ESE 531: Digital Signal Processing

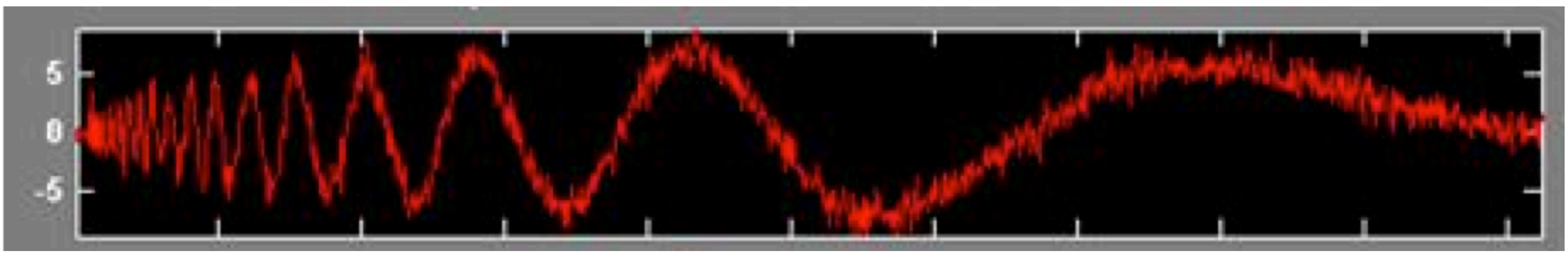
Lec 25: April 23, 2019

Wavelet Transform

Wavelet Transform

Motivation

- Some signals obviously have spectral characteristics that vary with time

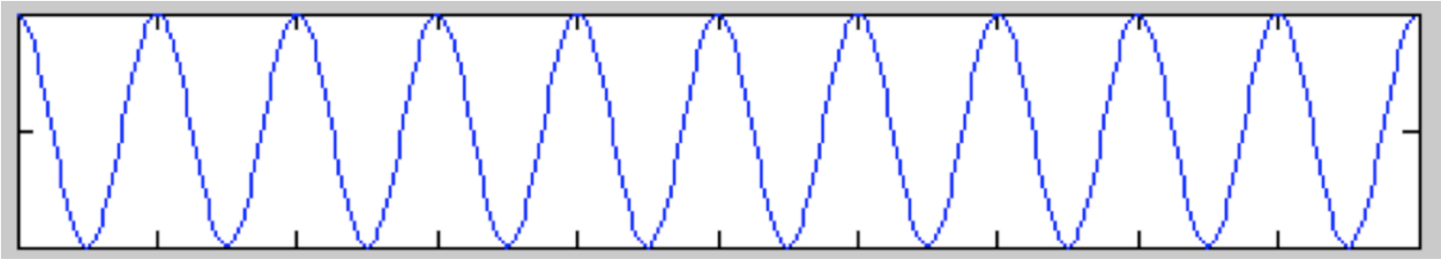


Criticism of Fourier Spectrum

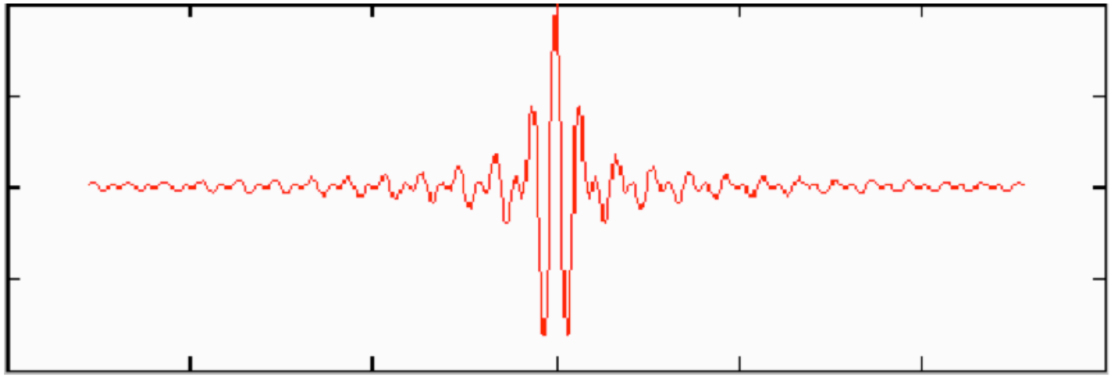
- It's giving you the spectrum of the 'whole time-series'
- Which is OK if the time-series is stationary. But what if its not?
- We need a technique that can "march along" a time series and that is capable of:
  - Analyzing spectral content in different places
  - Detecting sharp changes in spectral character

Fourier vs. Wavelet

- Fourier Analysis is based on an indefinitely long cosine wave of a specific frequency



- Wavelet Analysis is based on an short duration wavelet of a specific center frequency

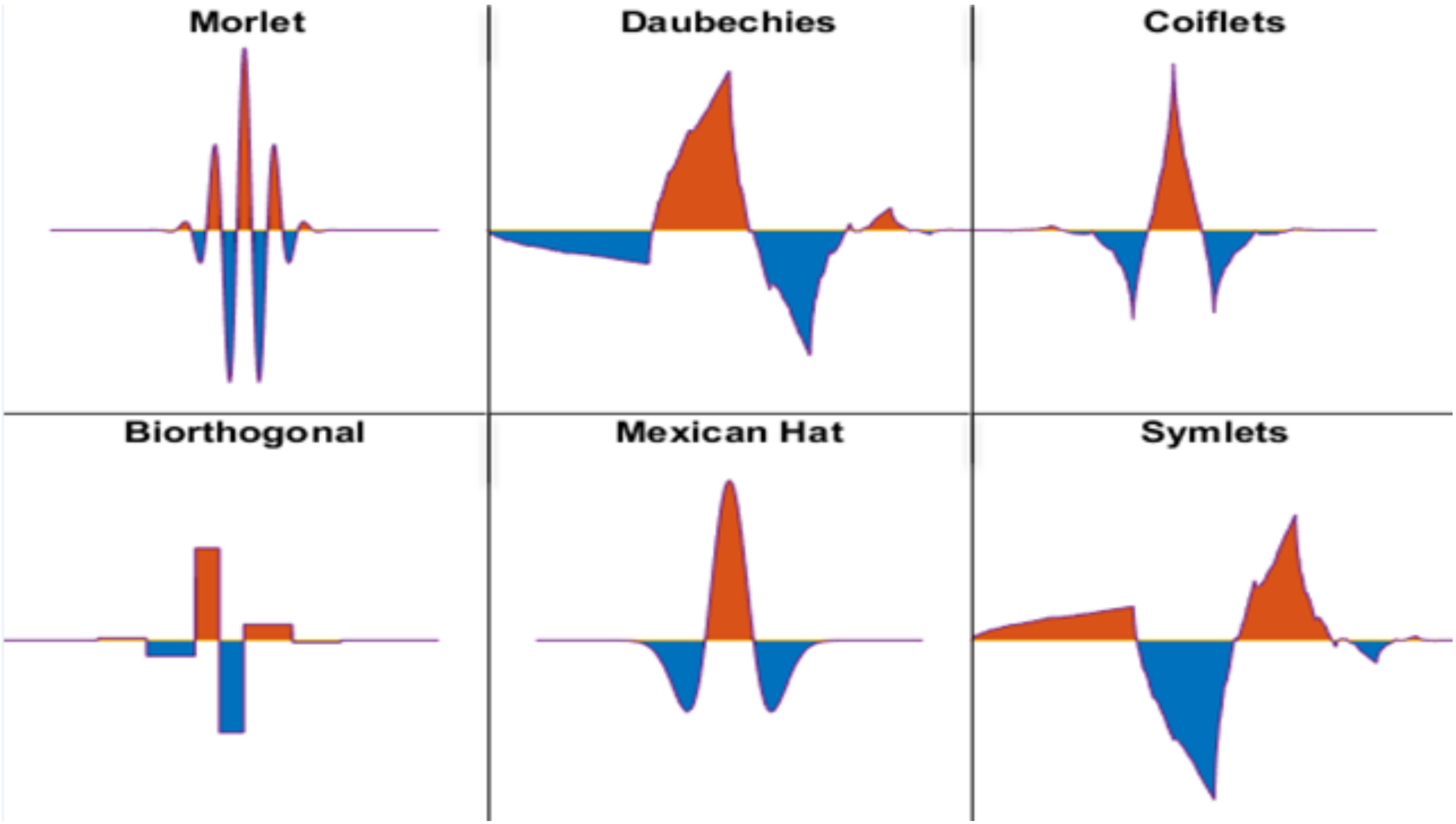


Wavelet Transform

- All wavelet derived from *mother* wavelet

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

Examples of Wavelets



Wavelet – Scaled and Shifted

normalization

shift in time

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\psi\!\left(\frac{t-\tau}{s}\right)$$

change in scale:
big s means long
wavelength

wavelet with
scale, s and time, τ

Mother wavelet

Continuous Wavelet Transform

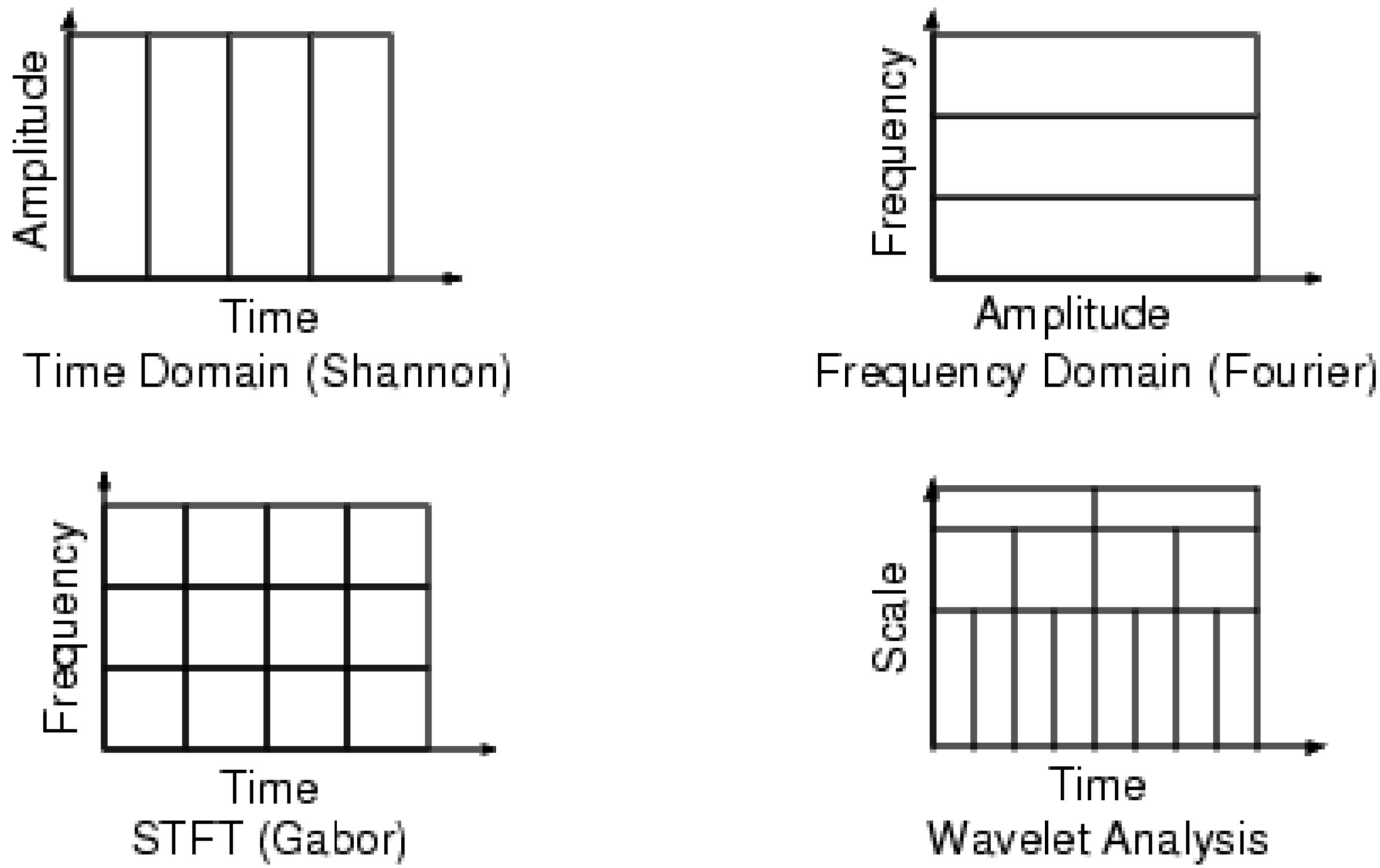
time-series

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

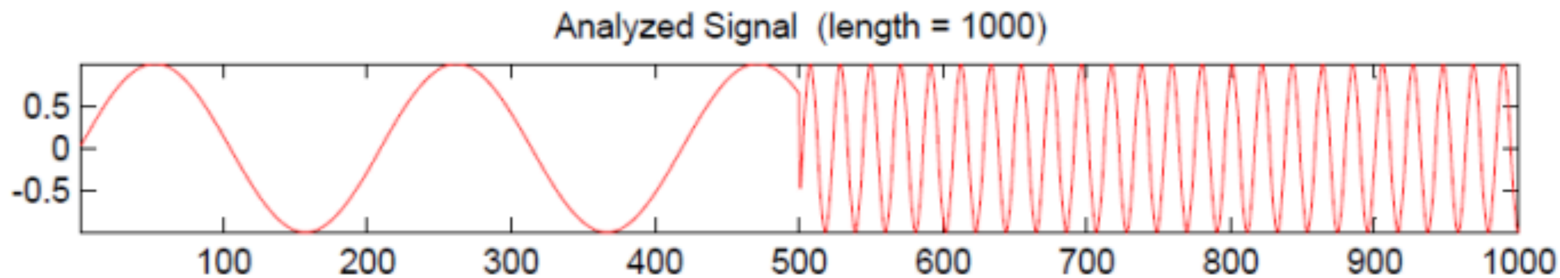
coefficient of wavelet with scale, s and time, τ

wavelet with scale, s, and shift, τ

Transform Comparison

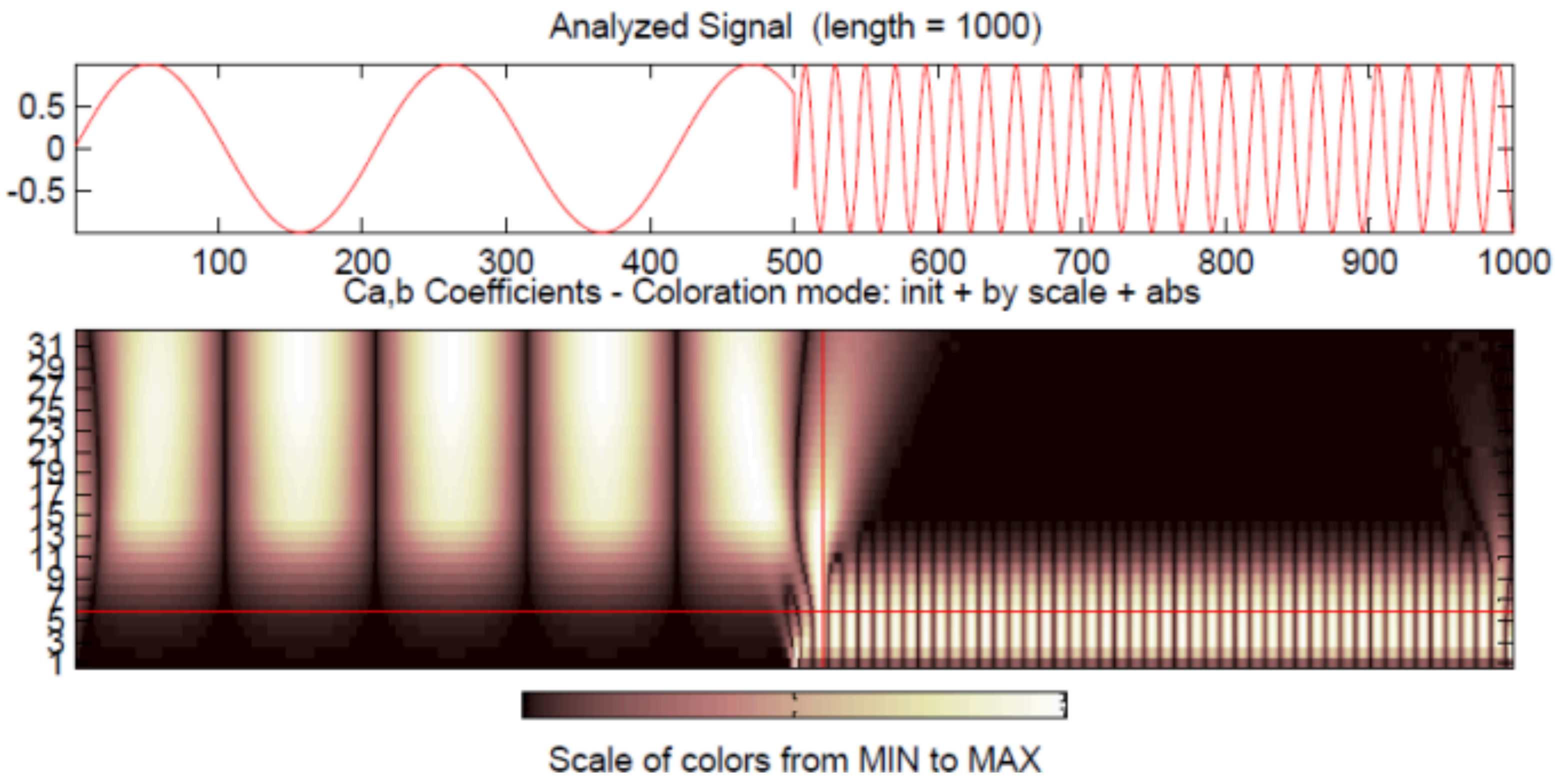


Wave Demo



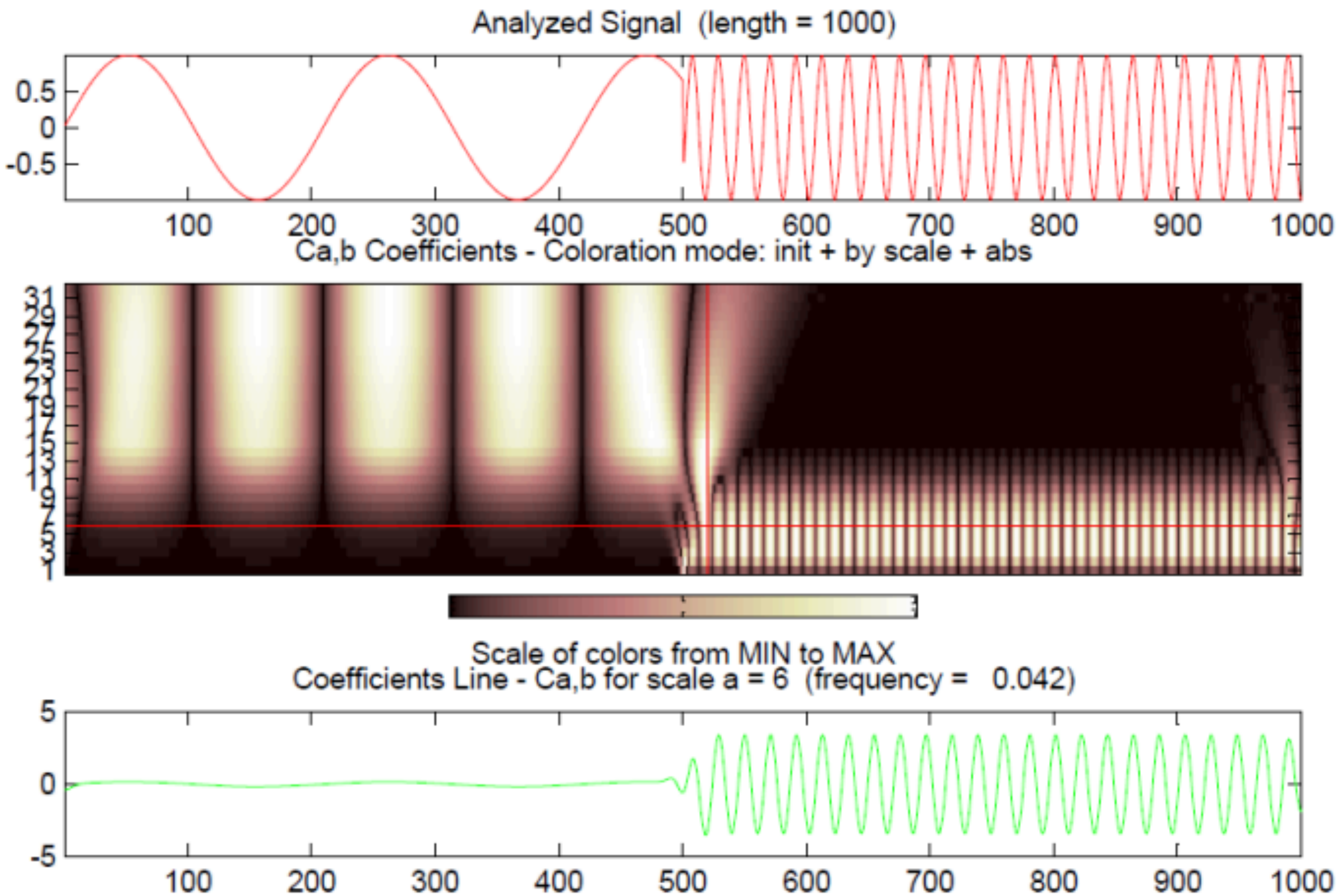
Analyzed Signal (length = 1000)
0.5
0
-0.5
100
200
300
400
500
600
700
800
900
1000

Wave Demo

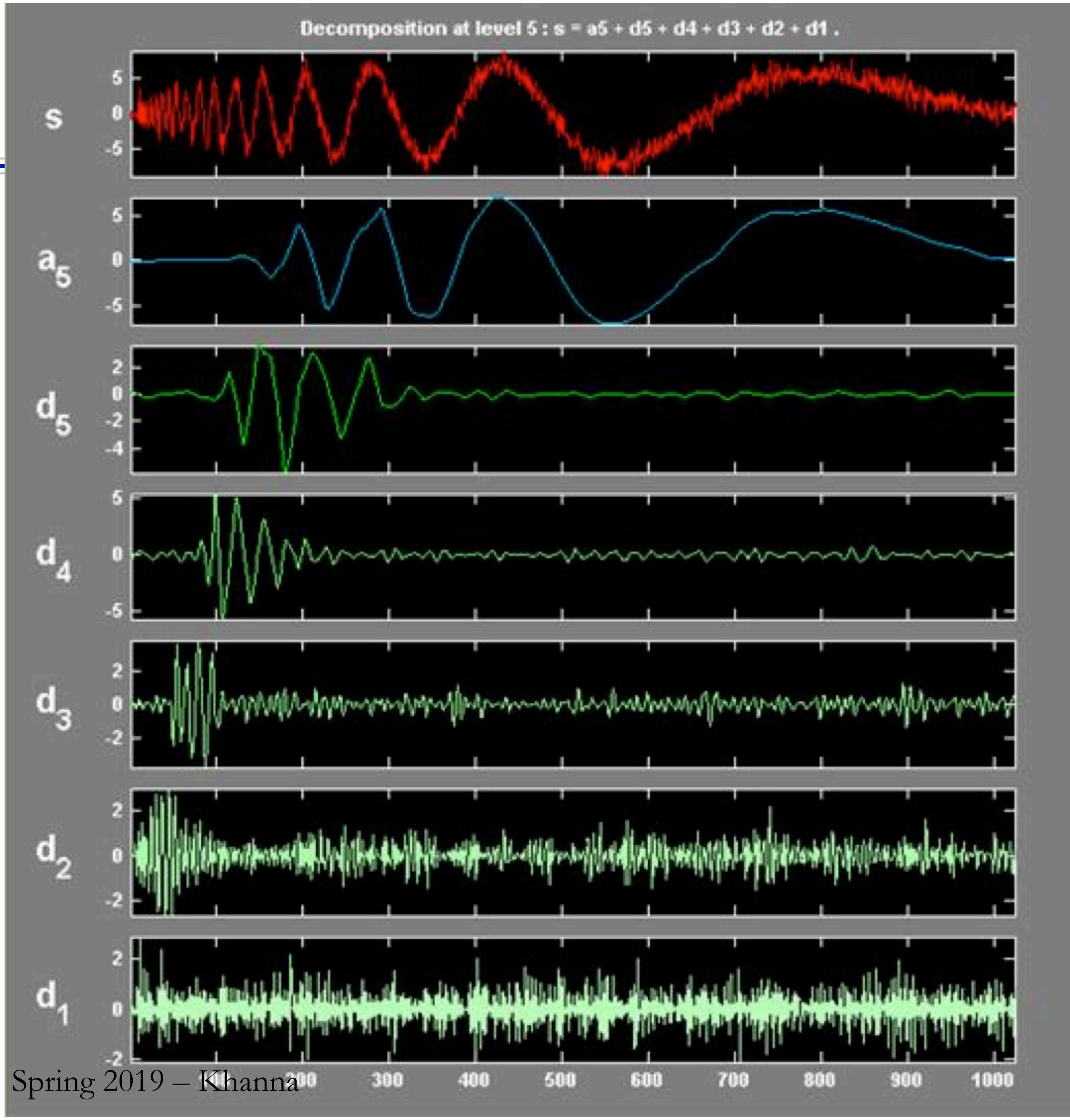


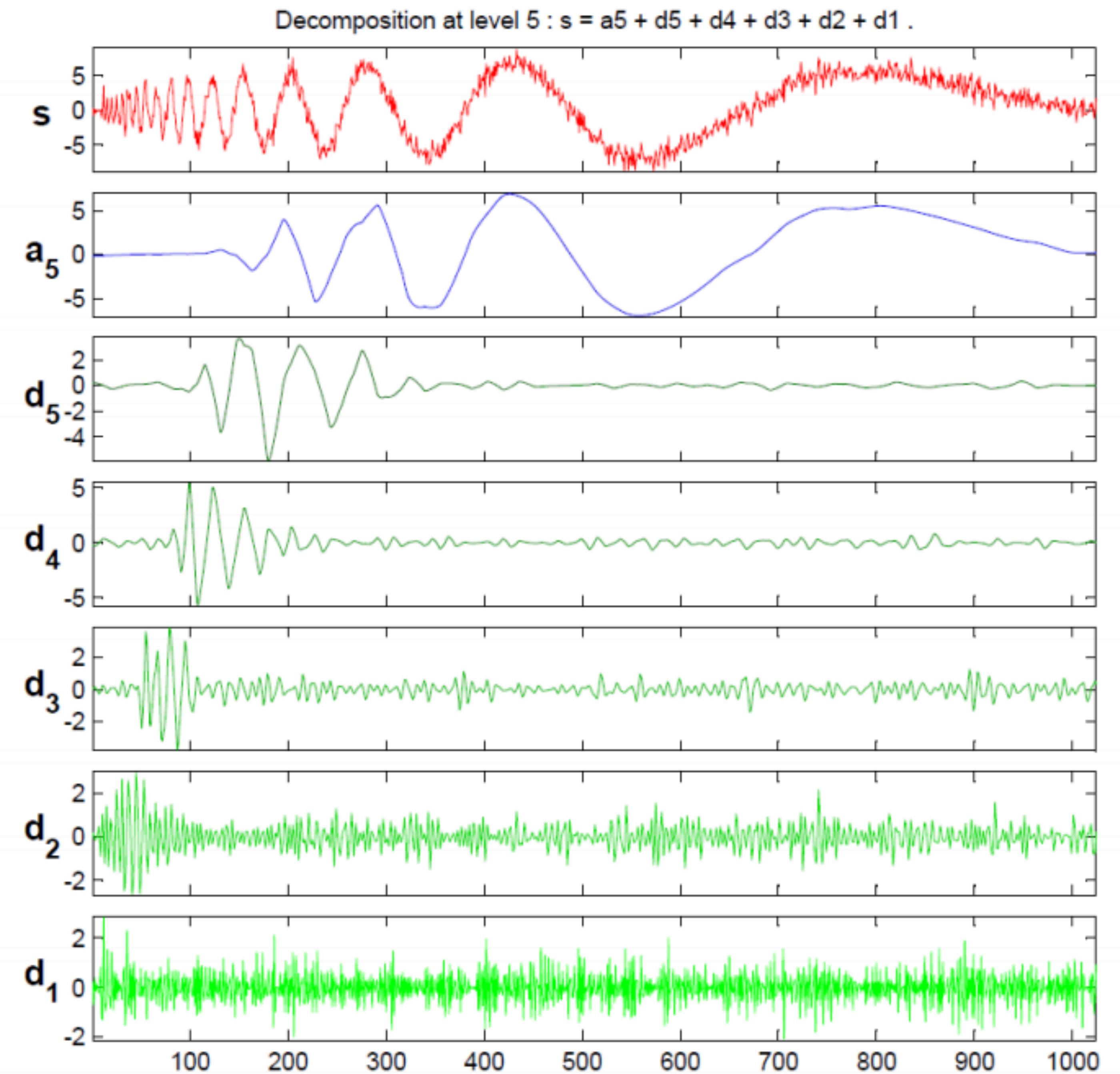
Analyzed Signal (length = 1000)
0.5
0
-0.5
100
200
300
400
500
600
700
800
900
1000
Ca,b Coefficients - Coloration mode: init + by scale + abs
31
29
27
25
23
21
19
17
15
13
11
9
7
5
3
1
Scale of colors from MIN to MAX

Wave Demo



Analyzed Signal (length = 1000)
0.5
0
-0.5
100
200
300
400
500
600
700
800
900
1000
Ca,b Coefficients - Coloration mode: init + by scale + abs
Scale of colors from MIN to MAX
Coefficients Line - Ca,b for scale a = 6 (frequency = 0.042)
5
0
-5


Decomposition at level 5 : s = a5 + d5 + d4 + d3 + d2 + d1 .
s
a5
d5
d4
d3
d2
d1
300
400
500
600
700
800
900
1000


Decomposition at level 5 : s = a5 + d5 + d4 + d3 + d2 + d1 .
s
a5
d5
d4
d3
d2
d1
100
200
300
400
500
600
700
800
900
1000

Inverse Wavelet Transform

- Build up a time-series as sum of wavelets of different scales, s, and positions, t

$$f(t) = \int\int \gamma(s,\tau)\psi_{s,\tau}(t)\,d\tau\, ds$$

time-series

wavelet with
scale, s and time, τ

coefficients
of wavelets

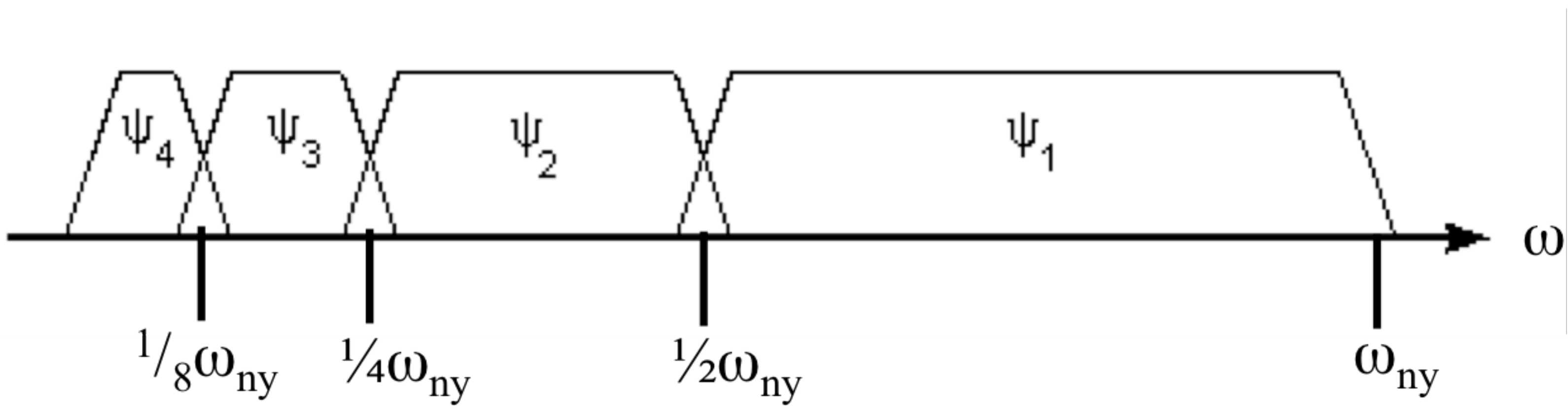
Discrete wavelets:

- Scale wavelets only by powers of 2
  - $s_j = 2^j$
- And shifting by multiples of s_j for each successive scale
  - $\tau_{j,k} = 2^j k$
- Then $\gamma(s_j, \tau_{j,k}) = \gamma_{jk}$
  - where $j = 1, 2, \ldots\infty$, $k = -\infty\ldots -2, -1, 0, 1, 2, \ldots\infty$

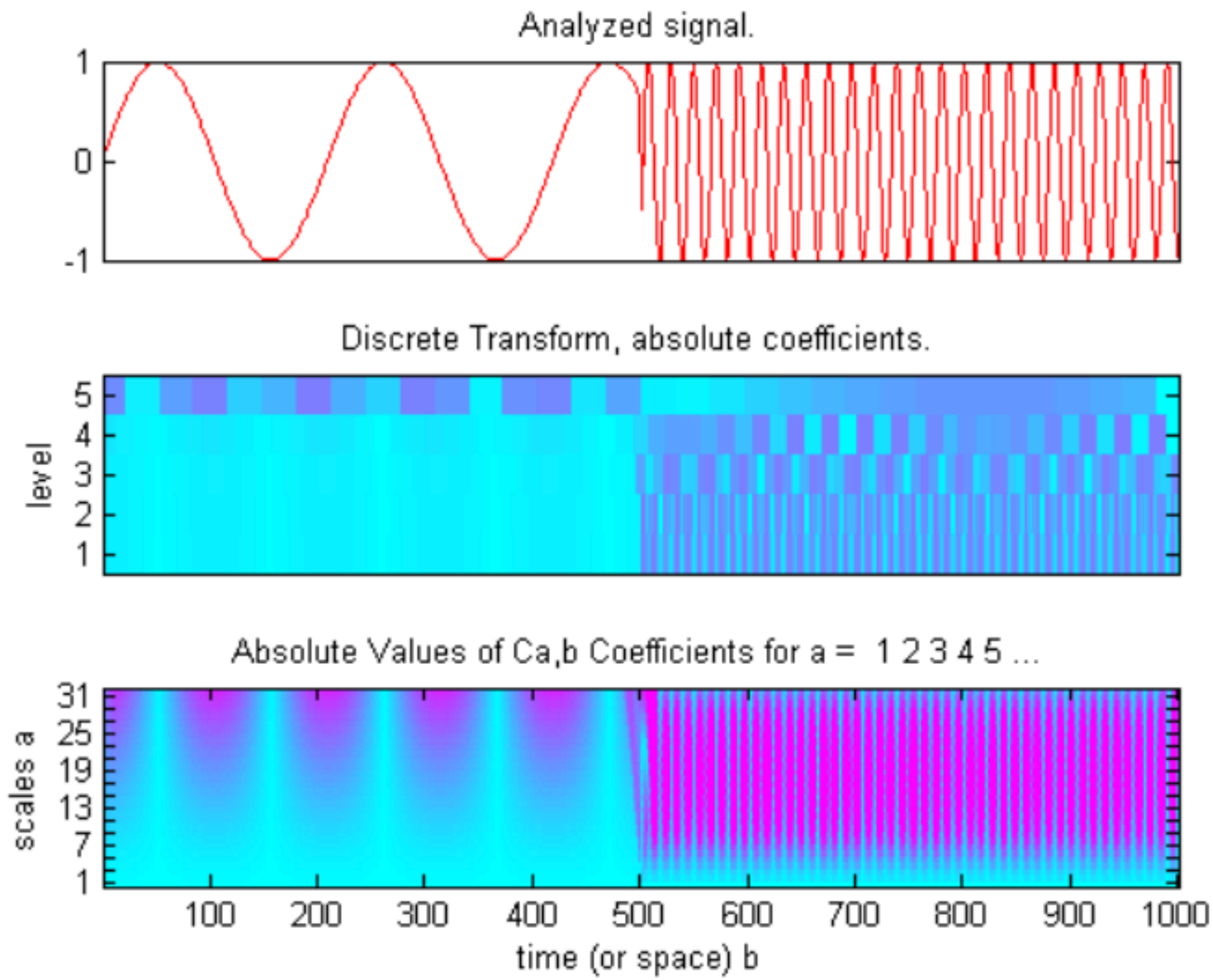
$$\gamma_{j,k} = \frac{1}{\sqrt{2^j}} \int f(t)\Psi\left(\frac{t - k2^j}{2^j}\right) dt$$

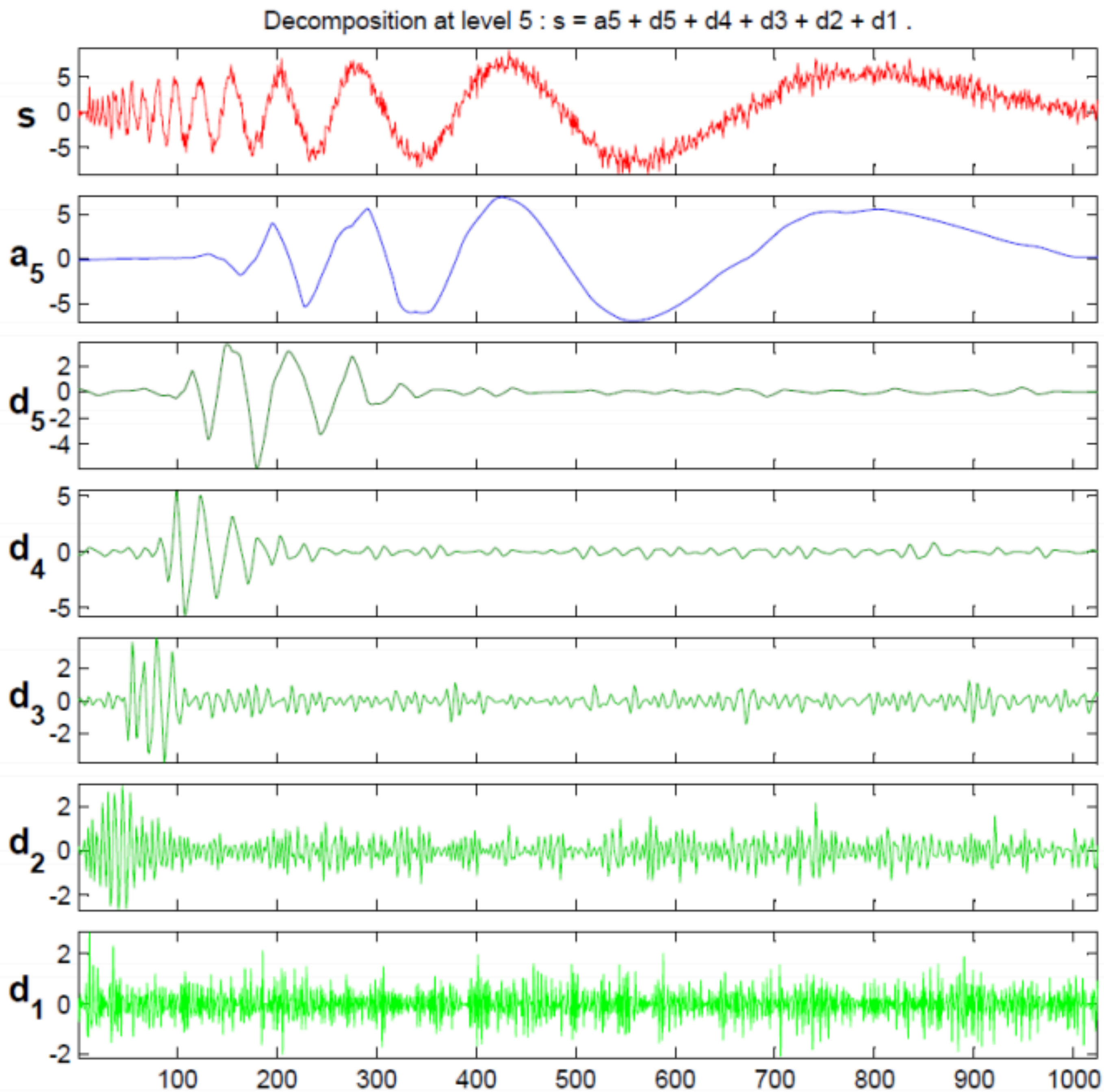
Discrete Wavelet Transform

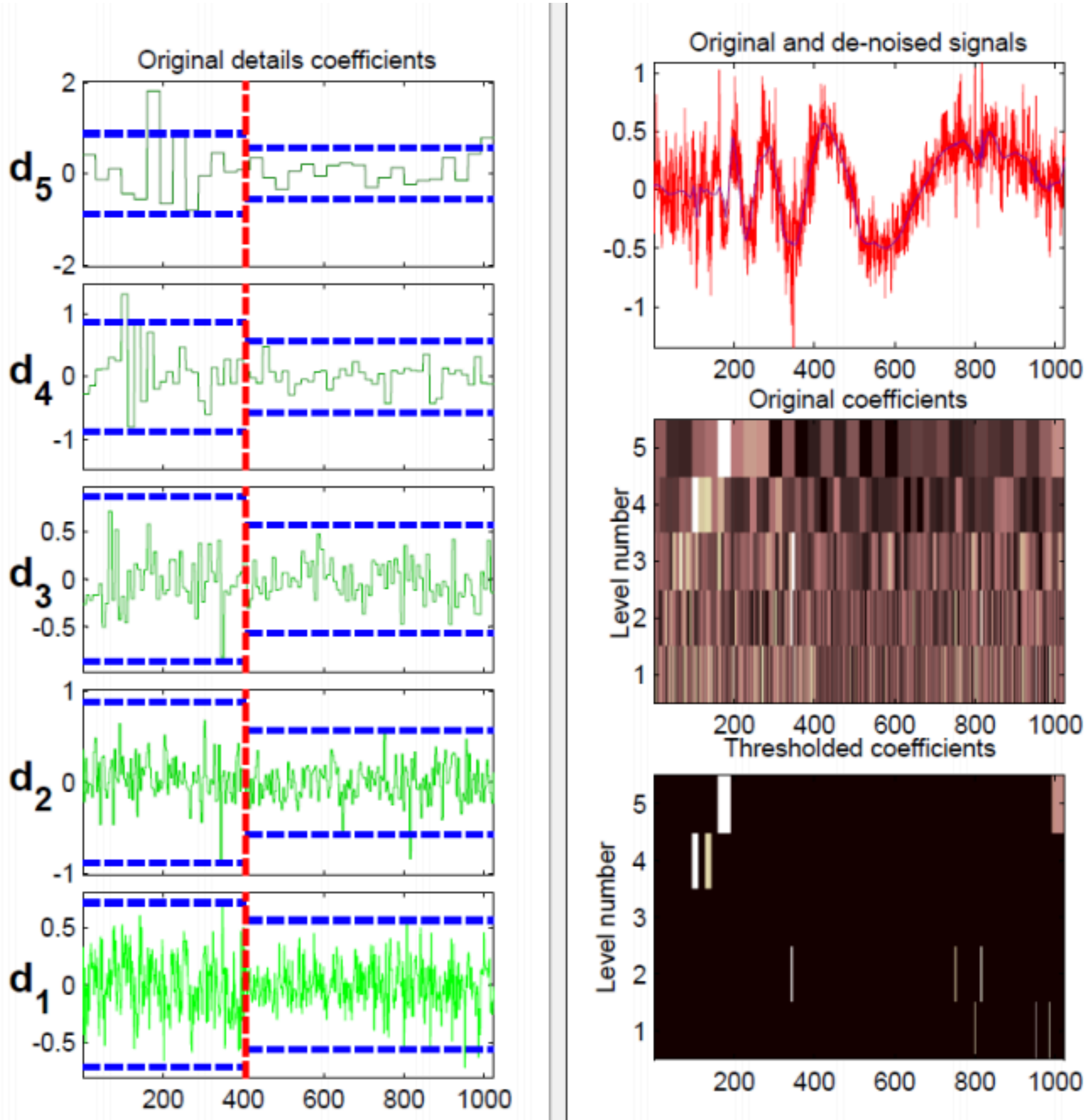
- The factor of two scaling means that the spectra of the wavelets divide up the frequency scale into octaves (frequency doubling intervals)



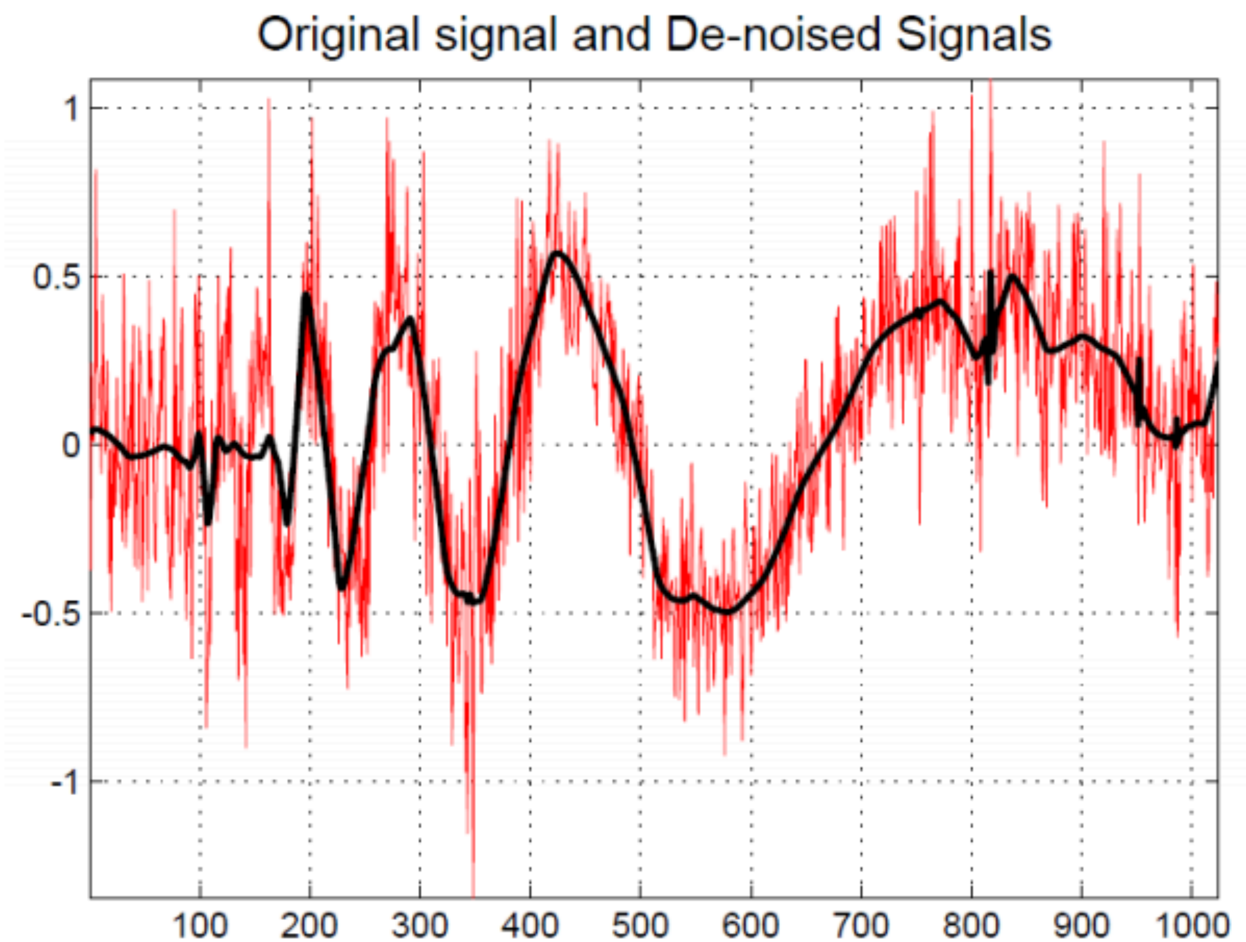
DWT vs CWT


Analyzed signal.
1
0
-1
Discrete Transform, absolute coefficients.
level
5
4
3
2
1
Absolute Values of Ca,b Coefficients for a = 1 2 3 4 5 ...
scales a
31
25
19
13
7
1
100
200
300
400
500
600
700
800
900
1000
time (or space) b


Decomposition at level 5 : s = a5 + d5 + d4 + d3 + d2 + d1 .
s
a5
d5
d4
d3
d2
d1
100 200 300 400 500 600 700 800 900 1000


Original details coefficients
d5
d4
d3
d2
d1
Original and de-noised signals
Original coefficients
Level number
Thresholded coefficients
Level number

De-noised Signal



Wavelet Transform

- Determining the wavelet coefficients for a fixed scale, *s*, can be thought of as a filtering operation

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

$$\gamma_s(t) = \int f(t)\Psi_s(t)\,dt = f(t) * \Psi_s(t)$$

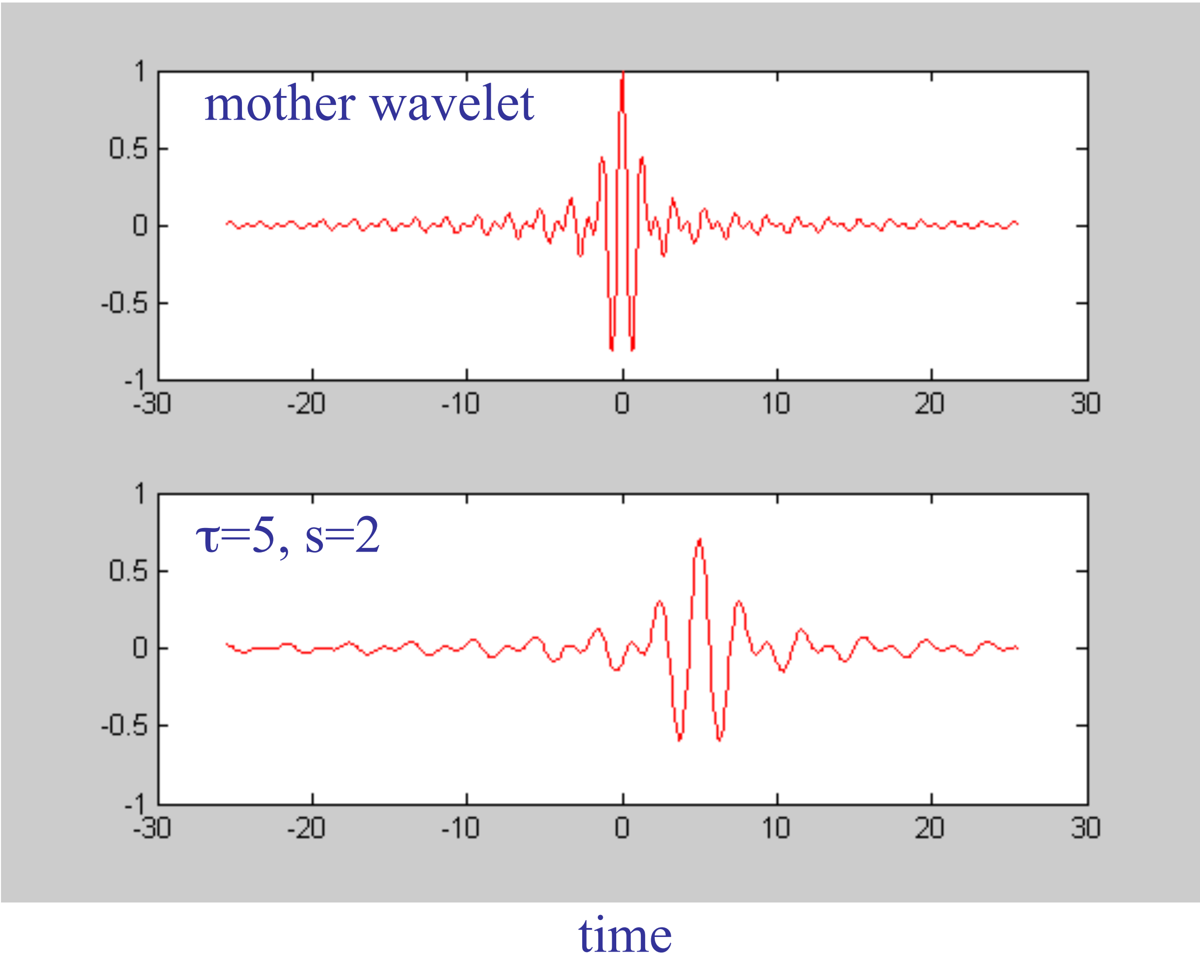
- where

$$\Psi_s(t) = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

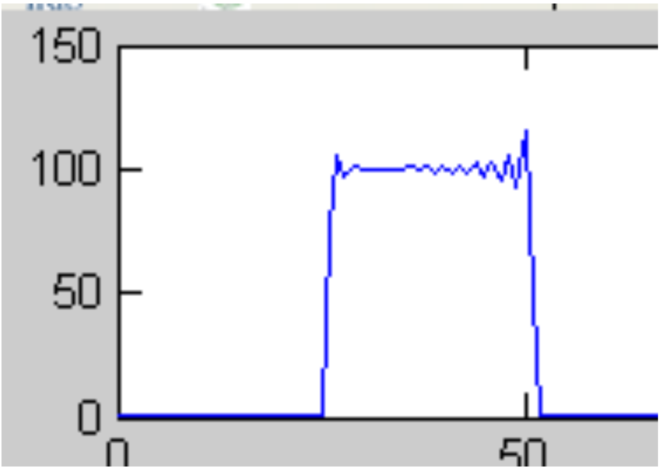
Shannon Wavelet

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}} \Psi\left(\frac{t-\tau}{s}\right)$$

- $\Psi(t) = 2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)$



Fourier spectrum of Shannon Wavelet



frequency, ω

- Wavelet coefficients are a result of bandpass filtering

Discrete Wavelet Transform

- The coefficients of Ψ is just the band-pass filtered time-series, where Ψ is the wavelet, now viewed as the impulse response of a bandpass filter.

Discrete Wavelet Transform

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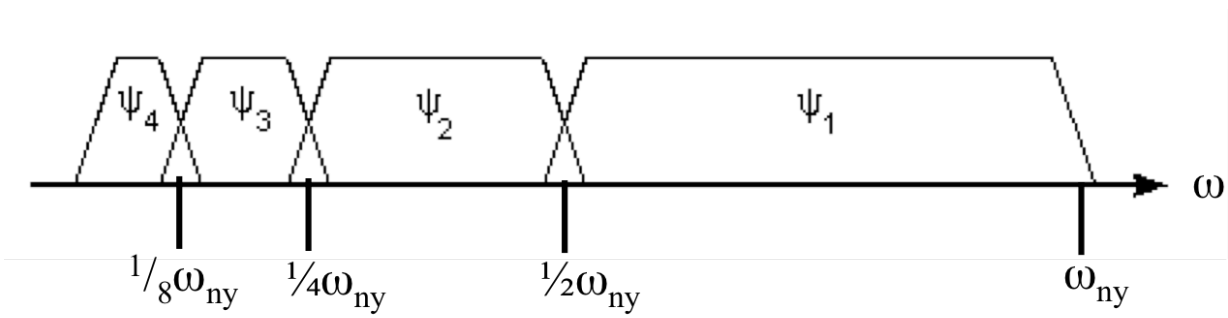
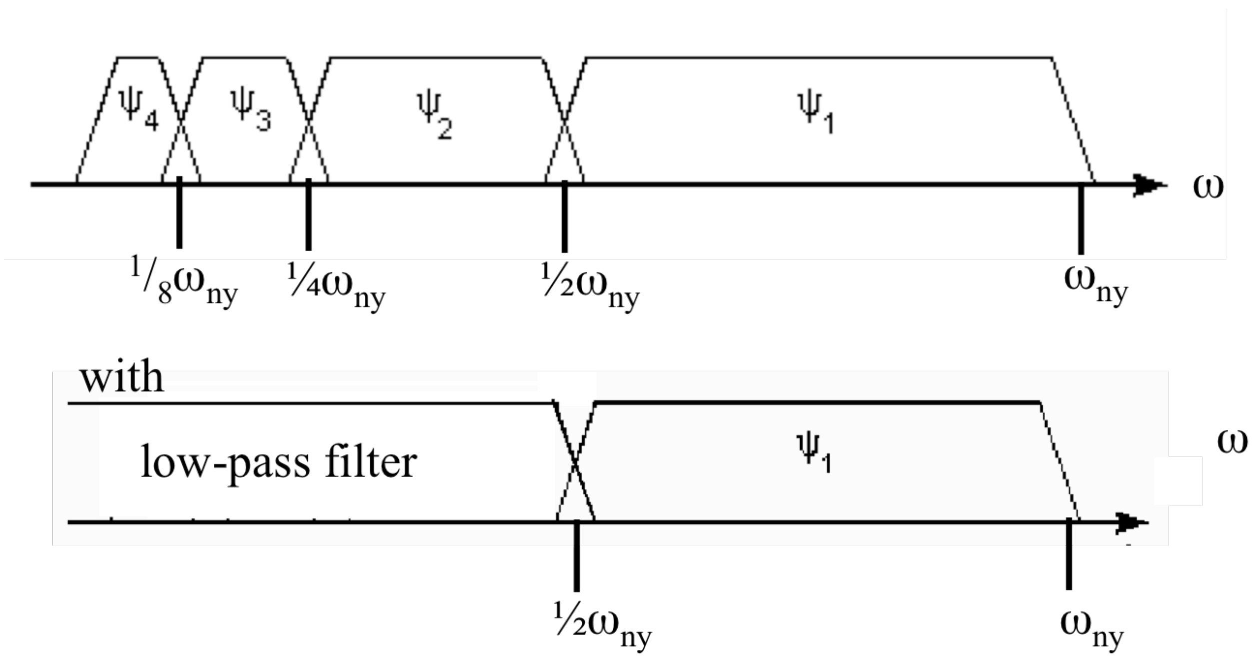


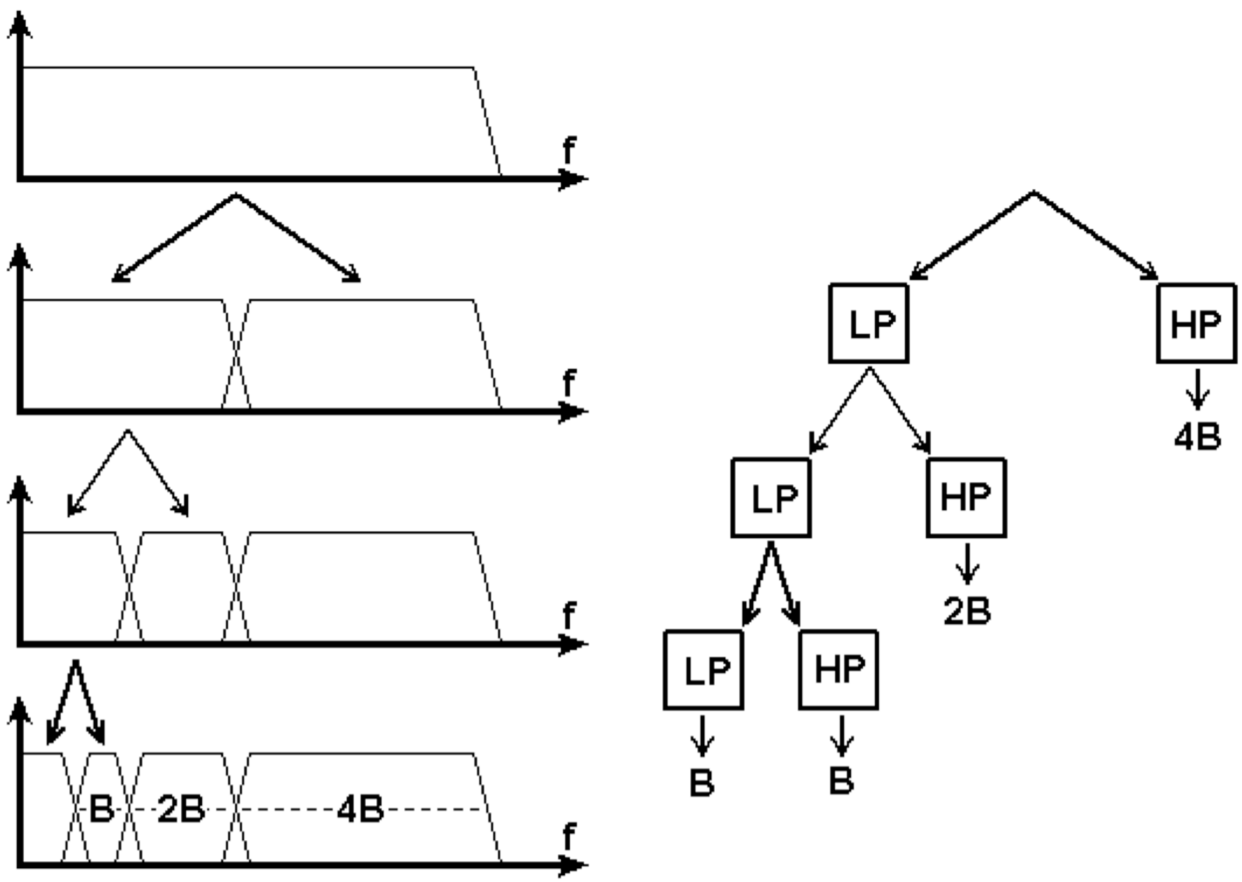
Discrete Wavelet Transform

- The coefficients of Ψ is just the band-pass filtered time-series, where Ψ is the wavelet, now viewed as the impulse response of a bandpass filter.

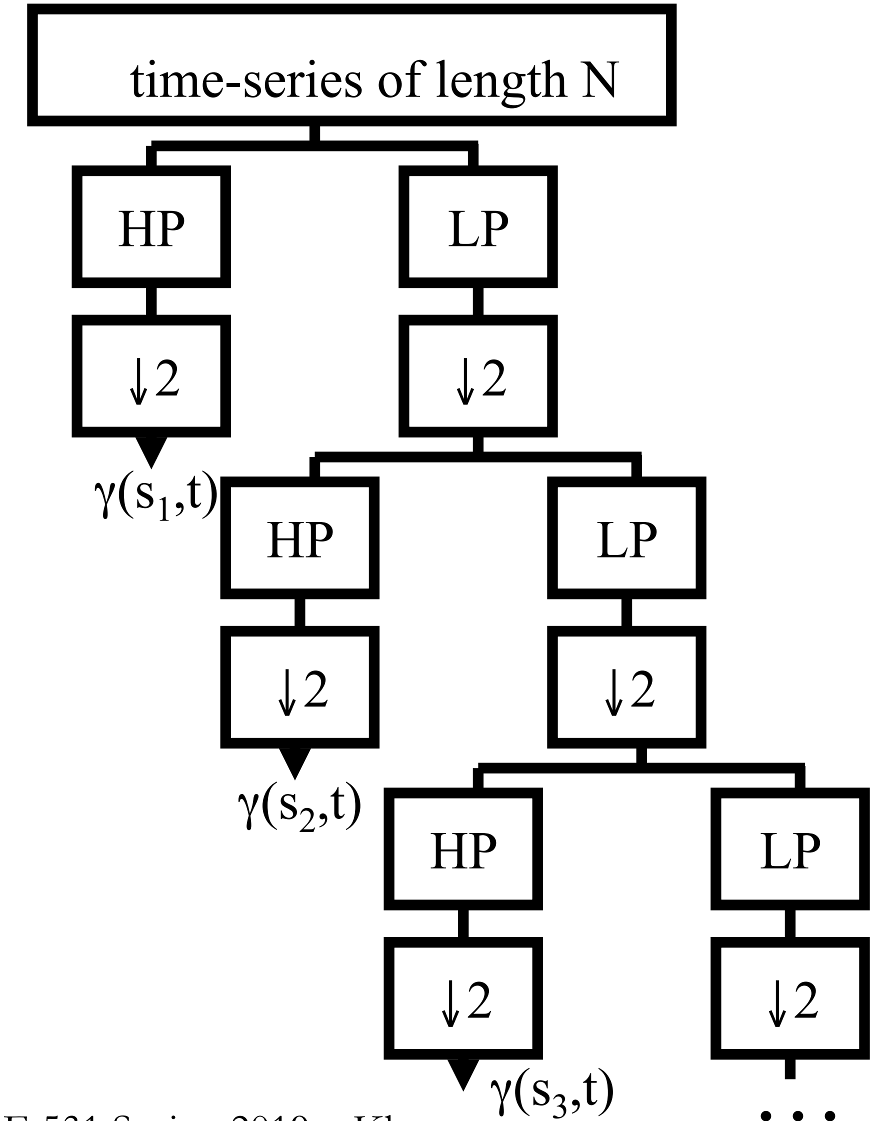


Digital Wavelet as Multirate Filter Bank

- Repeat recursively!



Digital Wavelet as Multirate Filter Bank



$\gamma(s_1,t)$: N/2 coefficients

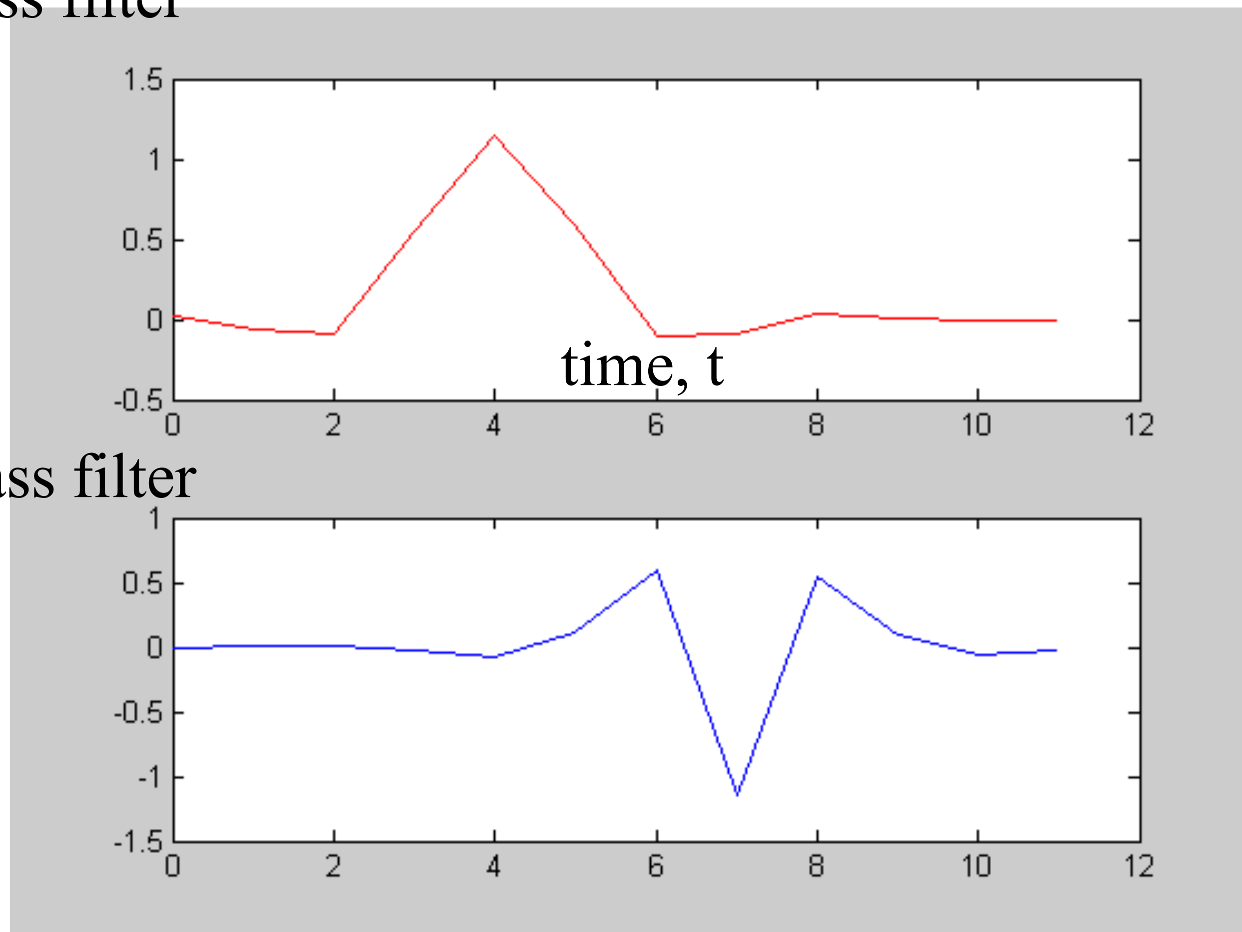
$\gamma(s_2,t)$: N/4 coefficients

$\gamma(s_2,t)$: N/8 coefficients

Total: N coefficients

Impulse Responses

Coiflet low pass filter



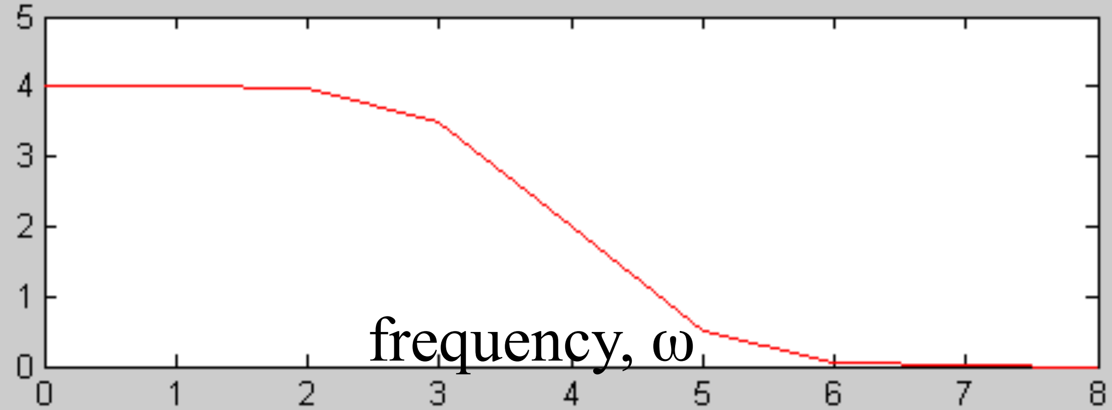
time, t

Coiflet high-pass filter

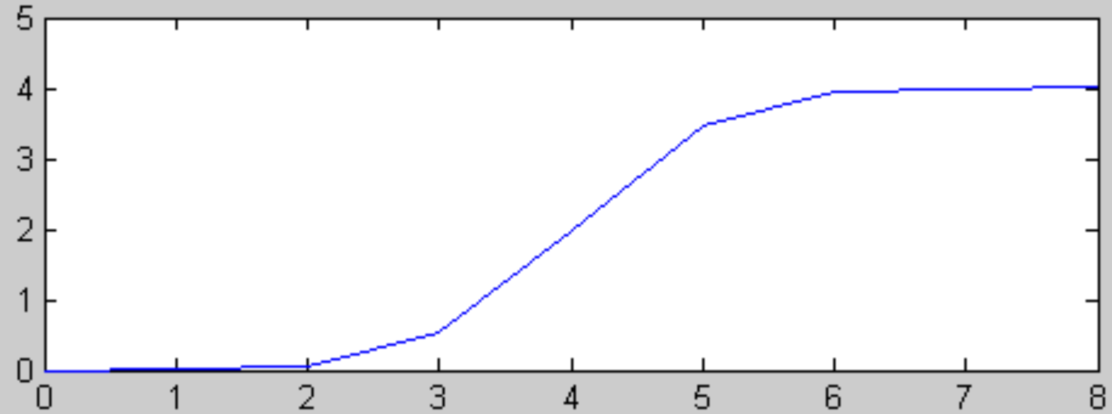
time, t

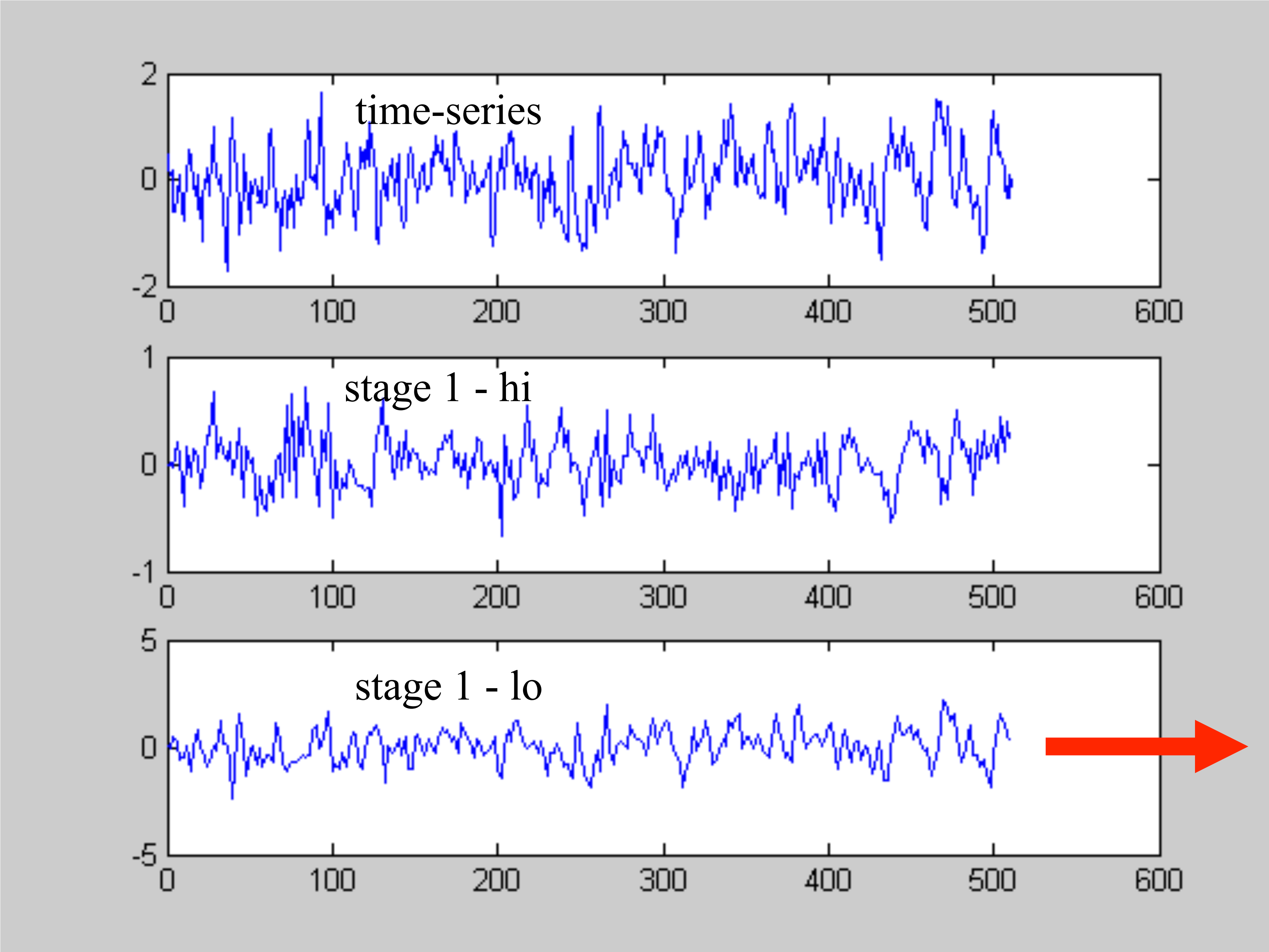
Filter Responses

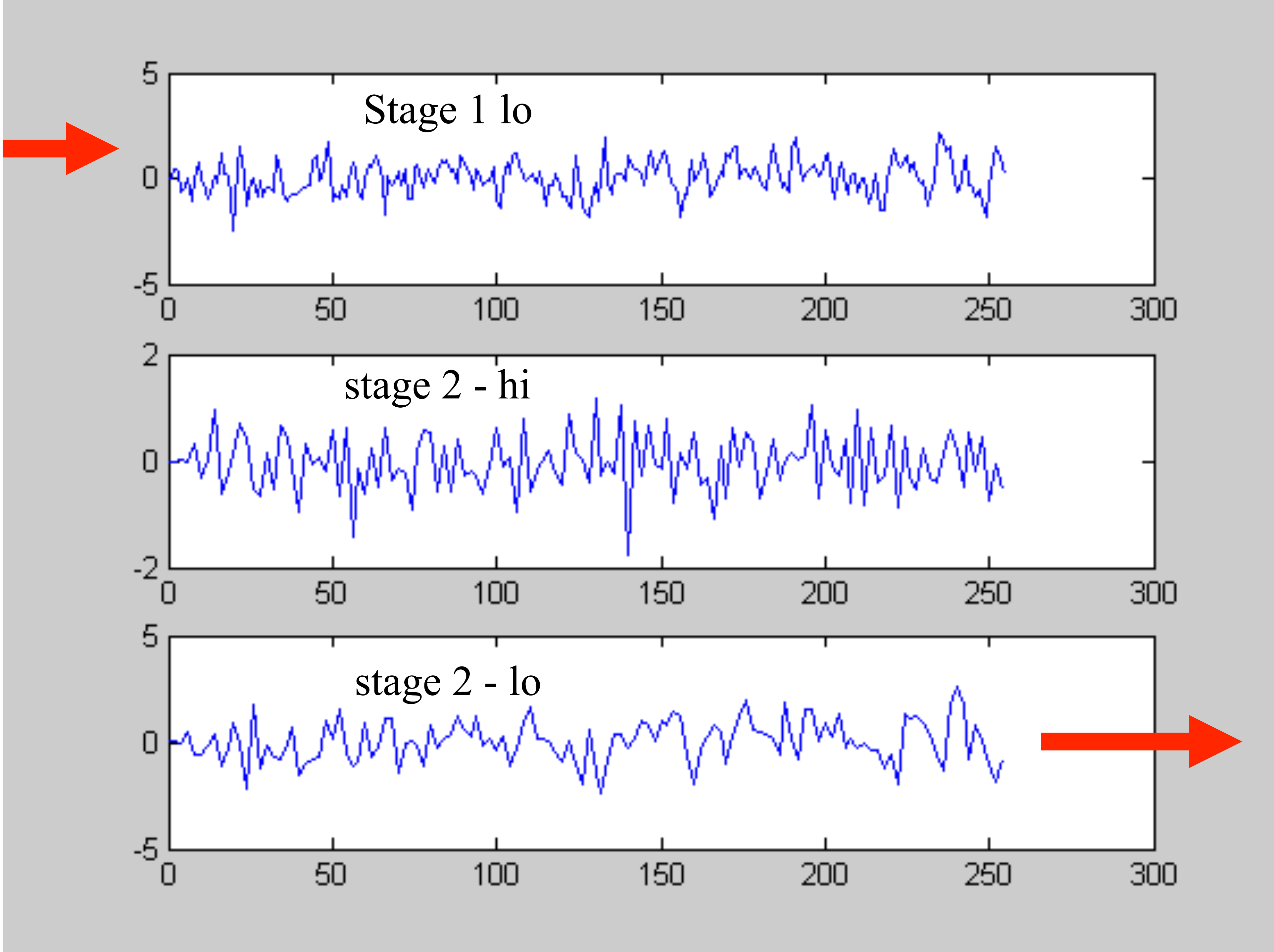
Spectrum of low pass filter

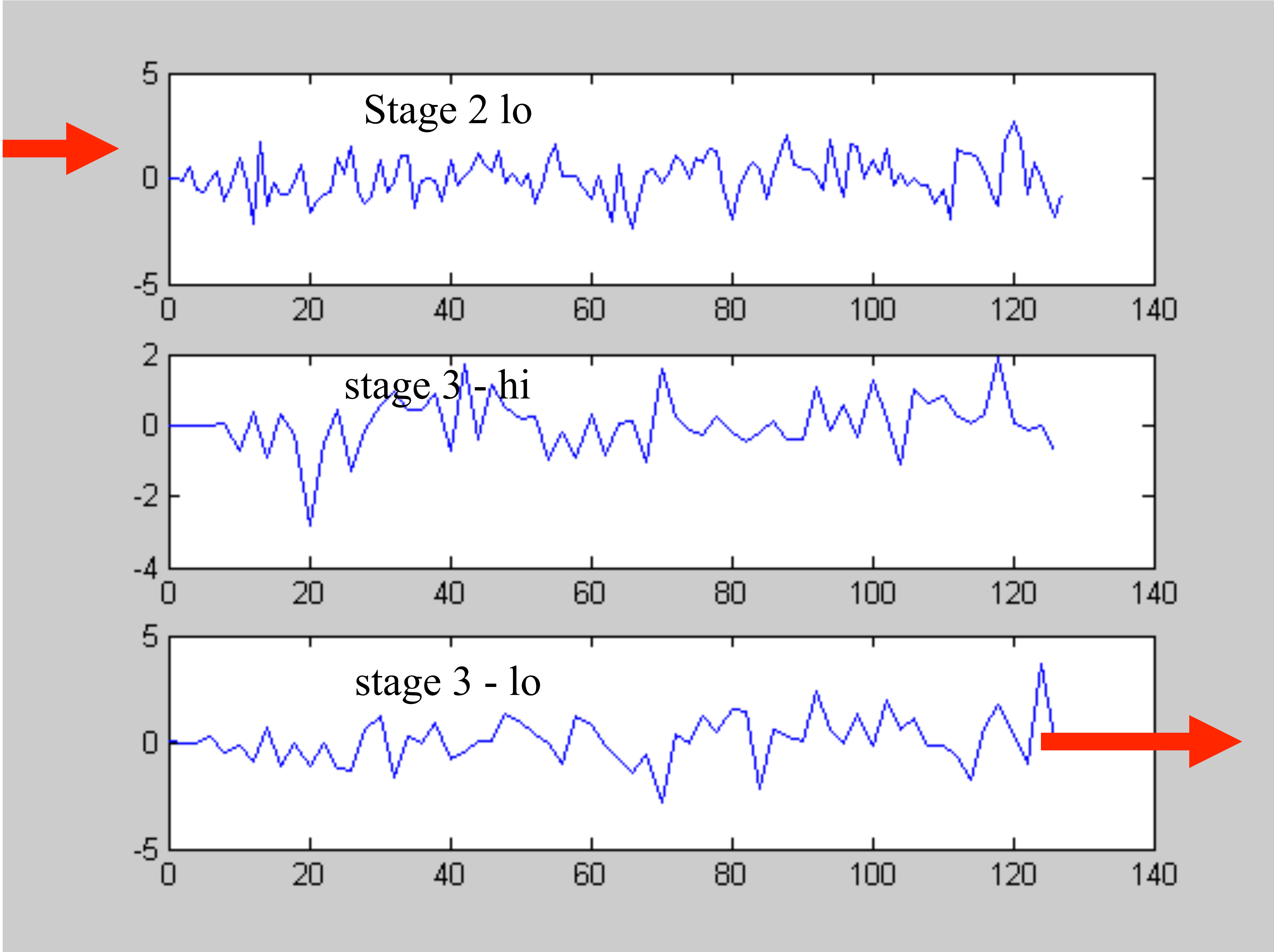


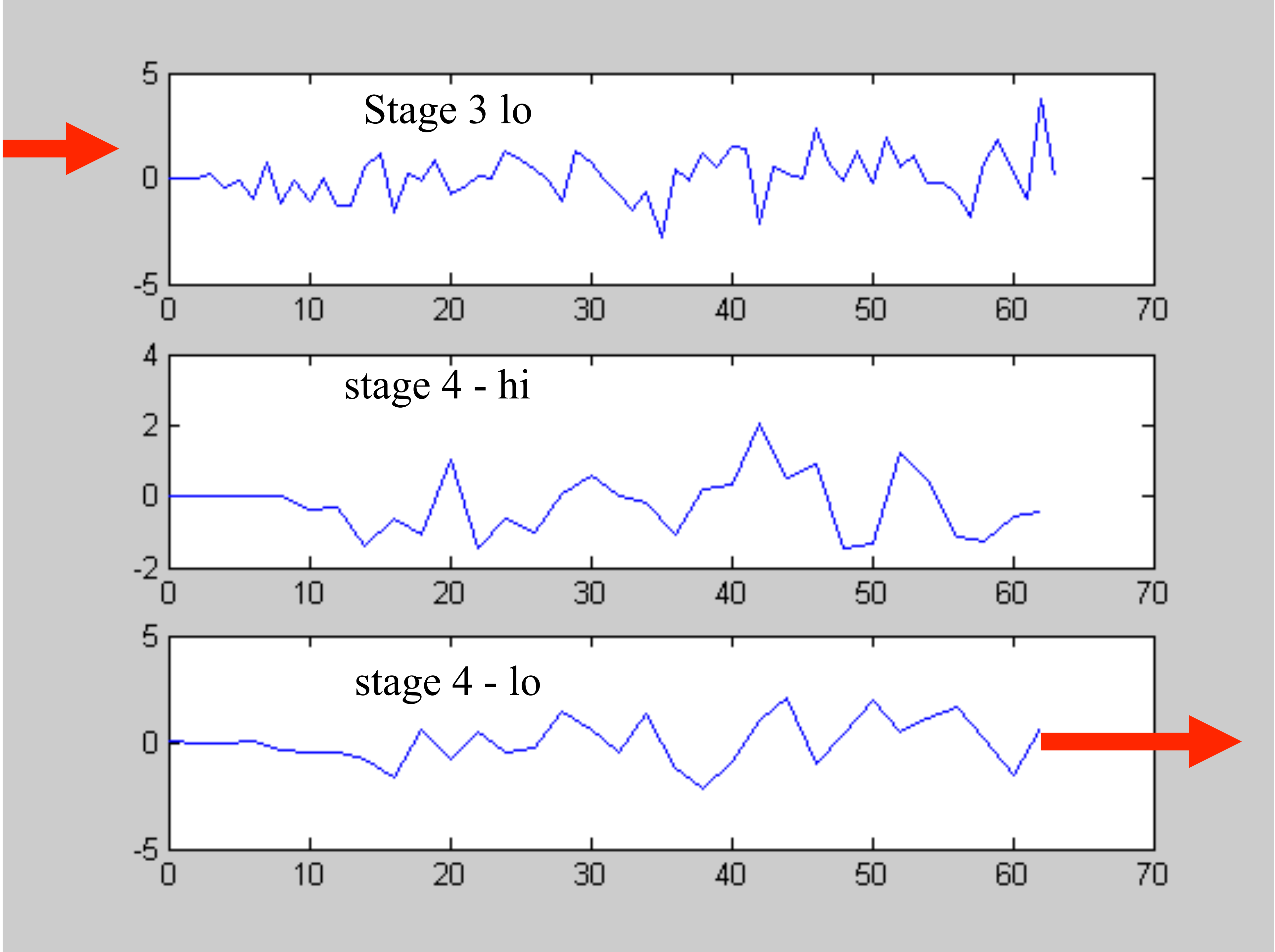
Spectrum of high pass filter



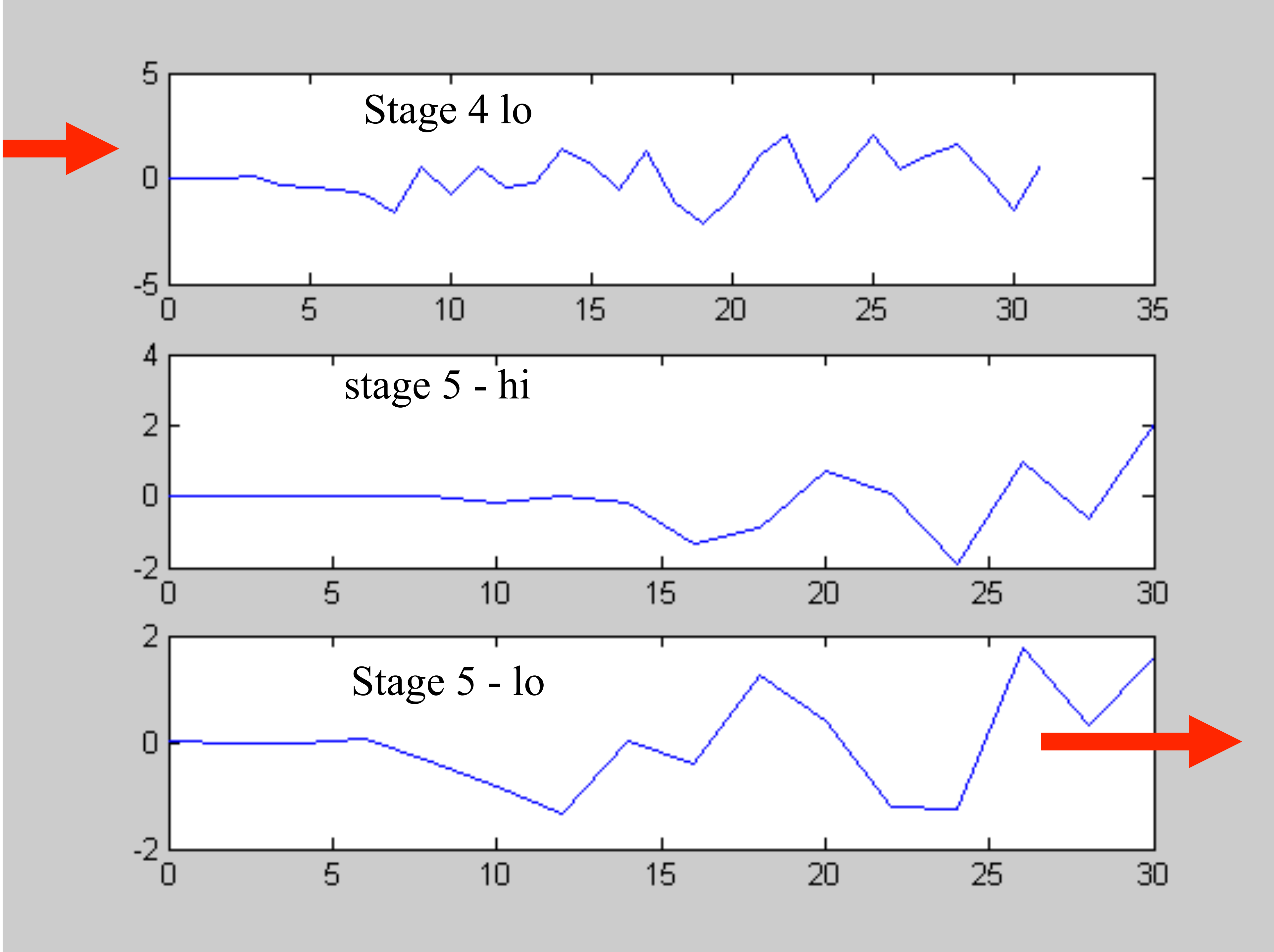

time-series
stage 1 - hi
stage 1 - lo
2
0
-2
1
-1
5
-5
0
100
200
300
400
500
600


Stage 1 lo
stage 2 - hi
stage 2 - lo
5
0
-5
2
-2
0
50
100
150
200
250
300

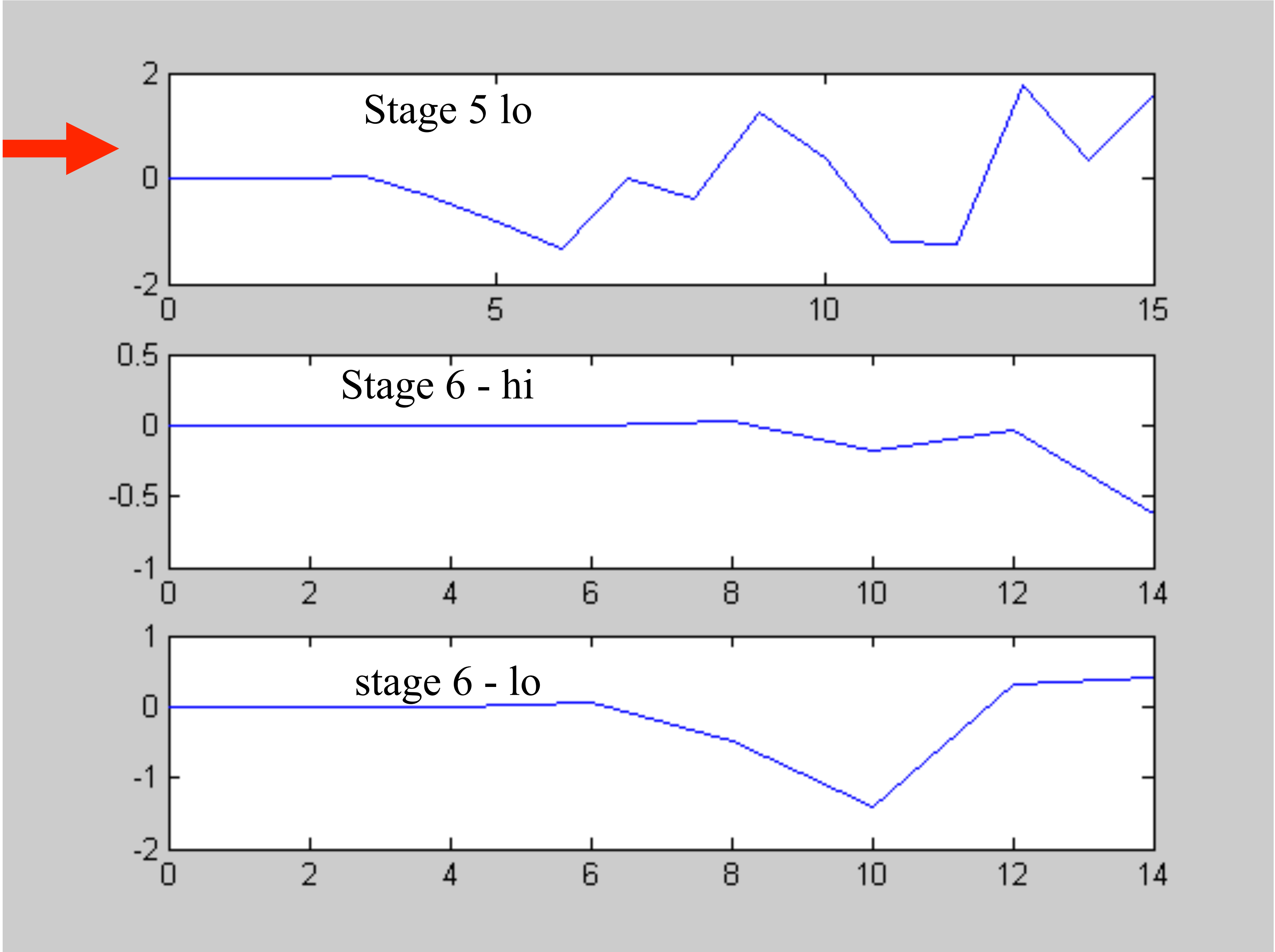

Stage 2 lo
stage 3 - hi
stage 3 - lo
5
0
-5
2
-2
-4
0 20 40 60 80 100 120 140



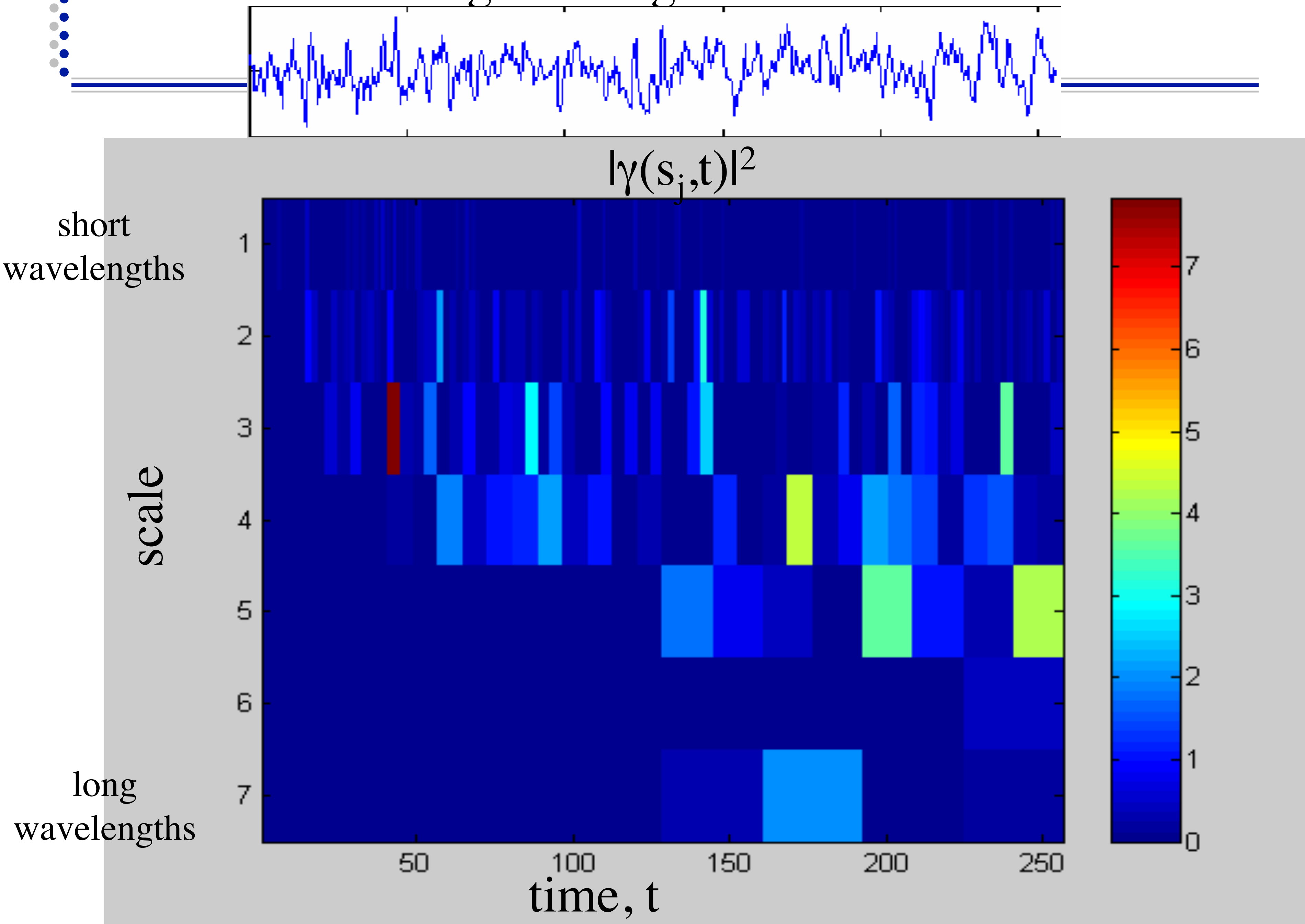
Stage 3 lo
stage 4 - hi
stage 4 - lo
5
0
-5
4
2
0
-2
0
10
20
30
40
50
60
70



Stage 4 lo
stage 5 - hi
Stage 5 - lo
5
0
-5
0 5 10 15 20 25 30 35
4
2
0
-2
0 5 10 15 20 25 30
2
0
-2
0 5 10 15 20 25 30


Stage 5 lo
2
0
-2
0
5
10
15
Stage 6 - hi
0.5
0
-0.5
-1
0
2
4
6
8
10
12
14
stage 6 - lo
1
0
-1
-2
0
2
4
6
8
10
12
14

Putting it all together …



Big Ideas

- Wavelet transform
  - Capture temporal data with fewer coefficients than STFT
  - Use scaling and translation to get different resolution at different levels

Admin

- Project
  - Due 4/30