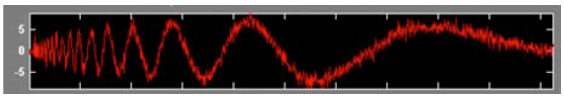
ESE 531: Digital Signal Processing

Lec 25: April 23, 2019
Wavelet Transform

Wavelet Transform

Motivation

- Some signals obviously have spectral characteristics that vary with time

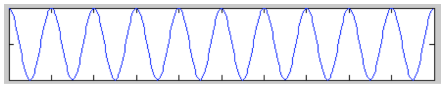


Criticism of Fourier Spectrum

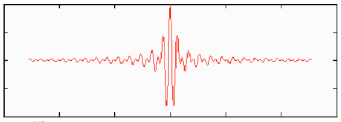
- It's giving you the spectrum of the 'whole time-series'
- Which is OK if the time-series is stationary. But what if its not?
- We need a technique that can "march along" a time series and that is capable of:
  - Analyzing spectral content in different places
  - Detecting sharp changes in spectral character

Fourier vs. Wavelet

- Fourier Analysis is based on an indefinitely long cosine wave of a specific frequency



- Wavelet Analysis is based on an short duration wavelet of a specific center frequency

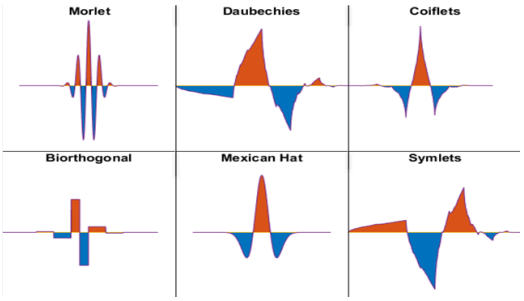


Wavelet Transform

- All wavelet derived from *mother* wavelet

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

Examples of Wavelets



Wavelet – Scaled and Shifted

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)$$

normalization

shift in time

change in scale: big s means long wavelength

wavelet with scale, s and time, τ

Mother wavelet

Continuous Wavelet Transform

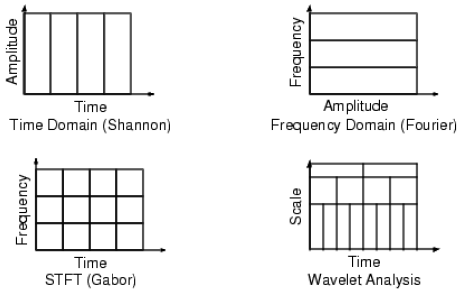
time-series

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

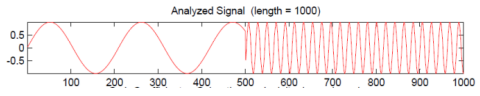
coefficient of wavelet with scale, s and time, τ

wavelet with scale, s, and shift, τ

Transform Comparison



Wave Demo



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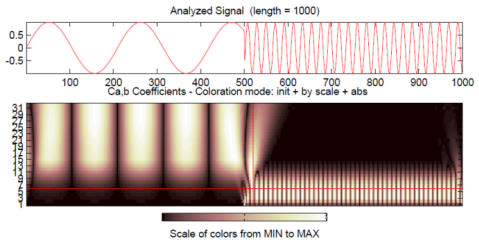

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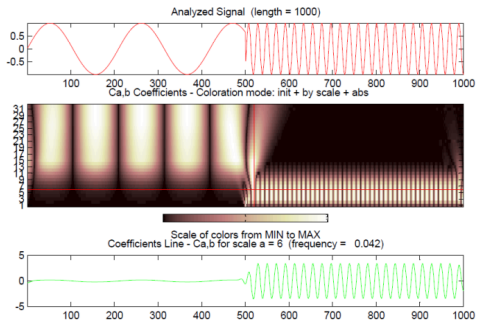
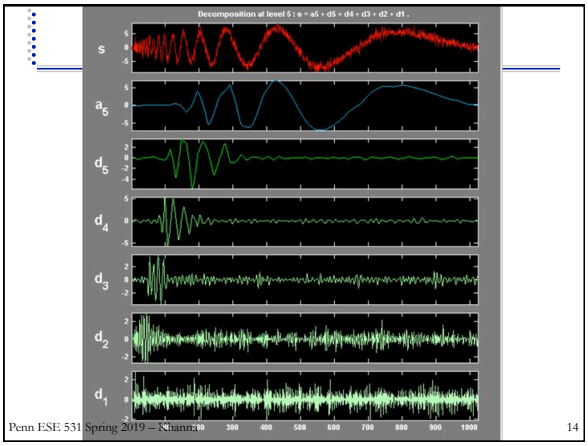
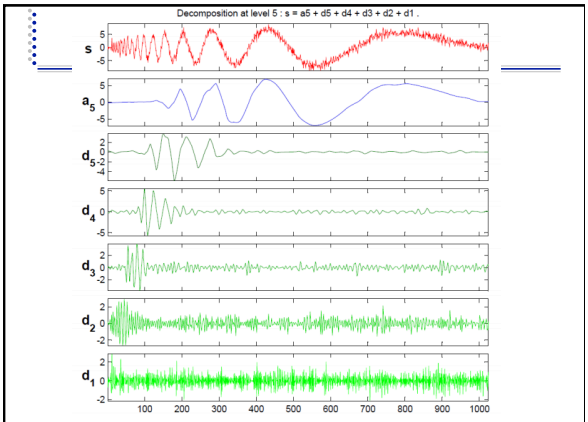






Inverse Wavelet Transform

- Build up a time-series as sum of wavelets of different scales, s, and positions, t

$$f(t) = \int\int \gamma(s,\tau)\psi_{s,\tau}(t)\,d\tau\,ds$$

time-series

coefficients of wavelets

wavelet with scale, s and time, τ

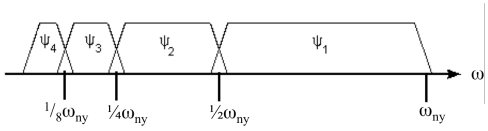
Discrete wavelets:

- Scale wavelets only by powers of 2
  - $s_j = 2^j$
- And shifting by multiples of s_j for each successive scale
  - $\tau_{j,k} = 2^j k$
- Then $\gamma(s_j, \tau_{j,k}) = \gamma_{jk}$
  - where j = 1, 2, …∞, k = -∞… -2, -1, 0, 1, 2, …∞

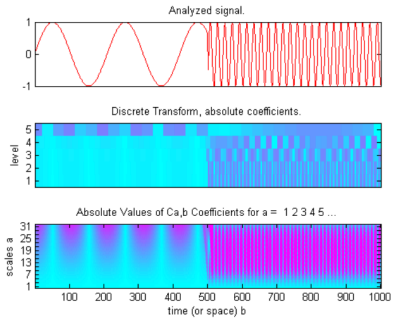
$$\gamma_{j,k} = \frac{1}{\sqrt{2^j}} \int f(t)\Psi\left(\frac{t - k2^j}{2^j}\right) dt$$

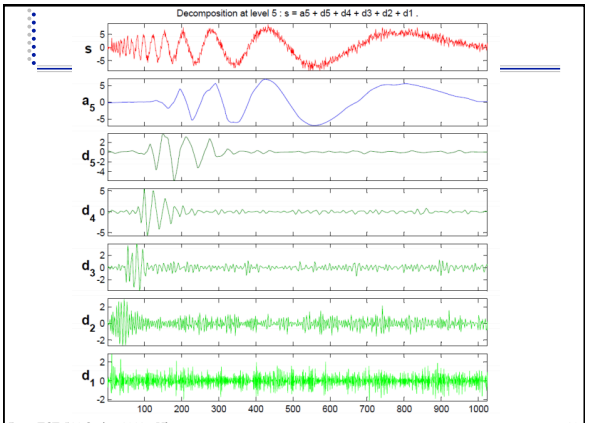
Discrete Wavelet Transform

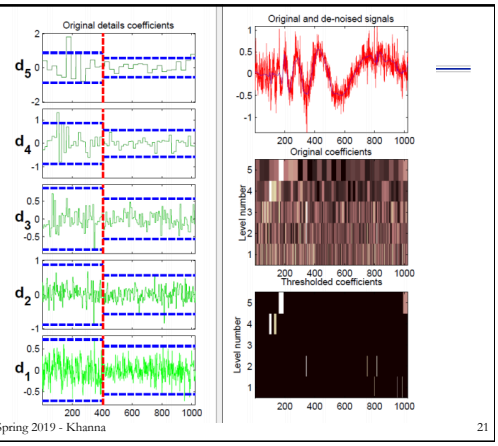
- The factor of two scaling means that the spectra of the wavelets divide up the frequency scale into octaves (frequency doubling intervals)



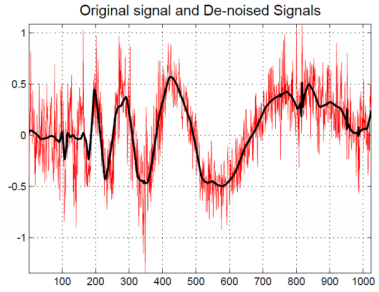
DWT vs CWT







De-noised Signal



Wavelet Transform

- Determining the wavelet coefficients for a fixed scale, *s*, can be thought of as a filtering operation

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

$$\gamma_s(t) = \int f(t)\Psi_s(t)dt = f(t) * \Psi_s(t)$$

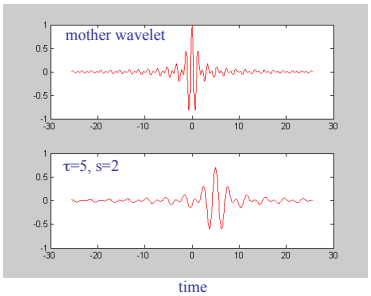
- where

$$\Psi_s(t) = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

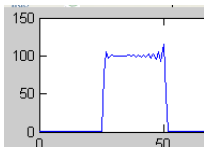
Shannon Wavelet

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)$$

- $\Psi(t) = 2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t)$



Fourier spectrum of Shannon Wavelet



frequency, ω

- Wavelet coefficients are a result of bandpass filtering

Discrete Wavelet Transform

- The coefficients of Ψ is just the band-pass filtered time-series, where Ψ is the wavelet, now viewed as the impulse response of a bandpass filter.

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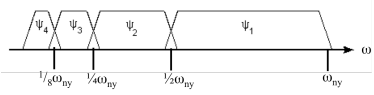

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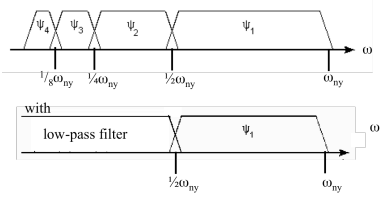
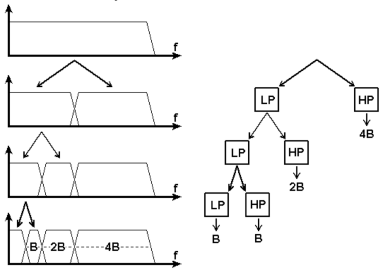


Digital Wavelet as Multirate Filter Bank

- Repeat recursively!



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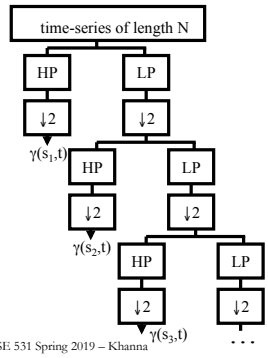


$\gamma(s_1,t)$: N/2 coefficients

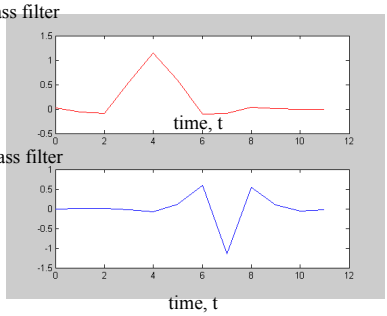
$\gamma(s_2,t)$: N/4 coefficients

$\gamma(s_2,t)$: N/8 coefficients

Total: N coefficients

Impulse Responses

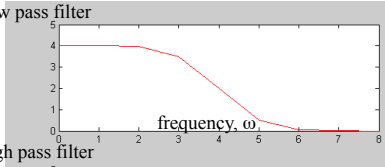
Coiflet low pass filter



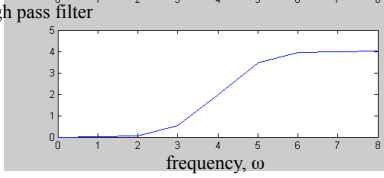
Coiflet high-pass filter

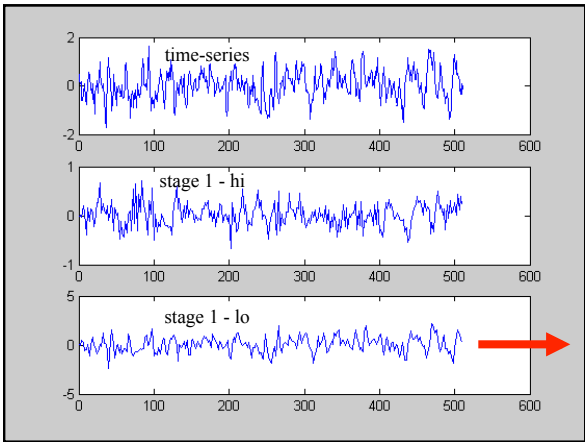
Filter Responses

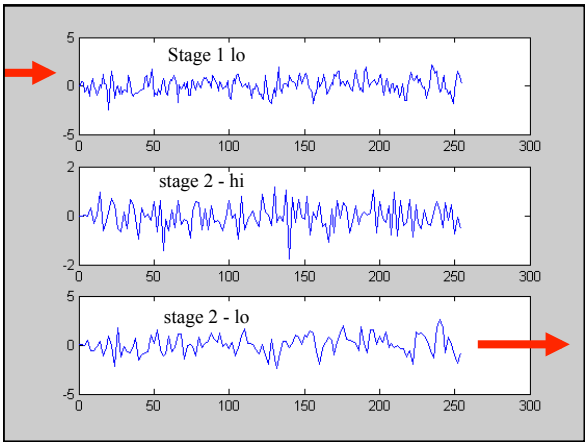
Spectrum of low pass filter

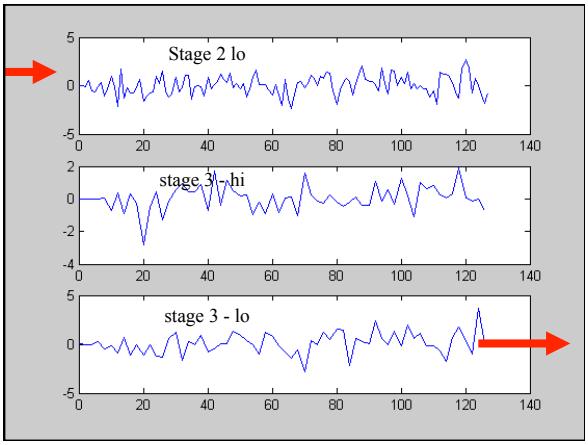


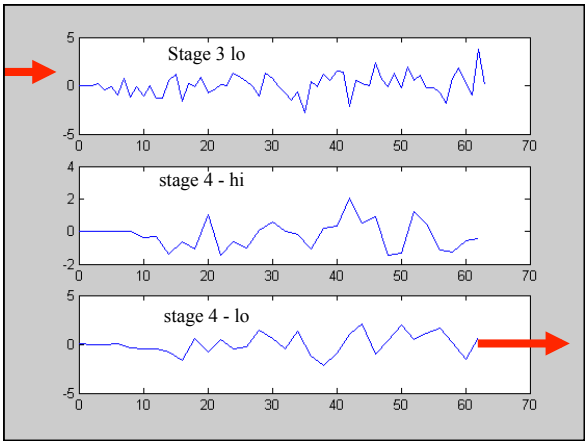
Spectrum of high pass filter

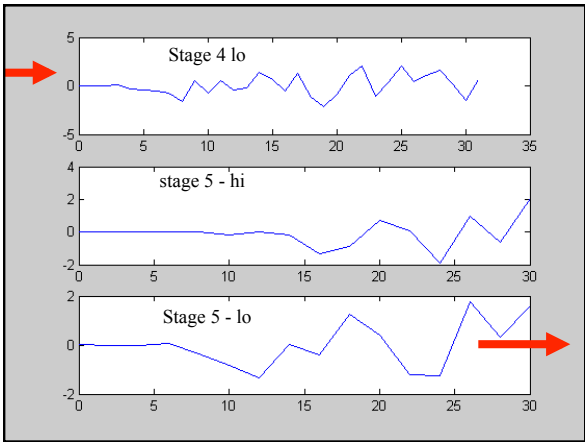


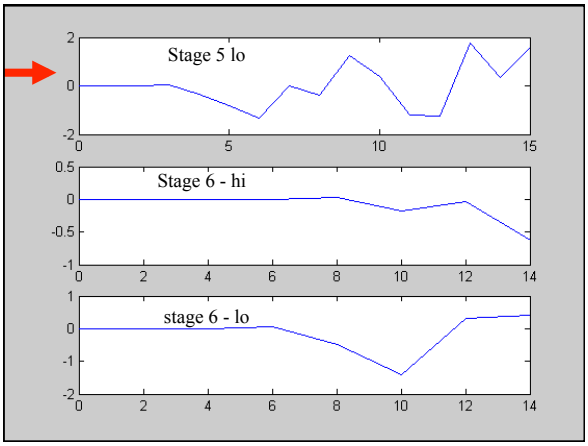


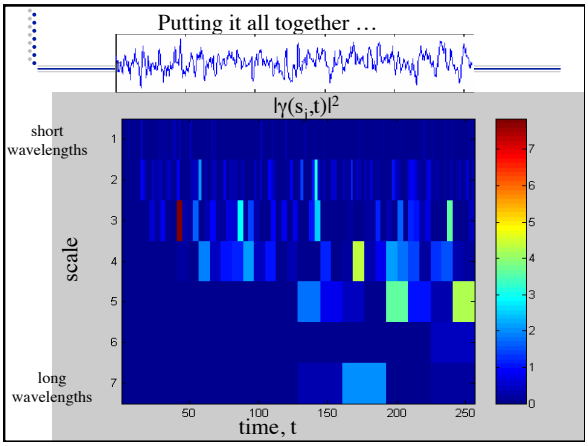












Big Ideas

- Wavelet transform
  - Capture temporal data with fewer coefficients than STFT
  - Use scaling and translation to get different resolution at different levels

Admin

- Project
  - Due 4/30