ESE 531: Digital Signal Processing

Lec 7: February 7, 2019

Sampling and Reconstruction

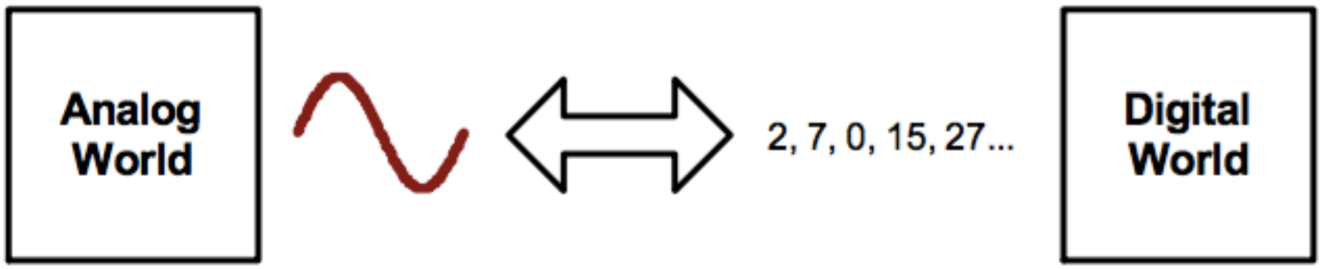
Lecture Outline

- Sampling
  - Frequency Response of Sampled Signal
- Reconstruction
- Anti-aliasing Filter

Video Example

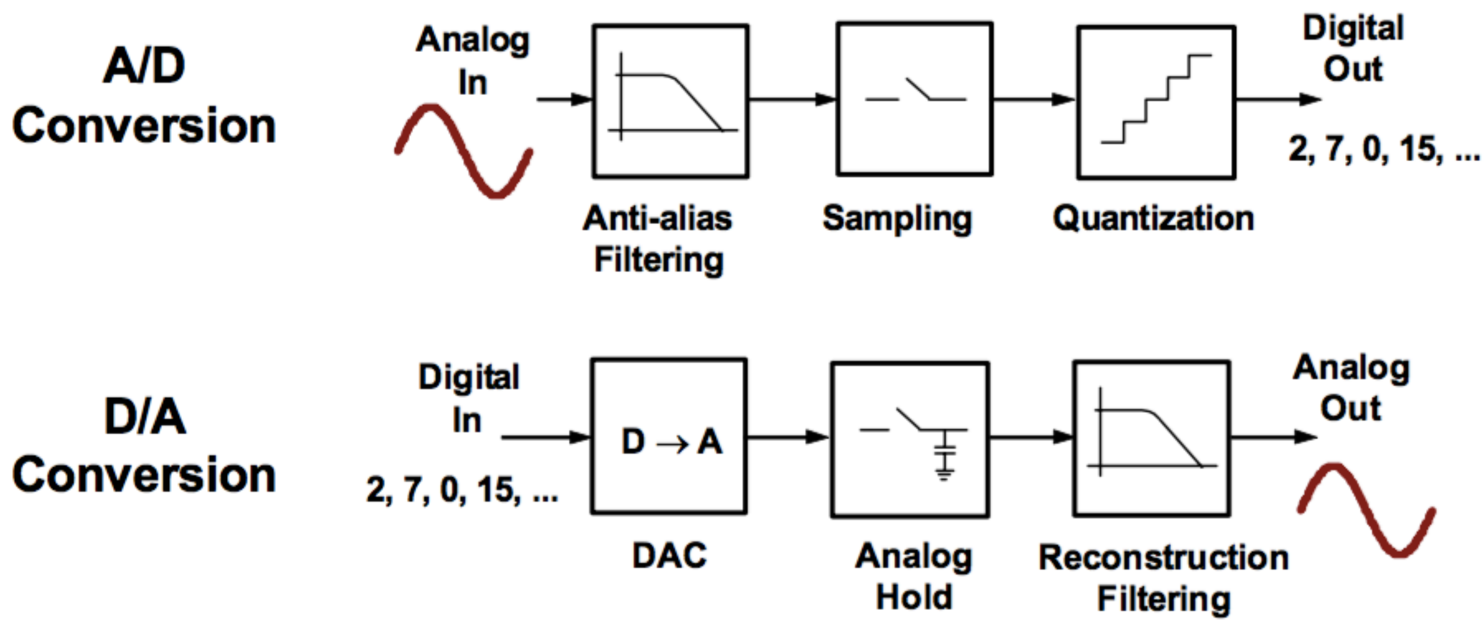
- https://www.youtube.com/watch?v=ByTsISFXUoY

The Data Conversion Problem



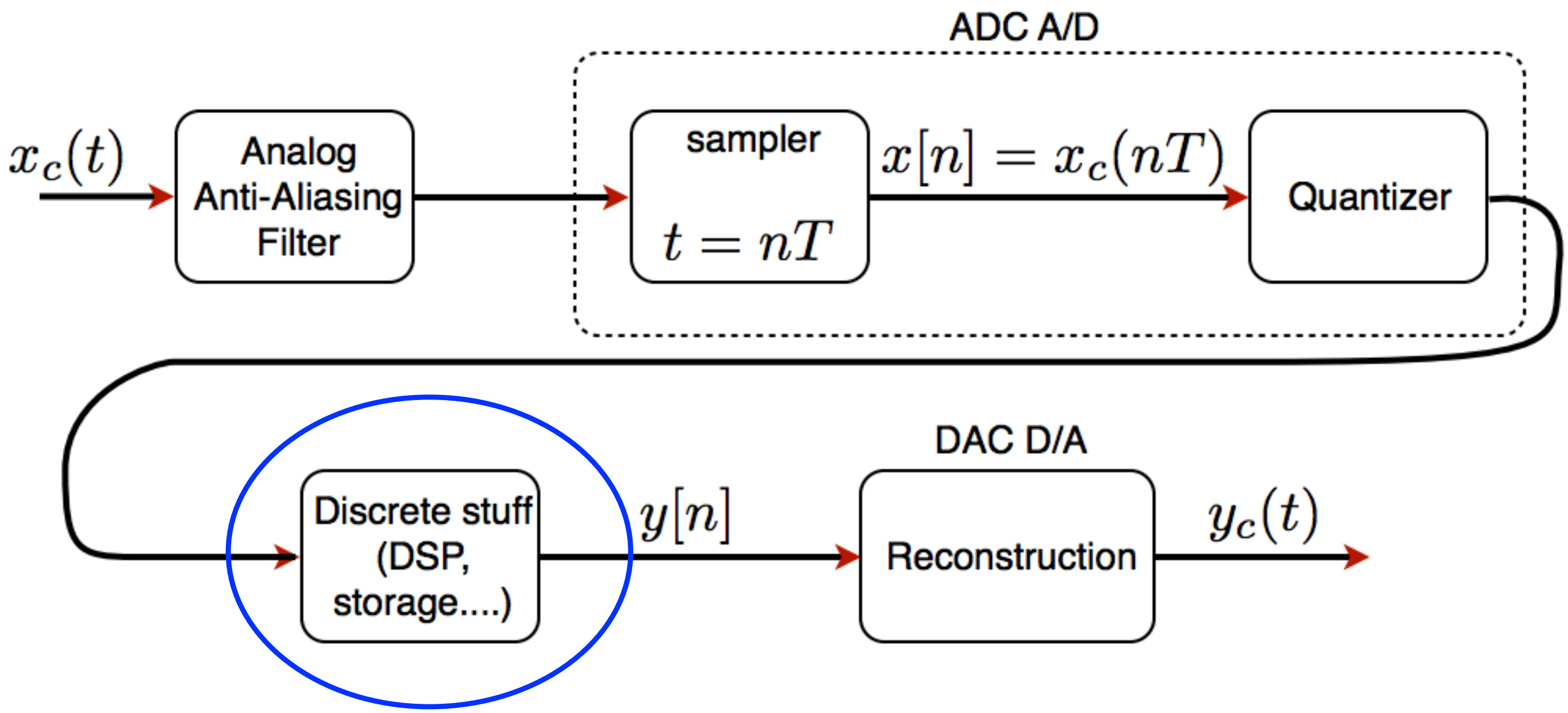
- Real world signals
  - Continuous time, continuous amplitude
- Digital abstraction
  - Discrete time, discrete amplitude
- Two problems
  - How to go discretize in time and amplitude
    - A/D conversion
  - How to "undescretize" in time and amplitude
    - D/A conversion

Overview



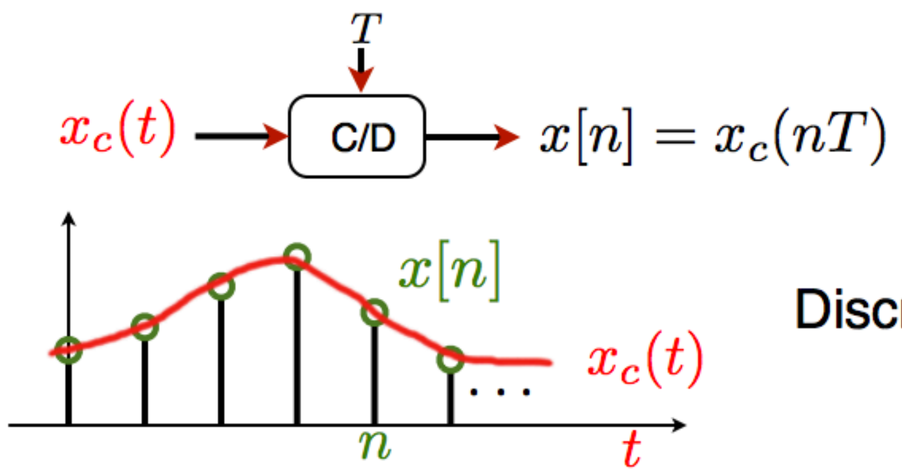
- We'll first look at these building blocks from a functional, "black box" perspective

DSP System



ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

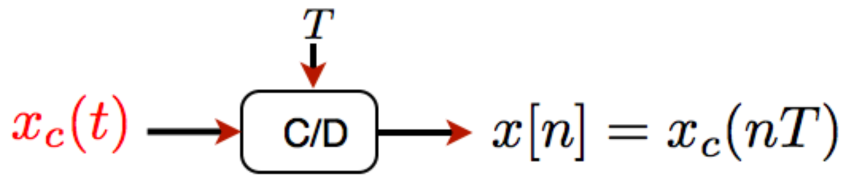
Ideal Sampling Model

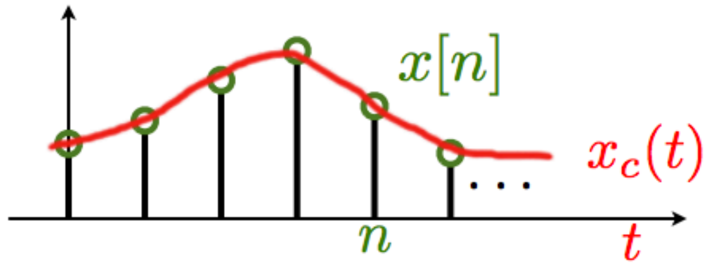


Discrete and Continuous

- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - f_s=1/T is the sampling frequency
  - $\Omega_s = 2\pi/T$

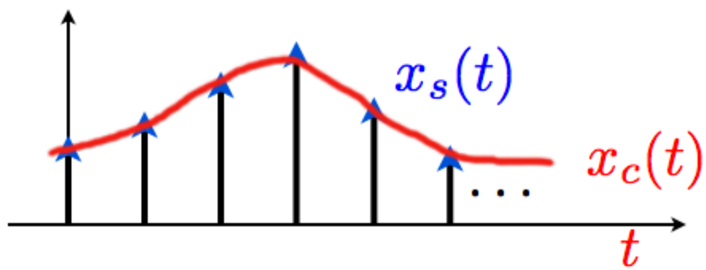
Ideal Sampling Model





Discrete and Continuous

define impulsive sampling:



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Dirac delta function, δ (t)
  - Infinitely high and thin, area of 1
  - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- Three signals. How are they related? In time? In frequency?

Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad\qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

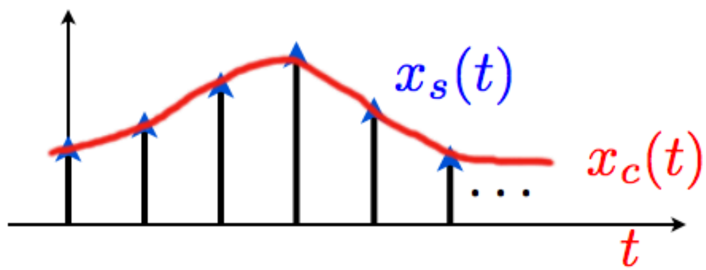
$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$x_s(t)$:C.T
$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

$x[n]$:D.T
$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \qquad \omega = \Omega T$$

$$\boxed{X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}} \qquad \boxed{X_s(j\Omega) = X(e^{j\omega})\big|_{\omega=\Omega T}}$$

Frequency Domain Analysis



Continuous

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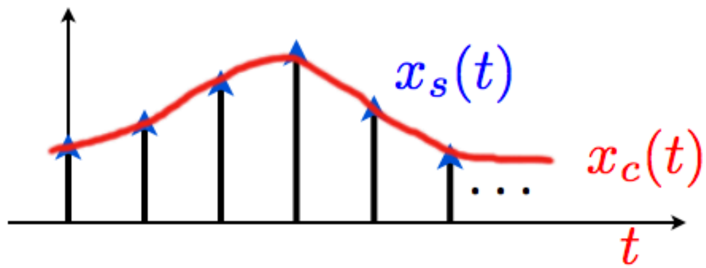
$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Frequency Domain Analysis



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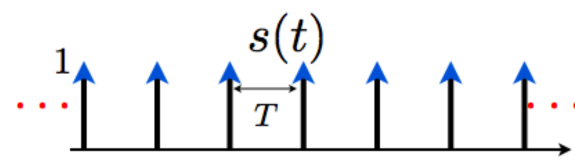
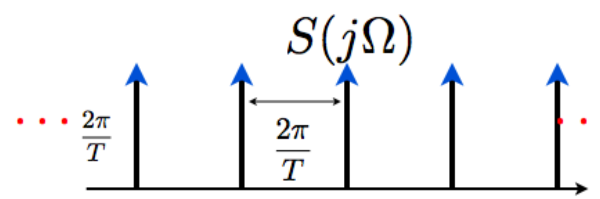
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$s(t) \quad \leftrightarrow \quad S(j\Omega)$$

$$S(j\Omega) \quad = \quad \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \underbrace{\frac{2\pi}{T}}_{=\Omega_s} k)$$





Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

$$X_s(j\Omega) \quad = \quad \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

Frequency Domain Analysis

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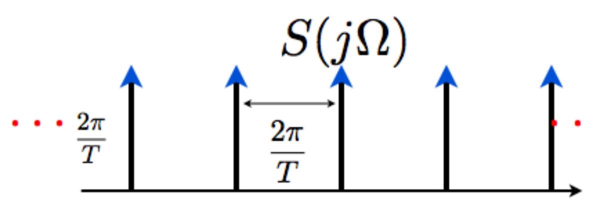

Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

$$\boxed{\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T} \end{aligned}}$$

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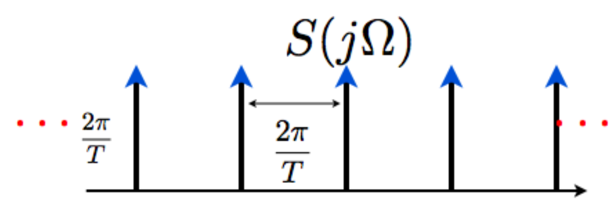
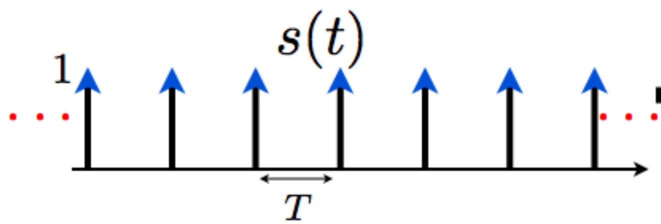
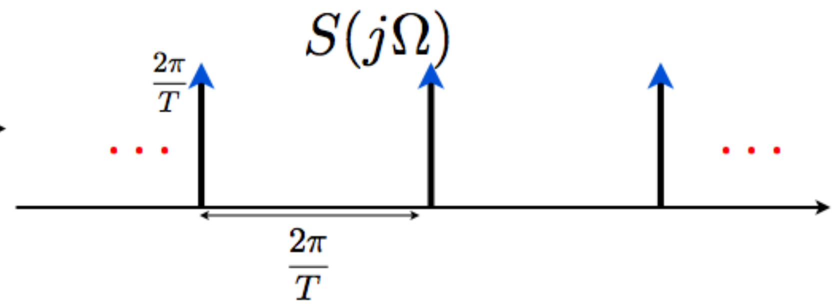
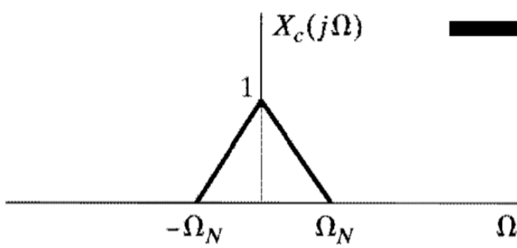


Frequency Domain Analysis



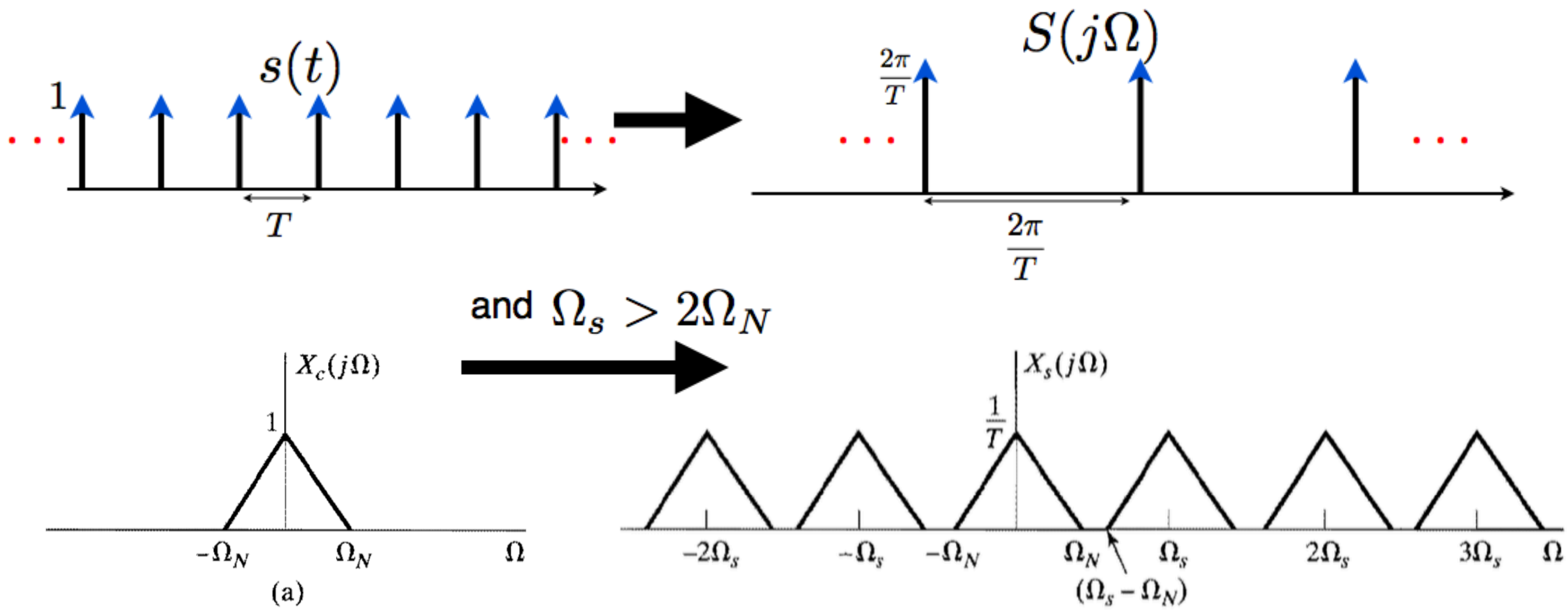


and $\Omega_s > 2\Omega_N$

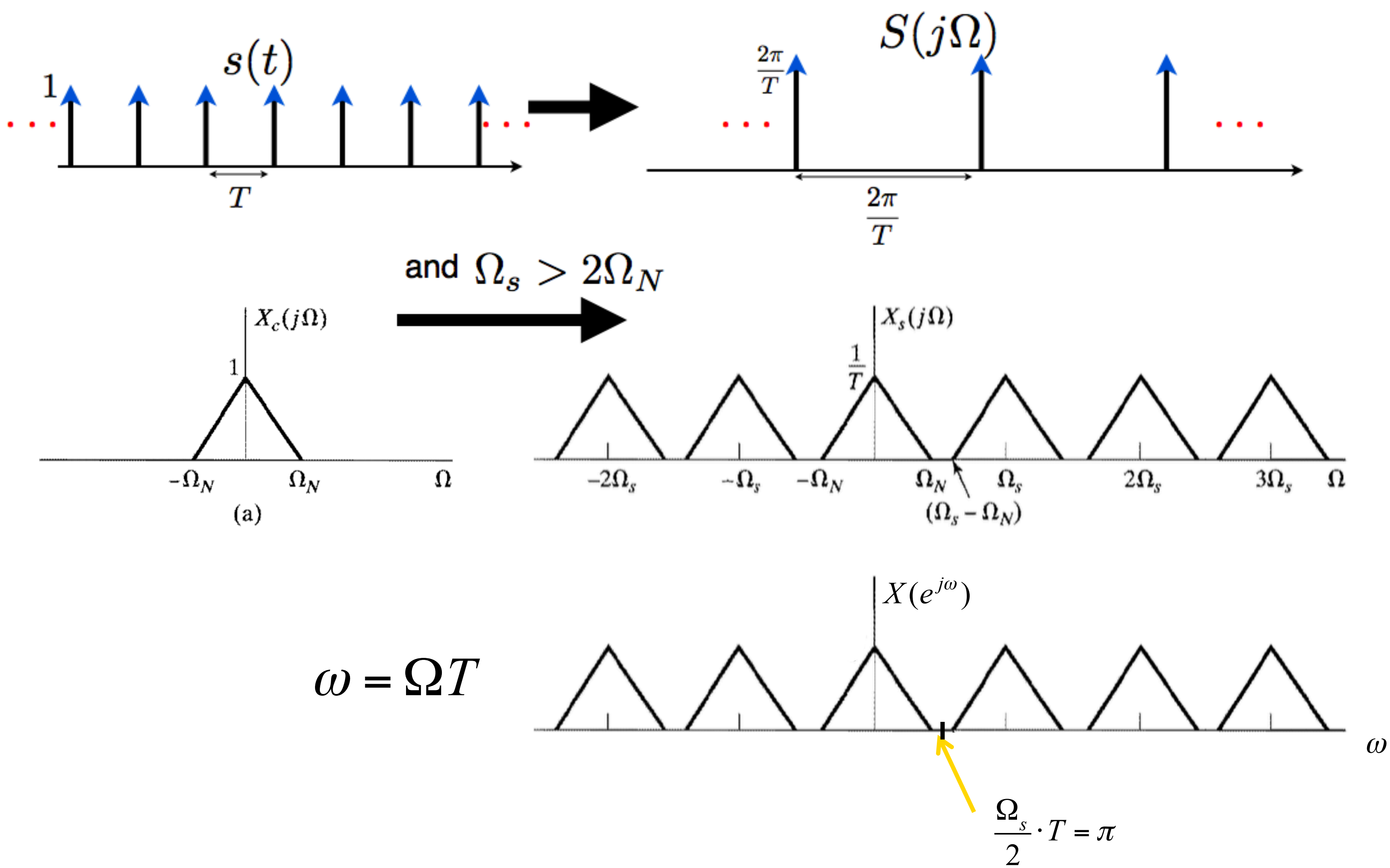


(a)

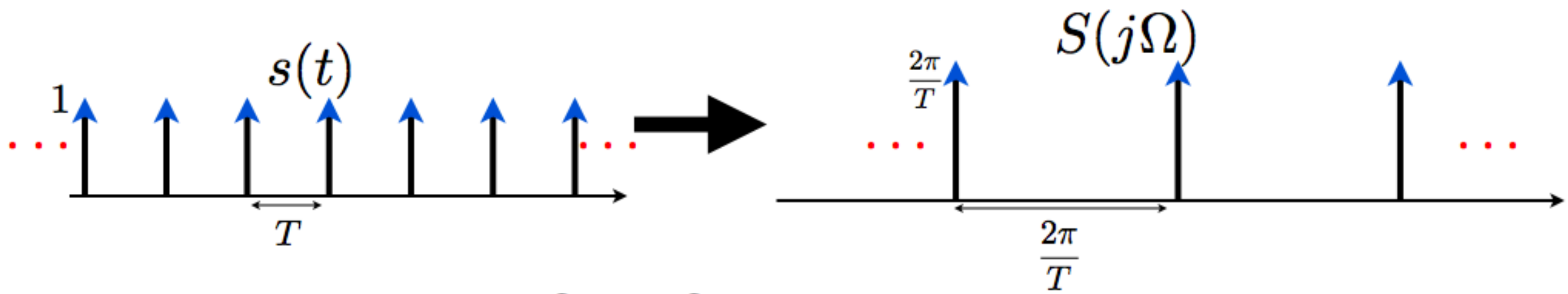
Frequency Domain Analysis

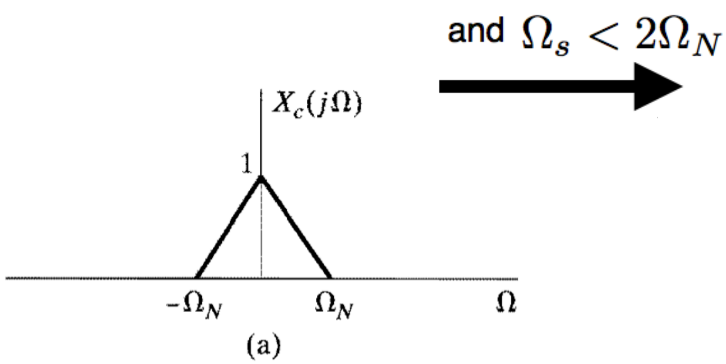


Frequency Domain Analysis



Frequency Domain Analysis w/ Aliasing





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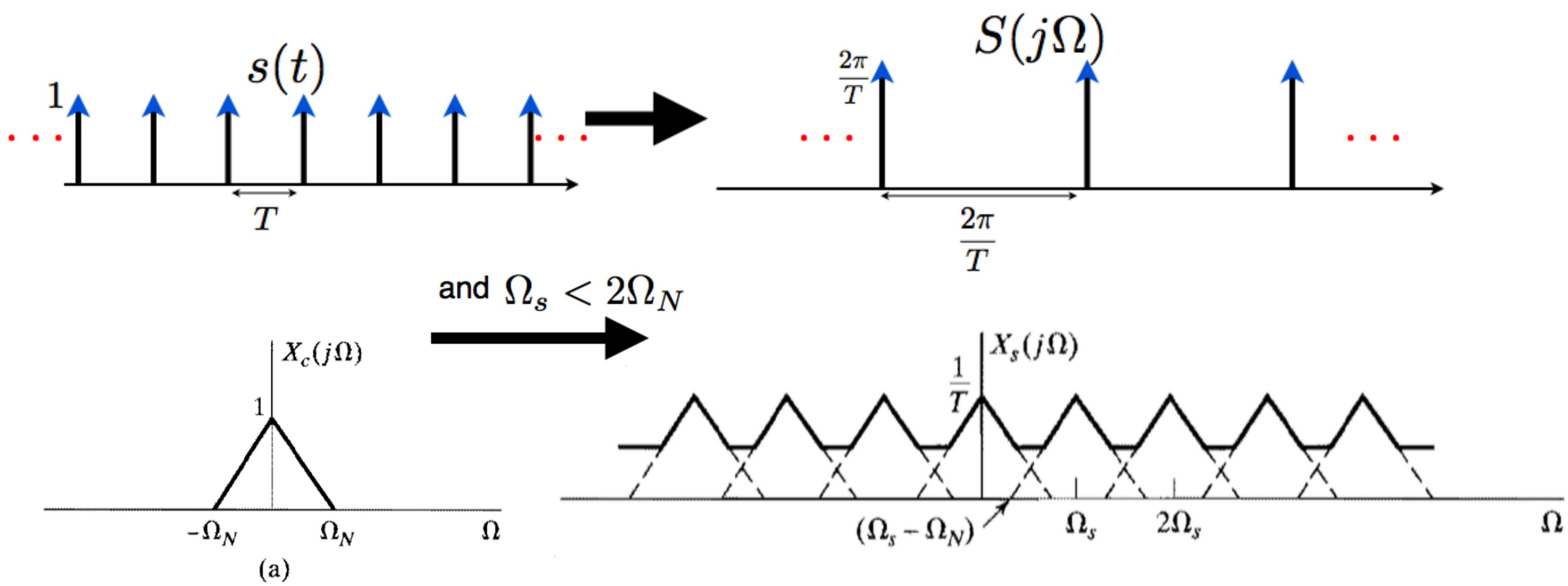
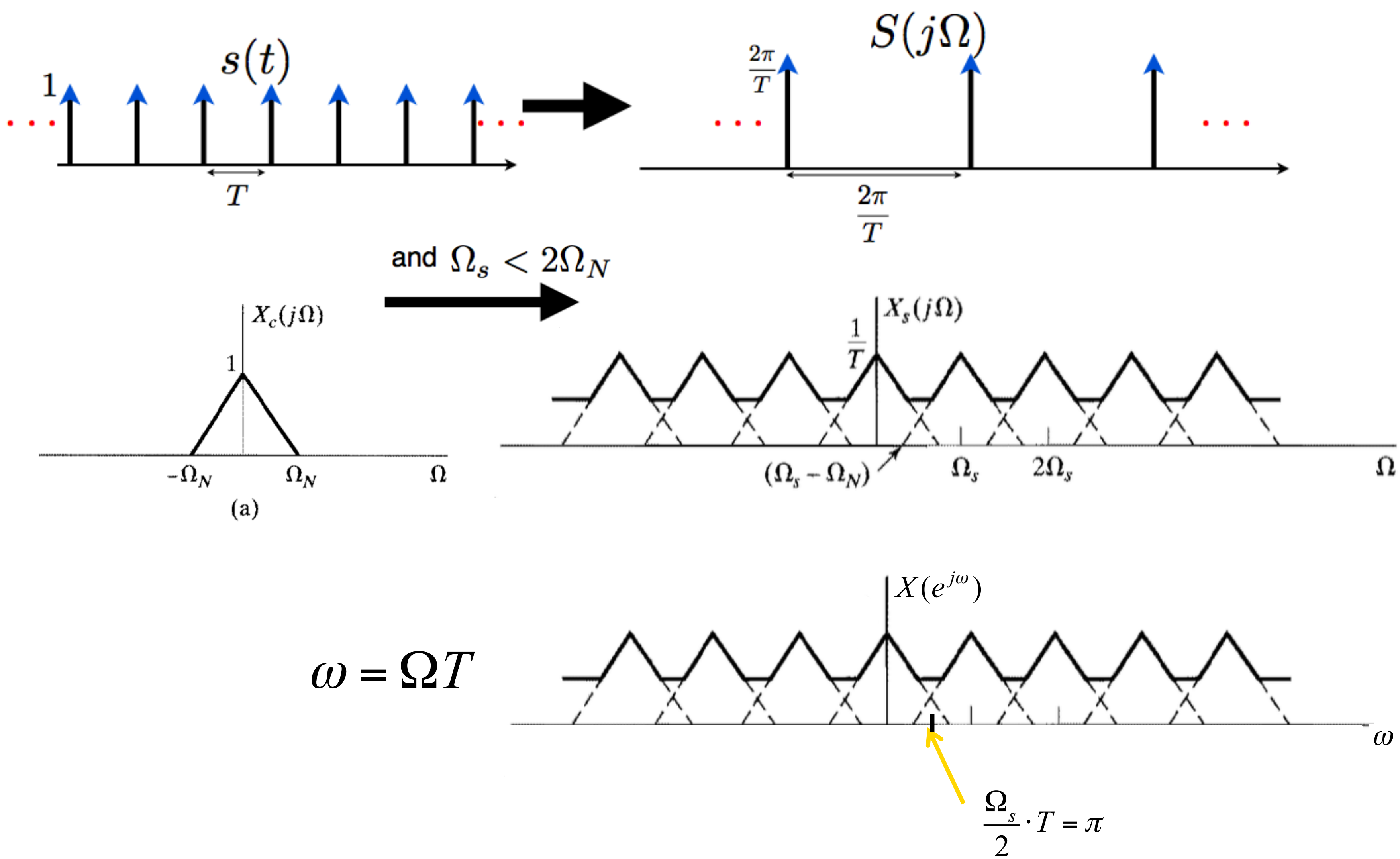


Frequency Domain Analysis w/ Aliasing

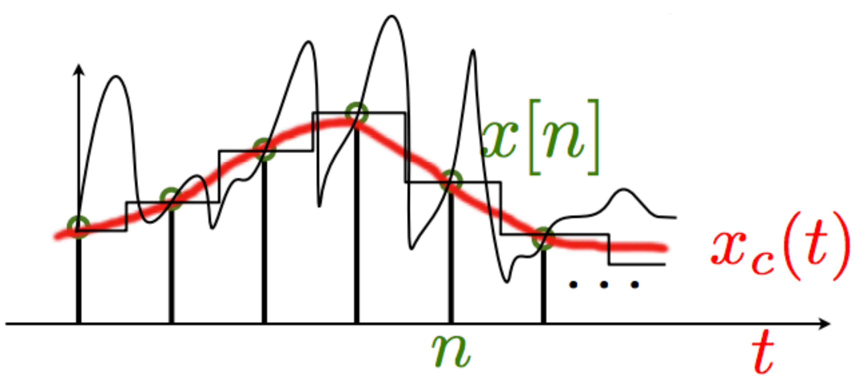


Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

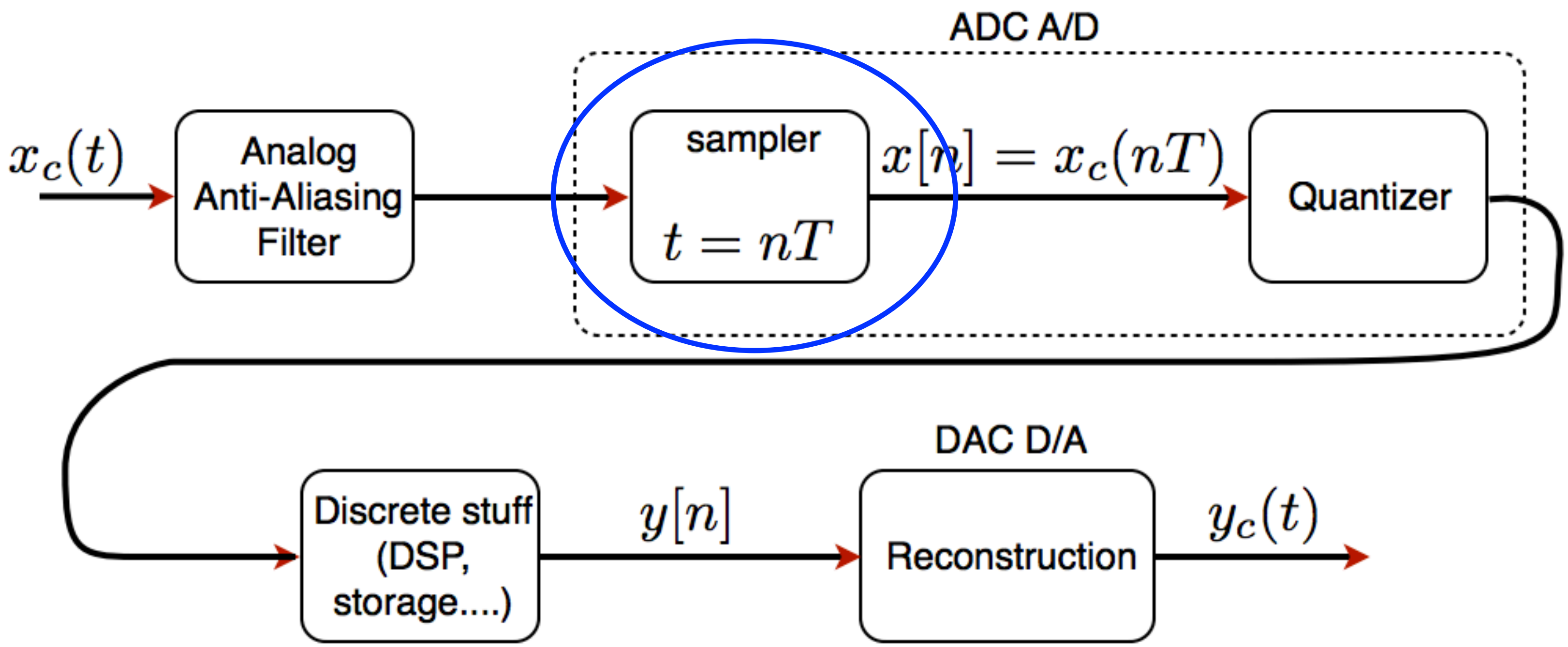
$$X_c(j\Omega) = 0 \ \ \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness

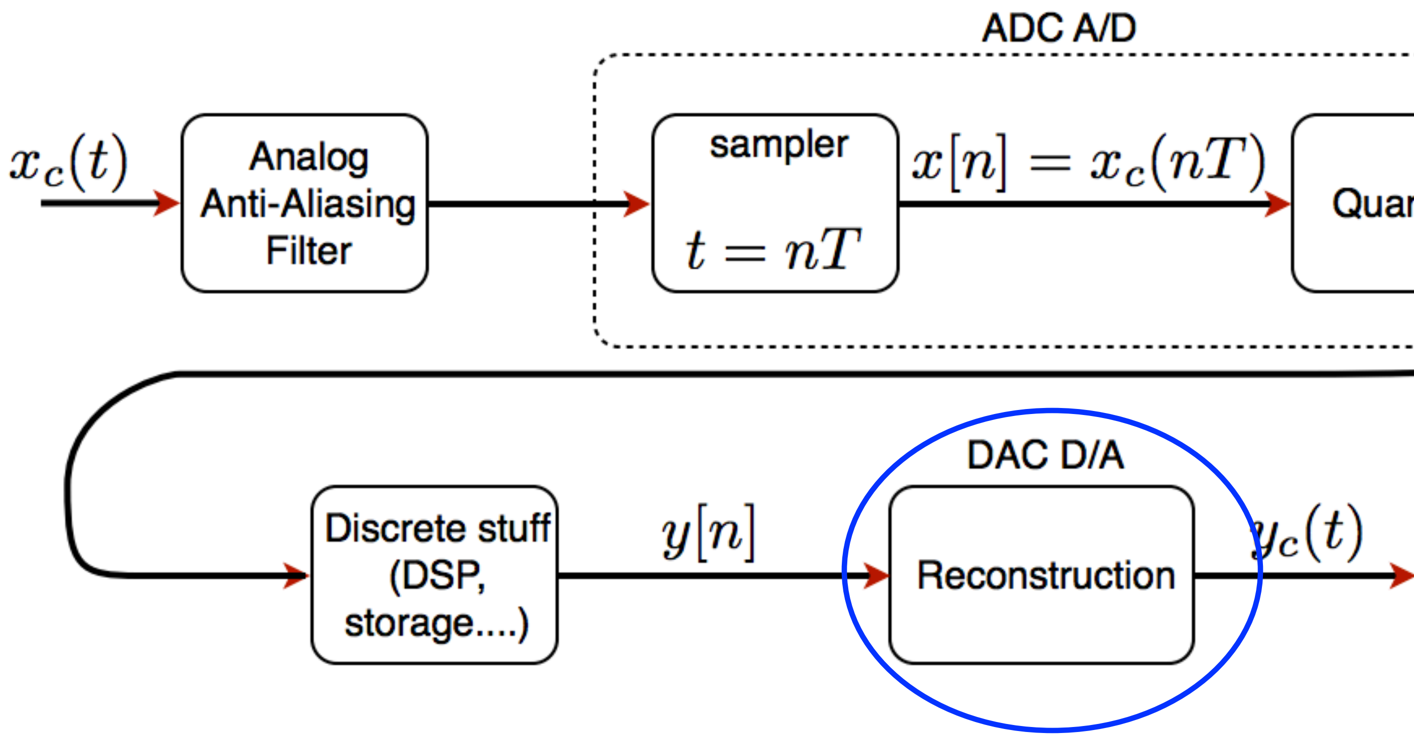


Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

DSP System


ADC A/D
$x_c(t)$
Analog
Anti-Aliasing
Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff
(DSP,
storage....)
$y[n]$
Reconstruction
$y_c(t)$

DSP System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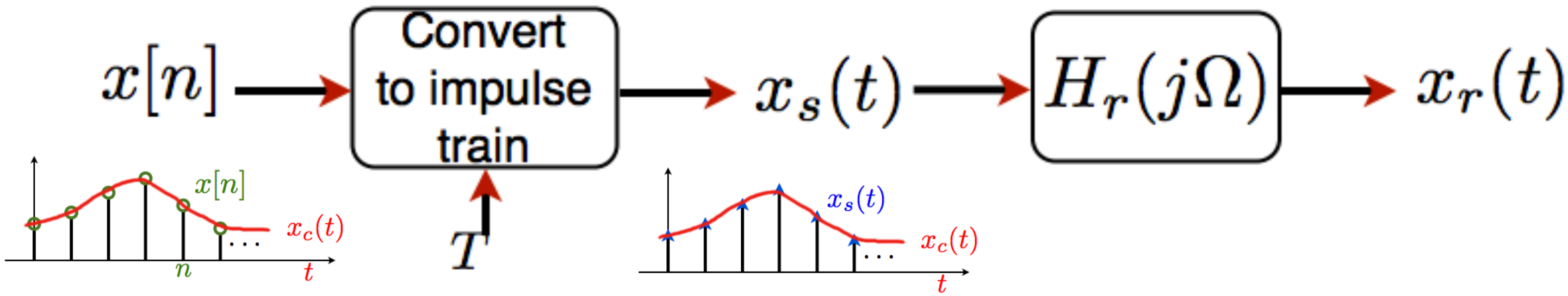
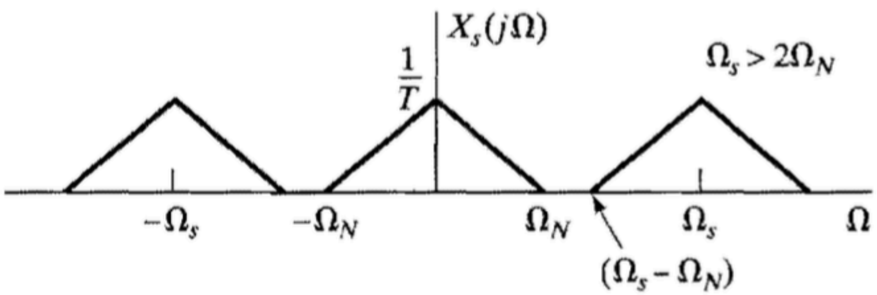
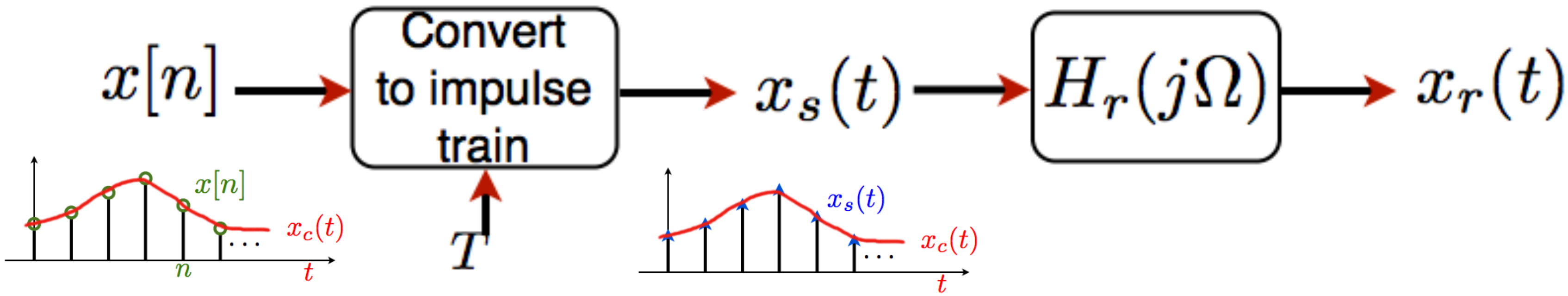
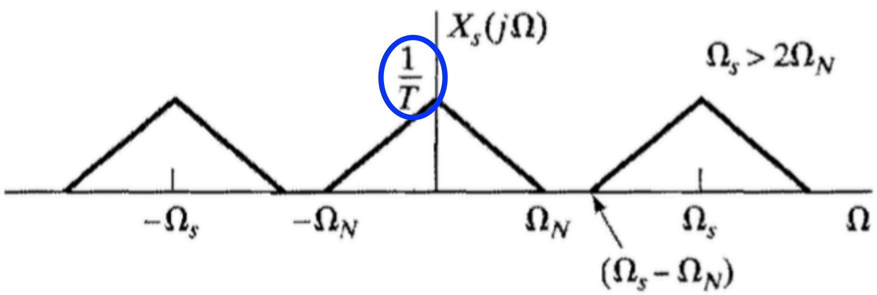
Reconstruction in Frequency Domain

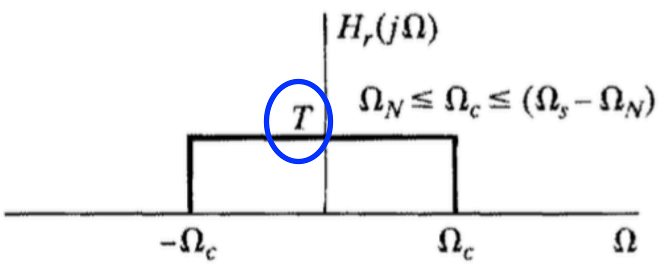




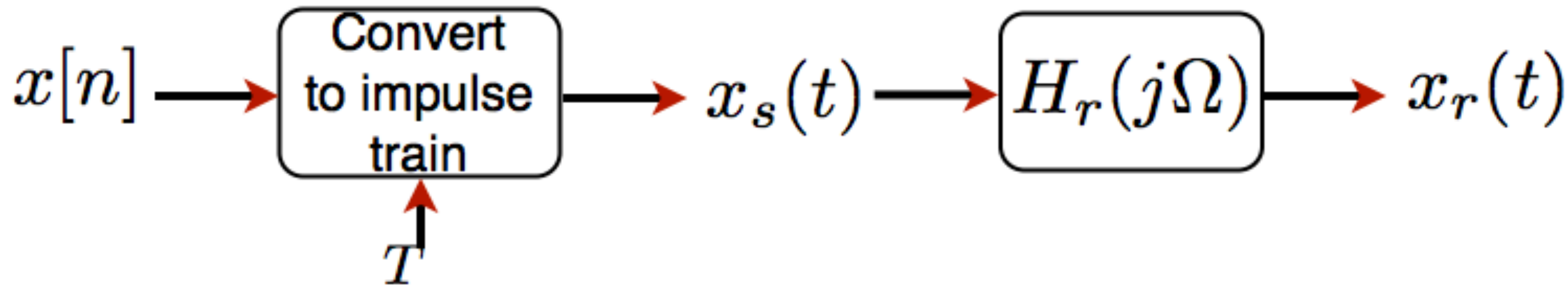
Reconstruction in Frequency Domain

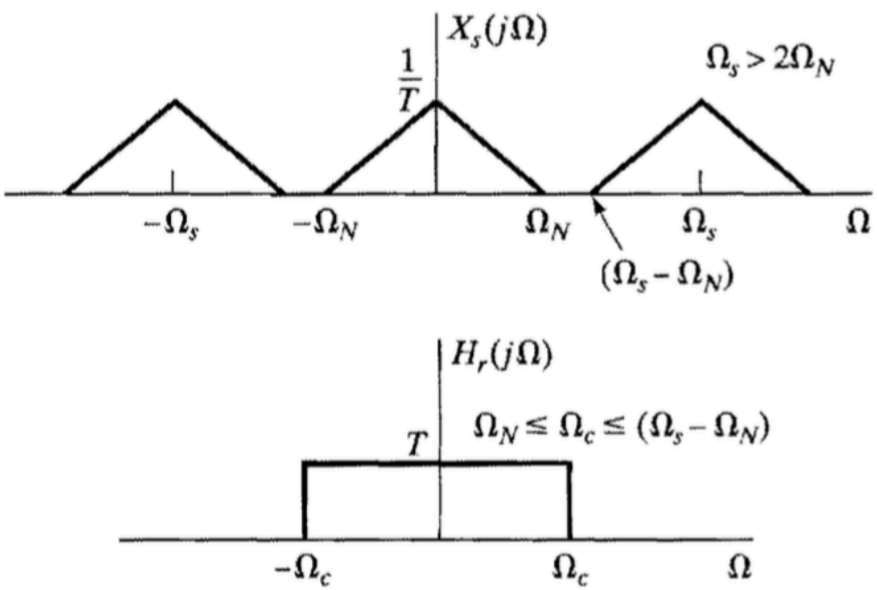


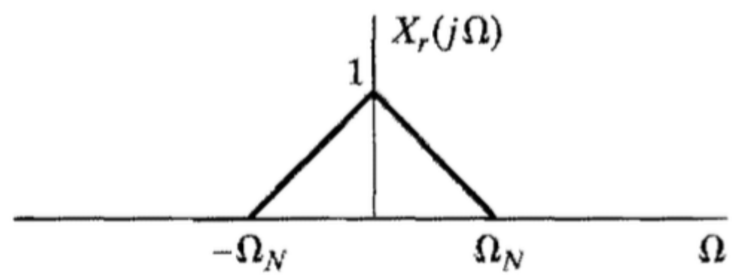




Reconstruction in Frequency Domain







Example: Cosine Input

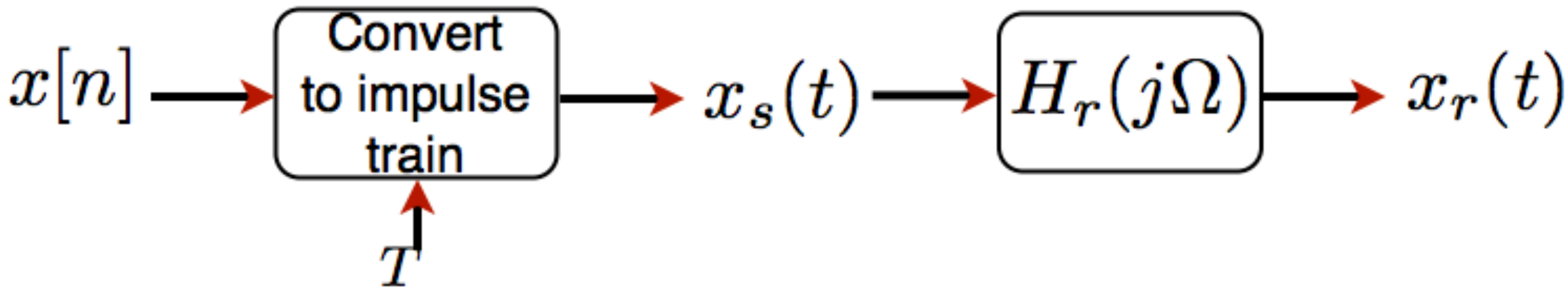
- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(4000\pi t)$ with sampling period $T=1/6000$ (f_s=6kHz)

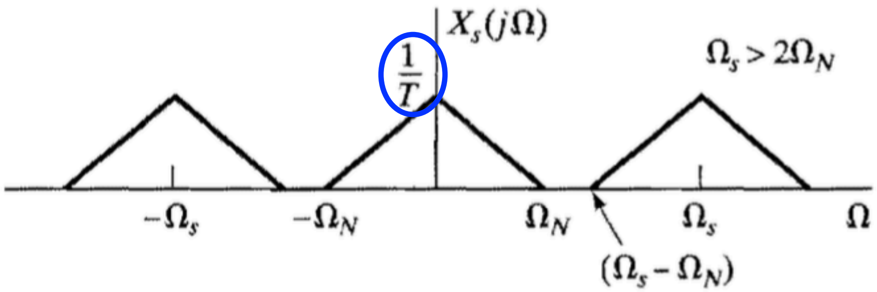
$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}
\end{aligned}
$$

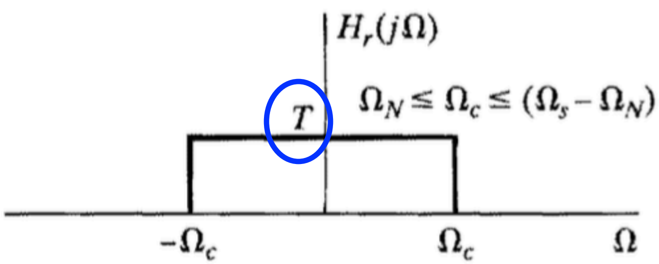
Example: Cosine Input

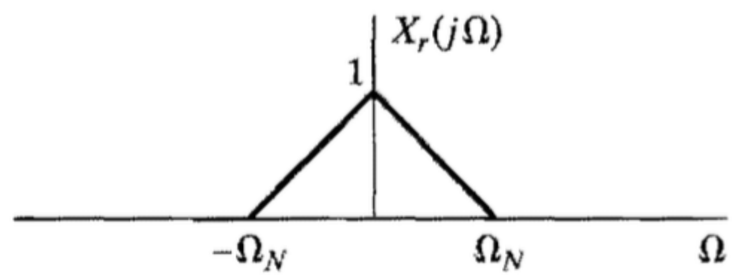
- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(16000\pi t)$ with sampling period $T=1/6000$ (f_s=6kHz)

Reconstruction in Frequency Domain









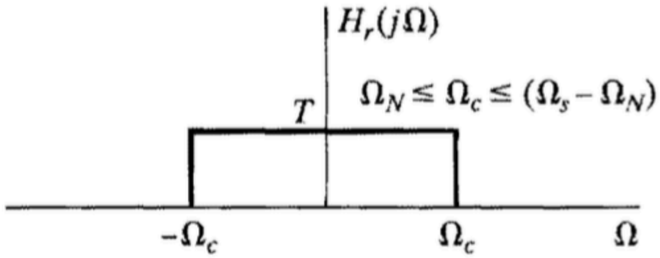
$$2\Omega_N = \Omega_s$$

$$\Omega_N = \Omega_C = \Omega_s/2$$

Sample at Nyquist and filter at signal badnwidth

Reconstruction in Time Domain

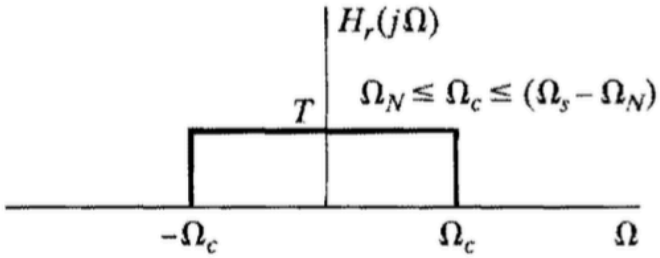
$\Omega_N = \Omega_C = \Omega_s/2$



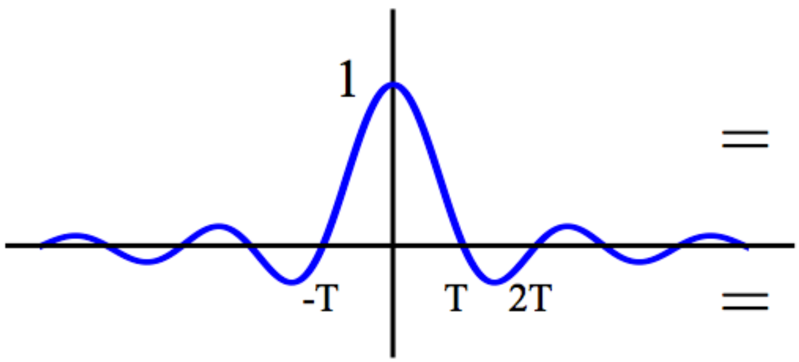
$$h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Reconstruction in Time Domain

$\Omega_N = \Omega_C = \Omega_s/2$

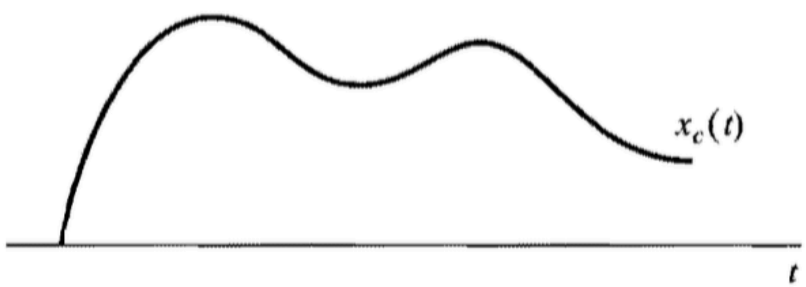


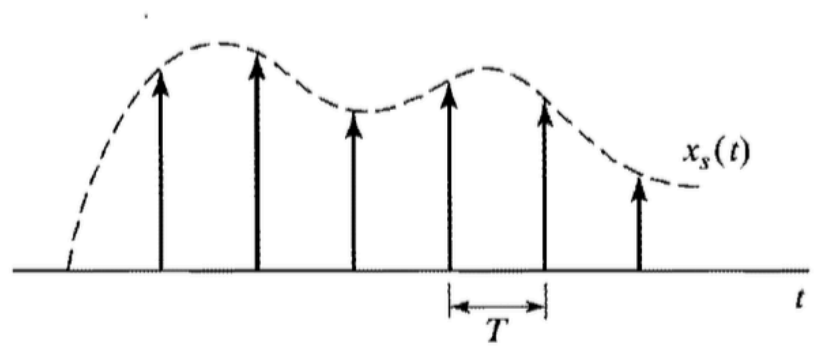
$$
\begin{aligned}
h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \\
&= \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\
&= \frac{T}{\pi t}\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\
&= \frac{T}{\pi t}\sin(\frac{\Omega_s}{2}t) = \frac{T}{\pi t}\sin(\frac{\pi}{T}t) \\
&= \text{sinc}(\frac{t}{T})
\end{aligned}
$$

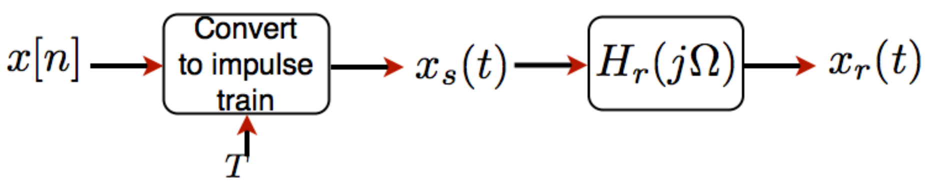


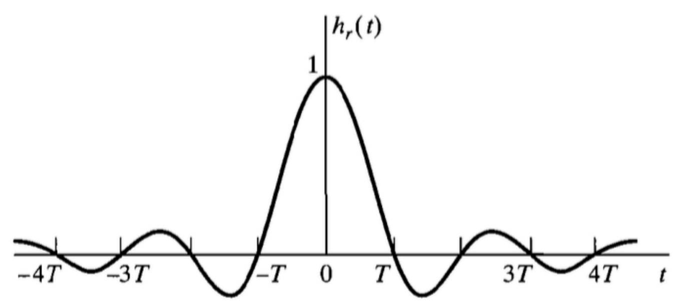
Reconstruction in Time Domain

$$
\begin{aligned}
x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t) \\
&= \sum_n x[n]h_r(t-nT)
\end{aligned}
$$





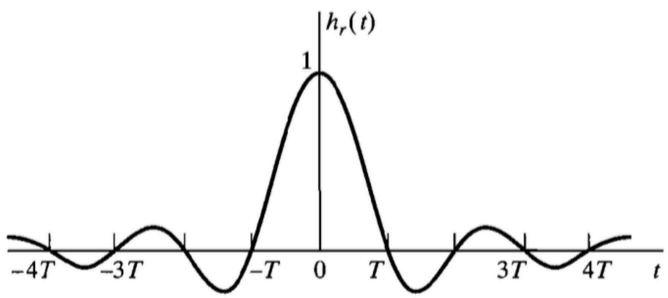




Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) \;=\; \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t)$$

$$= \sum_n x[n]h_r(t-nT)$$



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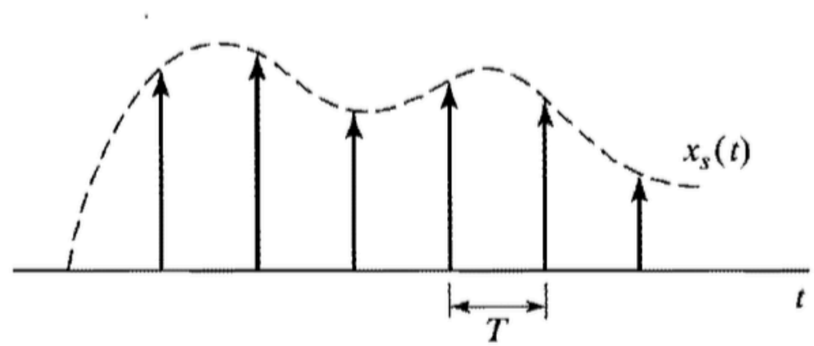

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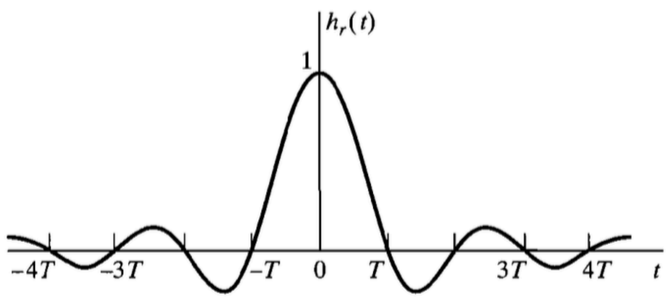
Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT)\right) * h_r(t)$$

$$= \sum_n x[n] h_r(t - nT)$$



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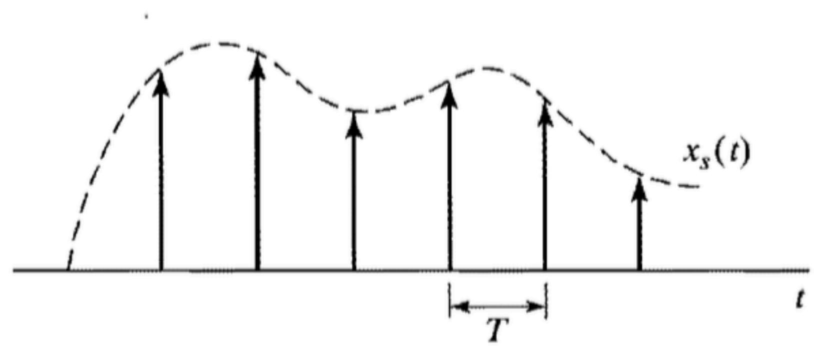

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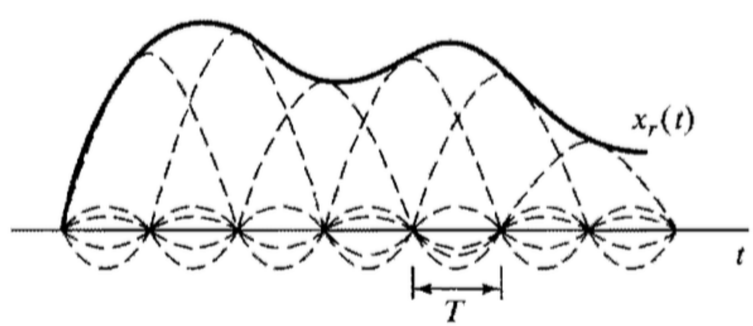
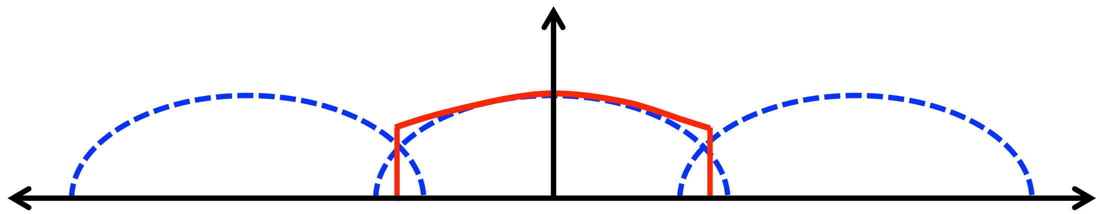


The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

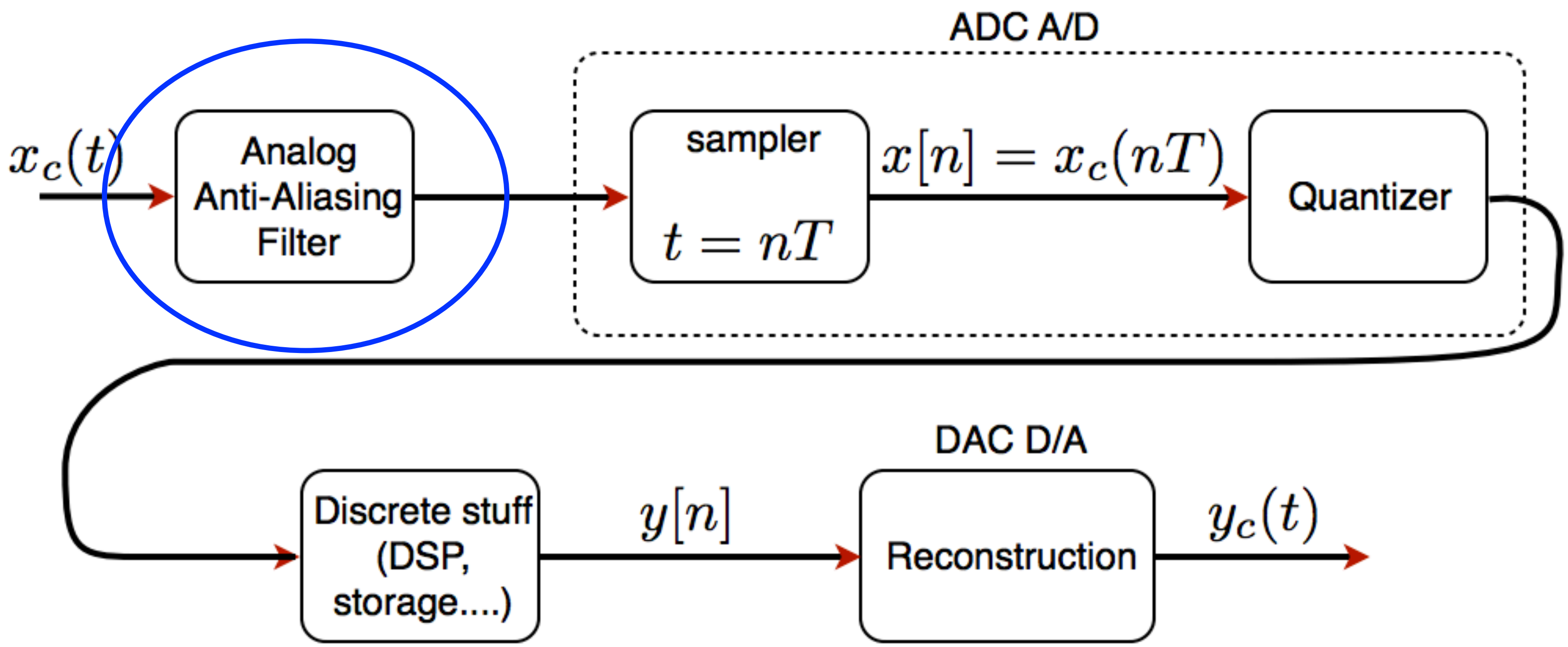
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$



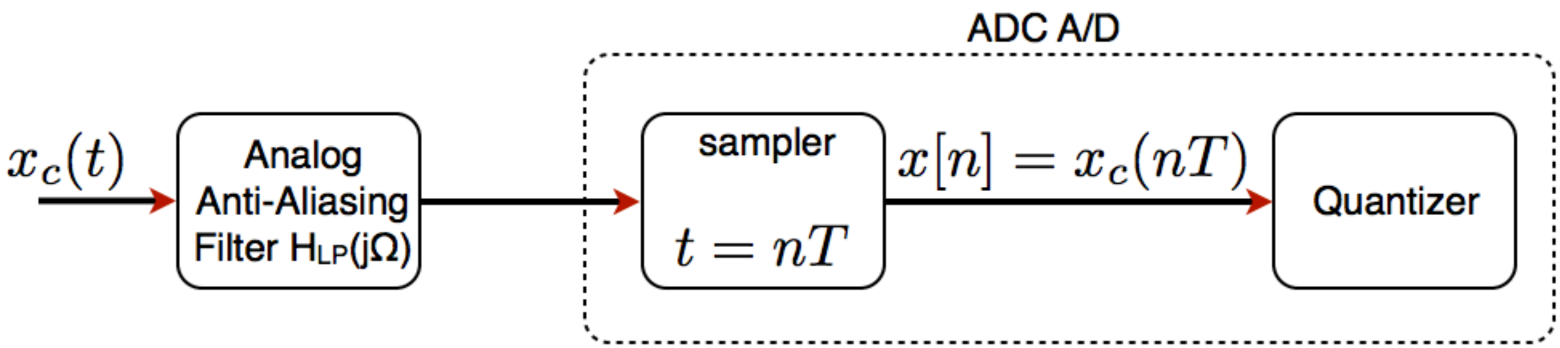
$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if} \quad |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

DSP System

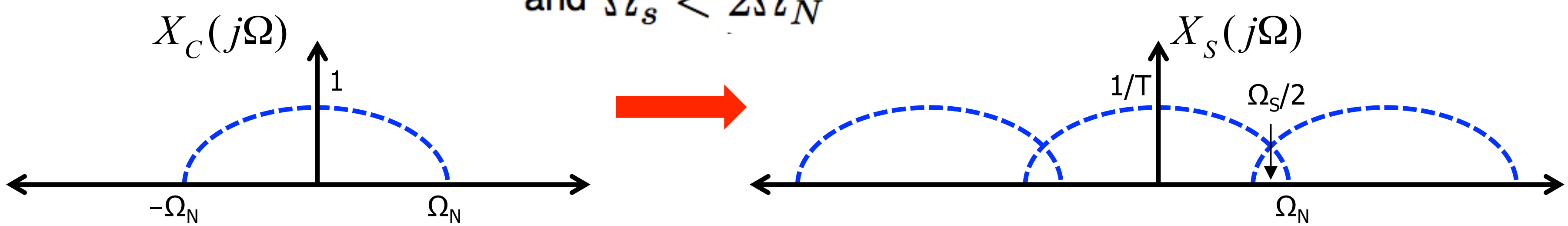


ADC A/D
$x_c(t)$
Analog
Anti-Aliasing
Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff
(DSP,
storage....)
$y[n]$
Reconstruction
$y_c(t)$

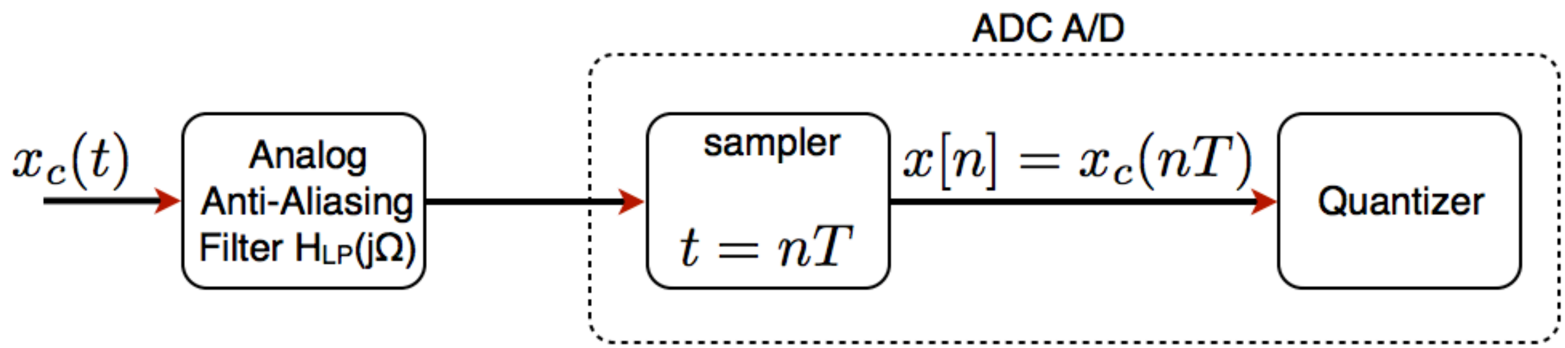
Anti-Aliasing Filter



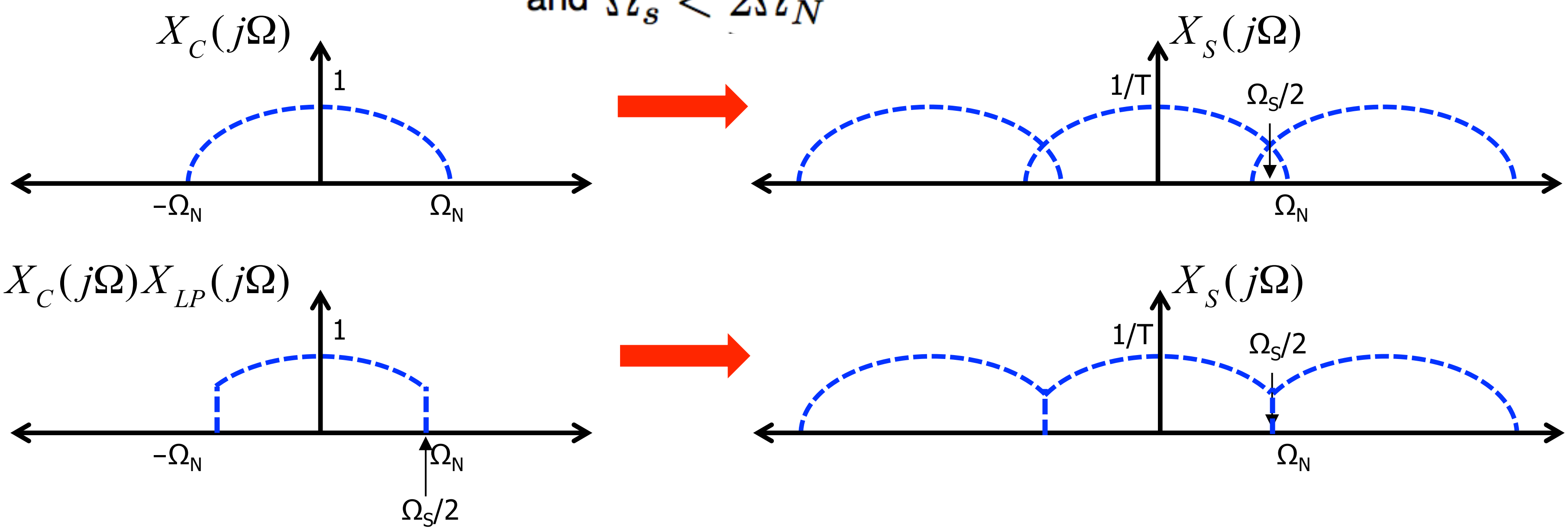
and $\Omega_s < 2\Omega_N$



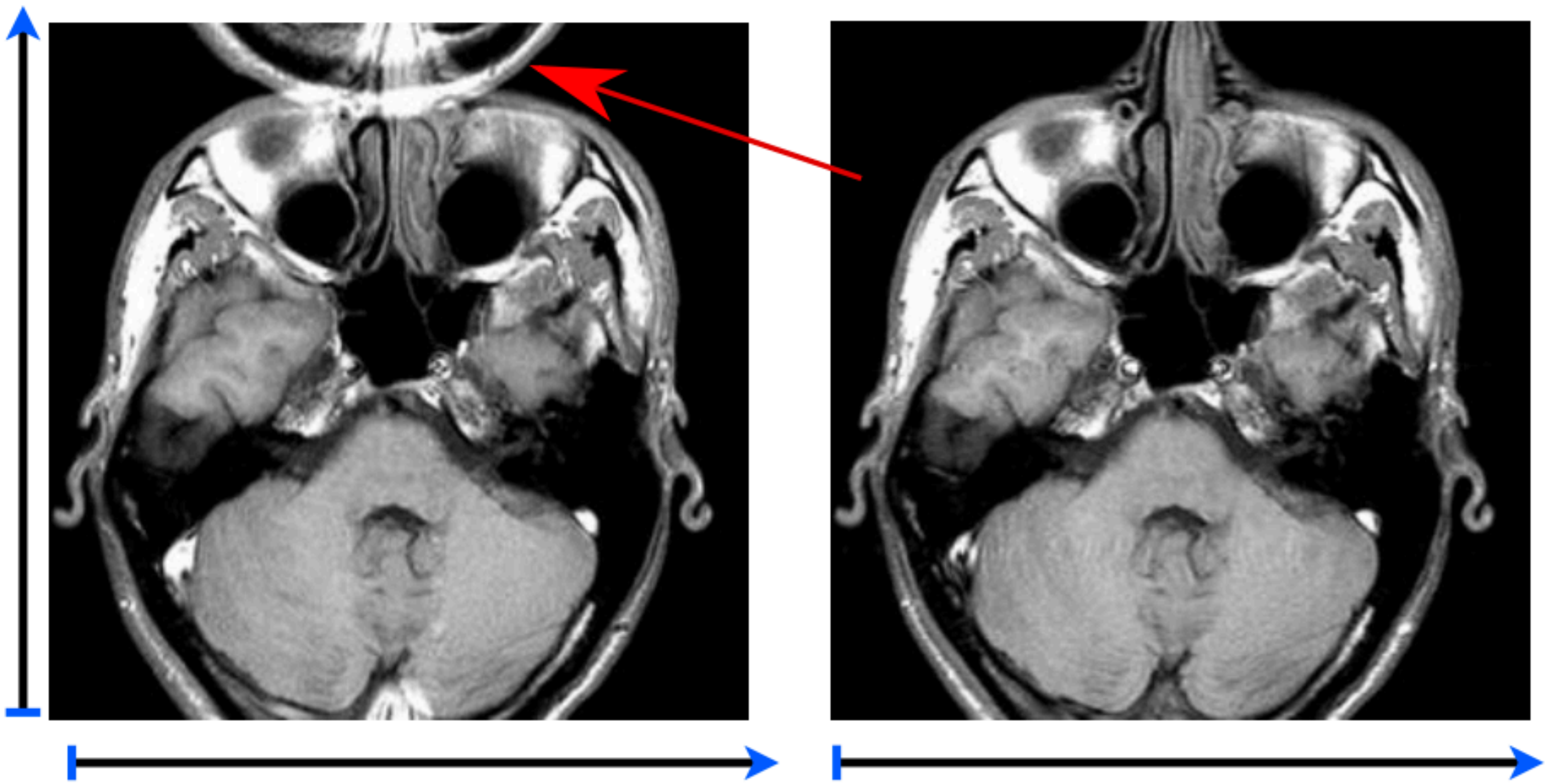
Anti-Aliasing Filter



and $\Omega_s < 2\Omega_N$



MRI aliasing example



MRI anti-aliasing example

MRI anti-aliasing example

Big Ideas

- Sampling
  - Ideal sampling modeled as impulsive sampling
  - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- Frequency Response of Sampled Signal
  - Sampled signal is period replicated input CT signal
- Reconstruction
  - Low pass/reconstruction filter results in sum of sincs
- Anti-aliasing filtering
  - Force input signal to be bandlimited

Admin

- Office hours location starting next week – Towne 211
  - Course webpage updated with this location

- HW 2 due Sunday at midnight
- HW 3 posted Sunday