

ESE 531: Digital Signal Processing

Lec 5: January 31, 2019
z-Transform



Lecture Outline

- ❑ LTI System Frequency Response
- ❑ z-Transform
 - Regions of convergence (ROC) & properties
 - z-Transform properties
- ❑ Inverse z-transform

LTI System Frequency Response

- (DT) Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

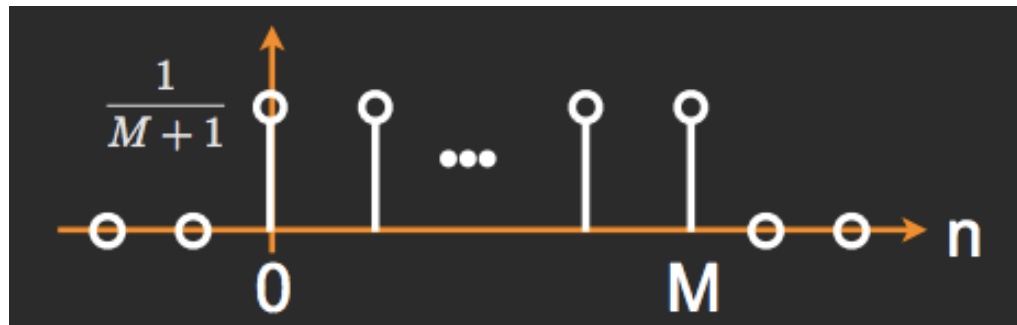
Example: Moving Average

□ Moving Average Filter

- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

Impulse
response



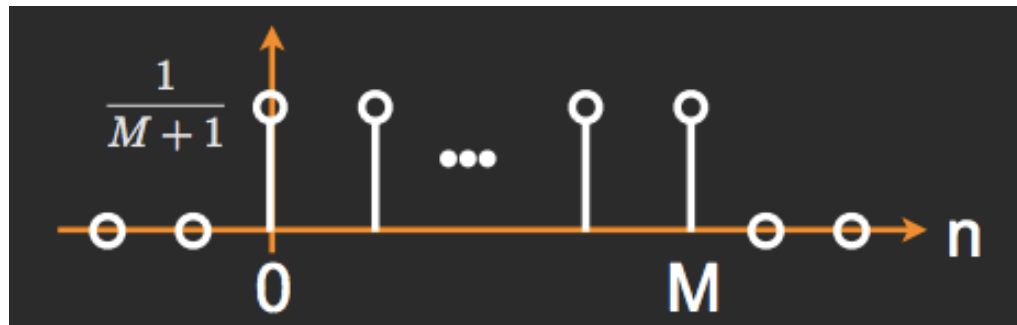
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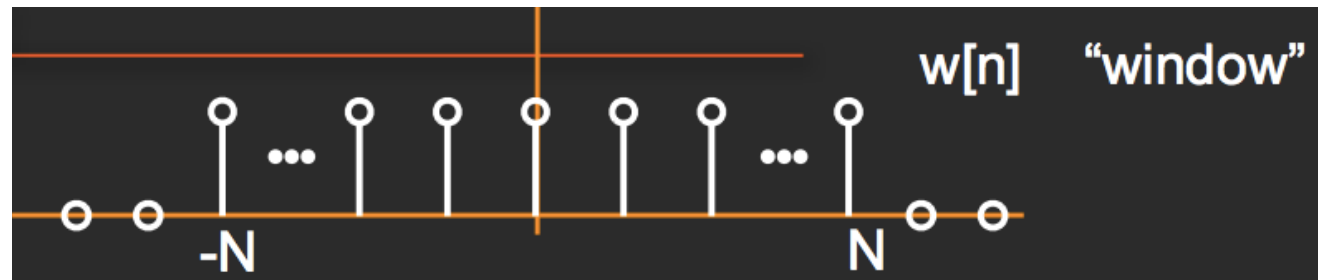
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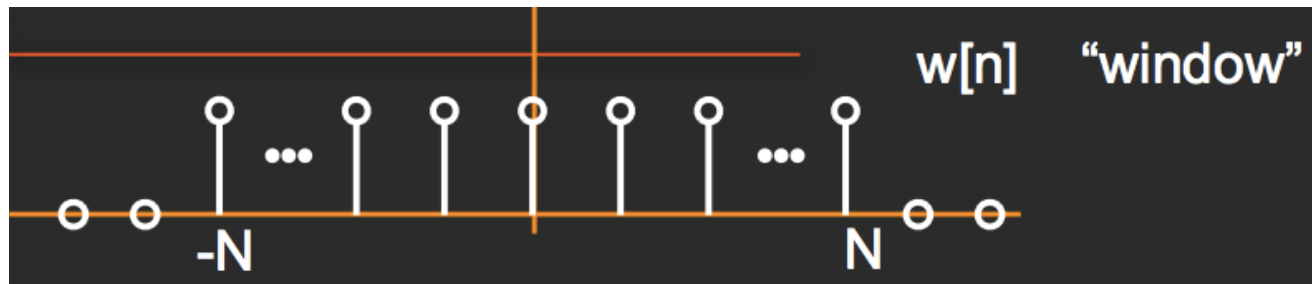
Impulse
response



Scaled & Time
Shifted window



Example: Moving Average



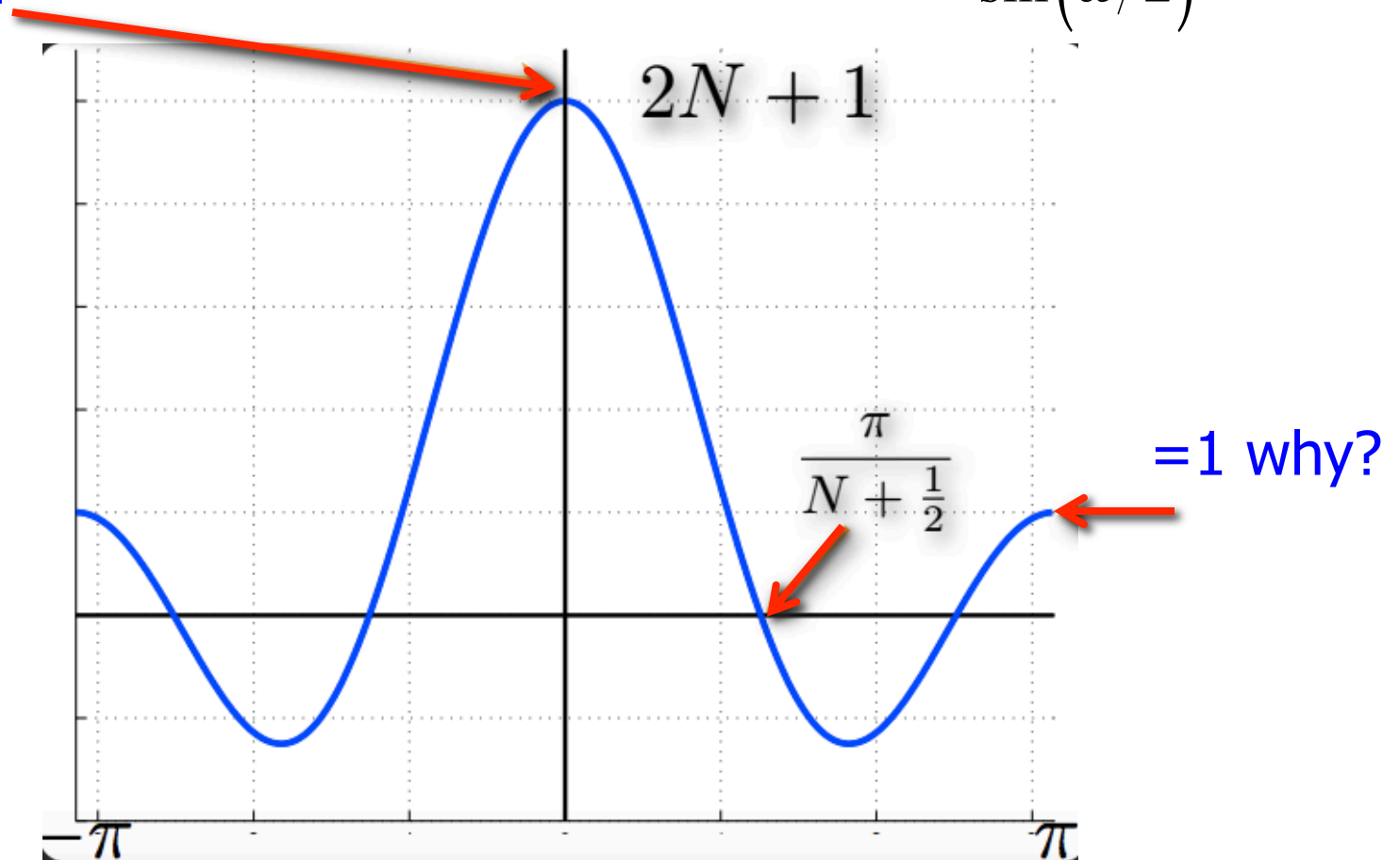
$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Window DTFT

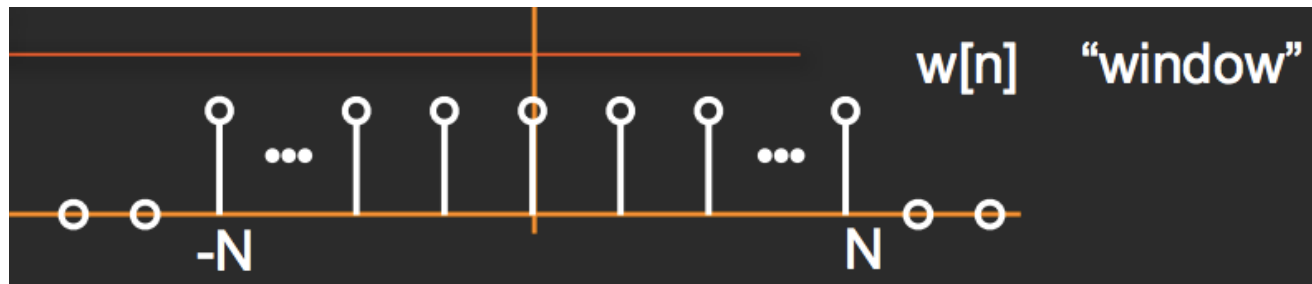
$$W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Also, $\sum x[n]$

Plot for $N=2$



Example: Moving Average

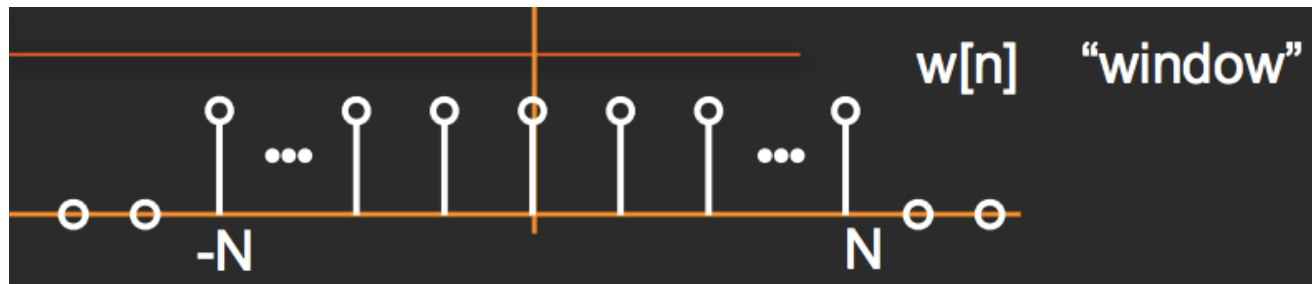


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N + 1/2)\omega\right)}{\sin(\omega/2)}$$



$$h[n] =$$

Example: Moving Average

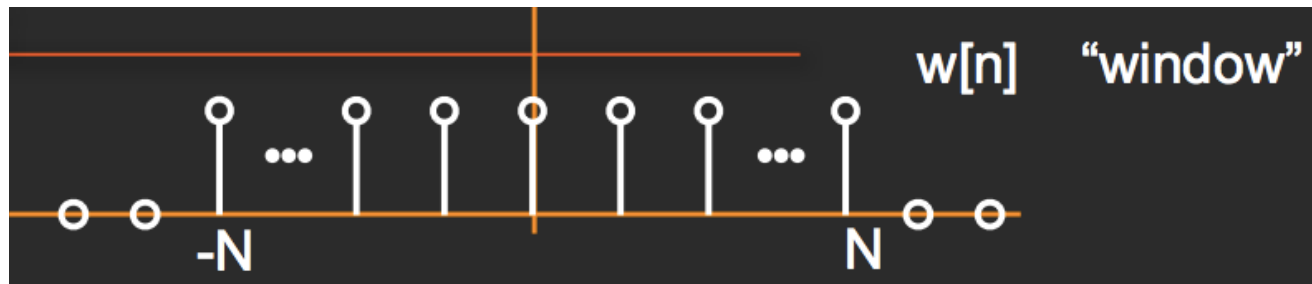


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) =$$

Example: Moving Average



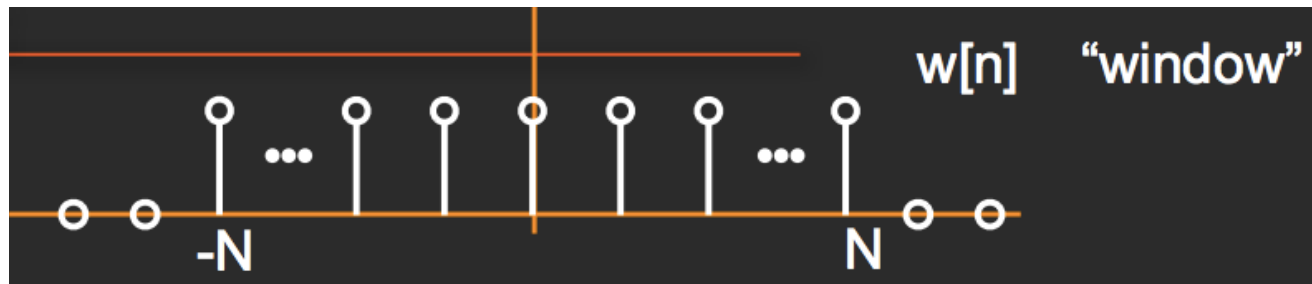
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$$x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

Example: Moving Average

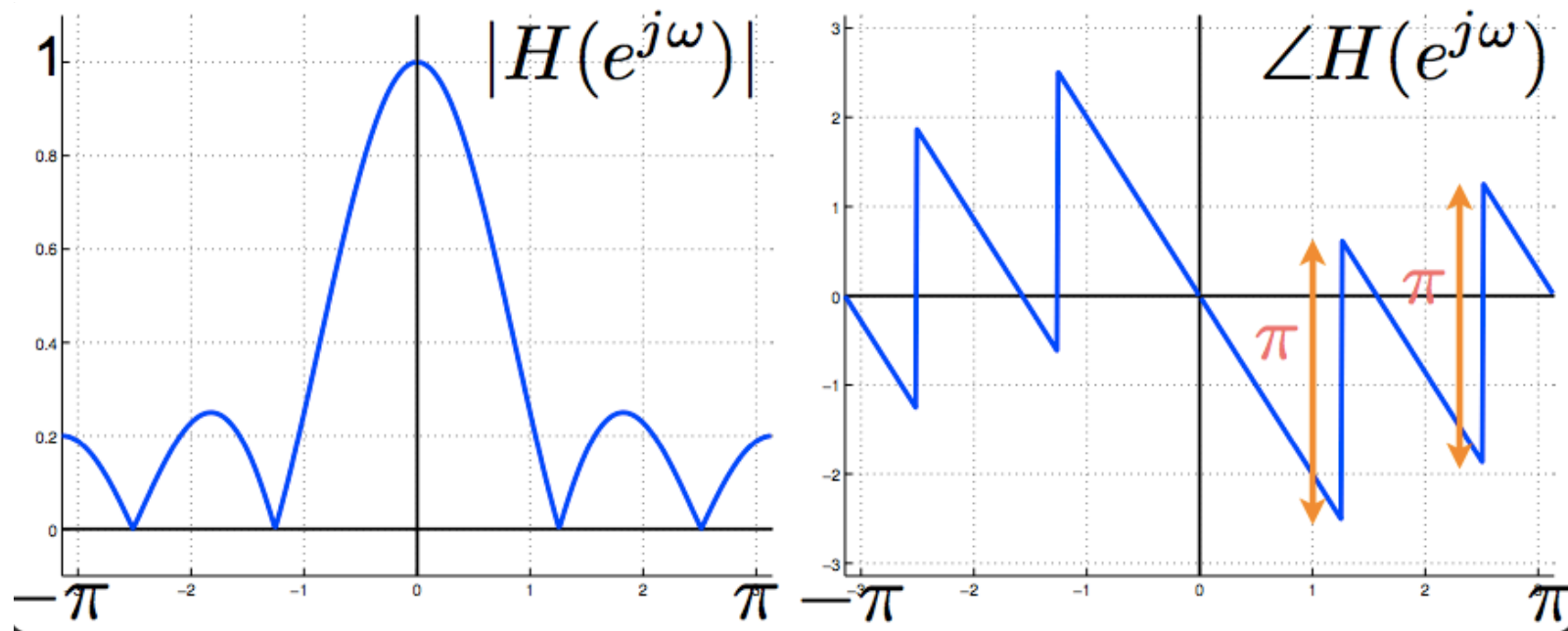


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$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Moving Average

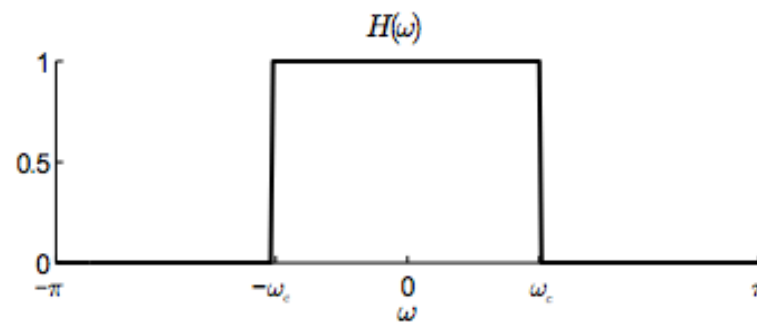


$M=4$
($N=2$)

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

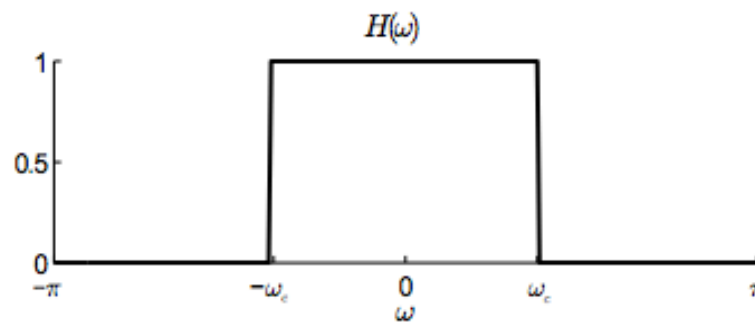
$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



Example: Ideal Low-Pass Filter

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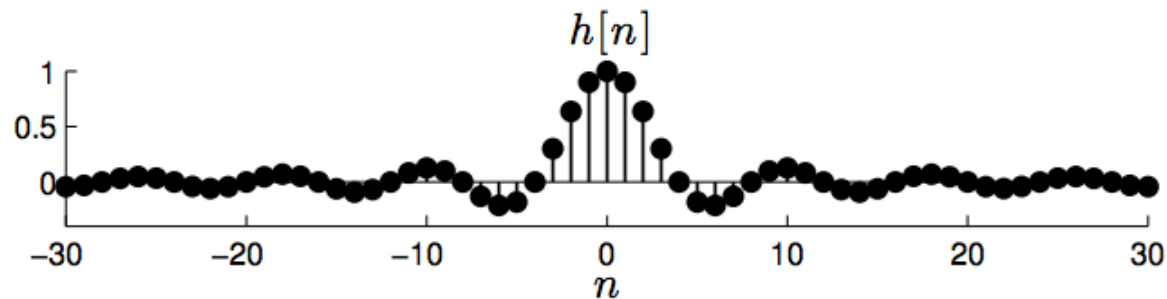
- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi j n} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

Example: Ideal Low-Pass Filter

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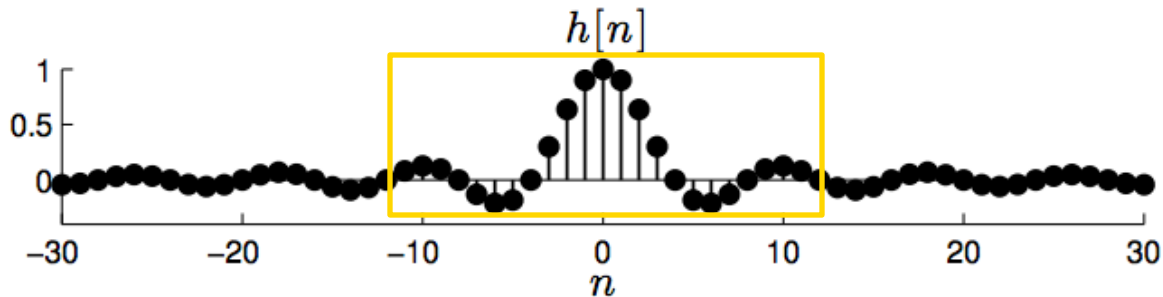


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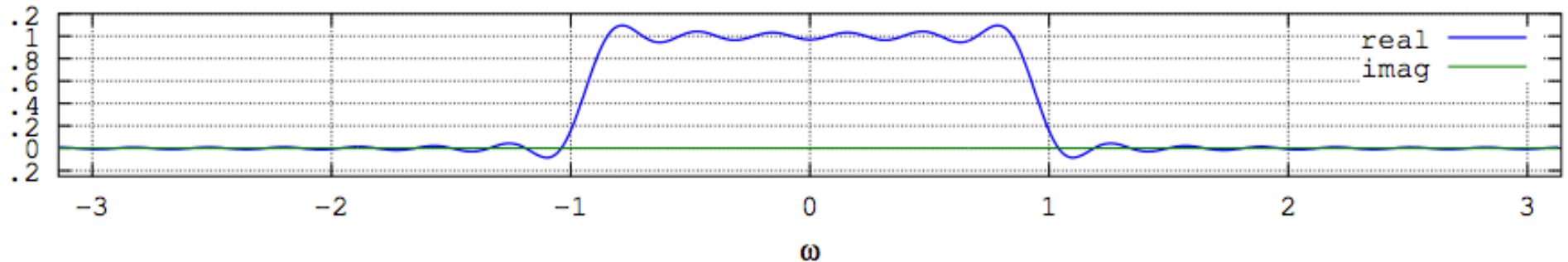
$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate
and shift

$$h_{LP}[n] = w_N[n - N] \cdot h[n - N]$$

Example: Practical LP Filter



- ❑ Pass band smeared and rippled
 - Smearing determined by width of main lobe
 - Rippling determined by size of main side lobes

z-Transform



z-Transform

- ❑ The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- ❑ Very useful for designing and analyzing signal processing systems
- ❑ Properties are very similar to the DTFT with a few caveats



Reminder: DTFT Definition

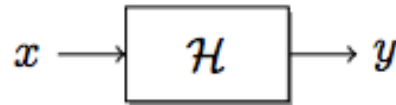
- ❑ The core “basis functions” (I.e eigenfunctions) of the DTFT are the complex sinusoids $e^{j\omega n}$ with arbitrary frequencies ω
- ❑ The sinusoids $e^{j\omega n}$ are eigenvectors of LTI systems for infinite-length signals

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \quad -\pi \leq \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega, \quad -\infty \leq n < \infty$$

Reminder: Frequency Response of LTI System

- We can use the DTFT to characterize an LTI system



$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

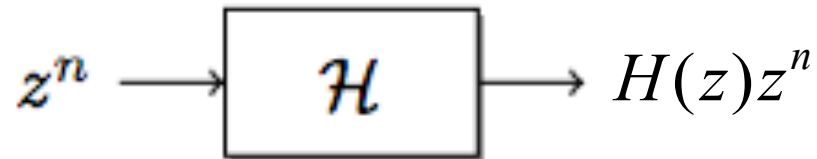
- and relate the DTFTs of the input and output

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad H(\omega) = \sum_{m=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$Y(\omega) = X(\omega)H(\omega)$$

Complex Exponentials as Eigenfunctions

- Fact: A more general set of eigenfunctions of an LTI system are the complex exponentials z^n , $z \in \mathbb{C}$

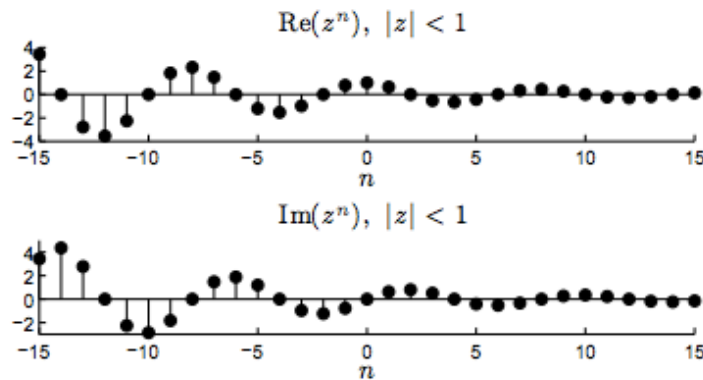


Reminder: Complex Exponentials

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

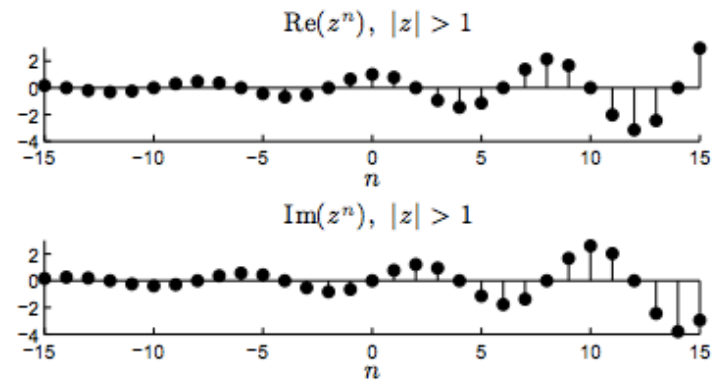
- $|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)
- $e^{j\omega n}$ is a **complex sinusoid**

$$|z| < 1$$



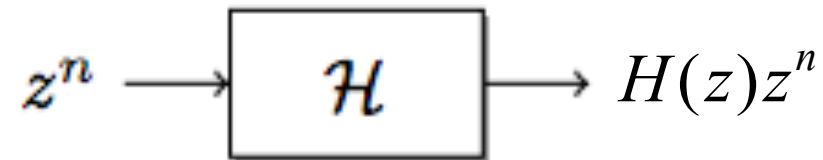
Bounded

$$|z| > 1$$



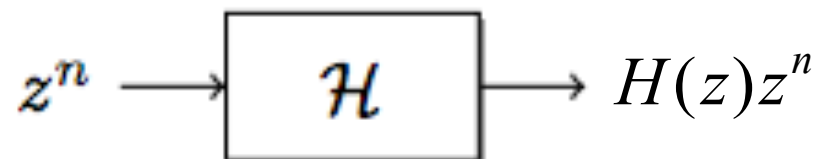
Unbounded

Proof: Complex Exponentials as Eigenfunctions



- Prove by computing the convolution with input $x[n] = z^n$

Proof: Complex Exponentials as Eigenfunctions



- Prove by computing the convolution with input $x[n] = z^n$

$$\begin{aligned} z^n * h[n] &= \sum_{m=-\infty}^{\infty} z^{n-m} h[m] = \sum_{m=-\infty}^{\infty} z^n z^{-m} h[m] \\ &= \left(\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right) z^n \\ &= H(z) z^n \checkmark \end{aligned}$$



z-Transform

- Define the **forward z-transform** of $x[n]$ as

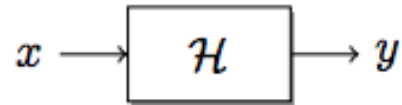
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The core “basis functions” of the z-transform are the complex exponentials z^n with arbitrary $z \in \mathbb{C}$; these are the eigenfunctions of LTI systems for infinite-length signals
- **Notation abuse alert:** We use $X(\bullet)$ to represent both the DTFT $X(\omega)$ and the z-transform $X(z)$; they are, in fact, intimately related

$$X_{\text{DTFT}}(\omega) = X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

Transfer Function of LTI System

- We can use the z-Transform to characterize an LTI system



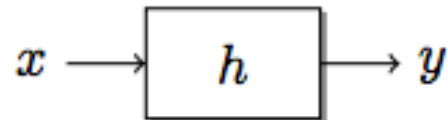
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- and relate the z-transforms of the input and output

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

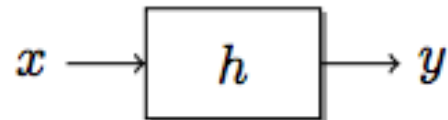
$$Y(z) = X(z) H(z)$$

Proof: Transfer Function of LTI Systems



- Compute z-transform of output by computing the convolution of impulse response with input $x[n] = z^n$

Proof: Transfer Function of LTI Systems



- Compute z-transform of output by computing the convolution of impulse response with input $x[n] = z^n$

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m] h[n-m] \right) z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{n=-\infty}^{\infty} h[n-m] z^{-n} \right) \quad (\text{let } r = n - m) \\ &= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{r=-\infty}^{\infty} h[r] z^{-r-m} \right) = \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m} \right) \left(\sum_{r=-\infty}^{\infty} h[r] z^{-r} \right) \\ &= X(z) H(z) \quad \checkmark \end{aligned}$$



Z-transform

What are we missing?

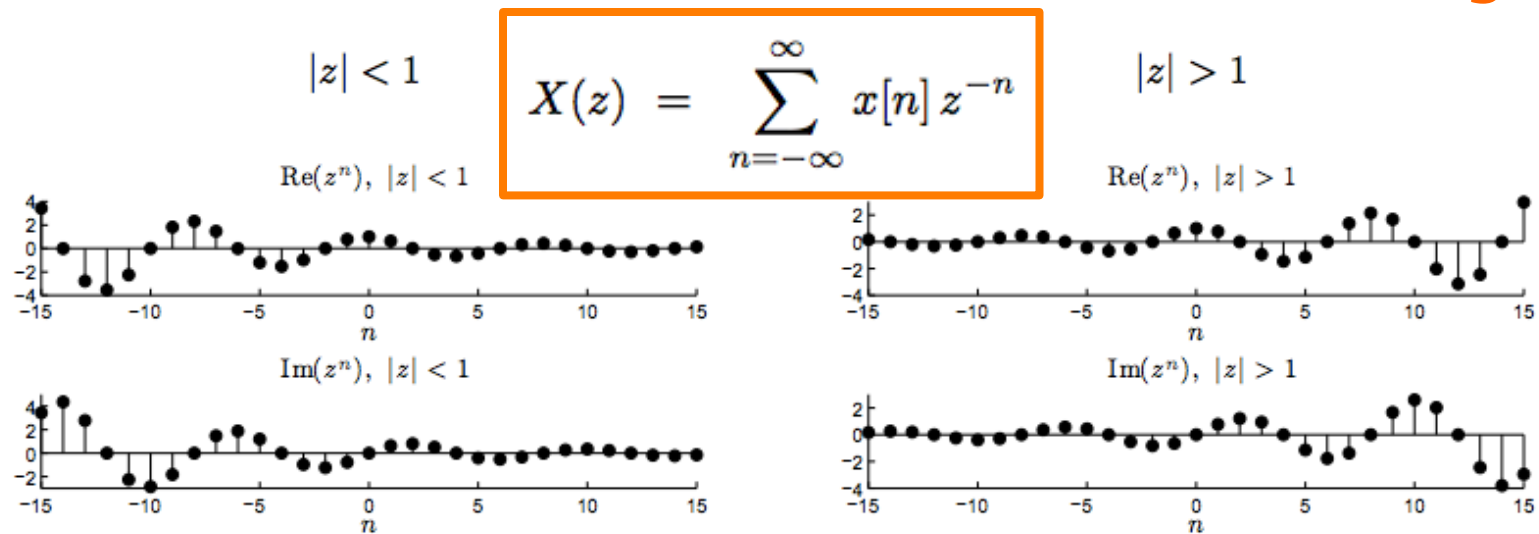
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Z-Transform

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

- $|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)
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What are we missing?



Bounded

Unbounded

Region of Convergence (ROC)

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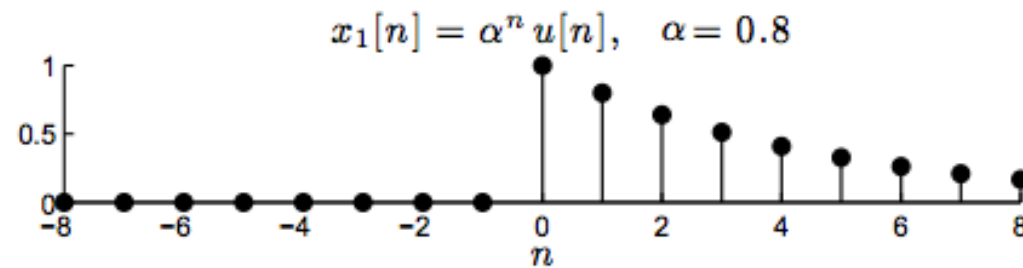
DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its z -transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n] z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n] z^{-n}| < \infty$$

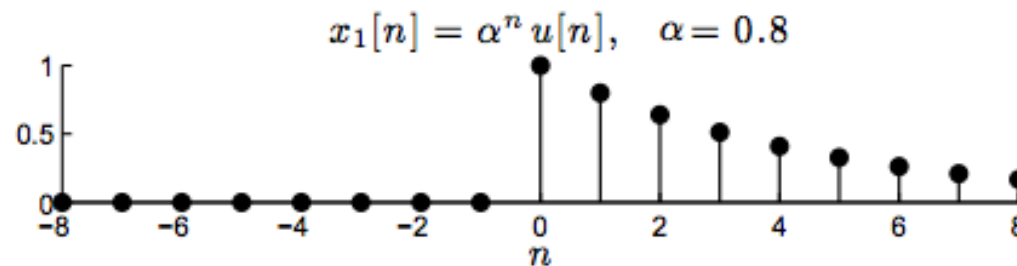
ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal) Right-sided sequence
- Example for $\alpha = 0.8$



ROC Example 1

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- The **forward** z -transform of $x_1[n]$

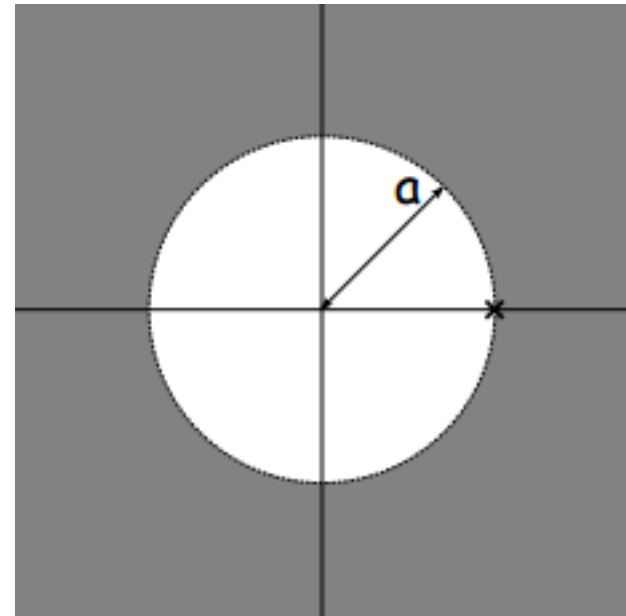
$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$

ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal)

$$ROC = \{z : |z| > |\alpha|\}$$



- The **forward** z -transform of $x_1[n]$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$



ROC Example 1

□ What is the DTFT of $x_1[n] = a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}, \quad -\pi \leq \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \leq n < \infty$$

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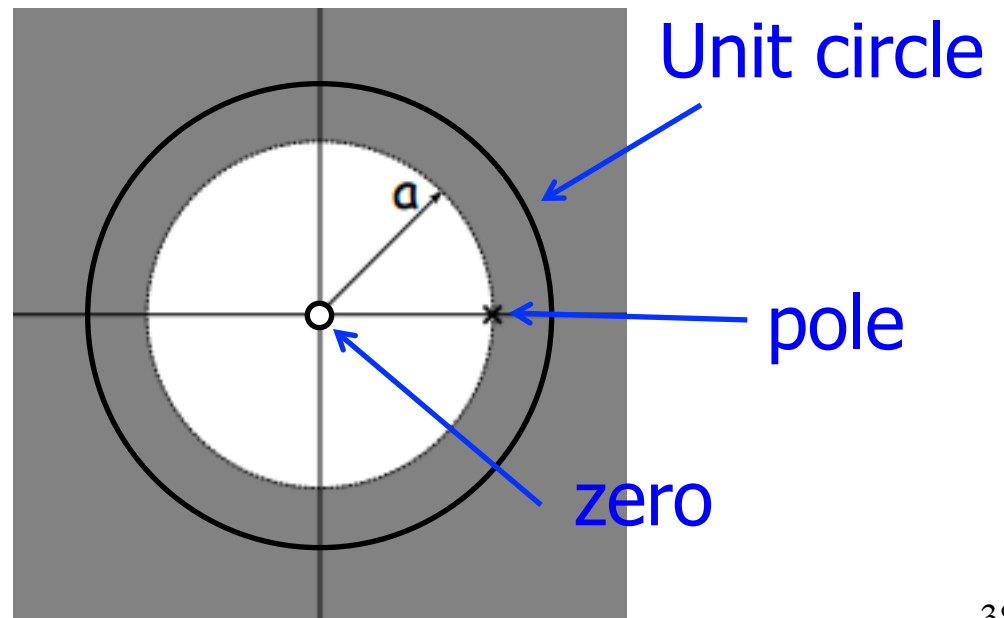
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$$X_1(z) = \frac{z}{z - a}$$

$$ROC = \{z : |z| > |a|\}$$





ROC Example 2

□ What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$



ROC Example 2

□ What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

□ Hint:

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}$$



ROC Example 3

□ What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

Left-sided sequence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The z-transform without ROC does not uniquely define a sequence!



ROC Example 4

□ What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$



ROC Example 4

□ What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

□ Hint:

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}, \quad ROC = \{z : |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}, \quad ROC = \{z : |z| < |a|\}$$



ROC Example 5

□ What is the z-transform of $x_5[n]$? ROC?

$$x_5[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

two-sided sequence



ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$



ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

finite length sequence

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}} \quad \text{Zero cancels pole}$$
$$= \prod_{k=1}^{M-1} (1 - ae^{j2\pi k/M} z^{-1})$$

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

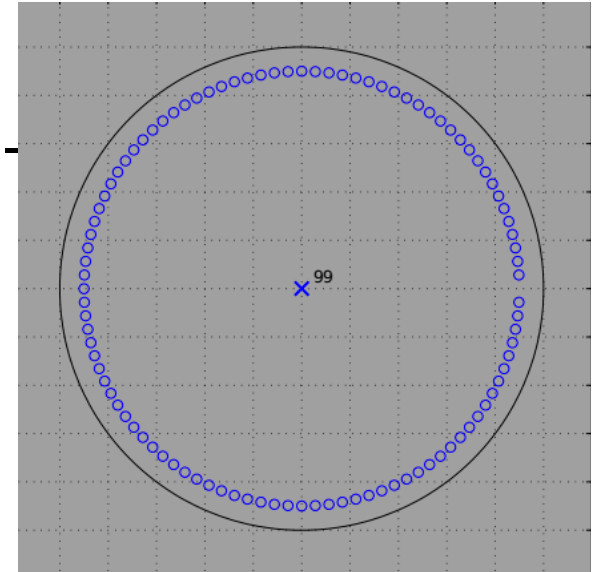
$$x_6[n] = a^n u[n] u[-n + M]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}}$$

Zero cancels pole

$$= \prod_{k=1}^{M-1} (1 - ae^{j2\pi k/M} z^{-1})$$

M=100



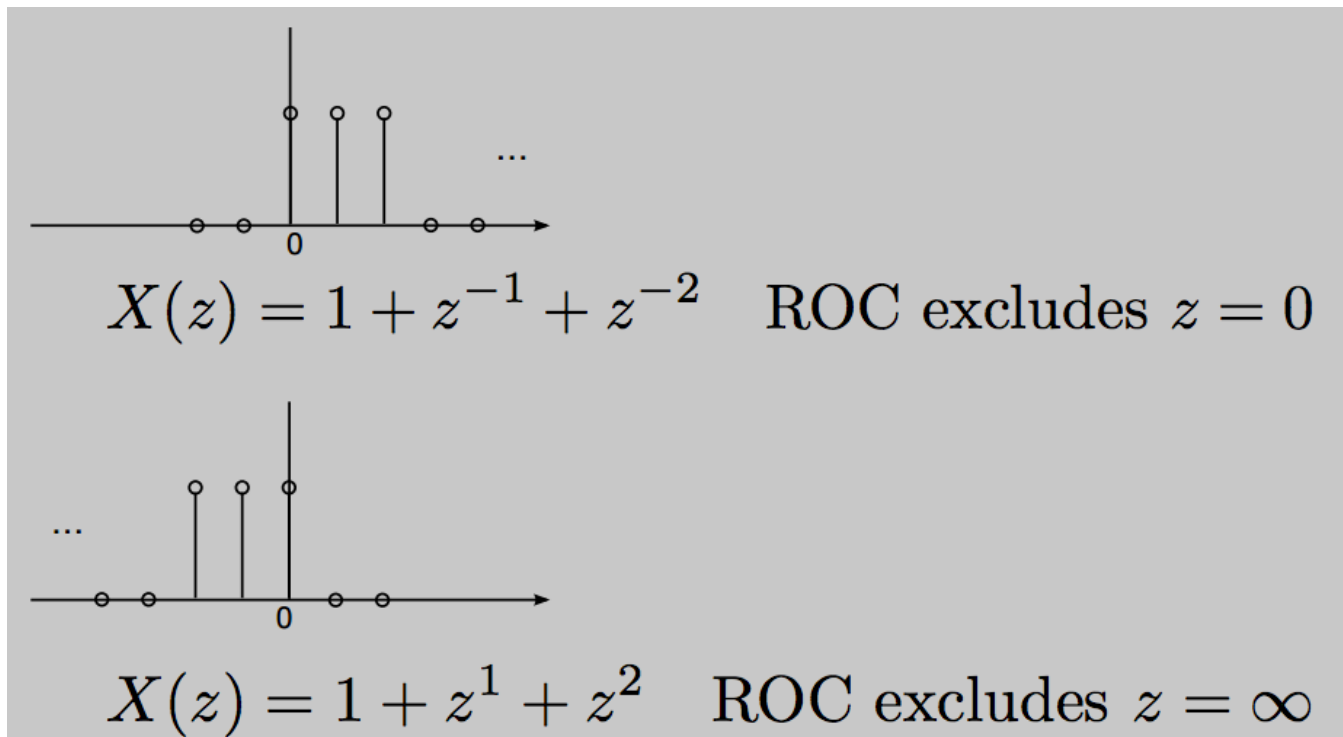


Properties of ROC

- ❑ For right-sided sequences: ROC extends outward from the outermost pole to infinity
 - Examples 1,2
- ❑ For left-sided: inwards from inner most pole to zero
 - Example 3
- ❑ For two-sided, ROC is a ring - or do not exist
 - Examples 4,5

Properties of ROC

- For finite duration sequences, ROC is the entire z -plane, except possibly $z=0$, $z=\infty$ (Example 6)





Formal Properties of the ROC

❑ PROPERTY 1:

- The ROC will either be of the form $0 < r_R < |z|$, or $|z| < r_L < \infty$, or, in general the annulus, i.e., $0 < r_R < |z| < r_L < \infty$.

❑ PROPERTY 2:

- The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z -transform of $x[n]$ includes the unit circle.

❑ PROPERTY 3:

- The ROC cannot contain any poles.

❑ PROPERTY 4:

- If $x[n]$ is *a finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 < n < N_2 < \infty$, then the ROC is the entire z -plane, except possibly $z = 0$ or $z = \infty$.



Formal Properties of the ROC

❑ PROPERTY 5:

- If $x[n]$ is a *right-sided sequence*, the ROC extends outward from the *outermost* finite pole in $X(z)$ to (and possibly including) $z = \infty$.

❑ PROPERTY 6:

- If $x[n]$ is a *left-sided sequence*, the ROC extends inward from the *innermost* nonzero pole in $X(z)$ to (and possibly including) $z=0$.

❑ PROPERTY 7:

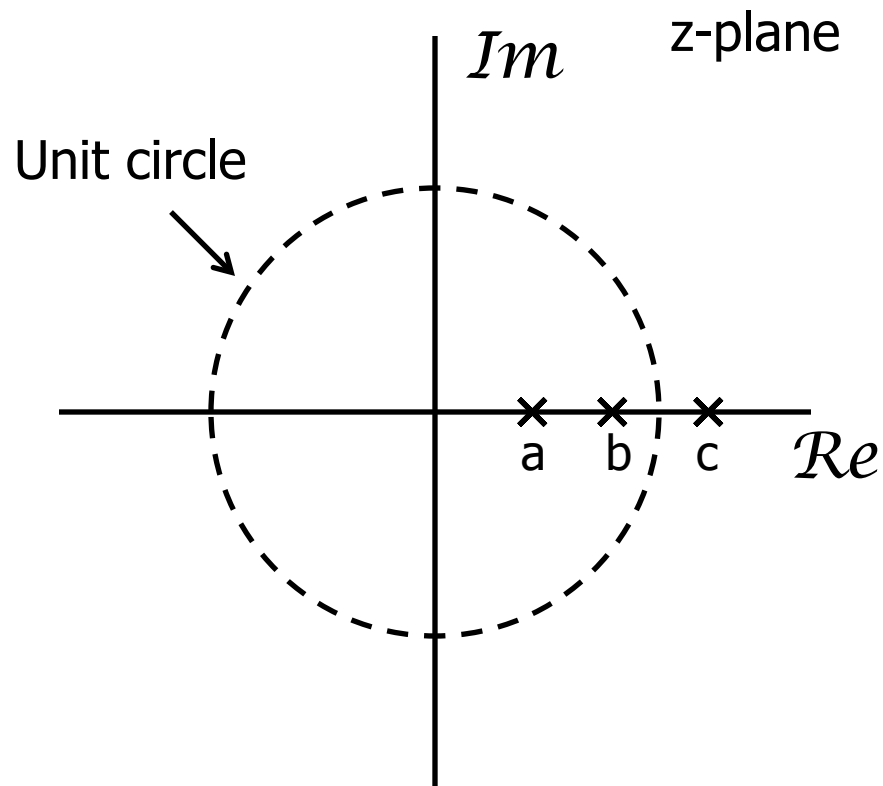
- A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z -plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

❑ PROPERTY 8:

- The ROC must be a connected region.

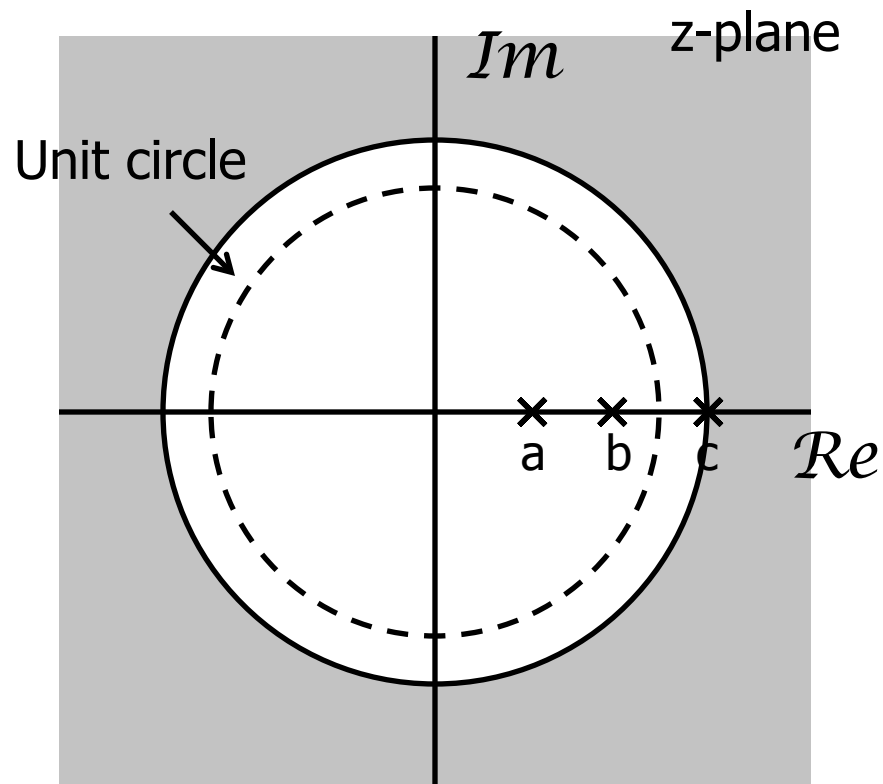
Example: ROC from Pole-Zero Plot

□ How many possible ROCs?



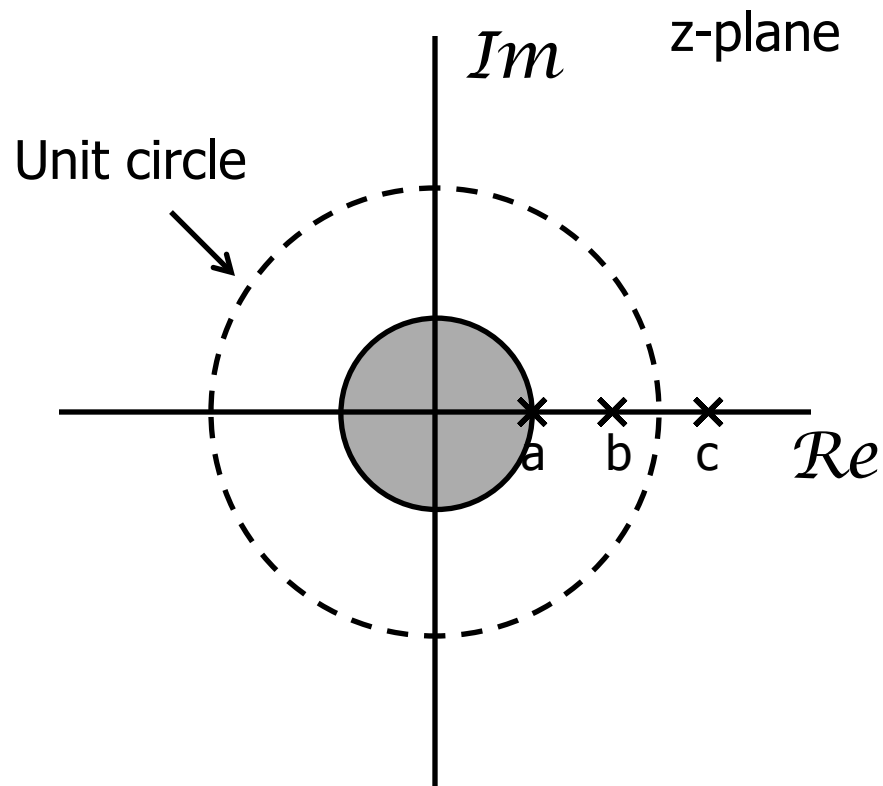
Example: ROC from Pole-Zero Plot

ROC 1: right-sided



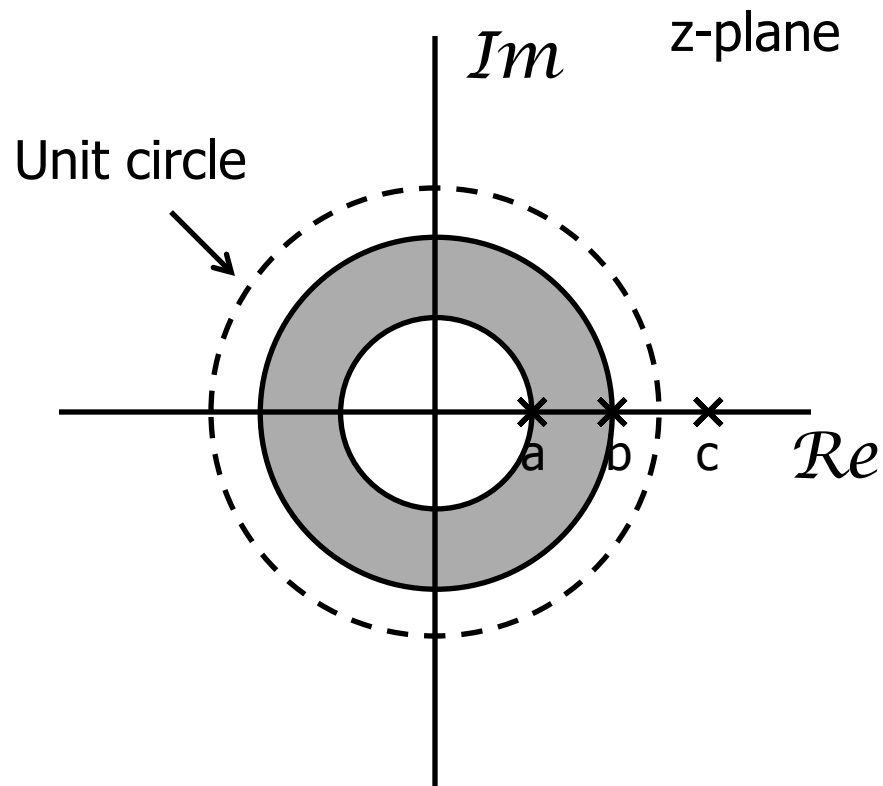
Example: ROC from Pole-Zero Plot

ROC 2: left-sided



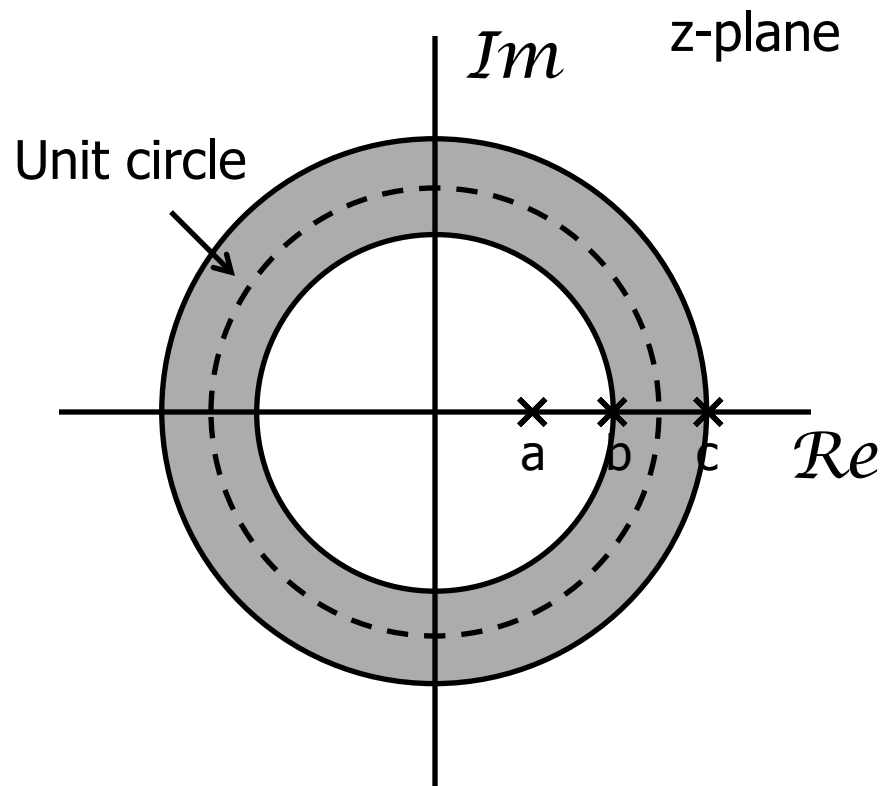
Example: ROC from Pole-Zero Plot

ROC 3: two-sided



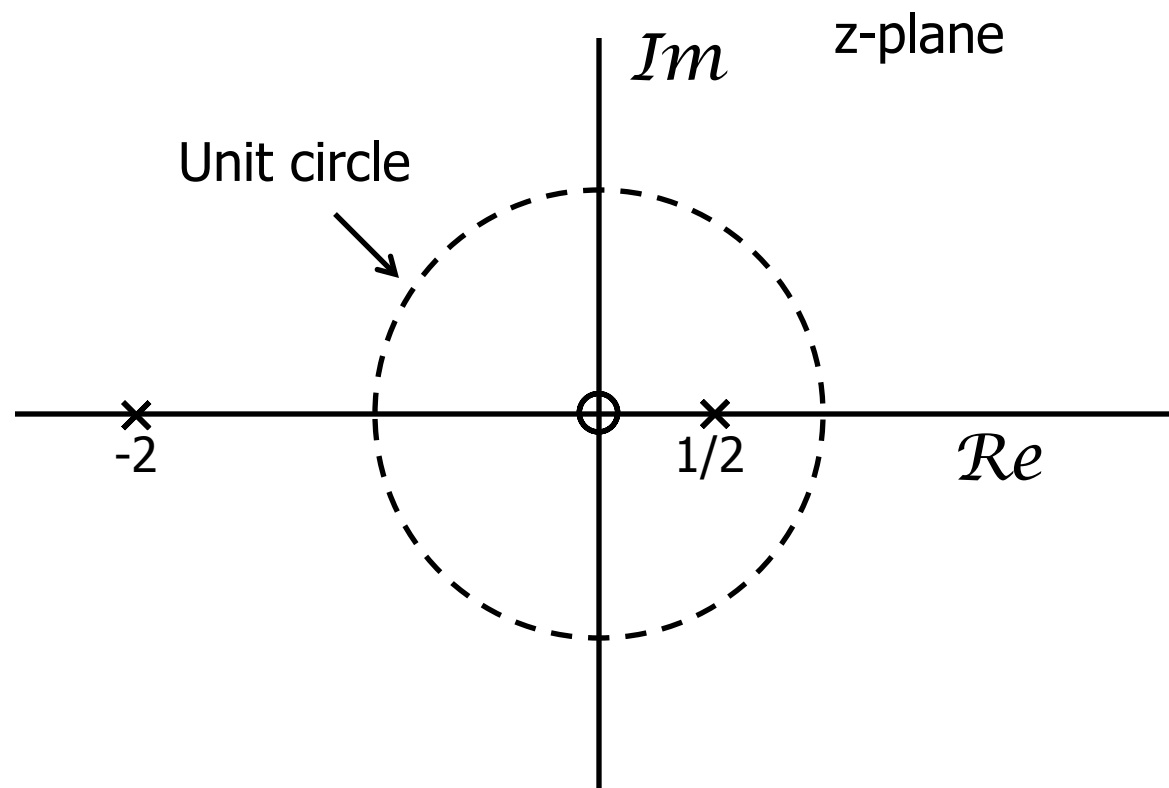
Example: ROC from Pole-Zero Plot

ROC 4: two-sided



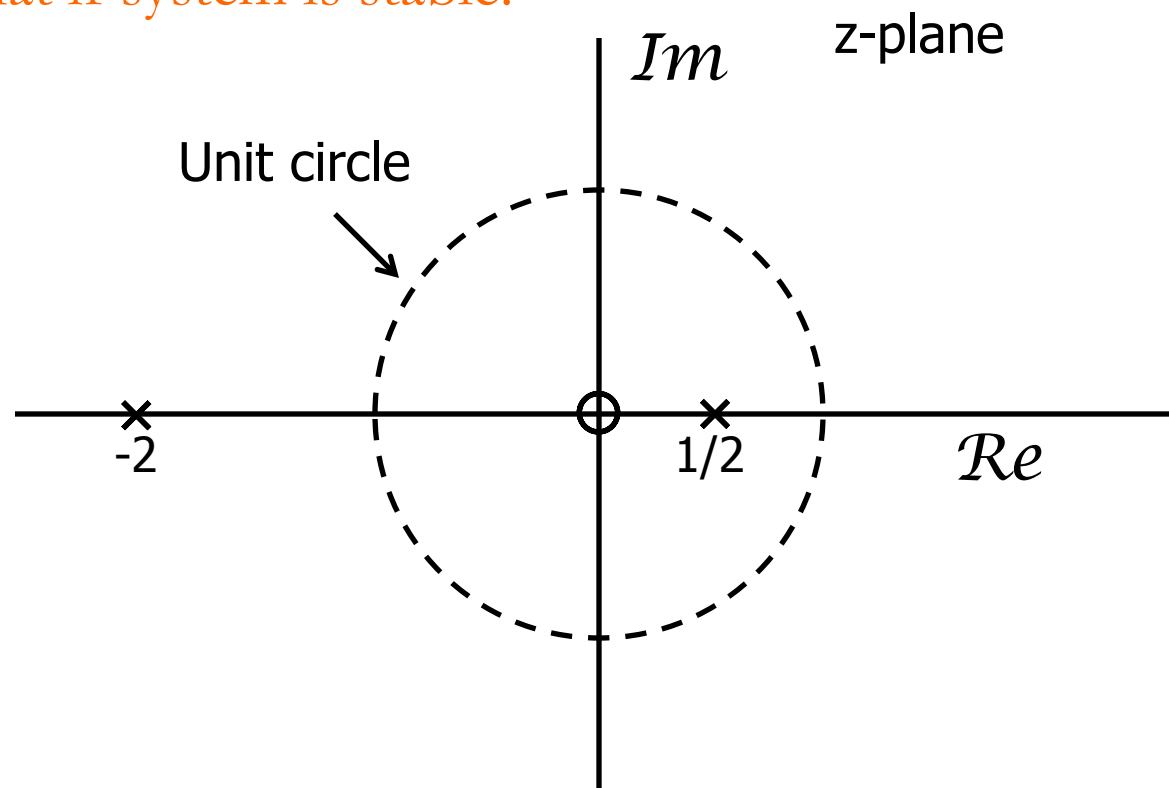
Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?



Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?
 - What if system is stable?

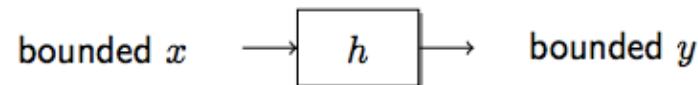


BIBO Stability Revisited

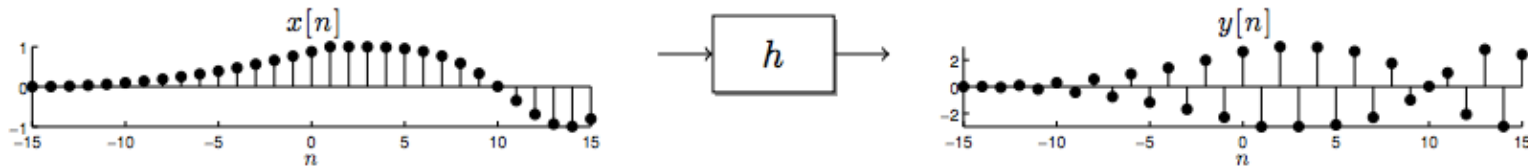
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

DEFINITION

An LTI system is **bounded-input bounded-output** (BIBO) stable if every bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_{\infty} < \infty$ and $\|y\|_{\infty} < \infty$

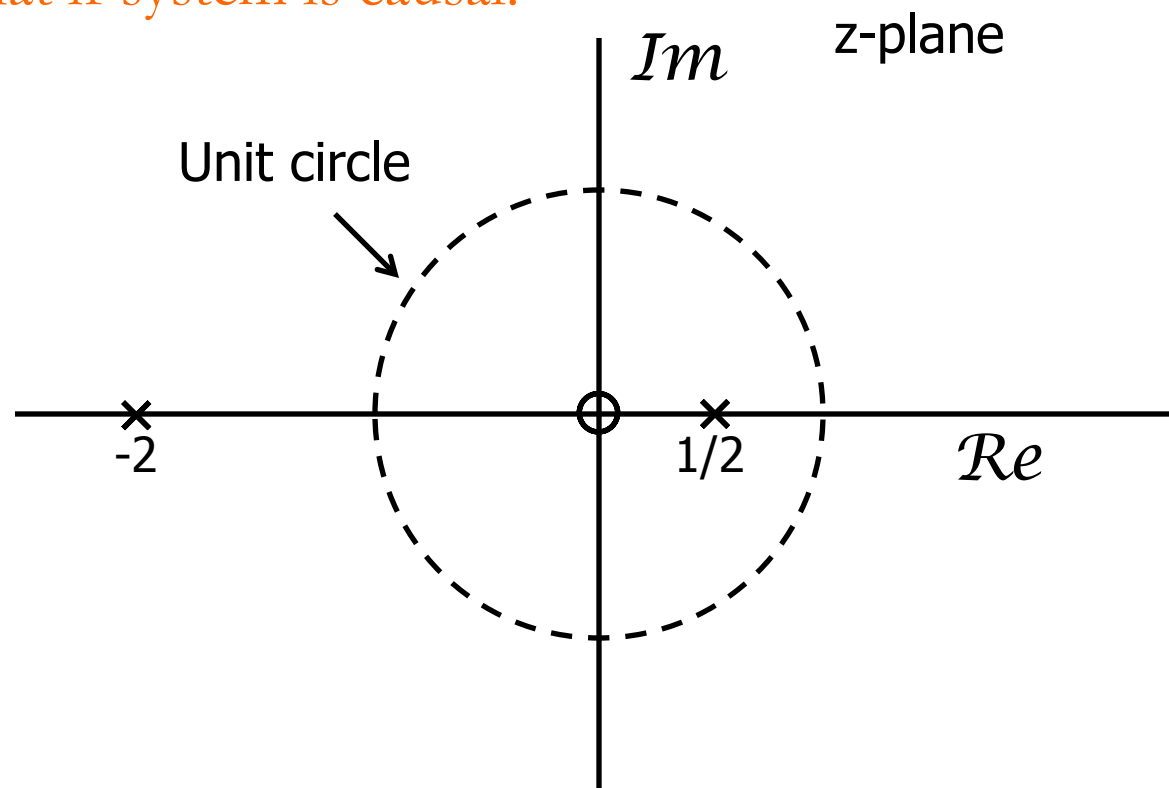


- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?
 - What if system is causal?





Properties of z-Transform

- Linearity:

$$ax_1[n] + bx_2[n] \Leftrightarrow aX_1(z) + bX_2(z)$$

- Time shifting:

$$x[n] \Leftrightarrow X(z)$$

$$x[n - n_d] \Leftrightarrow z^{-n_d} X(z)$$

- Multiplication by exponential sequence

$$x[n] \Leftrightarrow X(z)$$

$$z_0^n x[n] \Leftrightarrow X\left(\frac{z}{z_0}\right)$$



Properties of z-Transform

□ Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

□ Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz}$$

□ Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$$\text{ROC}_Y \text{ at least } \text{ROC}_X \wedge \text{ROC}_H$$



Big Ideas

□ z-Transform

- Uses complex exponential eigenfunctions to represent discrete time sequence
 - DTFT is z-Transform where $z=e^{j\omega}$, $|z|=1$
- Draw pole-zero plots
- Must specify region of convergence (ROC)

□ z-Transform properties

- Similar to DTFT



Admin

- ❑ HW 1 due Sunday at midnight
 - Submit **single pdf** in Canvas
 - Don't need to submit .m file, just the code in your pdf
- ❑ HW 2 posted Sunday
- ❑ Advice
 - Want to try and develop intuition
 - Practice problems at end of chapter with answers in the text