ESE 531: Digital Signal Processing

Lec 3: January 24, 2019

Discrete Time Signals and Systems

Lecture Outline

- ❑ Discrete Time Systems
- ❑ LTI Systems
- ❑ LTI System Properties
- ❑ Difference Equations

Discrete-Time Systems

Discrete Time Systems

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$

$$x \longrightarrow \boxed{\mathcal{H}} \longrightarrow y$$

❑ Systems manipulate the information in signals

❑ Examples

- Speech recognition system that converts acoustic waves into text
- Radar system transforms radar pulse into position and velocity
- fMRI system transform frequency into images of brain activity
- Moving average system smooths out the day-to-day variability in a stock price

System Properties

- ❑ Causality
  - ◼ y[n] only depends on x[m] for m<=n

- ❑ Linearity
  - ◼ Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - ◼ $Ax_1[n]+Bx_2[n]$ ➔ $Ay_1[n]+By_2[n]$

- ❑ Memoryless
  - ◼ y[n] depends only on x[n]

- ❑ Time Invariance
  - ◼ Shifted input results in shifted output
    - ◼ x[n-q] ➔ y[n-q]

- ❑ BIBO Stability
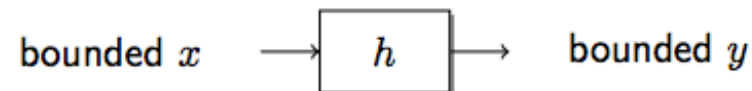 - ◼ A bounded input results in a bounded output (ie. max signal value exists for output if max)
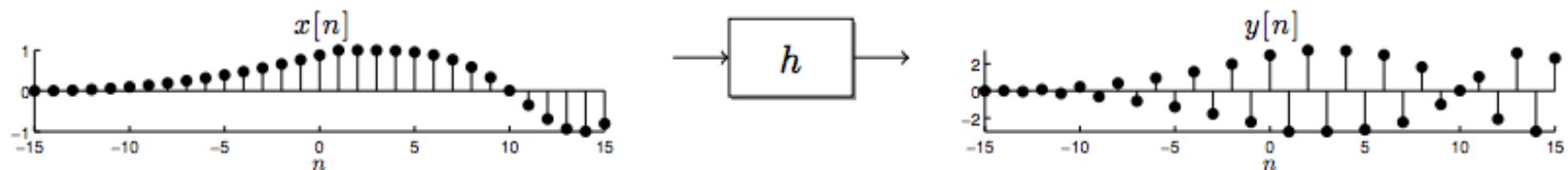
Stability

❑ BIBO Stability

■ Bounded-input bounded-output Stability

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y

$$\text{bounded } x \longrightarrow \boxed{h} \longrightarrow \text{bounded } y$$

■ Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$, or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all n

Examples

- Causal? Linear? Time-invariant? Memoryless? BIBO Stable?

- Time Shift:
  - $y[n] = x[n-m]$

- Accumulator:
  - $$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- Compressor (M>1):
  $$y[n] = x[Mn]$$

Non-Linear System Example

❑ Median Filter

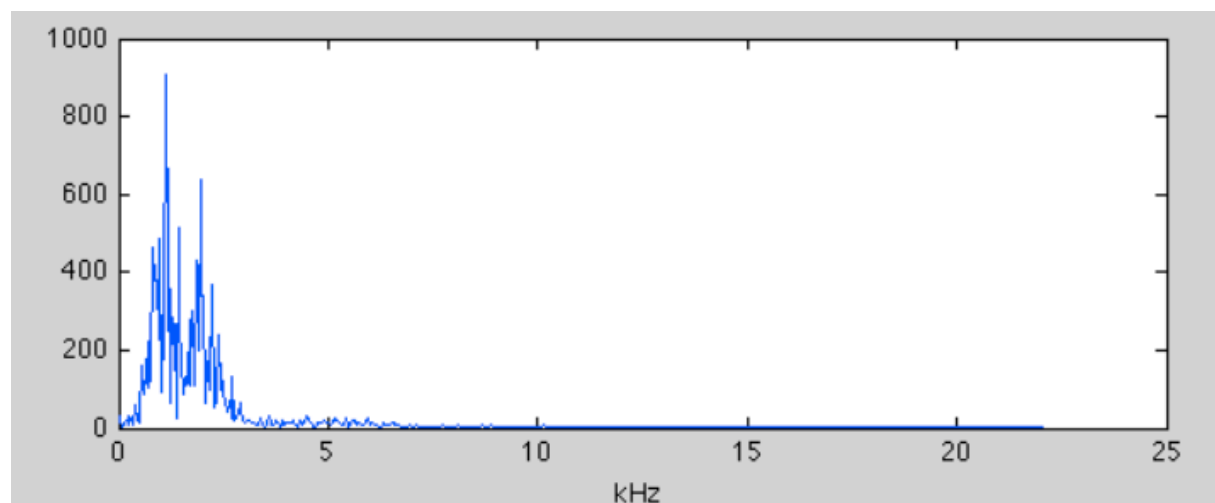 ▪ y[n]=MED{x[n-k], …x[n+k]}

 ▪ Let k=1

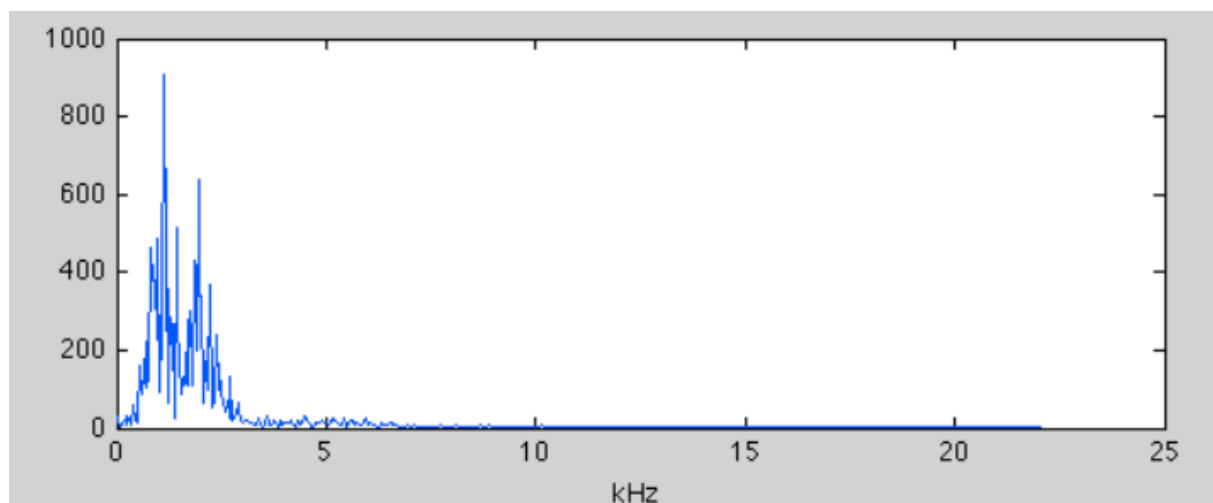 ▪ y[n]=MED{x[n-1], x[n], x[n+1]}
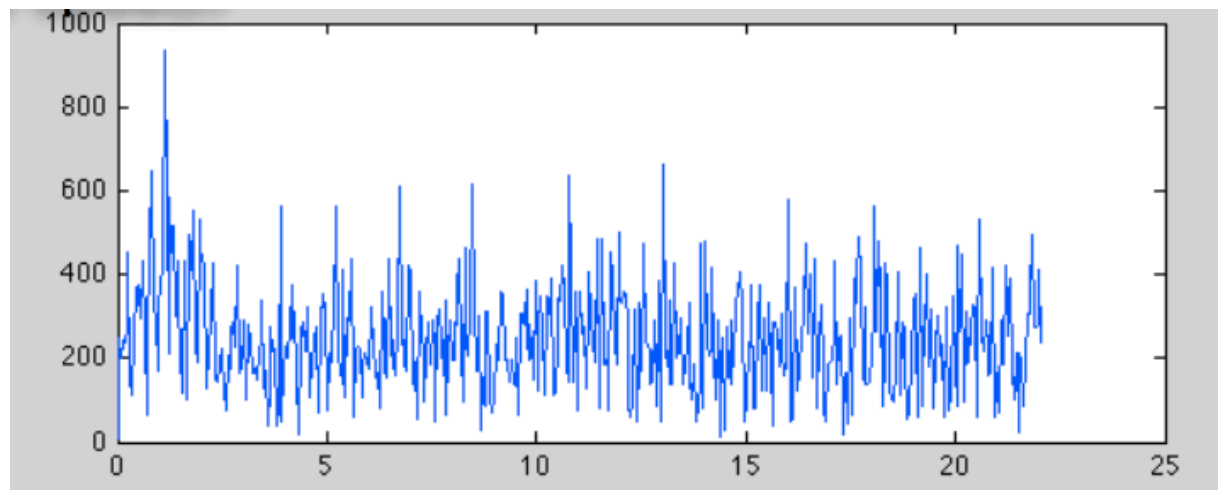
Spectrum of Speech
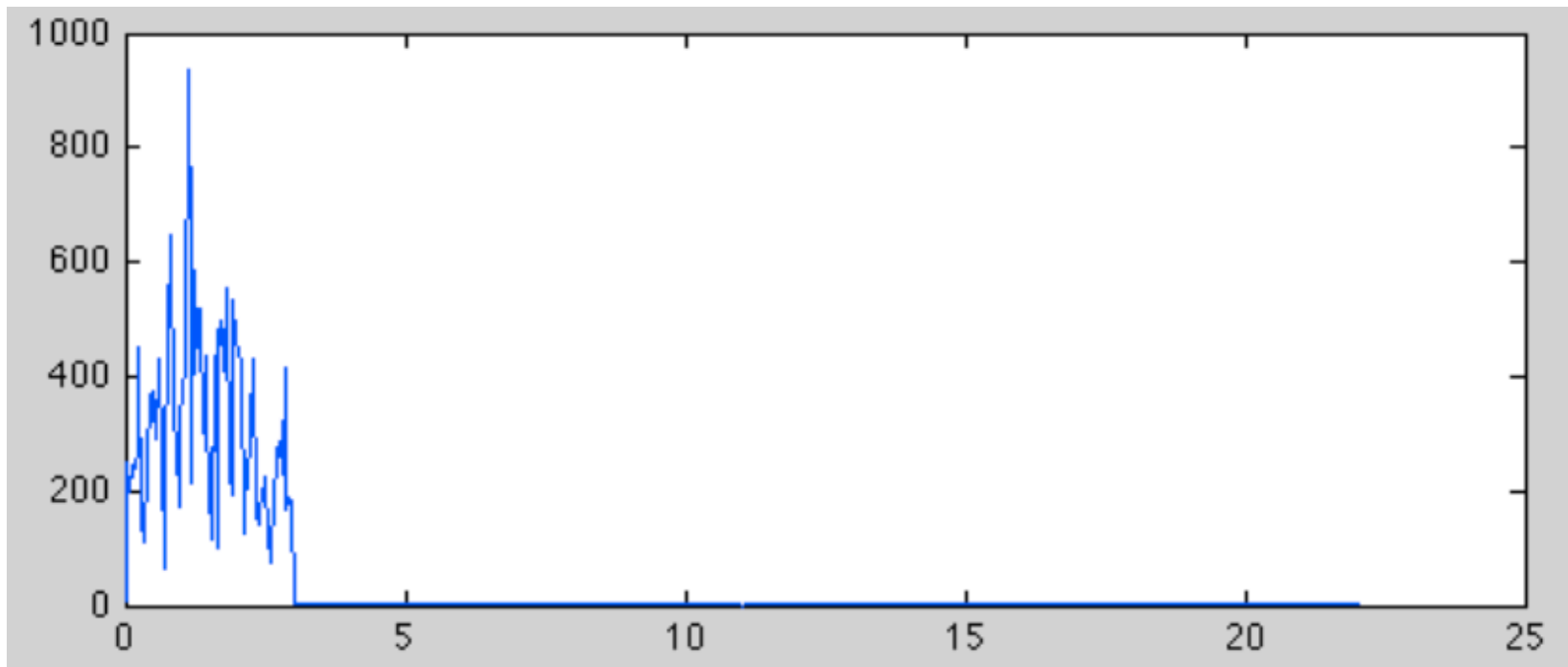
Speech

Spectrum of Speech

Speech

Corrupted Speech

Low Pass Filtering
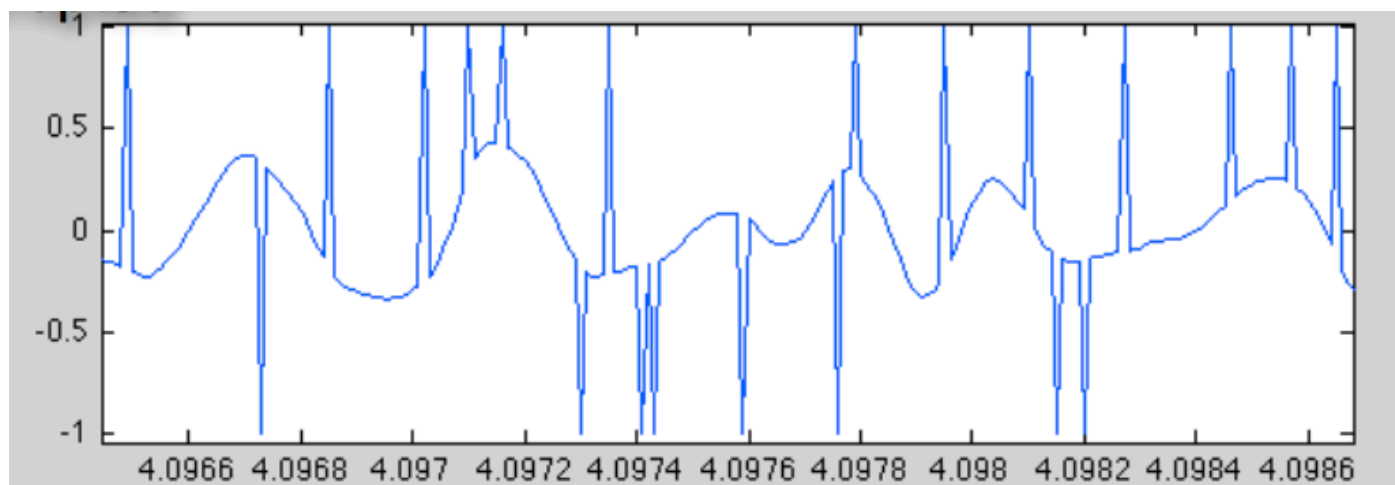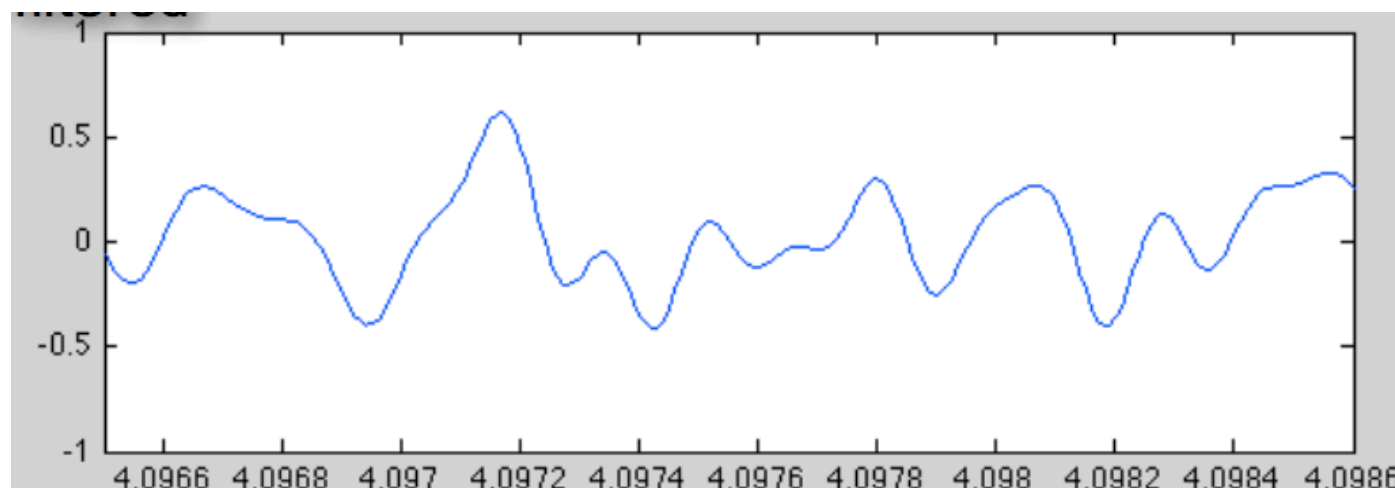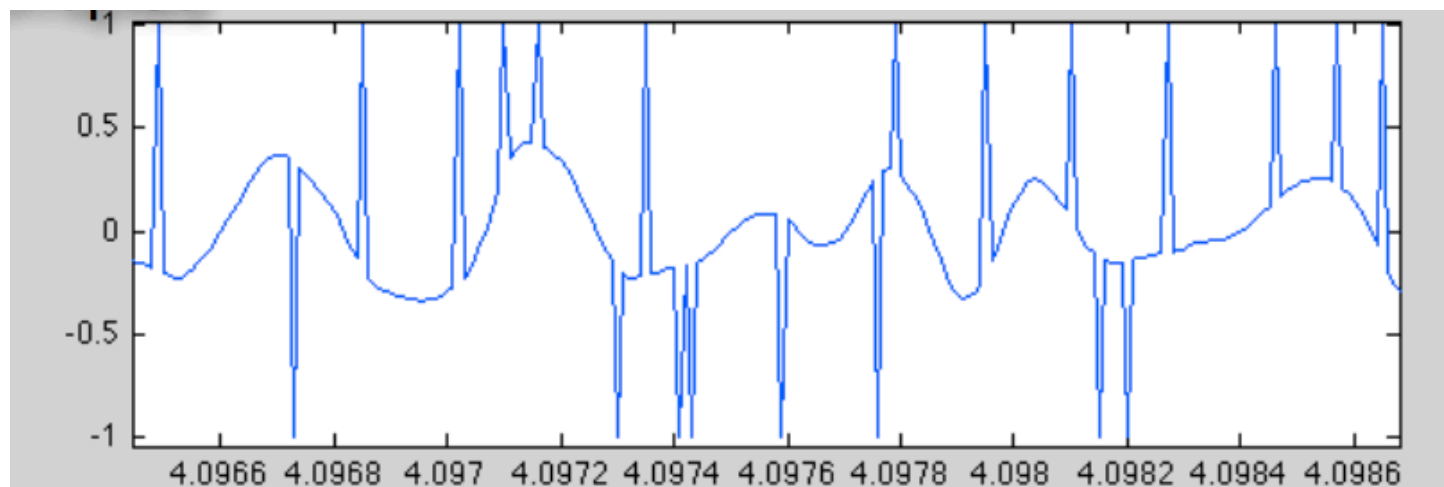
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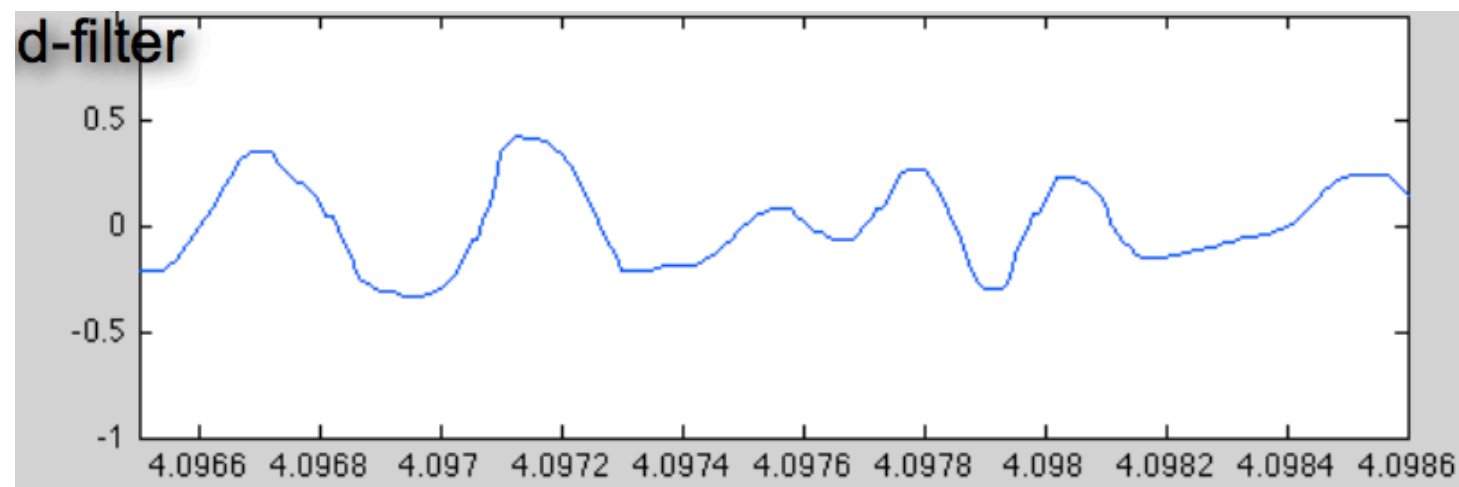
Corrupted Speech

LP-Filtered Speech

Median Filtering
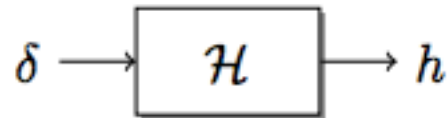

Corrupted Speech

Med-Filter Speech

LTI Systems

LTI Systems

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

❑ LTI system can be completely characterized by its impulse response

$$\delta \longrightarrow \boxed{\mathcal{H}} \longrightarrow h$$

❑ Then the output for an arbitrary input is a sum of weighted, delay impulse responses
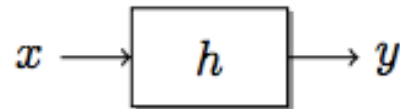
$$x \longrightarrow \boxed{h} \longrightarrow y \qquad y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m]$$

$$y[n] = x[n] * h[n]$$

Convolution

❏ Convolution formula:

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$
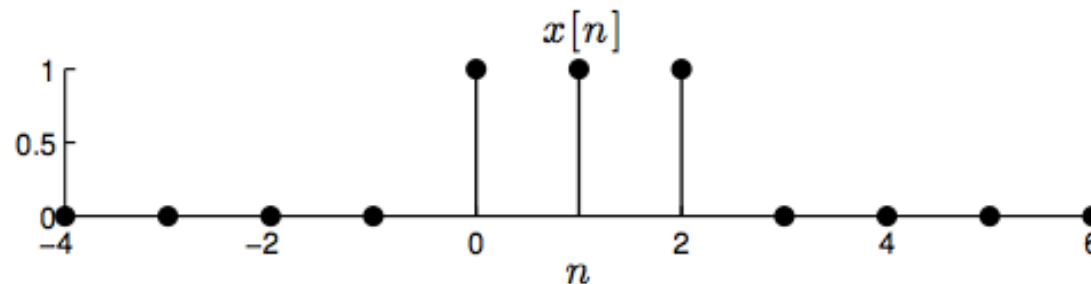
❏ Convolution method:

- 1) Time reverse the impulse response and shift it n time steps to the right
- 2) Compute the inner product between the shifted impulse response and the input vector
- Repeat for evey n

Convolution Example

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m]$$

❑ Convolve a unit pulse with itself

Convolution is Commutative

❑ Convolution is commutative:

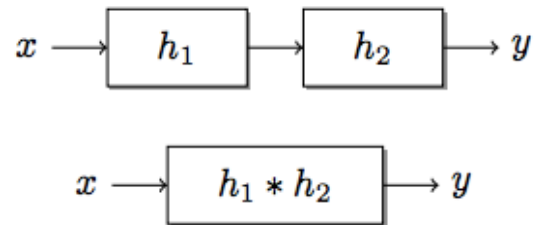$$x * h \;=\; h * x$$

❑ These block diagrams are equivalent

❑ Implication: pick either h or x to flip and shift when convolving
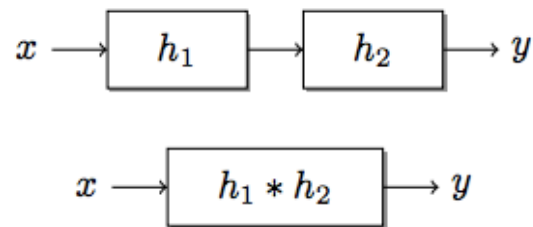
LTI Systems in Series

❑ Impulse response of the cascade of two LTI systems:

$$x \longrightarrow \boxed{h_1} \longrightarrow \boxed{h_2} \longrightarrow y$$

$$x \longrightarrow \boxed{h_1 * h_2} \longrightarrow y$$

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❑ Proof by picture

$$\delta \longrightarrow \boxed{h_1} \longrightarrow h_1 \longrightarrow \boxed{h_2} \longrightarrow h_1 * h_2$$
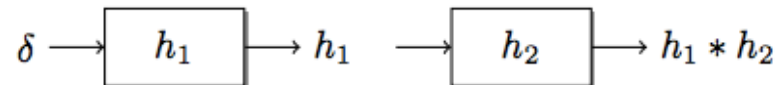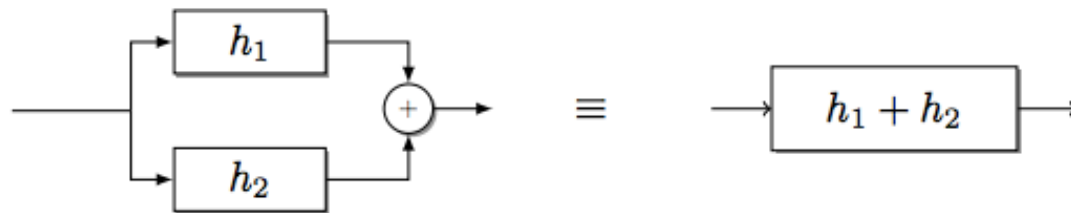
LTI Systems in Parallel

❑ Impulse response of the parallel connection of two LTI systems:

Causal System Revisited

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

❑ An LTI system is causal if its impulse response is causal:

$$h[n] = 0 \text{ for } n < 0$$

$$h[n] = \alpha^n u[n], \quad \alpha = 0.8$$
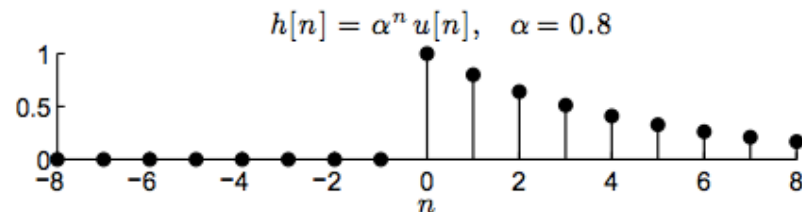
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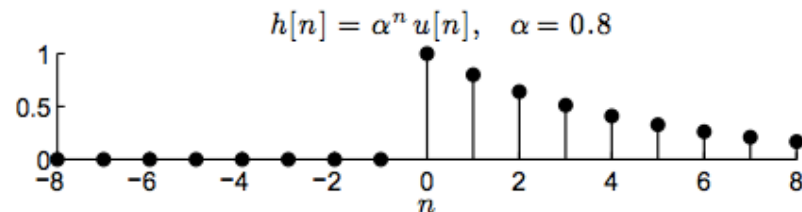
$$h[n] = 0 \text{ for } n < 0$$

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❑ To prove, note that the convolution does not look into the future if the impulse response is causal

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m] \qquad h[n-m] = 0 \text{ when } m > n;$$

Duration of Impulse

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

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❑ Example: Moving average

$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; \tfrac{1}{2}\left(x[n] + x[n-1]\right)$$

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❑ Example: Moving average

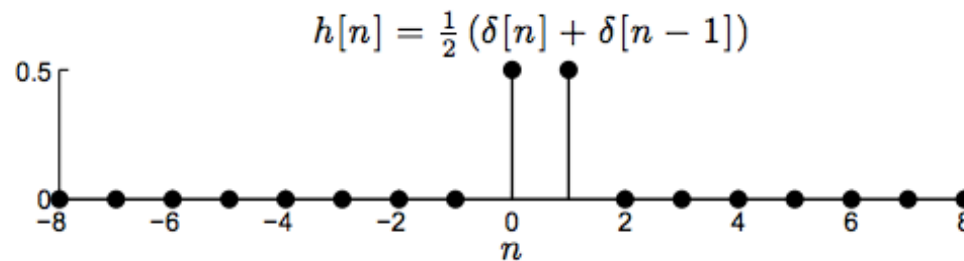$$y[n] = \mathcal{H}\{x[n]\} = \tfrac{1}{2}(x[n] + x[n-1])$$

$$h[n] = \tfrac{1}{2}(\delta[n] + \delta[n-1])$$

Duration of Impulse

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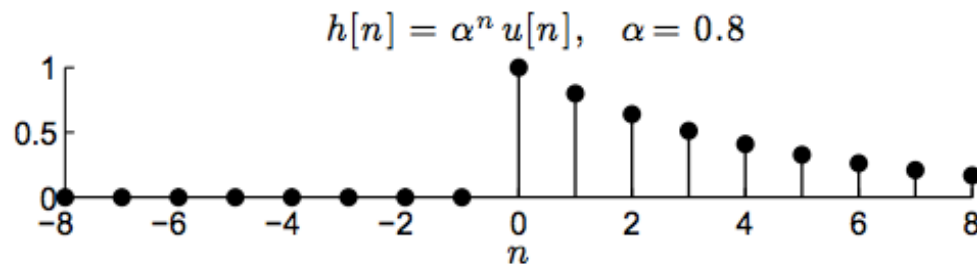
$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; x[n] + \alpha\,y[n-1]$$

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$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; x[n] + \alpha\, y[n-1]$$

$$h[n] = \alpha^n u[n], \quad \alpha = 0.8$$

BIBO Stability Revisited

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y

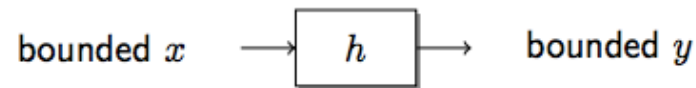$$\text{bounded } x \longrightarrow \boxed{h} \longrightarrow \text{bounded } y$$

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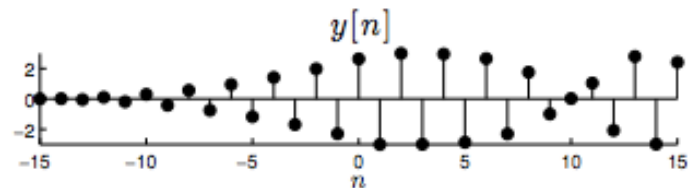
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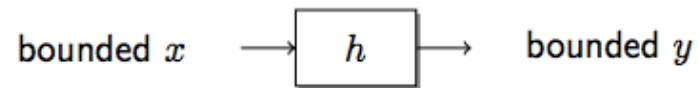
❏ Bounded input and output:

$$\|x\|_\infty < \infty \quad \text{and} \quad \|y\|_\infty < \infty$$

❏ Where

$$\|x\|_\infty = \max |x[n]|$$

$x[n]$

$$\longrightarrow \boxed{h} \longrightarrow$$

$y[n]$

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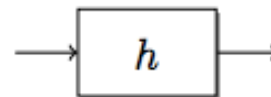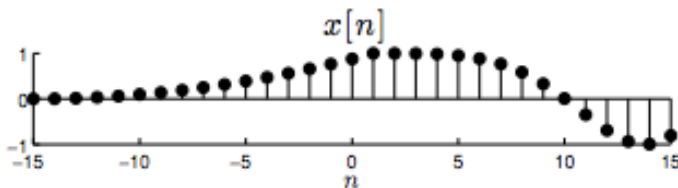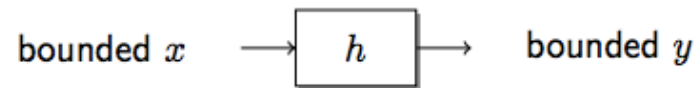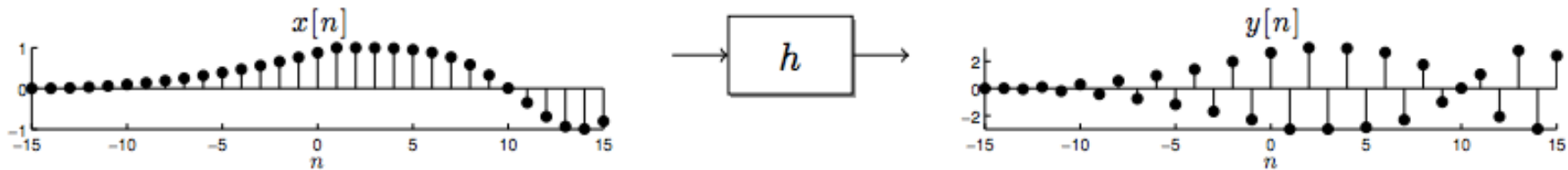$$\text{bounded } x \longrightarrow \boxed{h} \longrightarrow \text{bounded } y$$

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$x[n]$

$$\longrightarrow \boxed{h} \longrightarrow$$

$y[n]$

❑ An LTI system is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

BIBO Stability – Sufficient Condition

- Prove that <u>if $\|h\|_1 < \infty$</u> then the system is BIBO stable – for any input $\|x\|_\infty < \infty$ the output $\|y\|_\infty < \infty$

BIBO Stability – Sufficient Condition

- Prove that if $\|h\|_1 < \infty$ then the system is BIBO stable – for any input $\|x\|_\infty < \infty$ the output $\|y\|_\infty < \infty$

- Recall that $\|x\|_\infty < \infty$ means there exist a constant A such that $|x[n]| < A < \infty$ for all n

- Let $\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| = B < \infty$

- Compute a bound on $|y[n]|$ using the convolution of x and h and the bounds A and B

$$\begin{aligned} |y[n]| &= \left| \sum_{m=-\infty}^{\infty} h[n-m]\, x[m] \right| \leq \sum_{m=-\infty}^{\infty} |h[n-m]|\, |x[m]| \\ &< \sum_{m=-\infty}^{\infty} |h[n-m]|\, A = A \sum_{k=-\infty}^{\infty} |h[k]| = AB = C < \infty \end{aligned}$$

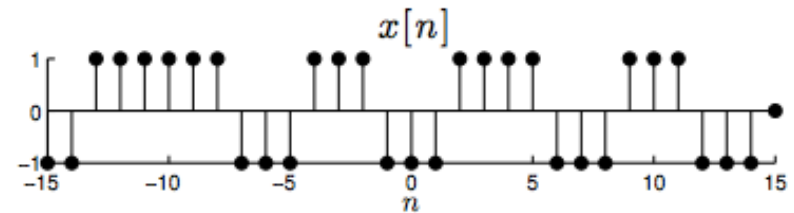- Since $|y[n]| < C < \infty$ for all n, $\|y\|_\infty < \infty$ ✓

BIBO Stability – Necessary Condition

- Prove that if $\|h\|_1 = \infty$ then the system is <u>not</u> BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
 - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical

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  - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical

- Given an impulse response h with $\|h\|_1 = \infty$ (assume complex-valued), form the tricky special signal $x[n] = \text{sgn}(h[-n])$
  - $x[n]$ is the $\pm$ sign of the time-reversed impulse response $h[-n]$
  - Note that x is bounded: $|x[n]| \leq 1$ for all n
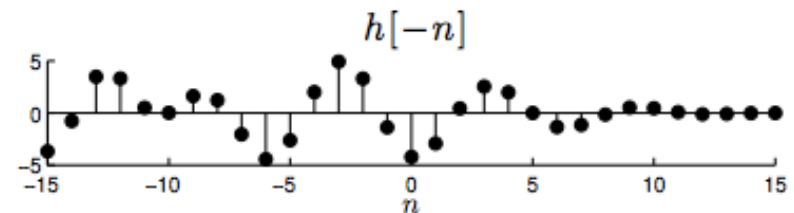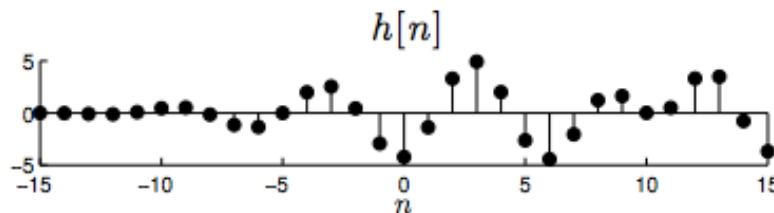
BIBO Stability – Necessary Condition

- We are proving that that if $\|h\|_1 = \infty$ then the system is <u>not</u> BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$

- Armed with the tricky special signal x, compute the output $y[n]$ at the time point $n = 0$

$$
\begin{aligned}
y[0] &= \sum_{m=-\infty}^{\infty} h[0-m]\,x[m] = \sum_{m=-\infty}^{\infty} h[-m]\,\mathrm{sgn}(h[-m]) \\
&= \sum_{m=-\infty}^{\infty} |h[-m]| = \sum_{k=-\infty}^{\infty} |h[k]| = \infty
\end{aligned}
$$

- So, even though x was bounded, y is <u>not</u> bounded; so system is not BIBO stable
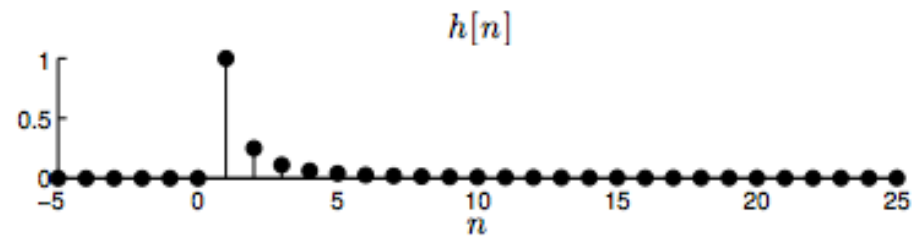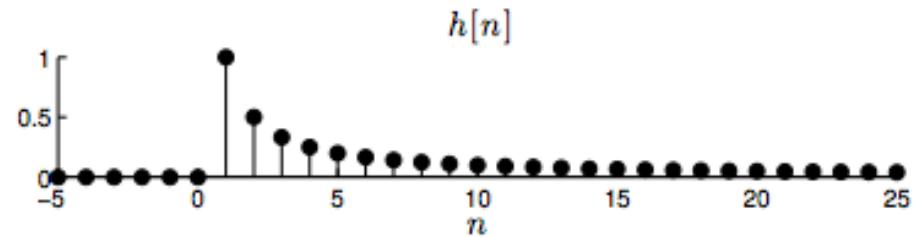
Examples

- Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n} \right| = \infty \Rightarrow$ not BIBO

- Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right| = \frac{\pi^2}{6} \Rightarrow$ BIBO
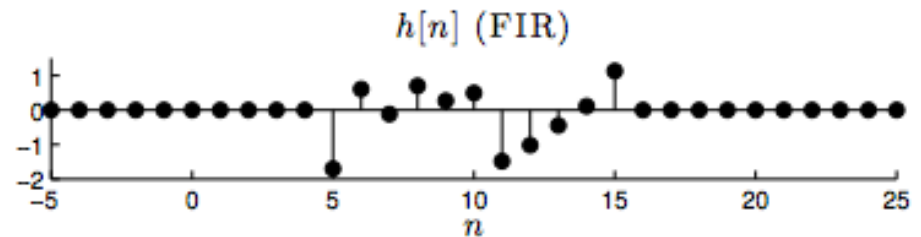
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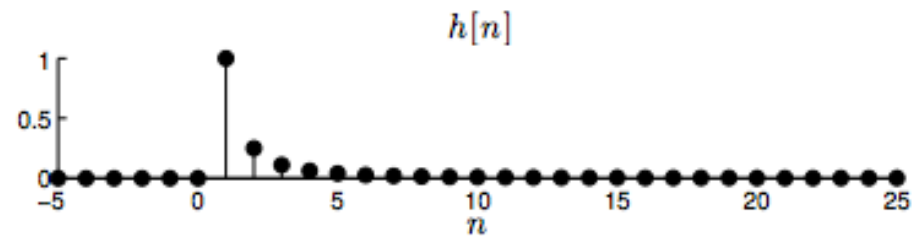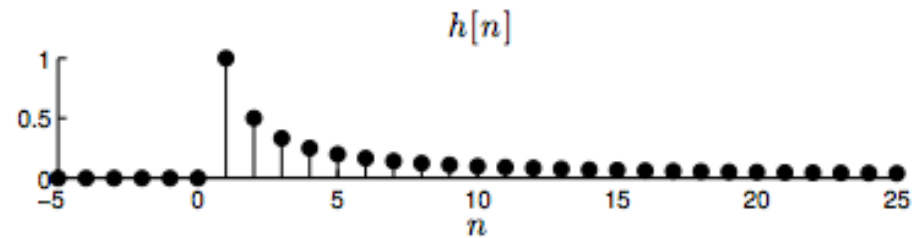
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$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right| = \frac{\pi^2}{6} \Rightarrow$ BIBO

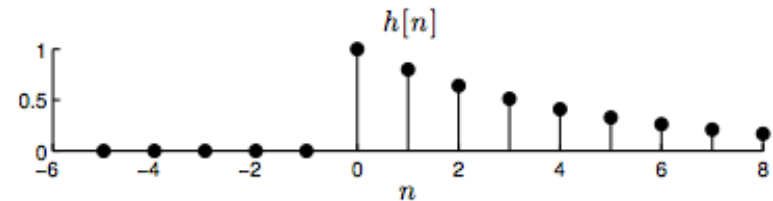- Example: h FIR $\Rightarrow$ BIBO

Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$

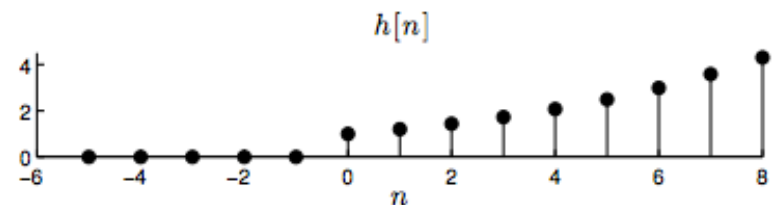- Impulse response: $h[n] = \alpha^n\, u[n]$

Example

- Example: Recall the recursive average system $\quad y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$

- Impulse response: $h[n] = \alpha^n\, u[n]$

- For $|\alpha| < 1$

$$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty \;\Rightarrow\; \text{BIBO}$$

- For $|\alpha| > 1$

$$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \infty \;\Rightarrow\; \text{not BIBO}$$

Difference Equations

❑ Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$

Difference Equations

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$$y[n] = x[n] + y[n-1]$$

$$\boxed{y[n] - y[n-1] = x[n]}$$

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$
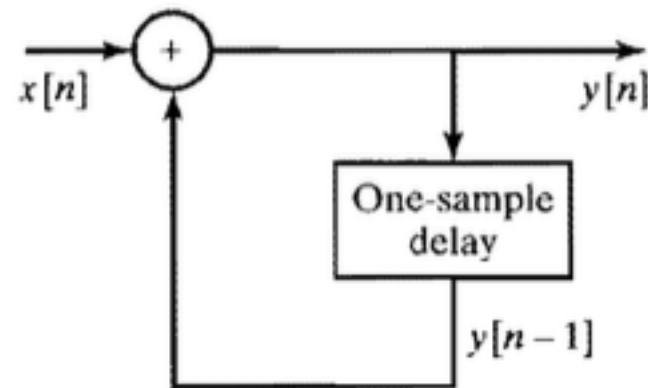
Difference Equations

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$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Example: Difference Equation

❑ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

❑ Let $M_1 = 0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Big Ideas

❏ LTI Systems are a special class of systems with significant signal processing applications

  ▪ Can be characterized by the impulse response

❏ LTI System Properties

  ▪ Causality and stability can be determined from impulse response

❏ Difference equations suggest implementation of systems

  ▪ Give insight into complexity of system

Admin

❑ HW 1 posted tomorrow

- Due 2/3 at midnight
- Submit in Canvas