# ESE 531: Digital Signal Processing

Lec 8: February 12th, 2019

Sampling and Reconstruction



## Lecture Outline

- Review
  - Ideal sampling
  - Frequency response of sampled signal
  - Reconstruction
  - Anti-aliasing filtering
- DT processing of CT signals
  - Impulse Invariance
- CT processing of DT signals (why??)

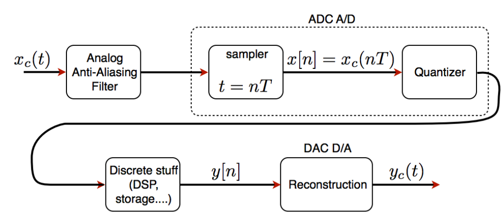


# Last Time…

Sampling, Frequency Response of Sampled Signal, Reconstruction, Anti-aliasing filtering

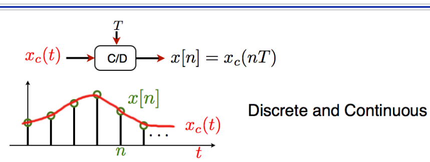


## DSP System

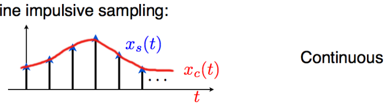





## Ideal Sampling Model




define impulsive sampling:




$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



## Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

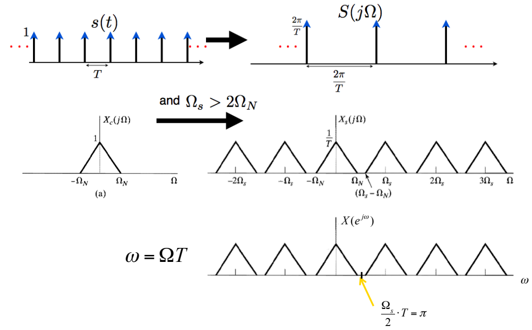
$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$x_s(t)$ :C.T $\qquad X_s(j\Omega) = \sum_n x_c(nT)e^{-j\Omega nT}$

$x[n]$ :D.T $\qquad X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n} \qquad \omega = \Omega T$

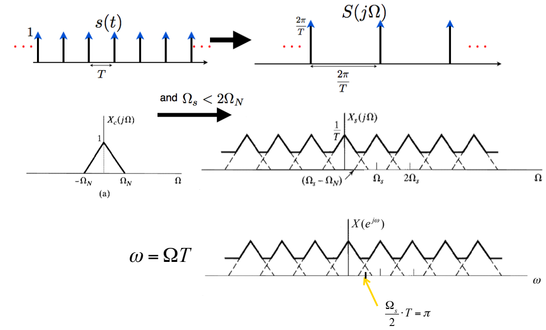
$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T} \qquad X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T}$$

## Frequency Domain Analysis






## Frequency Domain Analysis



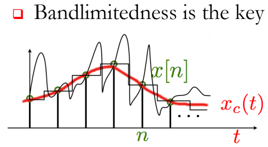



## Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

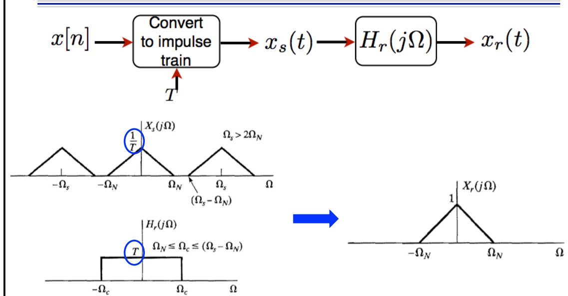
- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness




Mulitiple signals go through the samples, but only one is bandlimited within our sampling band



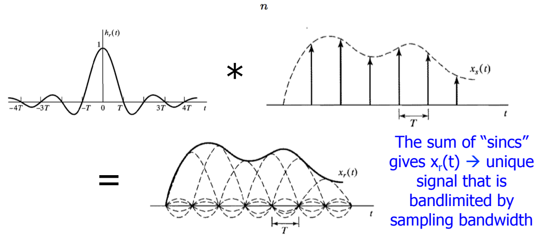
## Reconstruction in Frequency Domain






## Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT)\right) * h_r(t)$$
$$= \sum_n x[n]h_r(t - nT)$$

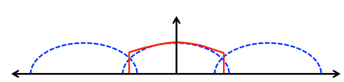



The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth



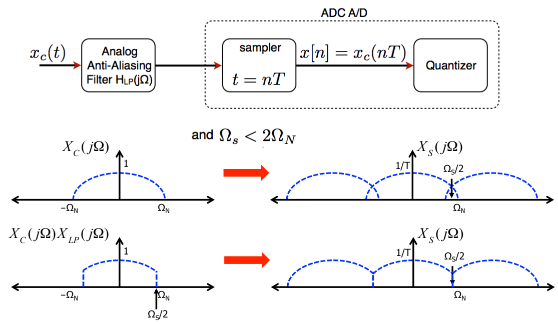
## Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$



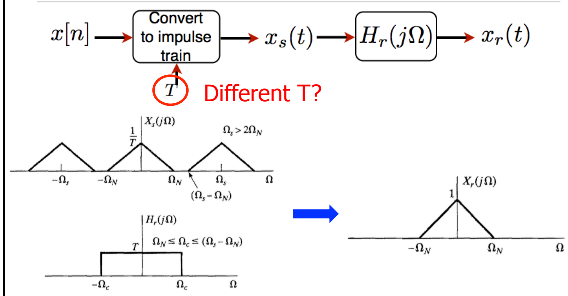
$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if} \quad |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

## Anti-Aliasing Filter





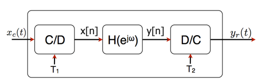

## Reconstruction in Frequency Domain






## Signal Processing

- Use theory of sampling (C/D) and reconstruction (D/C) to implement signal processing
- Two cases:
  - Discrete-time processing of continuous-time signals



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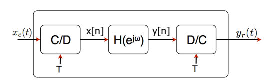
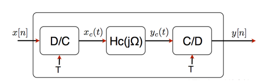
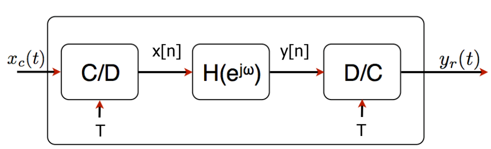



  - Continuous-time processing of discrete-time signals



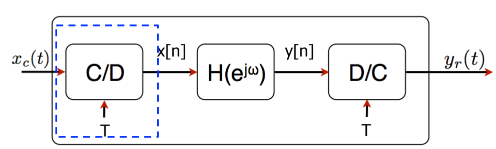



## Discrete-Time Processing of Continuous Time

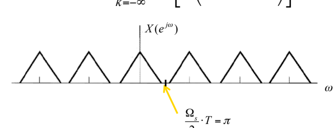





## Discrete-Time Processing of Continuous Time




$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$



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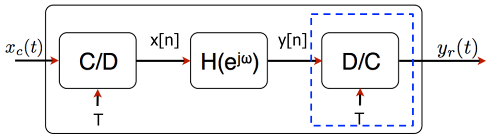


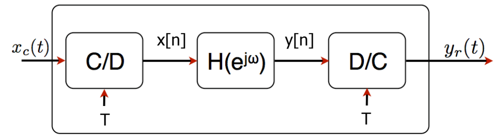

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

Sum of scaled shifted sincs

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$



## Discrete-Time Processing of Continuous Time

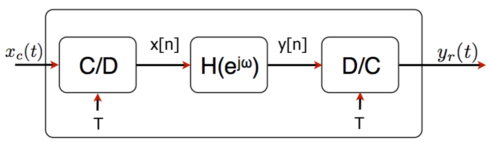



$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right] \qquad y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

- If h[n]/H($e^{j\omega}$) is LTI
  - Is the whole system from $x_c(t) \rightarrow y_c(t)$ LTI?



## Discrete-Time Processing of Continuous Time




$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right] \qquad y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

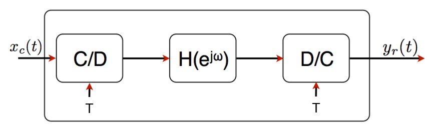
- If $x_c(t)$ is bandlimited by $\Omega_s/T = \pi/T$, then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{eff}(j\Omega) = \begin{cases} H(e^{j\omega})\big|_{\omega=\Omega T} & |\Omega| < \Omega_s/T \\ 0 & else \end{cases}$$



## Example 1

- Consider the following system




- Where

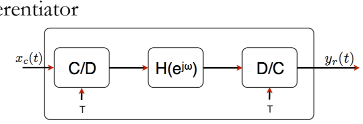
$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

- What is the effective frequency response of the system? What happens to a signal bandlimited by $\Omega_N$?



## Example 2

- DT implementation of an ideal CT bandlimited differentiator




- The ideal CT differentiator is defined by

$$y_C(t) = \frac{d}{dt}[x_C(t)]$$

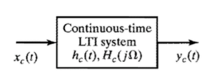
- With corresponding

$$H_C(j\Omega) = j\Omega$$



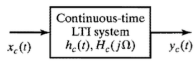
## Impulse Invariance

- Want to implement continuous-time system…

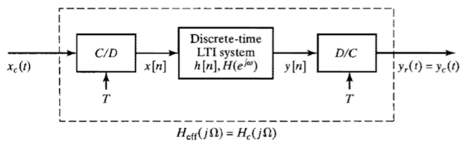


## Impulse Invariance

- Want to implement continuous-time system…




- …in discrete-time






## Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\Omega)\Big|_{\Omega=\frac{\omega}{T}}, \qquad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi / T$$



## Impulse Invariance

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$$\boxed{h[n] = Th_c(nT)}$$



## Impulse Invariance

- Let,

$$h[n] = h_c(nT)$$



## Impulse Invariance

- Let,

$$h[n] = h_c(nT)$$

- If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$



## Impulse Invariance

- Let,

$$h[n] = h_c(nT)$$

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- If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

$$H(e^{j\omega}) = \frac{1}{T} H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| < \pi$$

## Impulse Invariance

- Let,

$$h[n] = T h_c(nT)$$

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

- If sampling at Nyquist Rate then

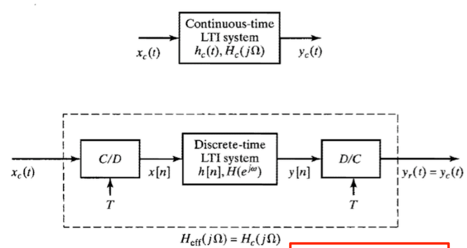
$$H(e^{j\omega}) = \frac{T}{T}\sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

$$H(e^{j\omega}) = \frac{T}{T} H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| < \pi$$



## Impulse Invariance

- Want to implement continuous-time system in discrete-time

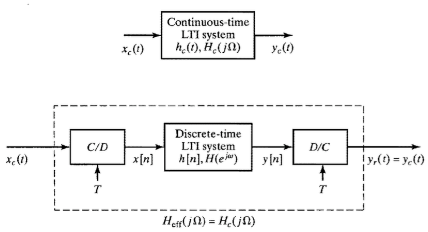



$$h[n] = T h_c(nT)$$



## Example 3: DT Lowpass Filter

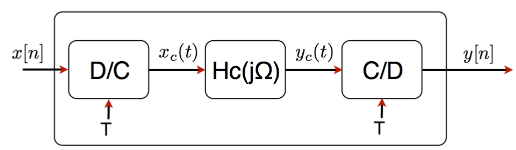
- We wish to implement a lowpass filter with cutoff frequency $\Omega_c$ on continuous time signal in discrete time with the following system






## Continuous-Time Processing of Discrete-Time

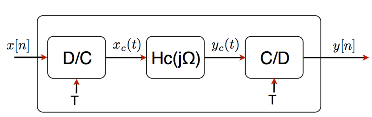
- Useful to interpret DT systems with no simple interpretation in discrete time




Is the effective $H(e^{j\omega})$ LTI?



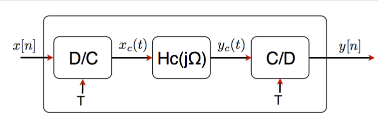
## Continuous-Time Processing of Discrete-Time




$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$



## Continuous-Time Processing of Discrete-Time

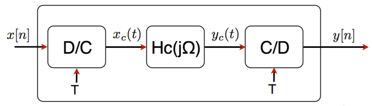



$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Also bandlimited

$$\rightarrow Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

## Continuous-Time Processing of Discrete-Time

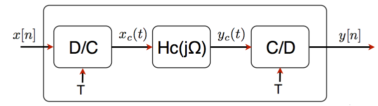



$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left[j\left(\Omega - k\Omega_s\right)\right]\Big|_{\Omega=\omega/T}$$



## Continuous-Time Processing of Discrete-Time

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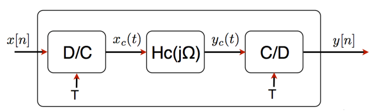
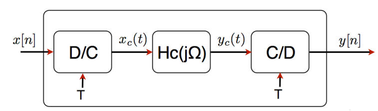
## Continuous-Time Processing of Discrete-Time




$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \qquad Y(e^{j\omega}) = \frac{1}{T}Y_c(j\Omega)\Big|_{\Omega=\omega/T}$$



## Continuous-Time Processing of Discrete-Time

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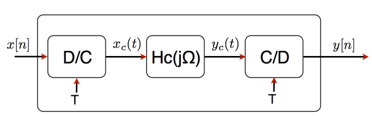
## Continuous-Time Processing of Discrete-Time

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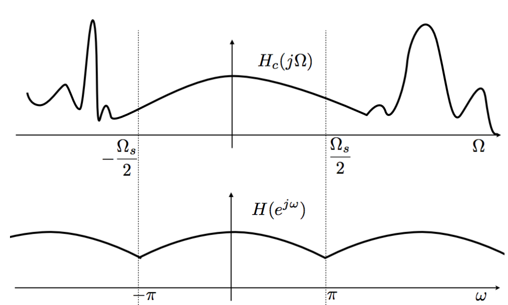
$$Y(e^{j\omega}) = \frac{1}{T}H_c(j\Omega)\Big|_{\Omega=\omega/T} X_c(j\Omega)\Big|_{\Omega=\omega/T}$$

$$= \frac{1}{T}H_c(j\Omega)\Big|_{\Omega=\omega/T}(TX(e^{j\omega})) \qquad |\omega| < \pi$$

$H(e^{j\omega})$



## Example



## Example: Non-integer Delay

- What is the time domain operation when $\Delta$ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

$$\delta[n] \leftrightarrow 1$$
$$\delta[n-n_d] \leftrightarrow e^{-j\omega n_d}$$



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Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ delay of ΔT in continuous time



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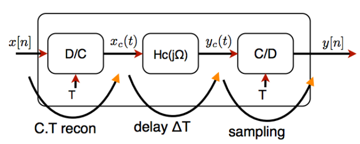




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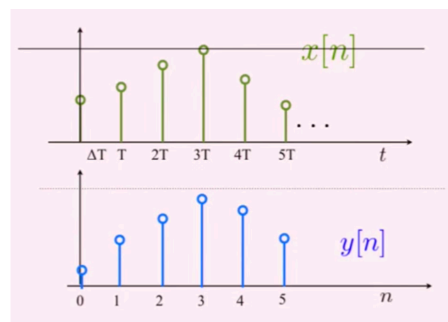




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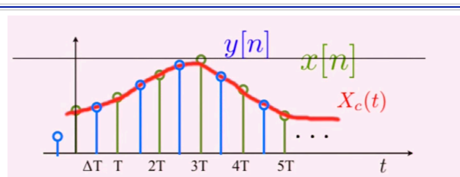




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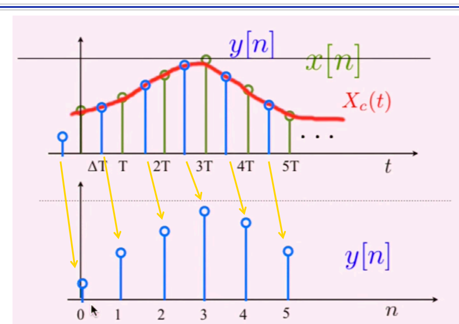
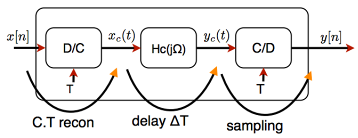


## Example: Non-integer Delay

- The block diagram is for interpretation/analysis only




$y_c(t) = x_c(t - T\Delta)$



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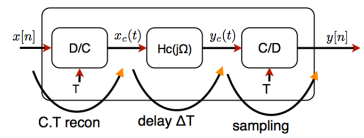
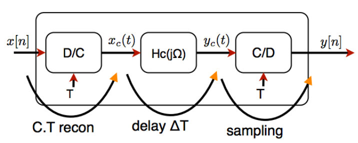



$y_c(t) = x_c(t - T\Delta)$ $\quad y[n] = y_c(nT) = x_c(nT - T\Delta)$



## Example: Non-integer Delay

- The block diagram is for interpretation/analysis only




$y_c(t) = x_c(t - T\Delta)$

$$y[n] = y_c(nT) = x_c(nT - T\Delta)$$
$$= \sum_k x[k]\,\mathrm{sinc}\left(\frac{t - kT - T\Delta}{T}\right)\Bigg|_{t=nT}$$
$$= \sum_k x[k]\,\mathrm{sinc}\left(n - k - \Delta\right)$$



## Example: Non-integer Delay

- My delay system has an impulse response of a sinc with a continuous time delay

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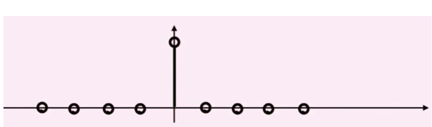



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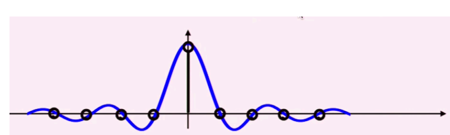

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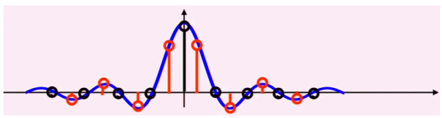



## Big Ideas

- Sampling and reconstruction
  - Rely on bandlimitedness for unique reconstruction
- CT processing of DT
  - Effectively LTI if no aliasing
- DT processing of CT
  - Always LTI
  - Useful for interpretation

- Changing the sampling rates next time
  - Upsampling, downsampling



## Admin

- HW 3 due Sunday