.} \end{cases}$$



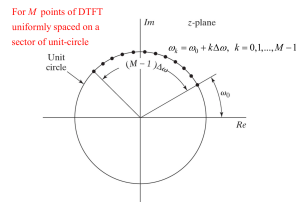
$$X(e^{j\omega_n}) = y_1[n + N - 1], \quad n = 0, 1, \dots, M-1.$$

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Example: Chirp Transform Parameters

- We have a finite-length sequence $x[n]$ that is nonzero only on the interval $n = 0, \dots, 25$, (Length $N=26$) and we wish to compute 16 samples of the DTFT $X(e^{j\omega})$ at the frequencies $\omega_k = 2\pi/27 + 2\pi k/1024$ for $k = 0, \dots, 15$.



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Circular Convolution

Linear Convolution with aliasing!



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Circular Convolution

- Circular Convolution:

$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[(n-m)_N]$$

For two signals of length N

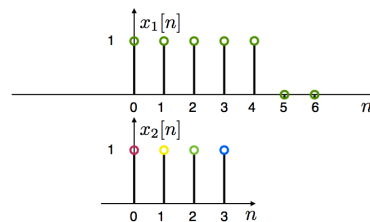
Note: Circular convolution is commutative

$$x_2[n] \circledast x_1[n] = x_1[n] \circledast x_2[n]$$

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Compute Circular Convolution Sum

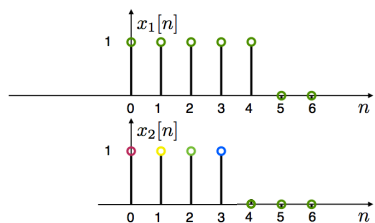


$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[(n-m)_N]$$

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Compute Circular Convolution Sum

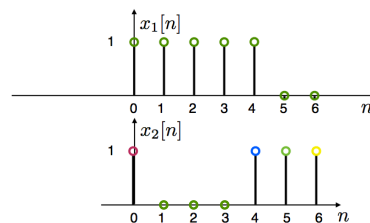


$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[(n-m)_N]$$

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Compute Circular Convolution Sum



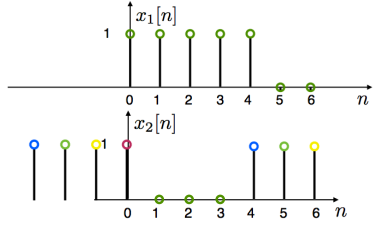
$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[(n-m)_N]$$

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Compute Circular Convolution Sum

$y[0]=2$



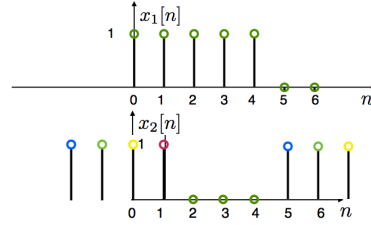
$$x_1[n] \otimes x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[(n-m)_N]$$

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Compute Circular Convolution Sum

$y[0]=2$
 $y[1]=2$



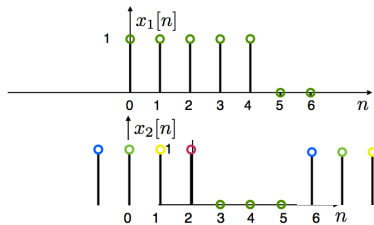
$$x_1[n] \otimes x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[(n-m)_N]$$

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Compute Circular Convolution Sum

$y[0]=2$
 $y[1]=2$
 $y[2]=3$



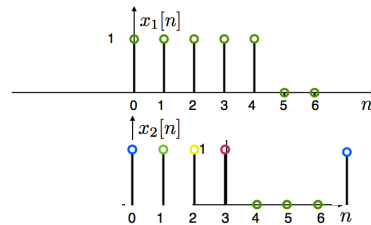
$$x_1[n] \otimes x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[(n-m)_N]$$

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Compute Circular Convolution Sum

$y[0]=2$
 $y[1]=2$
 $y[2]=3$
 $y[3]=4$



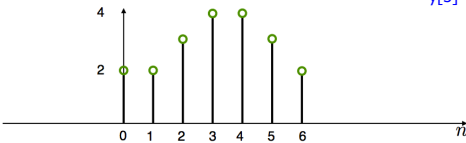
$$x_1[n] \otimes x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[(n-m)_N]$$

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Result

$y[0]=2$
 $y[1]=2$
 $y[2]=3$
 $y[3]=4$



$$x_1[n] \otimes x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[(n-m)_N]$$

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Linear Convolution

- We start with two non-periodic sequences:

$$x[n] \quad 0 \leq n \leq L-1$$

$$h[n] \quad 0 \leq n \leq P-1$$

- E.g. $x[n]$ is a signal and $h[n]$ a filter's impulse response
- We want to compute the linear convolution:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{L-1} x[m]h[n-m]$$

- $y[n]$ is nonzero for $0 \leq n \leq L+P-2$ with length $M=L+P-1$

Requires LP multiplications

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Linear Convolution via Circular Convolution

- Zero-pad $x[n]$ by $P-1$ zeros

$$x_{zp}[n] = \begin{cases} x[n] & 0 \leq n \leq L-1 \\ 0 & L \leq n \leq L+P-2 \end{cases}$$

- Zero-pad $h[n]$ by $L-1$ zeros

$$h_{zp}[n] = \begin{cases} h[n] & 0 \leq n \leq P-1 \\ 0 & P \leq n \leq L+P-2 \end{cases}$$

- Now, both sequences are length $M=L+P-1$

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Circular Conv. via Linear Conv. w/ Aliasing

- If the DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is sampled at N frequencies $\omega_k = 2\pi k/N$, then the resulting sequence $X[k]$ corresponds to the periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN].$$

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Circular Conv. via Linear Conv. w/ Aliasing

- If the DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is sampled at N frequencies $\omega_k = 2\pi k/N$, then the resulting sequence $X[k]$ corresponds to the periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN].$$

- And $x[k] = \begin{cases} X(e^{j(2\pi k/N)}), & 0 \leq k \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$ is the DFT of one period given as

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$$

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Circular Conv. via Linear Conv. w/ Aliasing

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$$

- If $x[n]$ has length less than or equal to N , then $x_p[n] = x[n]$

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Circular Conv. via Linear Conv. w/ Aliasing

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$$

- If $x[n]$ has length less than or equal to N , then $x_p[n] = x[n]$
- However if the length of $x[n]$ is greater than N , this might not be true and we get aliasing in time
 - N-point convolution results in N-point sequence

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Circular Conv. via Linear Conv. w/ Aliasing

- Given two N-point sequences ($x_1[n]$ and $x_2[n]$) and their N-point DFTs ($X_1[k]$ and $X_2[k]$)
- The N-point DFT of $x_3[n] = x_1[n] * x_2[n]$ is defined as

$$X_3[k] = X_1[k] X_2[k]$$

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Circular Conv. via Linear Conv. w/ Aliasing

- Given two N-point sequences ($x_1[n]$ and $x_2[n]$) and their N-point DFTs ($X_1[k]$ and $X_2[k]$)
- The N-point DFT of $x_3[n] = x_1[n] * x_2[n]$ is defined as

$$X_3[k] = X_3(e^{j(2\pi k/N)})$$

- And $X_3[k] = X_1[k] \cdot X_2[k]$, where the inverse DFT of $X_3[k]$ is

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \leq n \leq N-1, \\ 0, & \text{otherwise,} \end{cases}$$

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Circular Conv. as Linear Conv. w/ Aliasing

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \leq n \leq N-1, \\ 0, & \text{otherwise,} \end{cases}$$

- And

$$x_{3p}[n] = x_1[n] \otimes x_2[n]$$

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Circular Conv. as Linear Conv. w/ Aliasing

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \leq n \leq N-1, \\ 0, & \text{otherwise,} \end{cases}$$

- Thus

$$x_{3p}[n] = x_1[n] \otimes x_2[n]$$

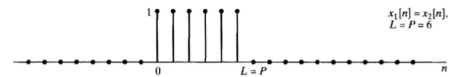
- The N-point circular convolution is the sum of linear convolutions shifted in time by N

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Example 1:

- Let



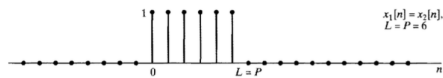
- The N=L=6-point circular convolution results in

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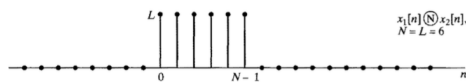
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Example 1:

- Let



- The N=L=6-point circular convolution results in

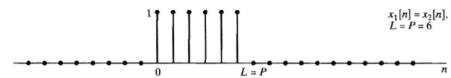


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Example 1:

- Let



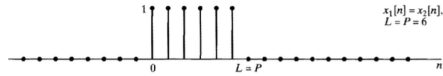
- The linear convolution results in

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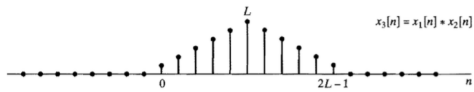
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Example 1:

- Let



- The linear convolution results in

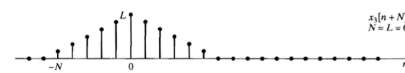
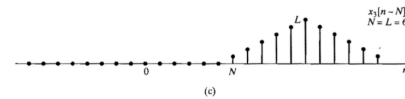
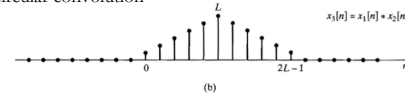


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Example 1:

- The sum of N-shifted linear convolutions equals the N-point circular convolution

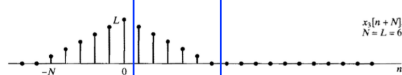
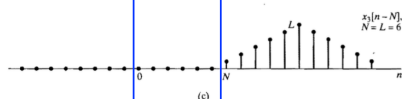
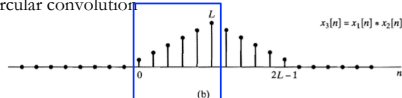


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Example 1:

- The sum of N-shifted linear convolutions equals the N-point circular convolution

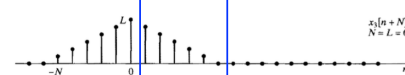
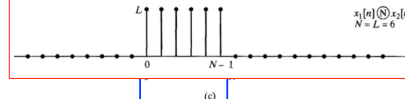
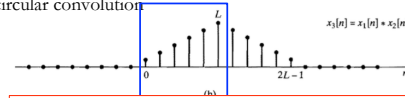


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Example 1:

- The sum of N-shifted linear convolutions equals the N-point circular convolution



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Example 1:

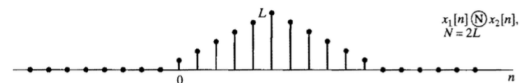
- If I want the circular convolution and linear convolution to be the same, what do I do?

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Example 1:

- If I want the circular convolution and linear convolution to be the same, what do I do?
 - Take the $N=2L$ -point circular convolution

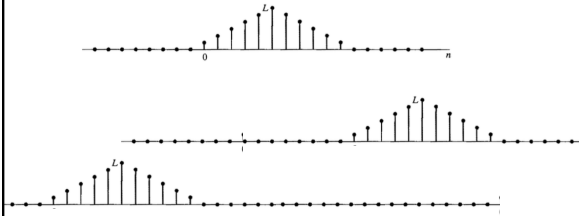


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Example 1:

- If I want the circular convolution and linear convolution to be the same, what do I do?
- Take the $N=2L$ -point circular convolution

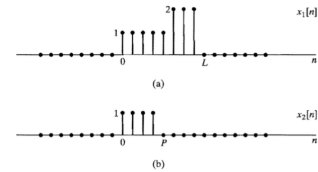


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Example 2:

- Let

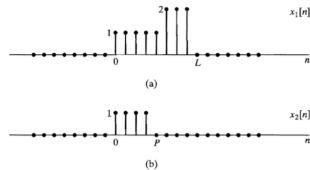


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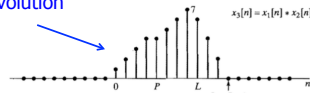
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Example 2:

- Let



Linear convolution



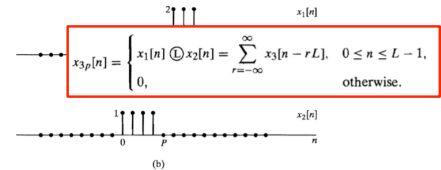
- What does the L -point circular convolution look like?

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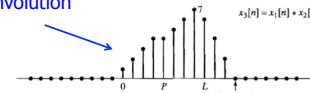
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Example 2:

- Let



Linear convolution



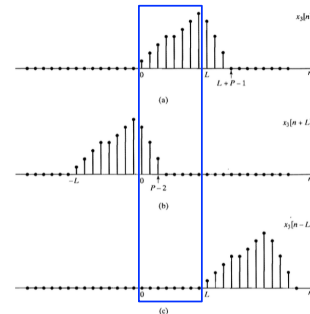
- What does the L -point circular convolution look like?

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Example 2:

- The L -shifted linear convolutions

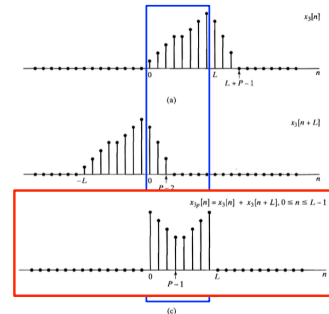


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Example 2:

- The L -shifted linear convolutions



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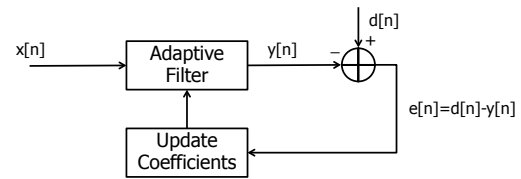
Adaptive Filters



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Adaptive Filters

- An adaptive filter is an adjustable filter that processes in time
 - It adapts...

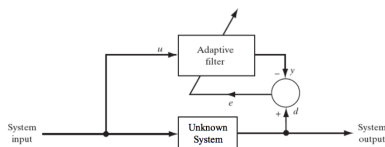


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Adaptive Filter Applications

- System Identification

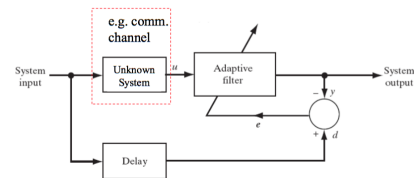


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Adaptive Filter Applications

- Identification of inverse system

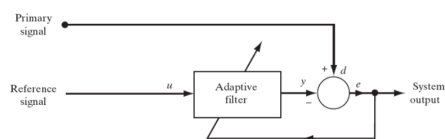


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Adaptive Filter Applications

- Adaptive Interference Cancellation

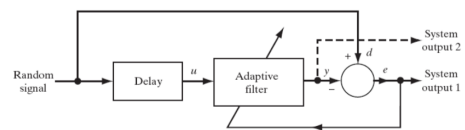


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Adaptive Filter Applications

- Adaptive Prediction



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Stochastic Gradient Approach

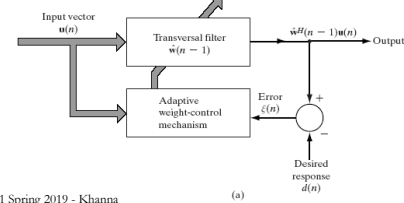
- Most commonly used type of Adaptive Filters
- Define cost function as mean-squared error
 - Difference between filter output and desired response
- Based on the method of steepest descent
 - Move towards the minimum on the error surface to get to minimum
 - Requires the gradient of the error surface to be known

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Least-Mean-Square (LMS) Algorithm

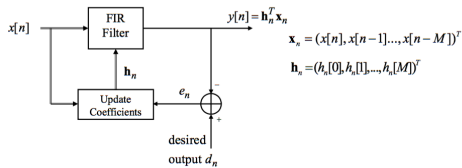
- The LMS Algorithm consists of two basic processes
 - Filtering process
 - Calculate the output of FIR filter by convolving input and taps
 - Calculate estimation error by comparing the output to desired signal
 - Adaptation process
 - Adjust tap weights based on the estimation error



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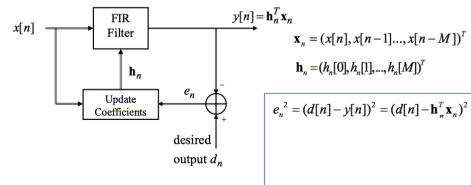
Adaptive FIR Filter: LMS



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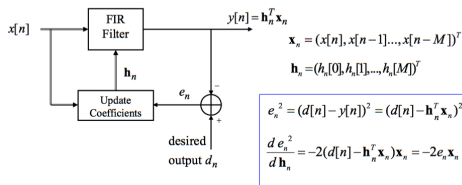
Adaptive FIR Filter: LMS



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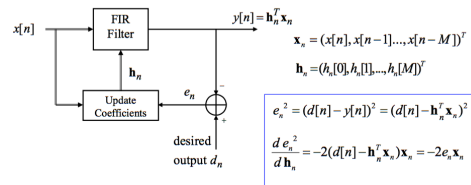
Adaptive FIR Filter: LMS



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Adaptive FIR Filter: LMS



Coefficient Update: Move in direction *opposite* to sign of gradient,
proportional to magnitude of gradient

$$\mathbf{h}_{n+1} = \mathbf{h}_n + 2\mu e_n \mathbf{x}_n$$

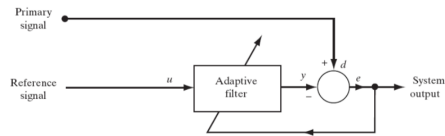
Stochastic Gradient Algorithm

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Adaptive Filter Applications

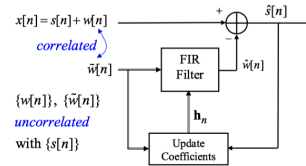
Adaptive Interference Cancellation



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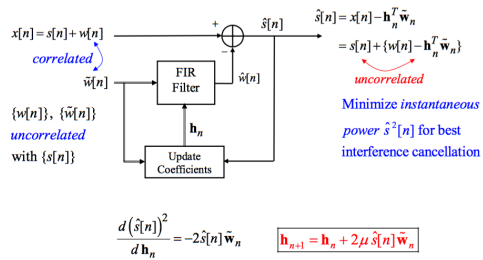
Adaptive Interference Cancellation



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Adaptive Interference Cancellation



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Stability of LMS

- The LMS algorithm is convergent in the mean square if and only if the step-size parameter satisfy

$$0 < \mu < \frac{2}{\lambda_{\max}}$$

- Here $\lambda_{\max}$ is the largest eigenvalue of the correlation matrix of the input data
- More practical test for stability is

$$0 < \mu < \frac{2}{\text{input signal power}}$$

- Larger values for step size
 - Increases adaptation rate (faster adaptation)
 - Increases residual mean-squared error

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Big Ideas

- Chirp Transform Algorithm
 - More flexible than the FFT, since it can be used to compute *any* set of equally spaced samples of the Fourier transform on the unit circle.
- Linear vs. Circular Convolution
 - Use circular convolution (i.e. DFT) to perform fast linear convolution
 - Overlap-Add, Overlap-Save
 - Circular convolution is linear convolution with aliasing
- Adaptive Filters
 - Use LMS algorithm to update filter coefficients
 - applications like system ID, channel equalization, and signal prediction

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- Project
 - Due 4/30

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