

ESE 531: Digital Signal Processing

Lec 6: February 5, 2019

Inverse z-Transform



Lecture Outline

- ❑ z-Transform
 - Tie up loose ends
 - Regions of convergence properties
 - Transform properties
- ❑ Inverse z-transform
 - Inspection
 - Partial fraction
 - Power series expansion
- ❑ z-transform of difference equations

z-Transform



z-Transform

- Define the **forward z-transform** of $x[n]$ as

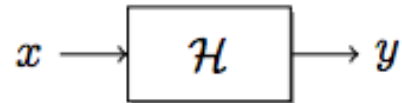
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The core “basis functions” of the z-transform are the complex exponentials z^n with arbitrary $z \in \mathbb{C}$; these are the eigenfunctions of LTI systems for infinite-length signals
- **Notation abuse alert:** We use $X(\bullet)$ to represent both the DTFT $X(\omega)$ and the z-transform $X(z)$; they are, in fact, intimately related

$$X_{\text{DTFT}}(\omega) = X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

Transfer Function of LTI System

- We can use the z-Transform to characterize an LTI system



$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- and relate the z-transforms of the input and output

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$Y(z) = X(z) H(z)$$

Region of Convergence (ROC)

Region of Convergence (ROC)

DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its z -transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n] z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n] z^{-n}| < \infty$$



Summary: Properties of ROC

- ❑ For right-sided sequences: ROC extends outward from the outermost pole to infinity
 - Examples 1,2
- ❑ For left-sided: inwards from inner most pole to zero
 - Example 3
- ❑ For two-sided, ROC is a ring - or does not exist
 - Examples 4,5



ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

finite length sequence

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}} \quad \text{Zero cancels pole}$$
$$= \prod_{k=1}^{M-1} (1 - ae^{j2\pi k/M} z^{-1})$$

Revisit: ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

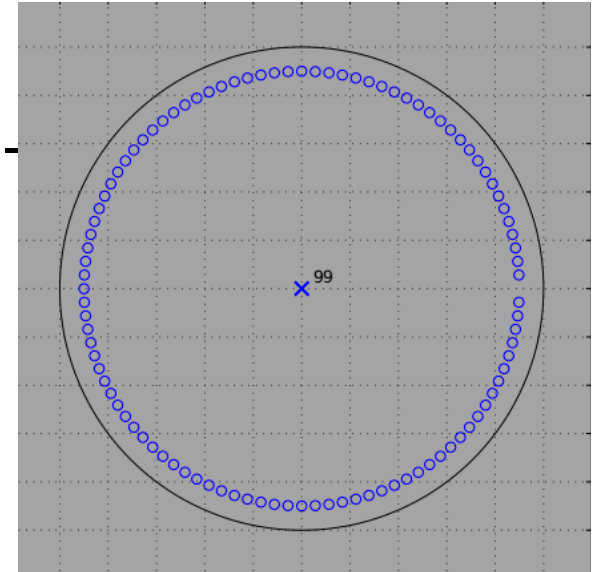
$$x_6[n] = a^n u[n] u[-n + M]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}}$$

Zero cancels pole

$$= \prod_{k=1}^{M-1} (1 - ae^{j2\pi k/M} z^{-1})$$

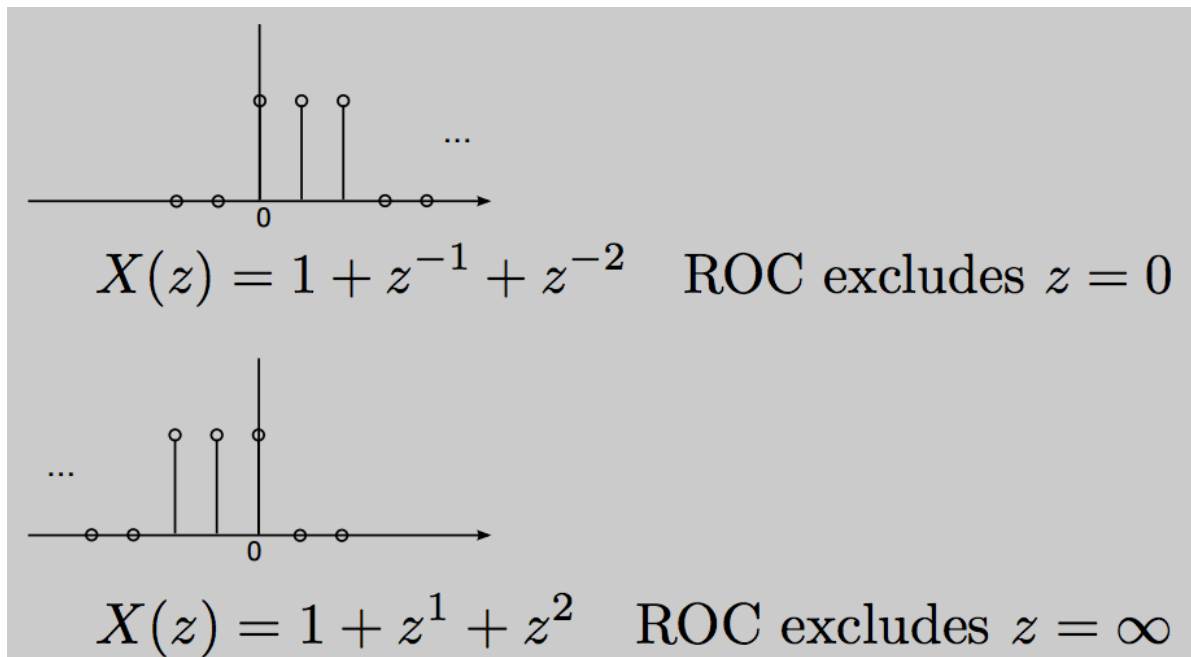
M=100



Properties of ROC

- For finite duration sequences, ROC is the entire z -plane, except possibly $z=0$, $z=\infty$

- Example 6





Formal Properties of the ROC

❑ PROPERTY 1:

- The ROC will either be of the form $0 < r_R < |z|$, or $|z| < r_L < \infty$, or, in general the annulus, i.e., $0 < r_R < |z| < r_L < \infty$.

❑ PROPERTY 2:

- The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z -transform of $x[n]$ includes the unit circle.

❑ PROPERTY 3:

- The ROC cannot contain any poles.

❑ PROPERTY 4:

- If $x[n]$ is *a finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 < n < N_2 < \infty$, then the ROC is the entire z -plane, except possibly $z = 0$ or $z = \infty$.



Formal Properties of the ROC

❑ PROPERTY 5:

- If $x[n]$ is a *right-sided sequence*, the ROC extends outward from the *outermost* finite pole in $X(z)$ to (and possibly including) $z = \infty$.

❑ PROPERTY 6:

- If $x[n]$ is a *left-sided sequence*, the ROC extends inward from the *innermost* nonzero pole in $X(z)$ to (and possibly including) $z=0$.

❑ PROPERTY 7:

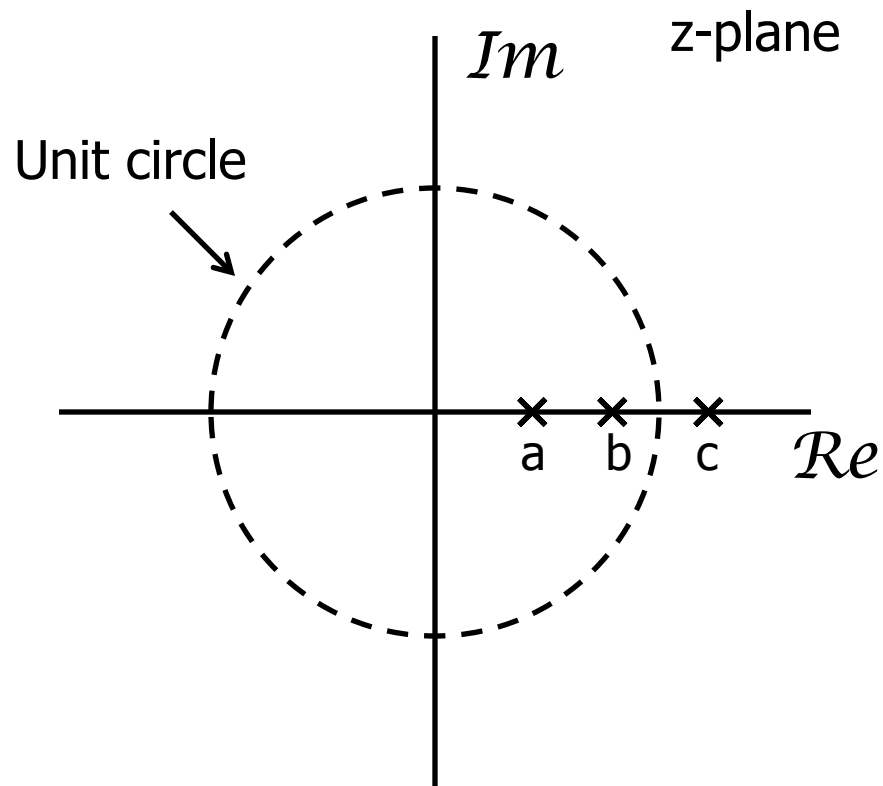
- A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z -plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

❑ PROPERTY 8:

- The ROC must be a connected region.

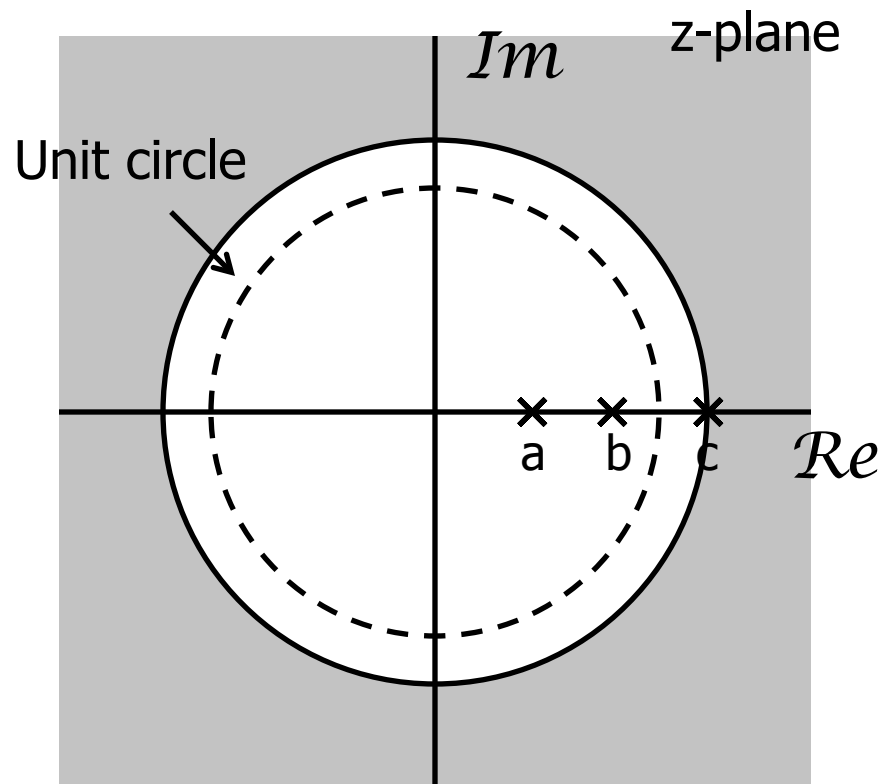
Example: ROC from Pole-Zero Plot

□ How many possible ROCs?



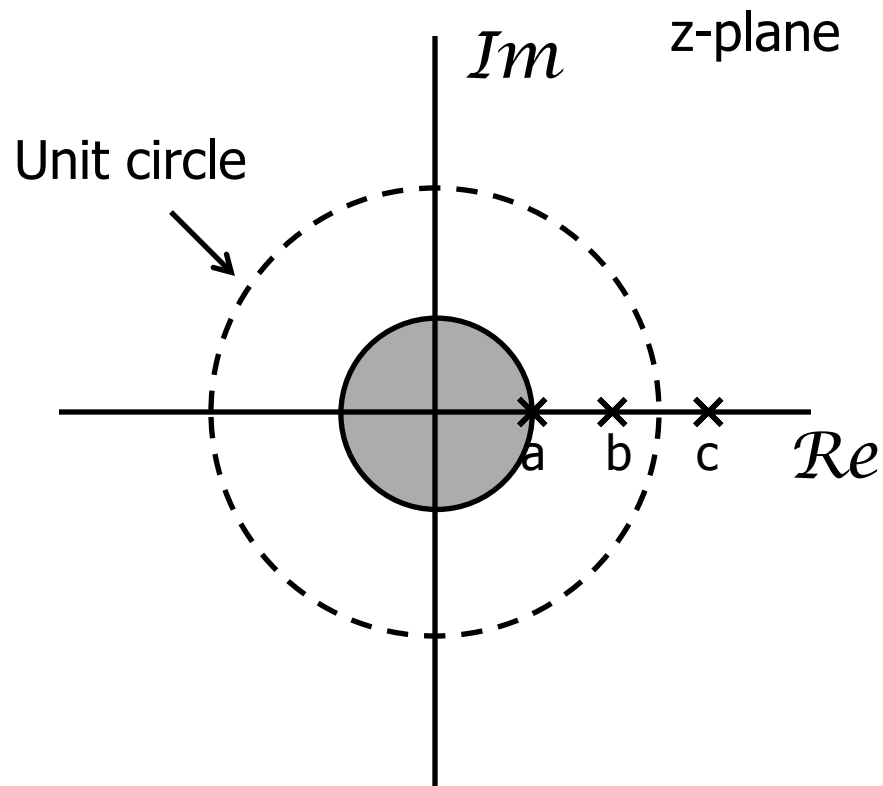
Example: ROC from Pole-Zero Plot

ROC 1: right-sided



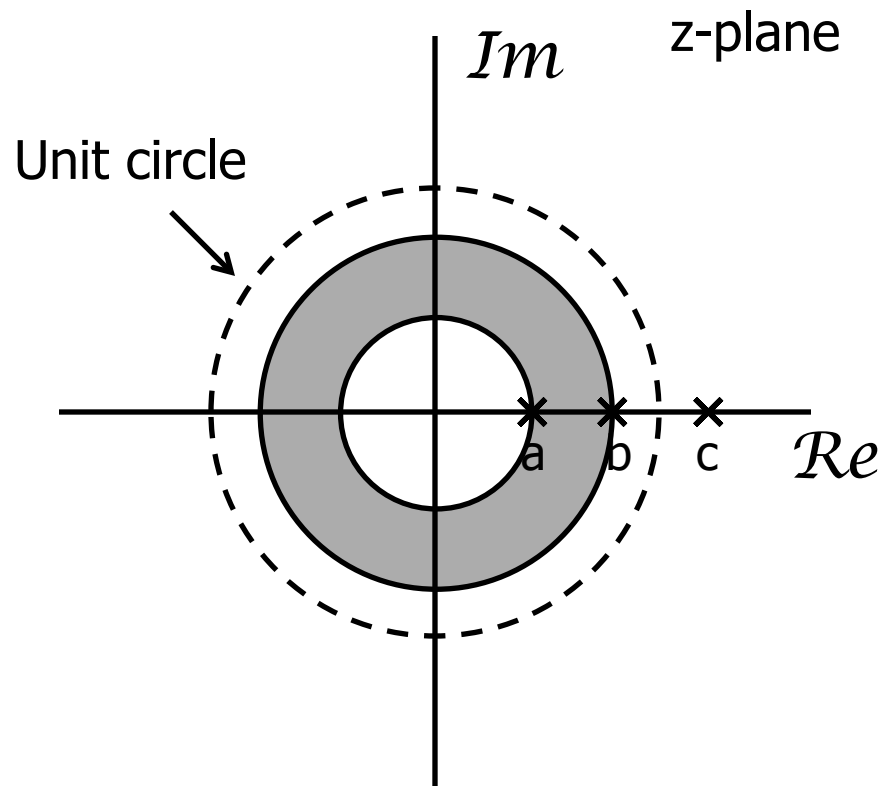
Example: ROC from Pole-Zero Plot

ROC 2: left-sided



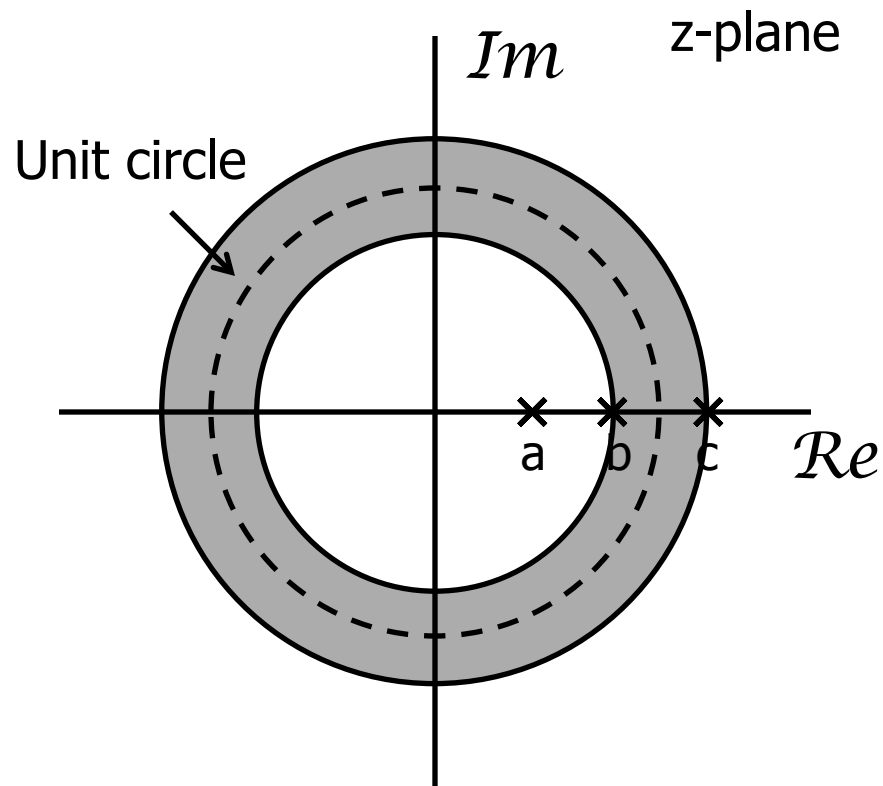
Example: ROC from Pole-Zero Plot

ROC 3: two-sided



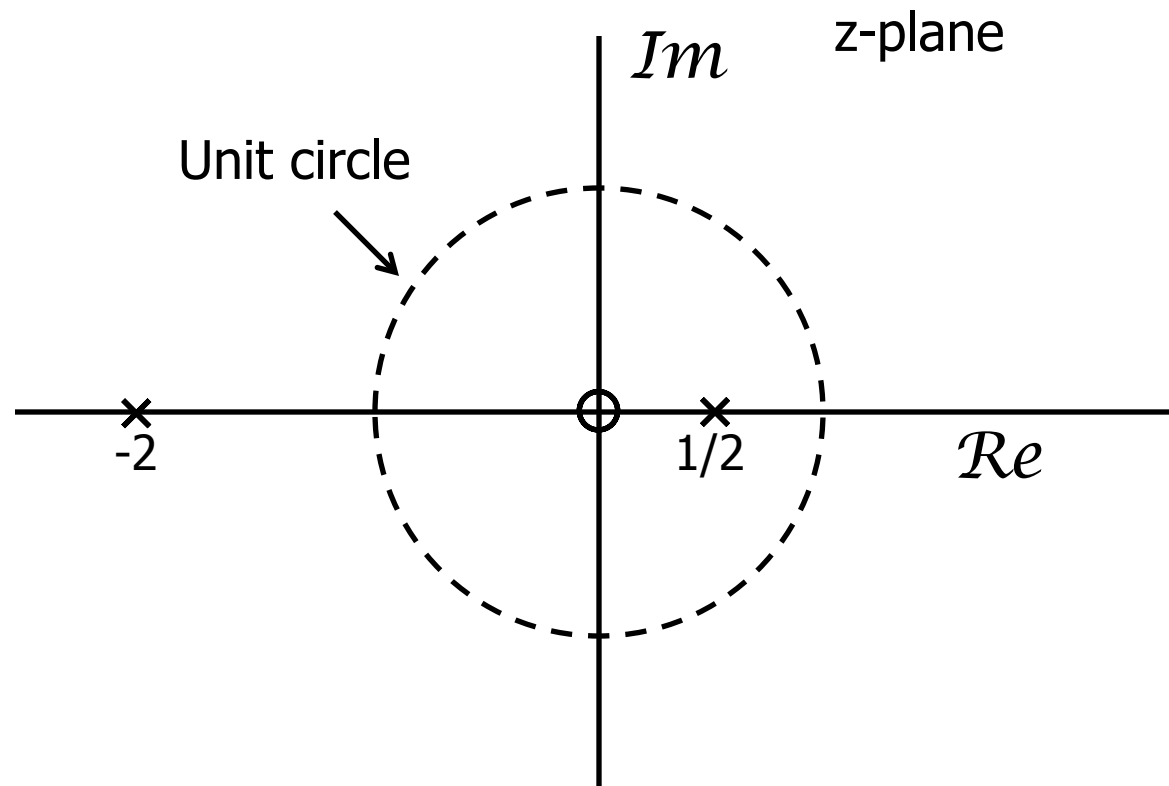
Example: ROC from Pole-Zero Plot

ROC 4: two-sided



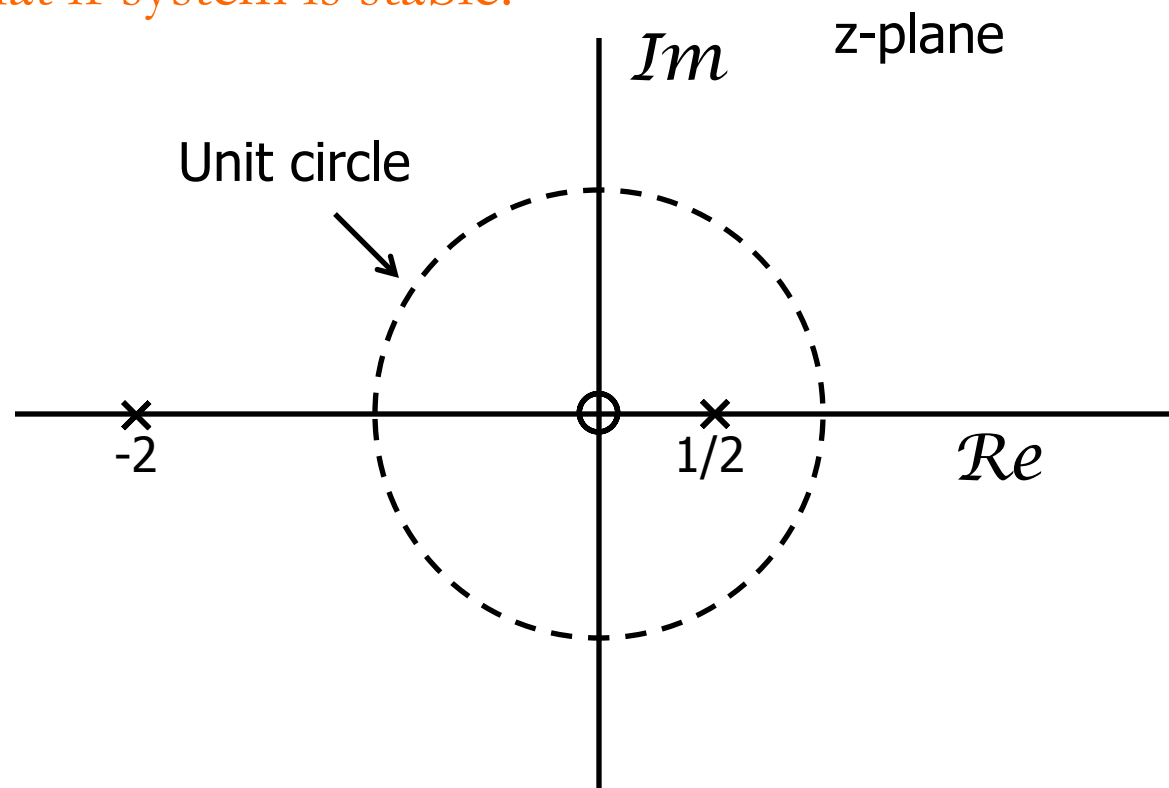
Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?



Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?
 - What if system is stable?

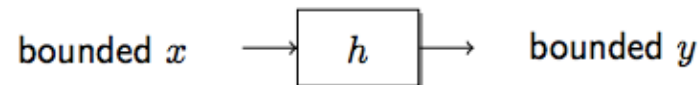


BIBO Stability Revisited

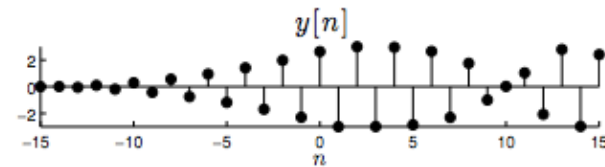
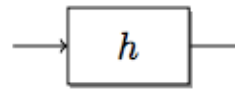
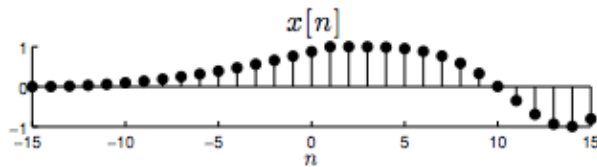
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

DEFINITION

An LTI system is **bounded-input bounded-output** (BIBO) stable if every bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_{\infty} < \infty$ and $\|y\|_{\infty} < \infty$

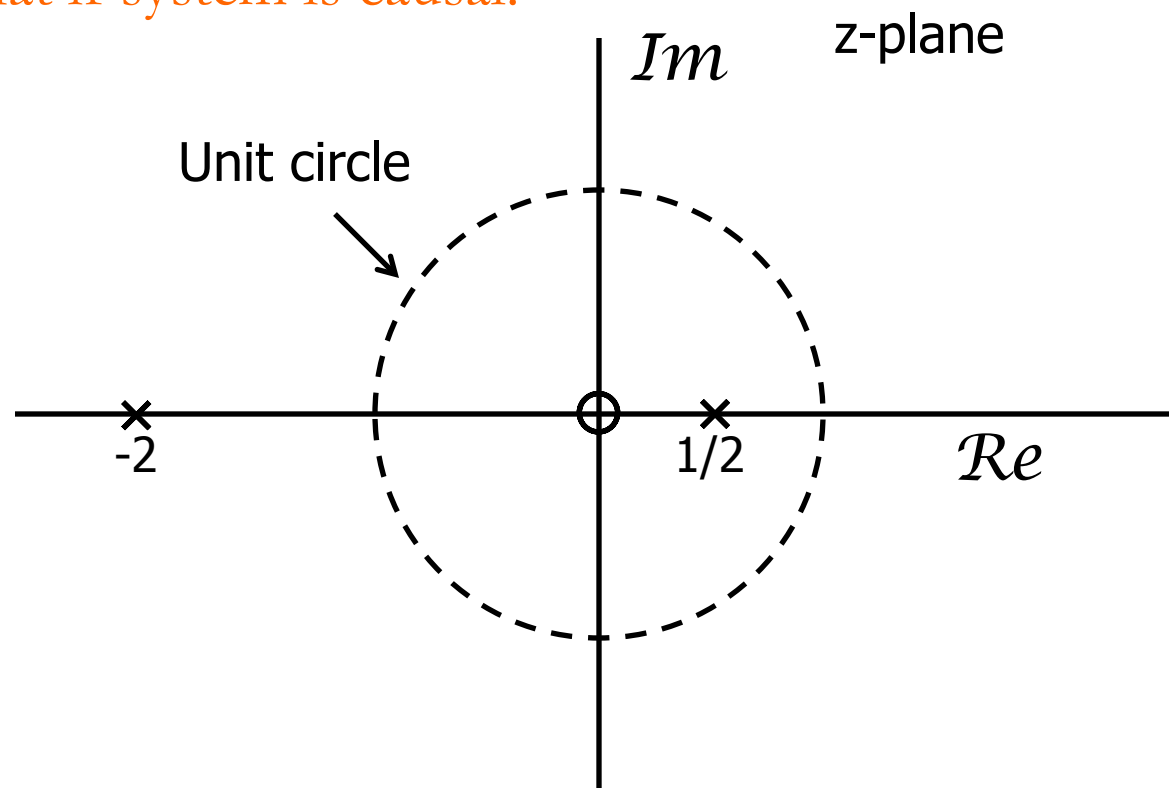


- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?
 - What if system is causal?





Properties of z-Transform

- Linearity:

$$ax_1[n] + bx_2[n] \Leftrightarrow aX_1(z) + bX_2(z)$$

- Time shifting:

$$x[n] \Leftrightarrow X(z)$$

$$x[n - n_d] \Leftrightarrow z^{-n_d} X(z)$$

- Multiplication by exponential sequence

$$x[n] \Leftrightarrow X(z)$$

$$z_0^n x[n] \Leftrightarrow X\left(\frac{z}{z_0}\right)$$



Properties of z-Transform

□ Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

□ Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz}$$

□ Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$$\text{ROC}_Y \text{ at least } \text{ROC}_X \wedge \text{ROC}_H$$

Inverse z-Transform





Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \frac{d\omega}{2\pi}$$

- There is a similar formula for the inverse z-transform using a contour integral

$$x[n] = \oint_{\mathcal{C}} X(z) z^n \frac{dz}{j2\pi z}$$

- Integrals are fun but beyond the scope of this course!



Inverse z-Transform

- Ways to avoid it:
 - Inspection (known transforms)
 - Properties of the z-transform
 - Partial fraction expansion
 - Power series expansion

Inspection

TABLE 3.1 SOME COMMON z -TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z > a $
6. $-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z < a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z > a $
8. $-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z < a $
9. $\cos(\omega_0 n)u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z > 0$



Properties of z-Transform

- Linearity:

$$ax_1[n] + bx_2[n] \Leftrightarrow aX_1(z) + bX_2(z)$$

- Time shifting:

$$x[n] \Leftrightarrow X(z)$$

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$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$$\text{ROC}_Y \text{ at least } \text{ROC}_X \wedge \text{ROC}_H$$



Partial Fraction Expansion

□ Let

$$X(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{z^N \sum_{k=0}^M b_k z^{M-k}}{z^M \sum_{k=0}^N a_k z^{N-k}}$$

□ M zeros and N poles at nonzero locations



Partial Fraction Expansion

$$X(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{z^N \sum_{k=0}^M b_k z^{M-k}}{z^M \sum_{k=0}^N a_k z^{N-k}}$$

□ Factored numerator/denominator

$$X(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$



Partial Fraction Expansion

- If $M < N$ and the poles are 1st order

$$X(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})} = \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}}$$

- where

$$A_k = (1 - d_k z^{-1}) X(z) \Big|_{z=d_k}$$



Example: 2nd-Order z-Transform

□ 2nd-order = two poles

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \quad ROC = \left\{z : \frac{1}{2} < |z|\right\}$$

Example: 2nd-Order z-Transform

□ 2nd-order = two poles

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \quad ROC = \left\{z : \frac{1}{2} < |z|\right\}$$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

Example: 2nd-Order z-Transform

□ 2nd-order = two poles $A_k = (1 - d_k z^{-1})X(z) \Big|_{z=d_k}$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$A_1 = (1 - \frac{1}{4}z^{-1})X(z) \Big|_{z=1/4} = \frac{(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})} \Big|_{z=1/4} = -1$$

$$A_2 = (1 - \frac{1}{2}z^{-1})X(z) \Big|_{z=1/2} = \frac{(1 - \frac{1}{2}z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})} \Big|_{z=1/2} = 2$$

Example: 2nd-Order z-Transform

□ 2nd-order = two poles

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)},$$

Right sided
↓

$$ROC = \left\{ z : \frac{1}{2} < |z| \right\}$$

Example: 2nd-Order z-Transform

□ 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)},$$

$$ROC = \left\{ z : \frac{1}{2} < |z| \right\}$$

5. $a^n u[n]$

$$\frac{1}{1 - az^{-1}}$$

$$|z| > |a|$$

Example: 2nd-Order z-Transform

□ 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}, \quad ROC = \left\{z : \frac{1}{2} < |z|\right\}$$

$$5. \quad a^n u[n] \quad \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$



Partial Fraction Expansion

- If $M \geq N$ and the poles are 1st order

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}}$$

- Where B_k is found by long division

$$A_k = (1 - d_k z^{-1}) X(z) \Big|_{z=d_k}$$



Example: Partial Fractions

- $M=N=2$ and poles are first order

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, \quad ROC = \{z : 1 < |z|\}$$
$$= \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

Example: Partial Fractions

- $M=N=2$ and poles are first order

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, \quad ROC = \{z : 1 < |z|\}$$
$$= \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

Example: Partial Fractions

- $M=N=2$ and poles are first order

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}, \quad ROC = \{z : 1 < |z|\}$$
$$\left(\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \right) \frac{z^{-2} + 2z^{-1} + 1}{z^{-2} - 3z^{-1} + 2} \frac{2}{5z^{-1} - 1}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}, \quad ROC = \{z : 1 < |z|\}$$

$$\left(\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \right) \frac{2}{z^{-2} - 3z^{-1} + 2}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$



Example: Partial Fractions

- $M=N=2$ and poles are first order

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}, \quad ROC = \{z : 1 < |z|\}$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$



Power Series Expansion

- Expansion of the z-transform definition

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \end{aligned}$$



Example: Finite-Length Sequence

□ Poles and zeros?

$$\begin{aligned} X(z) &= z^2 \left(1 - \frac{1}{2} z^{-1} \right) (1 + z^{-1}) (1 - z^{-1}) \\ &= z^2 - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1} \end{aligned}$$

Example: Finite-Length Sequence

□ Poles and zeros?

$$X(z) = z^2 \left(1 - \frac{1}{2} z^{-1} \right) (1 + z^{-1}) (1 - z^{-1})$$

$$= z^2 - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Example: Finite-Length Sequence

□ Poles and zeros?

$$X(z) = z^2 \left(1 - \frac{1}{2} z^{-1} \right) (1 + z^{-1}) (1 - z^{-1})$$
$$= z^2 - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1}$$
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$= \cdots + x[-2] z^2 + x[-1] z + x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$
$$x[n] = \begin{cases} 1, & n = -2 \\ -1/2, & n = -1 \\ -1, & n = 0 \\ 1/2, & n = 1 \\ 0, & \text{else} \end{cases}$$



Example: Finite-Length Sequence

□ Poles and zeros?

$$\begin{aligned} X(z) &= z^2 \left(1 - \frac{1}{2} z^{-1} \right) (1 + z^{-1}) (1 - z^{-1}) \\ &= z^2 - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1} \end{aligned}$$

$$x[n] = \begin{cases} 1, & n = -2 \\ -1/2, & n = -1 \\ -1, & n = 0 \\ 1/2, & n = 1 \\ 0, & \text{else} \end{cases} = \delta[n+2] - \frac{1}{2} \delta[n+1] - \delta[n] + \frac{1}{2} \delta[n-1]$$

Example: Finite-Length Sequence

□ Poles and zeros?

$$\begin{aligned} X(z) &= z^2 \left(1 - \frac{1}{2} z^{-1} \right) (1 + z^{-1}) (1 - z^{-1}) \\ &= z^2 - \frac{1}{2} z - 1 + \frac{1}{2} z^{-1} \end{aligned}$$

$$x[n] = \begin{cases} 1, & n = -2 \\ -1/2, & n = -1 \\ -1, & n = 0 \\ 1/2, & n = 1 \\ 0, & \text{else} \end{cases} = \delta[n+2] - \frac{1}{2} \delta[n+1] - \delta[n] + \frac{1}{2} \delta[n-1]$$

4. $\delta[n-m]$ z^{-m}

Reminder: Difference Equations

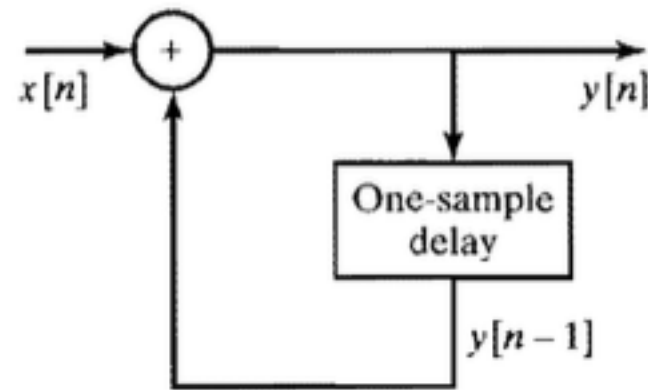
□ Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$



Difference Equation to z-Transform

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

$$y[n] = -\sum_{k=1}^N \left(\frac{a_k}{a_0} \right) y[n-k] + \sum_{m=0}^M \left(\frac{b_m}{a_0} \right) x[n-m]$$

- ❑ Difference equations of this form behave as causal LTI systems
 - when the input is zero prior to $n=0$
 - Initial rest equations are imposed prior to the time when input becomes nonzero
 - i.e $y[-N]=y[-N+1]=\dots=y[-1]=0$



Difference Equation to z-Transform

$$y[n] = -\sum_{k=1}^N \left(\frac{a_k}{a_0} \right) y[n-k] + \sum_{m=0}^M \left(\frac{b_m}{a_0} \right) x[n-m]$$

$$Y(z) = -\sum_{k=1}^N \left(\frac{a_k}{a_0} \right) z^{-k} Y(z) + \sum_{m=0}^M \left(\frac{b_m}{a_0} \right) z^{-m} X(z)$$



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$$\sum_{k=0}^N \left(\frac{a_k}{a_0} \right) z^{-k} Y(z) = \sum_{m=0}^M \left(\frac{b_m}{a_0} \right) z^{-m} X(z) \Rightarrow Y(z) = \frac{\sum_{m=0}^M (b_m) z^{-m}}{\sum_{k=0}^N (a_k) z^{-k}} X(z)$$

Difference Equation to z-Transform

$$y[n] = -\sum_{k=1}^N \left(\frac{a_k}{a_0} \right) y[n-k] + \sum_{m=0}^M \left(\frac{b_m}{a_0} \right) x[n-m]$$

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$$\sum_{k=0}^N \left(\frac{a_k}{a_0} \right) z^{-k} Y(z) = \sum_{m=0}^M \left(\frac{b_m}{a_0} \right) z^{-m} X(z) \Rightarrow Y(z) = \frac{\sum_{m=0}^M (b_m) z^{-m}}{\sum_{k=0}^N (a_k) z^{-k}} X(z)$$

$$H(z) = \frac{\sum_{m=0}^M (b_m) z^{-m}}{\sum_{k=0}^N (a_k) z^{-k}}$$



Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

$$H(z) = \frac{\sum_{m=0}^M (b_m) z^{-m}}{\sum_{k=0}^N (a_k) z^{-k}}$$

Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$H(z) = \frac{1}{1 - az^{-1}}$$

Diagram illustrating the transfer function $H(z) = \frac{1}{1 - az^{-1}}$ with blue arrows pointing to the coefficients: b_0 points to the numerator constant 1, a_0 points to the constant term 1 in the denominator, and a_1 points to the coefficient a in the denominator.

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

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$$h[n] = a^n u[n]$$

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

$$H(z) = \frac{\sum_{m=0}^M (b_m) z^{-m}}{\sum_{k=0}^N (a_k) z^{-k}}$$

Why right sided?



Big Ideas

- ❑ z-Transform
 - Draw pole-zero plots
 - Must specify region of convergence (ROC)
 - ROC properties
- ❑ z-Transform properties
 - Similar to DTFT
- ❑ Inverse z-transform
 - Avoid it!
 - Inspection, properties, partial fractions, power series
- ❑ Difference equations easy to transform



Admin

- ❑ HW 2 due Sunday at midnight