

ESE 531: Digital Signal Processing

Lec 7: February 7, 2019
Sampling and Reconstruction



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Lecture Outline

- Sampling
 - Frequency Response of Sampled Signal
- Reconstruction
- Anti-aliasing Filter

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2

Video Example

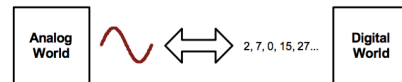


□ <https://www.youtube.com/watch?v=ByTsISFXUoY>

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3

The Data Conversion Problem

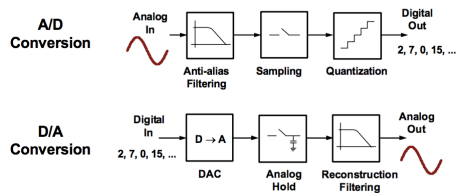


- Real world signals
 - Continuous time, continuous amplitude
- Digital abstraction
 - Discrete time, discrete amplitude
- Two problems
 - How to go discrete in time and amplitude
 - A/D conversion
 - How to "undescretize" in time and amplitude
 - D/A conversion

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4

Overview

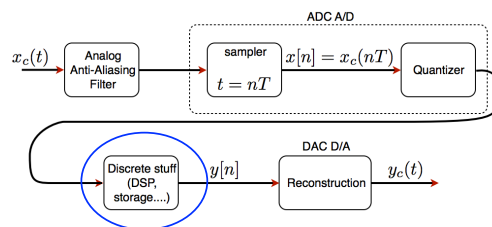


- We'll first look at these building blocks from a functional, "black box" perspective

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5

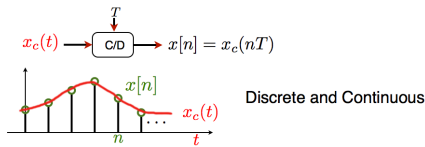
DSP System



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6

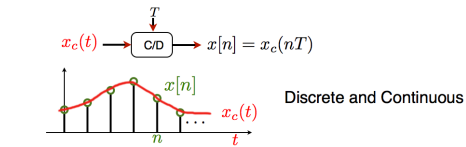
Ideal Sampling Model



□ Ideal continuous-to-discrete time (C/D) converter

- T is the sampling period
- $f_s = 1/T$ is the sampling frequency
- $\Omega_s = 2\pi/T$

Ideal Sampling Model



define impulsive sampling:

Continuous

$$x_s(t) = \dots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \dots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

- ### □ Dirac delta function, $\delta(t)$
- Infinitely high and thin, area of 1
 - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- ### □ Three signals. How are they related? In time? In frequency?

Frequency Domain Analysis

- ### □ How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \quad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Frequency Domain Analysis

- ### □ How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \quad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$\begin{aligned} x_s(t) &: \text{C.T} & X_s(j\Omega) &= \sum_n x_c(nT) e^{-j\Omega nT} \\ x[n] &: \text{D.T} & X(e^{j\omega}) &= \sum_n x[n] e^{-j\omega n} \end{aligned} \quad \omega = \Omega T$$

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$$

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T}$$

Frequency Domain Analysis

Continuous

$$x_s(t) = \dots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \dots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT) \triangleq s(t)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Frequency Domain Analysis

Continuous

$$x_s(t) = \dots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \dots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$s(t) \leftrightarrow S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$

$\Omega_s = \frac{2\pi}{T}$

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Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

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Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

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Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

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$\Omega_s = \frac{2\pi}{T}$

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Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T}$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$

$\Omega_s = \frac{2\pi}{T}$

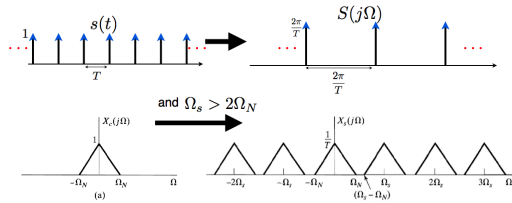
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Frequency Domain Analysis

$\Omega_s = \frac{2\pi}{T}$

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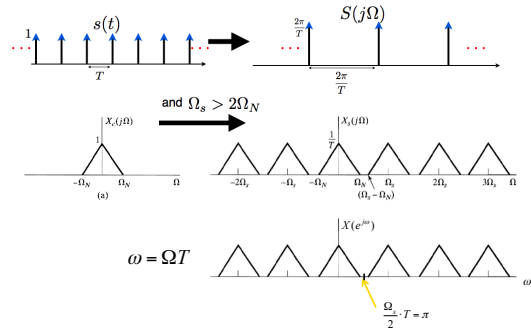
Frequency Domain Analysis



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19

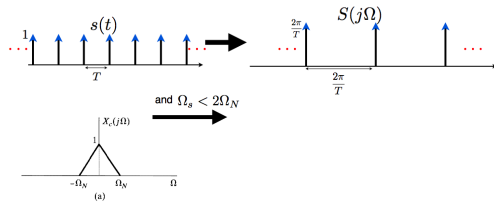
Frequency Domain Analysis



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20

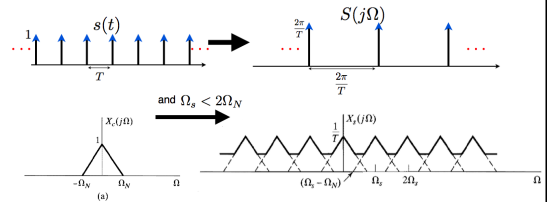
Frequency Domain Analysis w/ Aliasing



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21

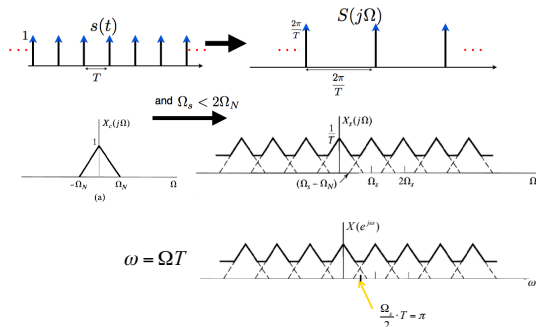
Frequency Domain Analysis w/ Aliasing



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22

Frequency Domain Analysis w/ Aliasing



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23

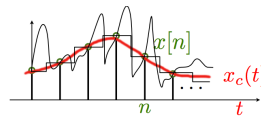
Reconstruction of Bandlimited Signals

□ Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

□ If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$

□ Bandlimitedness is the key to uniqueness

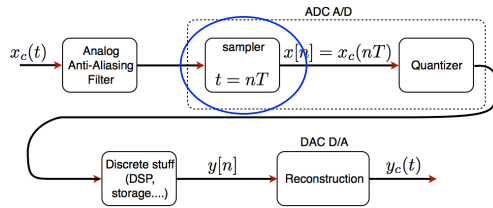


Multiple signals go through the samples, but only one is bandlimited within our sampling band

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24

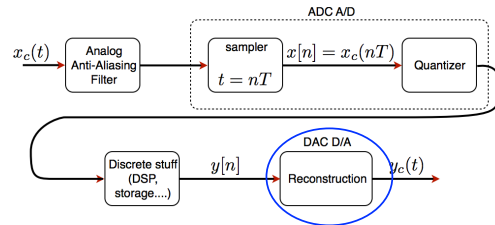
DSP System



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25

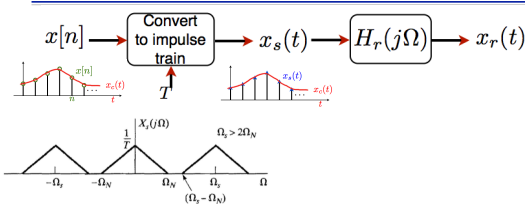
DSP System



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26

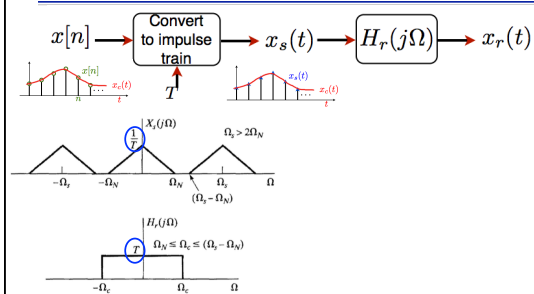
Reconstruction in Frequency Domain



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27

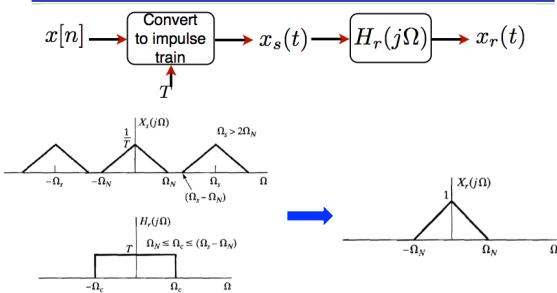
Reconstruction in Frequency Domain



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28

Reconstruction in Frequency Domain



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29

Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t) = \cos(4000\pi t)$ with sampling period $T = 1/6000$ ($f_s = 6\text{kHz}$)

$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T} \end{aligned}$$

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30

Example: Cosine Input

Sample and reconstruct the continuous-time signal $x_c(t) = \cos(16000\pi t)$ with sampling period $T = 1/6000$ ($f_s = 6\text{kHz}$)

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Reconstruction in Frequency Domain

Block diagram: $x[n] \rightarrow \text{Convert to impulse train} \rightarrow x_s(t) \rightarrow H_r(j\Omega) \rightarrow x_r(t)$

Frequency domain plots: $|X_c(j\Omega)|$ (triangular spectrum), $|H_r(j\Omega)|$ (rectangular filter), and $|X_r(j\Omega)|$ (reconstructed spectrum).

Annotations: $2\Omega_N = \Omega_s$, $\Omega_N = \Omega_C = \Omega_s/2$. Sample at Nyquist and filter at signal bandwidth.

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Reconstruction in Time Domain $\Omega_N = \Omega_C = \Omega_s/2$

Equation:
$$h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Plot of $|H_r(j\Omega)|$ vs Ω showing the rectangular filter response.

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Reconstruction in Time Domain $\Omega_N = \Omega_C = \Omega_s/2$

Equation:
$$h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega = \frac{T}{2\pi} \frac{1}{jt} \left[e^{j\Omega_s/2 t} - e^{-j\Omega_s/2 t} \right] = \frac{T}{\pi t} \frac{2j \sin(\frac{\Omega_s}{2} t)}{2j} = \frac{T}{\pi t} \sin(\frac{\Omega_s}{2} t) = \frac{T}{\pi t} \sin(\frac{\pi}{T} t) = \text{sinc}\left(\frac{t}{T}\right)$$

Plot of $|h_r(t)|$ vs t showing the sinc reconstruction kernel.

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Reconstruction in Time Domain

Equation:
$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n] \delta(t - nT) \right) * h_r(t) = \sum_n x[n] h_r(t - nT)$$

Block diagram: $x[n] \rightarrow \text{Convert to impulse train} \rightarrow x_s(t) \rightarrow H_r(j\Omega) \rightarrow x_r(t)$

Plots showing the impulse train $x_s(t)$, the sinc kernel $h_r(t)$, and the reconstructed signal $x_r(t)$.

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Reconstruction in Time Domain

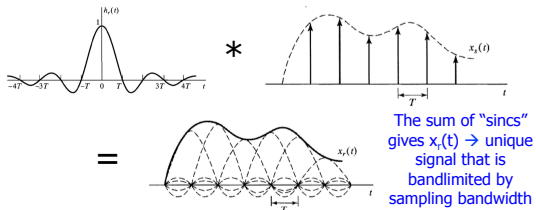
Equation:
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Plots showing the impulse train $x_s(t)$, the sinc kernel $h_r(t)$, and the reconstructed signal $x_r(t)$.

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Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n] \delta(t - nT) \right) * h_r(t) = \sum_n x[n] h_r(t - nT)$$

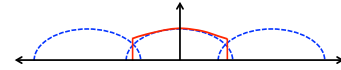


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37

Aliasing

□ If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

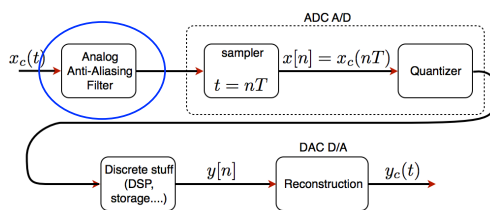


$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

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38

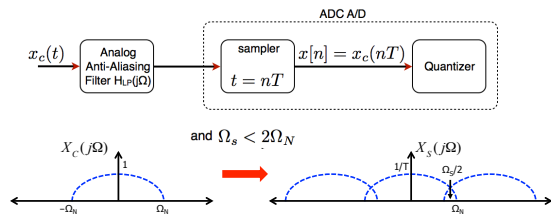
DSP System



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39

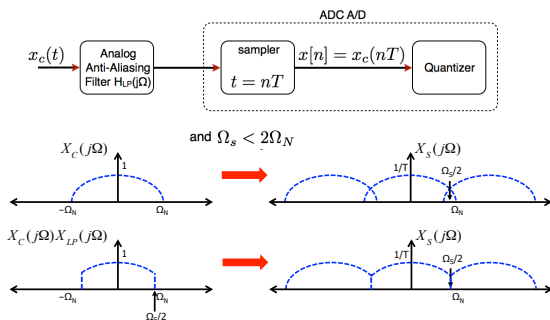
Anti-Aliasing Filter



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40

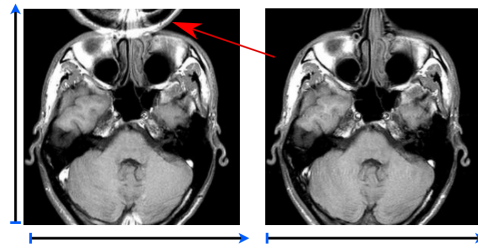
Anti-Aliasing Filter



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41

MRI aliasing example



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42

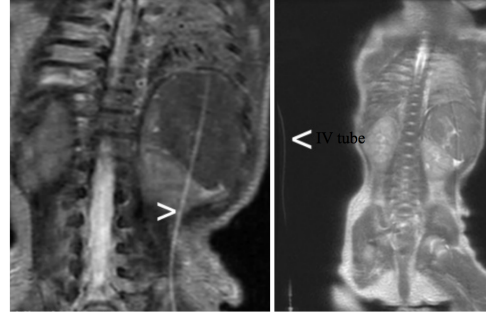
MRI anti-aliasing example



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43

MRI anti-aliasing example



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44

Big Ideas

- ❑ Sampling
 - Ideal sampling modeled as impulsive sampling
 - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- ❑ Frequency Response of Sampled Signal
 - Sampled signal is period replicated input CT signal
- ❑ Reconstruction
 - Low pass/reconstruction filter results in sum of sines
- ❑ Anti-aliasing filtering
 - Force input signal to be bandlimited

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45

Admin

- ❑ Office hours location starting next week – Towne 211
 - Course webpage updated with this location
- ❑ HW 2 due Sunday at midnight
- ❑ HW 3 posted Sunday

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46