ESE 531: Digital Signal Processing

Lec 9: February 14th, 2019

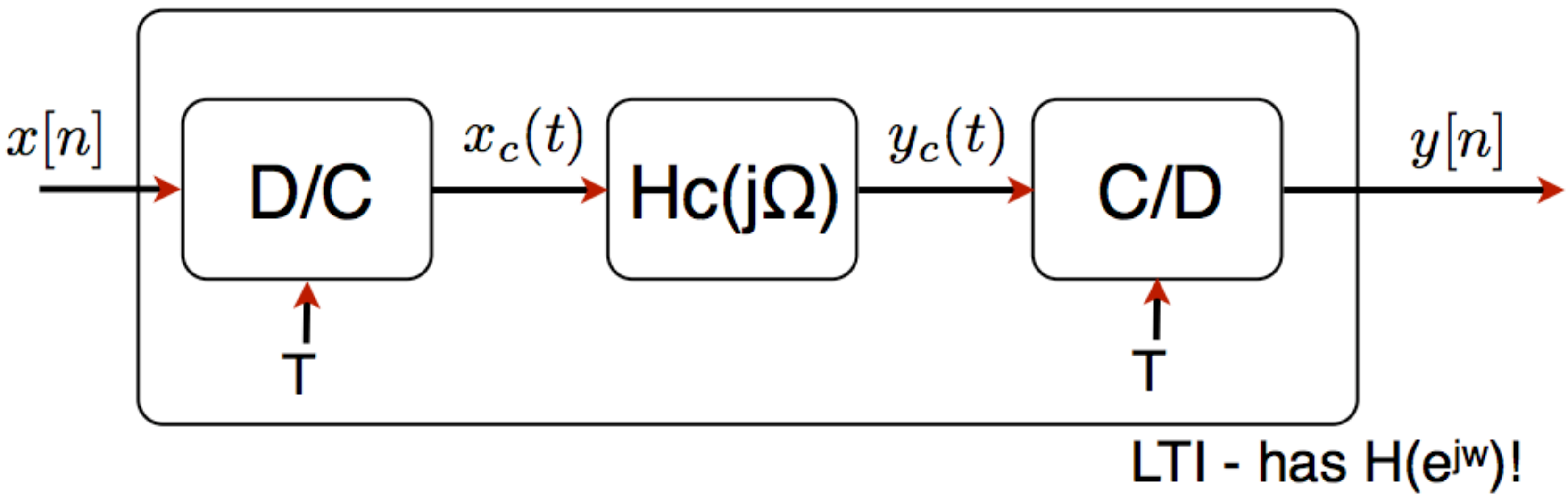
Downsampling/Upsampling and Practical Interpolation

Lecture Outline

- CT processing of DT signals
- Downsampling
- Upsampling
- Practical Interpolation (time permitting)

Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time

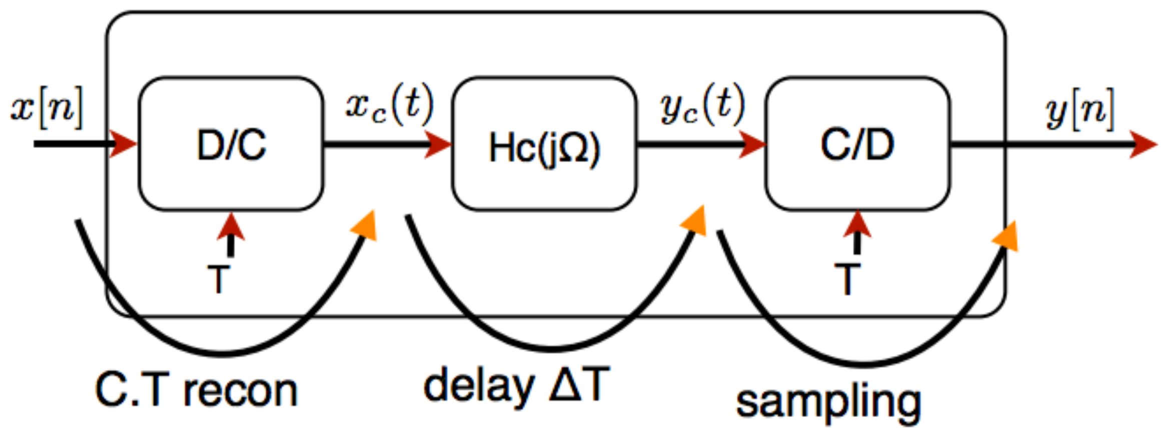


Example: Non-integer Delay

- What is the time domain operation when Δ is non-integer? I.e $\Delta=1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ delay of ΔT in time



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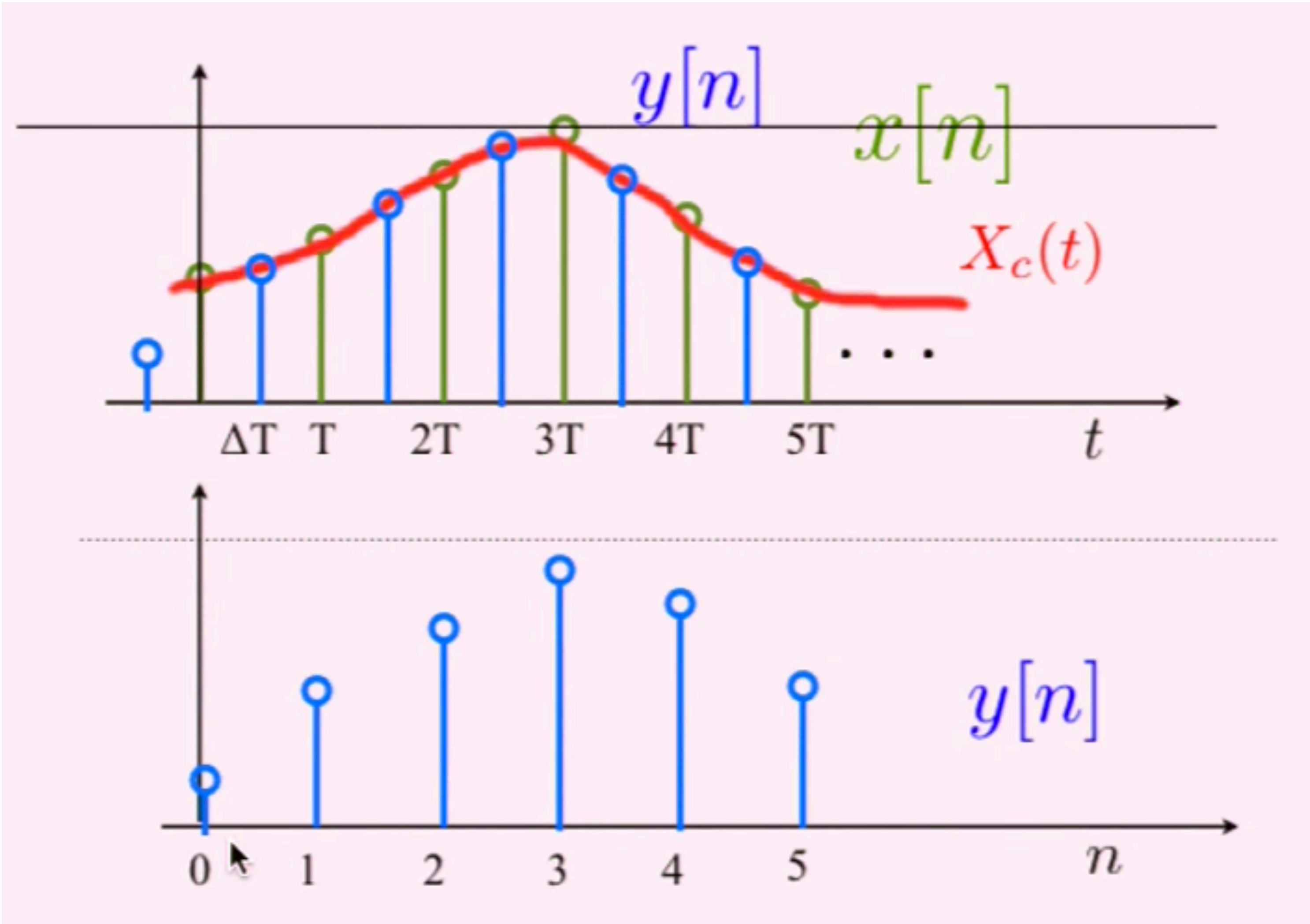
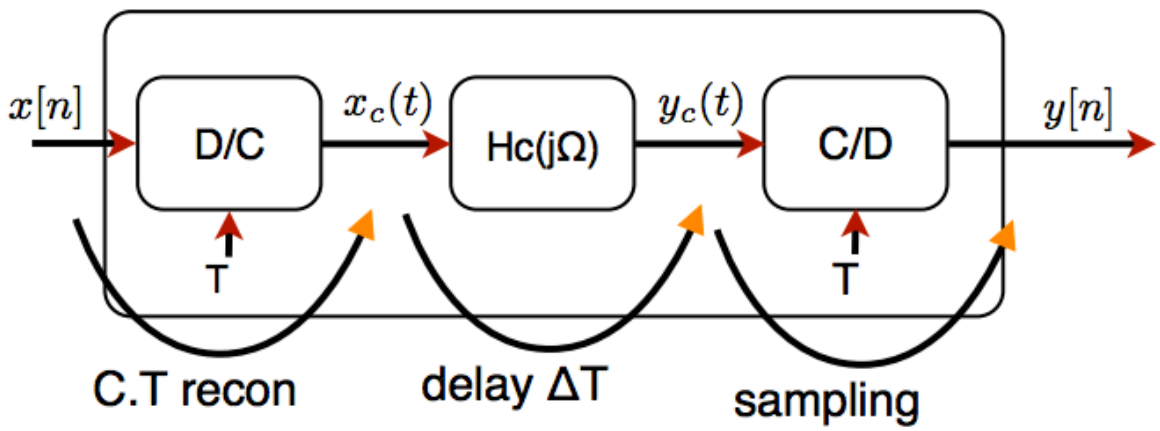


Example: Non-integer Delay

- The block diagram is for interpretation/analysis only



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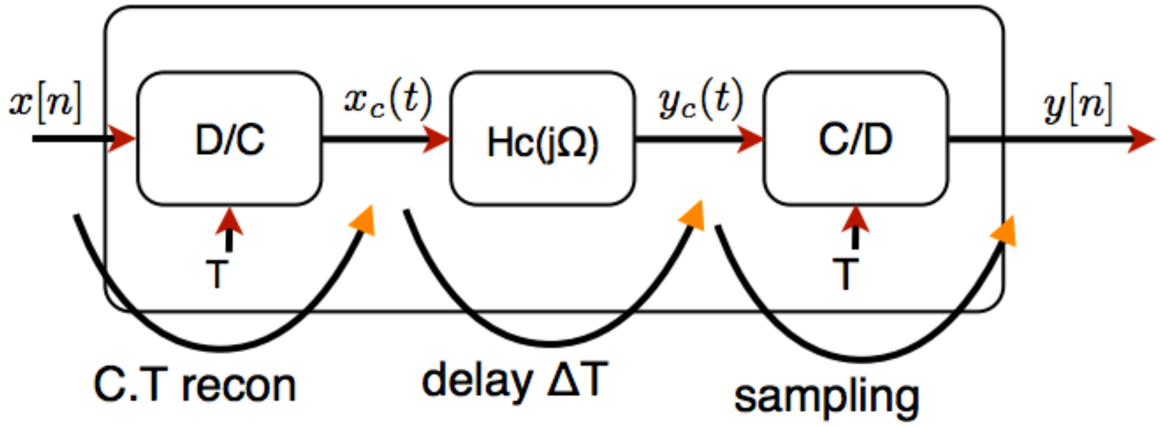

$$y[n] = y_c(nT) = x_c(nT - T \cdot \Delta) \qquad x_c(t) = \sum_k x[k] \mathrm{sinc}\left(\frac{t - kT}{T}\right)$$

$$x_c(nT - T \cdot \Delta) = \sum_k x[k] \mathrm{sinc}\left(\frac{nT - T \cdot \Delta - kT}{T}\right) = \sum_k x[k] \mathrm{sinc}(n - \Delta - k)$$

Example: Non-integer Delay

- Delay system has an impulse response of a sinc with a continuous time delay

$$y[n] = \sum_k x[k]\,\mathrm{sinc}\left(n - \Delta - k\right)$$
$$= x[n] * \mathrm{sinc}\left(n - \Delta\right)$$

$$\boxed{\Rightarrow h[n] = \mathrm{sinc}(n - \Delta)}$$

Example: Non-integer Delay

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$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

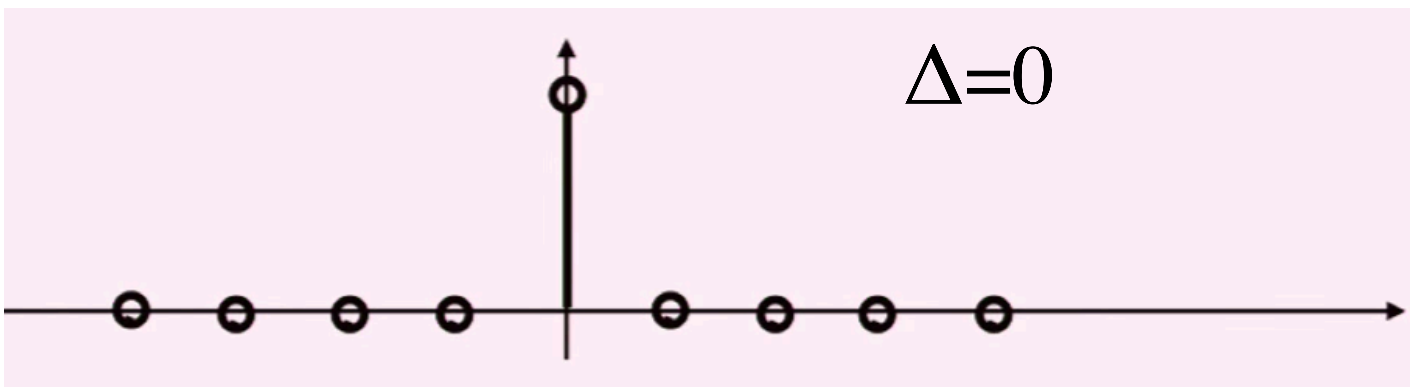
$$\boxed{\begin{aligned} \delta[n] &\leftrightarrow 1 \\ \delta[n-n_d] &\leftrightarrow e^{-j\omega n_d} \end{aligned}}$$

$$\boxed{\Rightarrow h[n] = \mathrm{sinc}(n-\Delta)}$$

Example: Non-integer Delay

- My delay system has an impulse response of a sinc with a continuous time delay

$$h[n] = \text{sinc}(n - \Delta)$$



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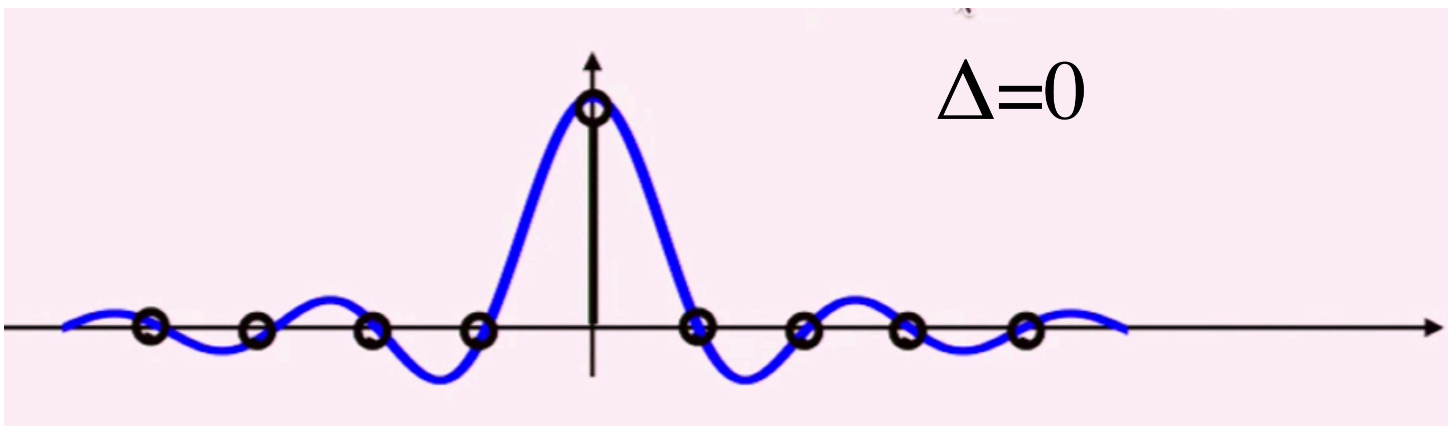

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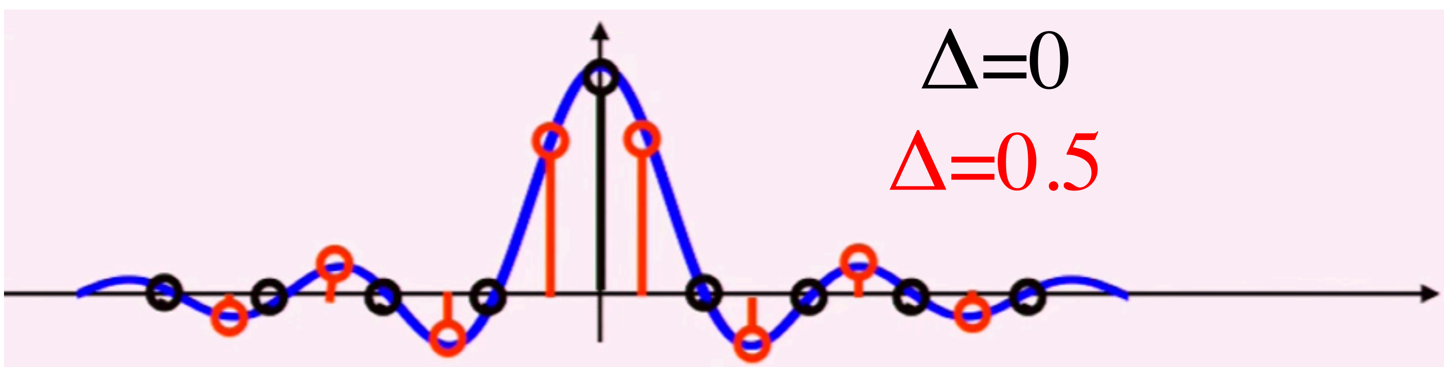

Downsampling

Downsampling

- Definition: Reducing the sampling rate by an integer number (M>1)

$$x[n] \longrightarrow \boxed{\downarrow \mathrm{M}} \longrightarrow x_d[n] = x[nM]$$

$$x[n] = x_c(nT)$$

$$x[nM] = x_c(n\underbrace{MT}_{T'})$$

Downsampling

- Similar to C/D conversion
  - Need to worry about aliasing
  - Use anti-aliasing filter to mitigate effects

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  - Need to worry about aliasing
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- If your discrete time signal is finely sampled (i.e oversampled) almost like a CT signal
  - Downsampling is just like sampling (C/D conversion)

Downsampling

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$$x[n] = x_c(nT)$$

$$x_d[n] = x_c(n\underbrace{MT}_{T'})$$

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

Downsampling

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Downsampling

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$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

- Want to relate $X_d(e^{j\omega})$ to $X(e^{j\omega})$ not $X_c(j\Omega)$

- Separate sum into two sums—fine sum and coarse sum (i.e like counting minutes within hours)

Downsampling

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

- k=rM+i
  - i = 0, 1, …, M-1
  - r = -∞, ..., ∞

Downsampling

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}i - \frac{2\pi}{T}r\right)\right)}_{X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})}$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}} - \underbrace{\frac{2\pi}{T}}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$

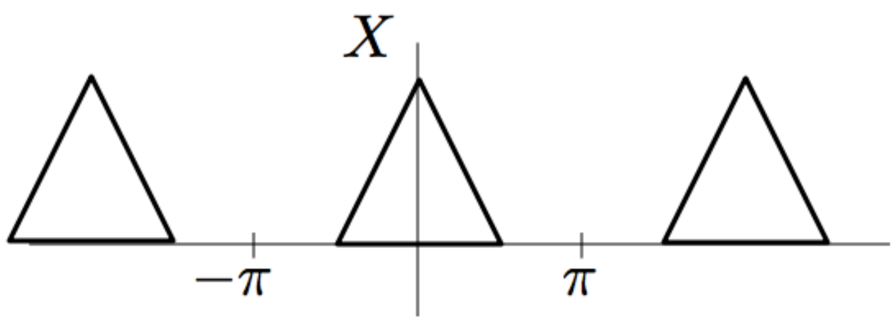
scale by 1/M

stretch by M

replicate

Example: M=2

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



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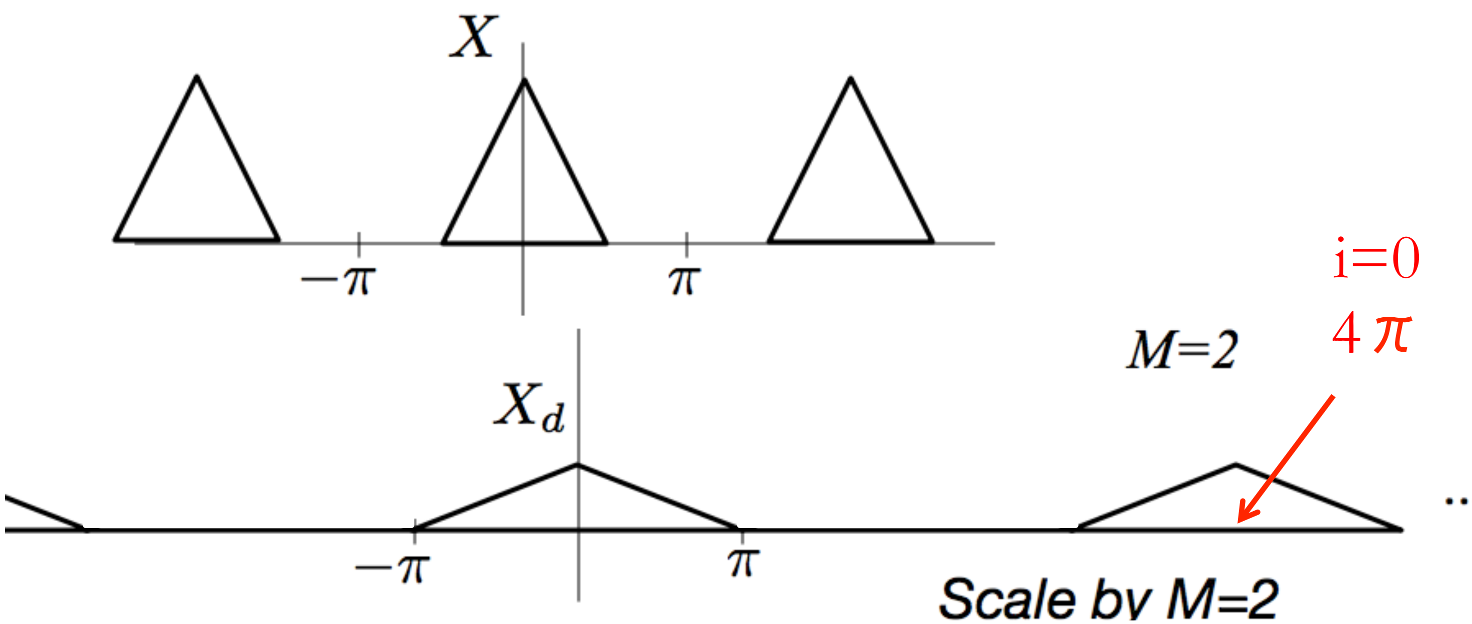

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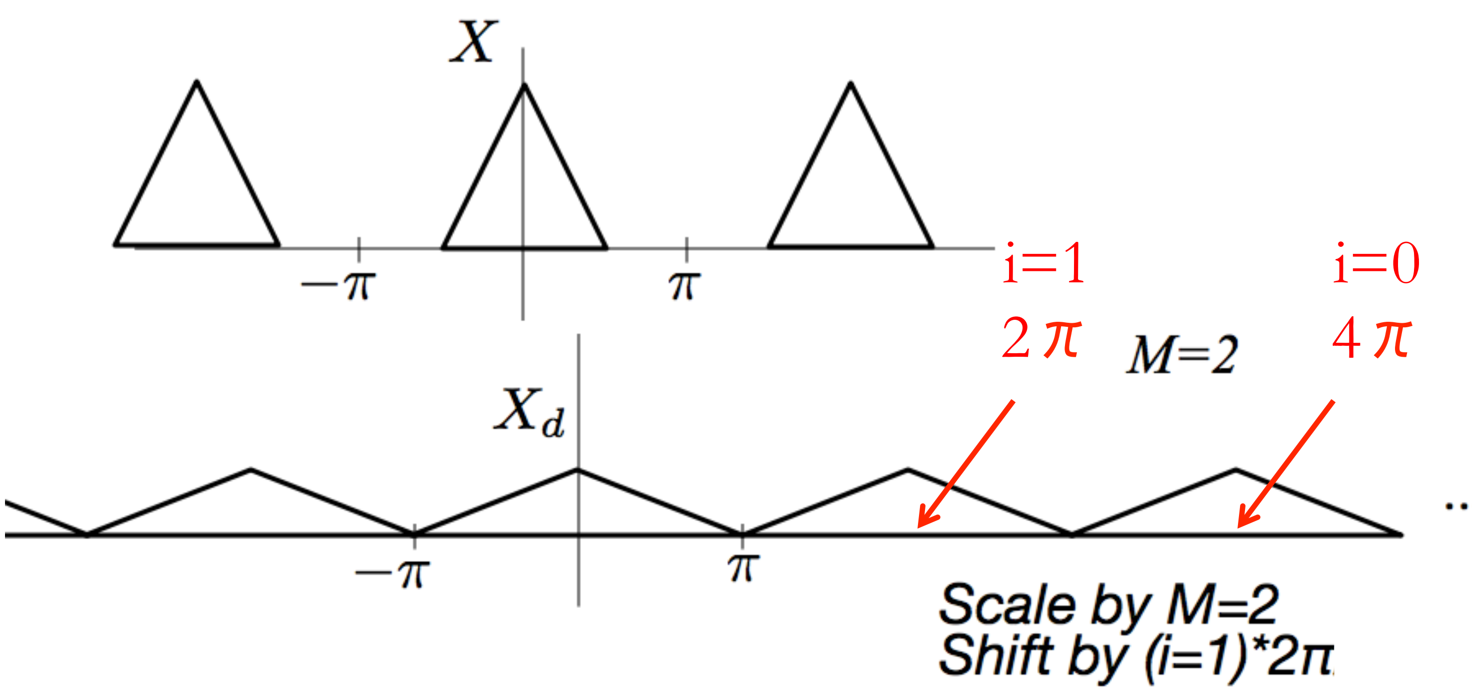

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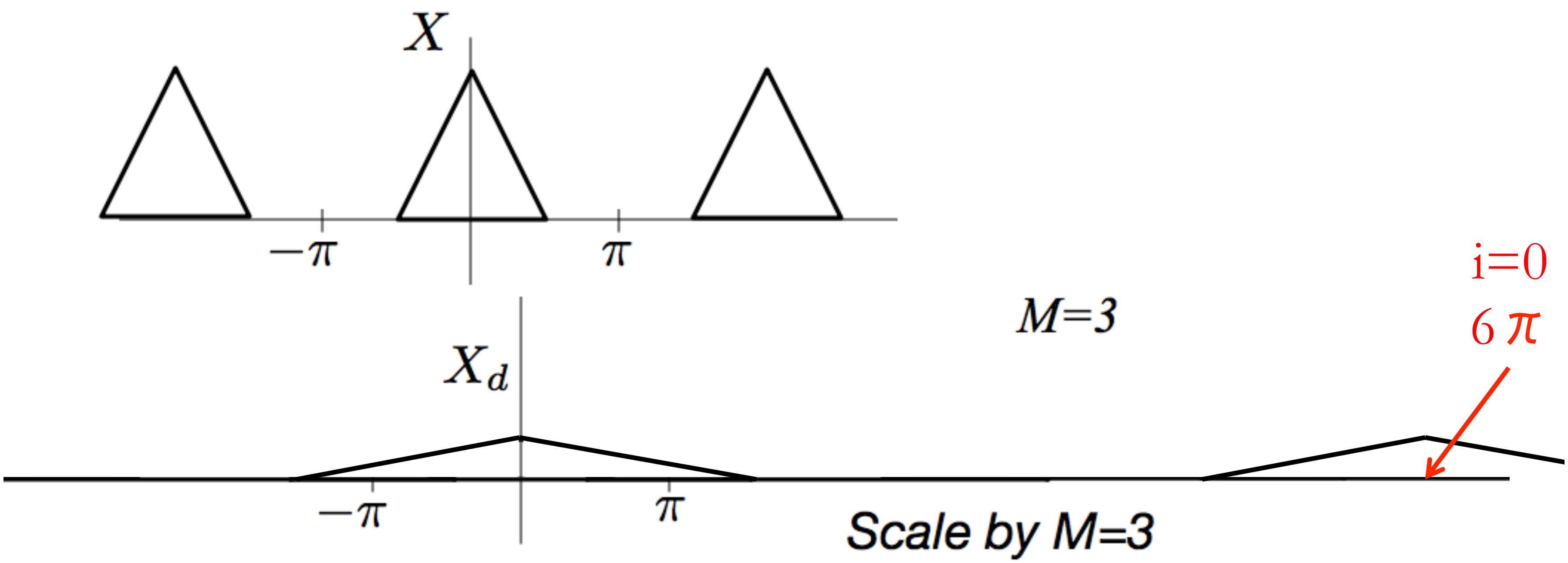

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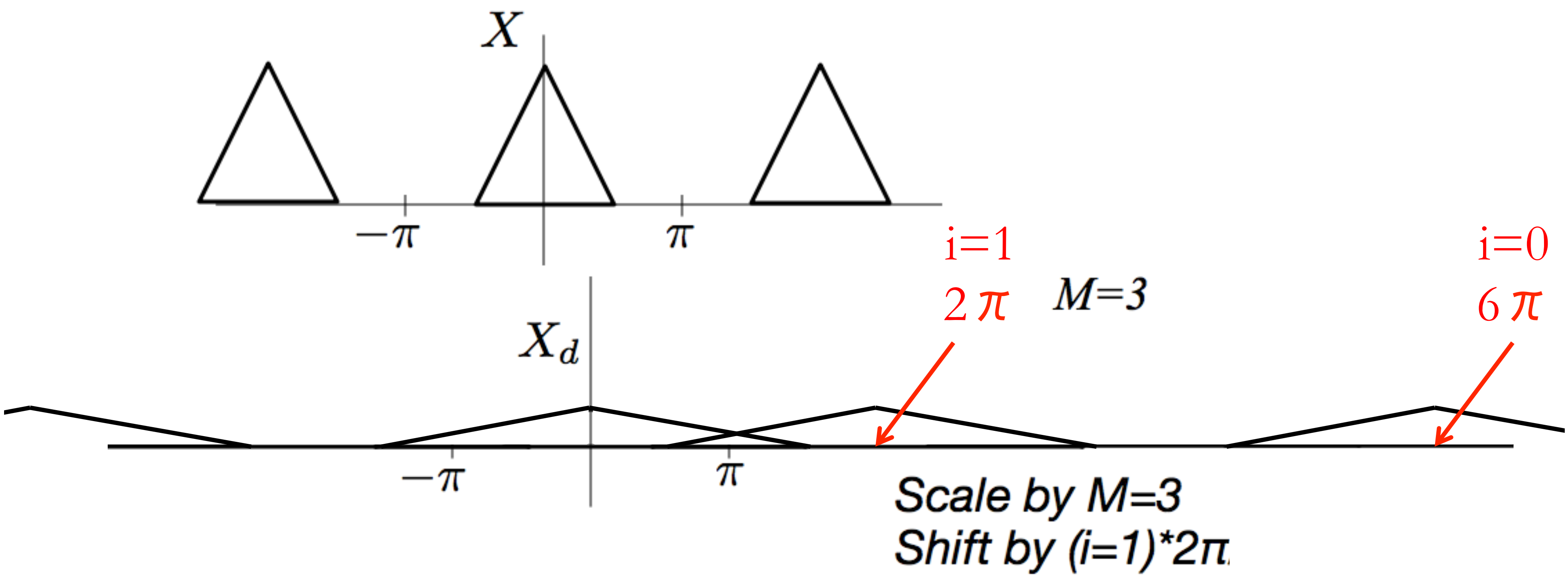

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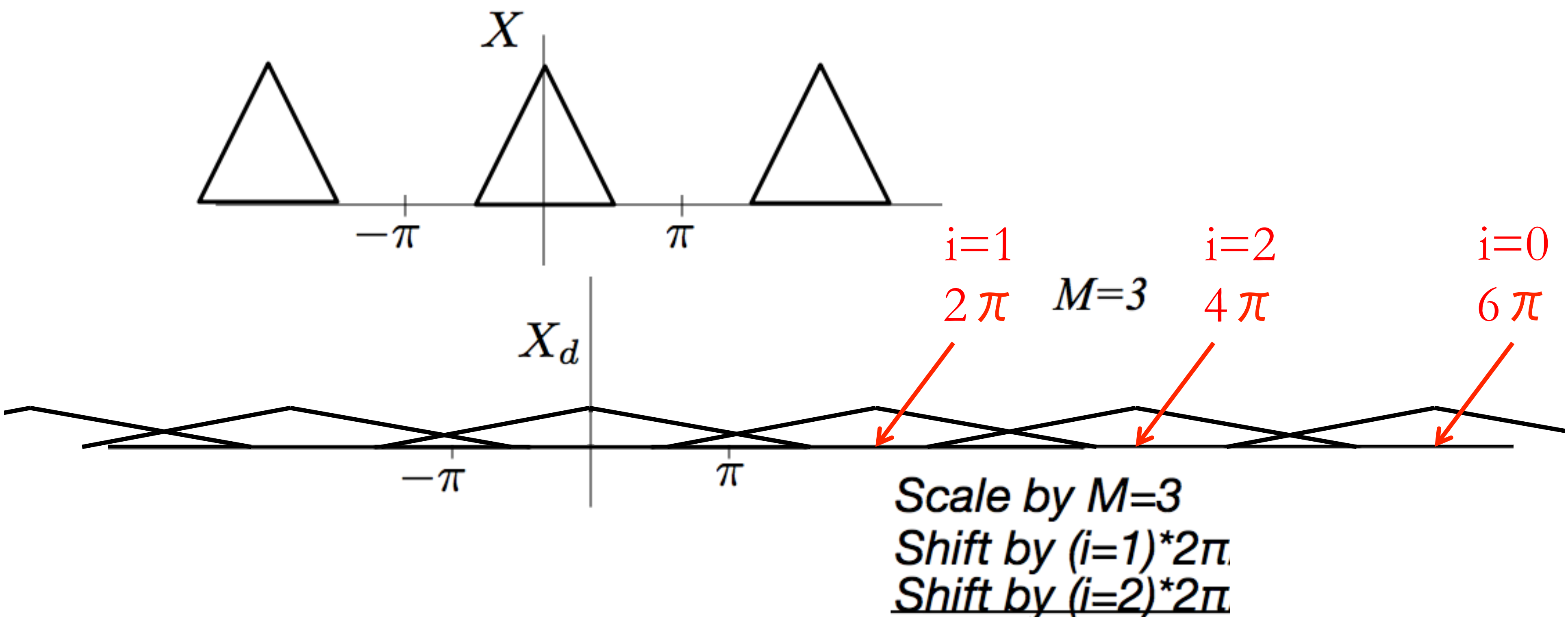

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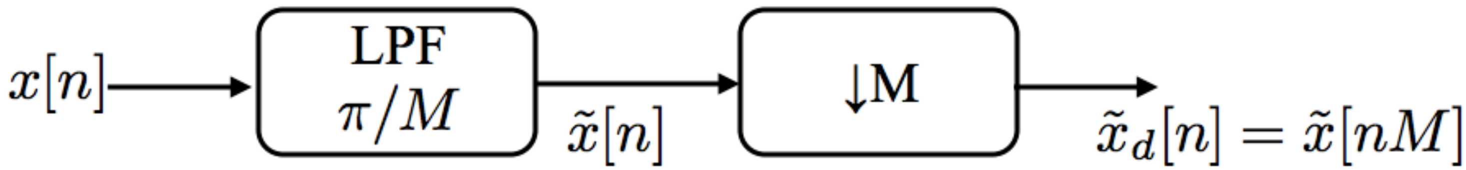

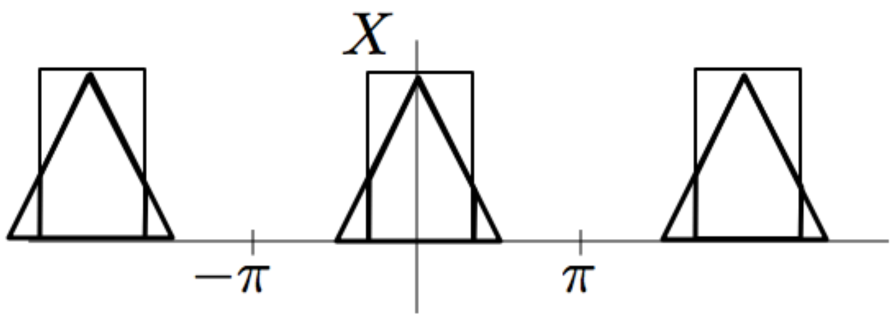

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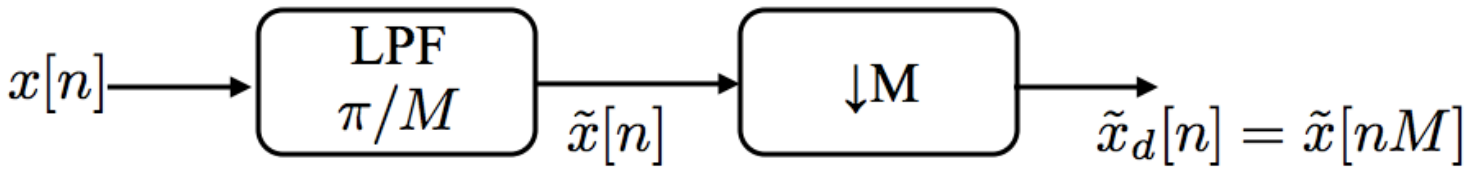

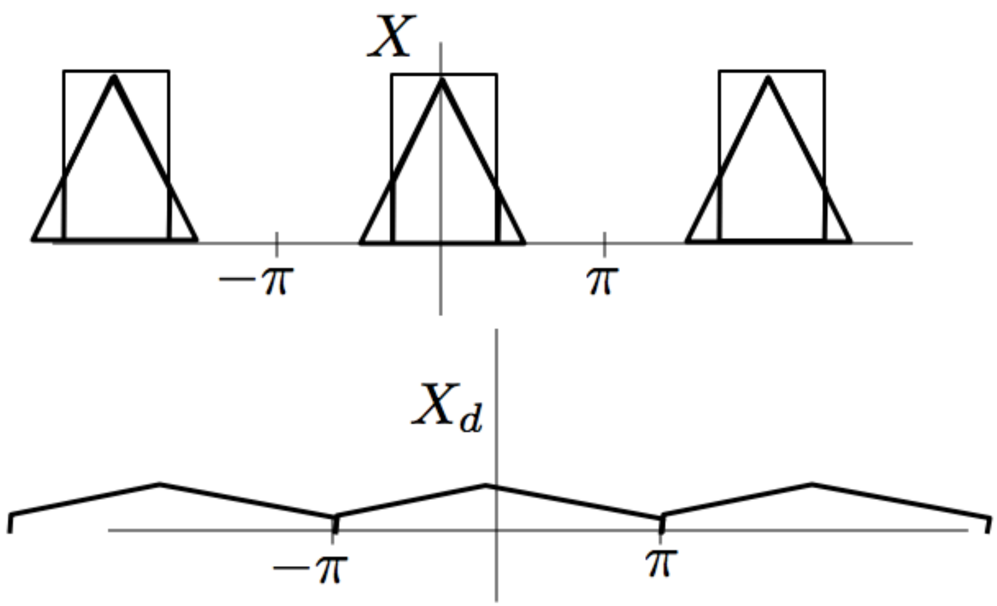

M=3

Upsampling

Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

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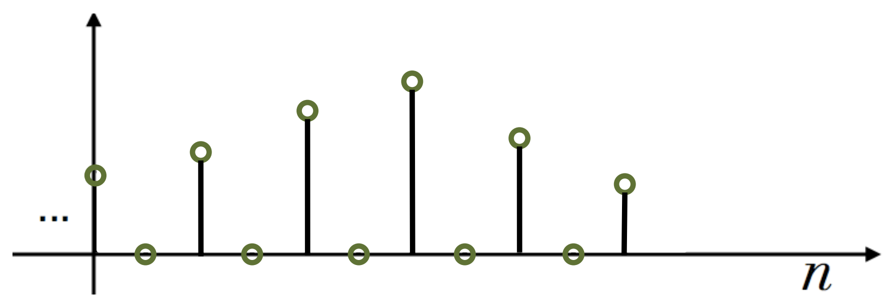
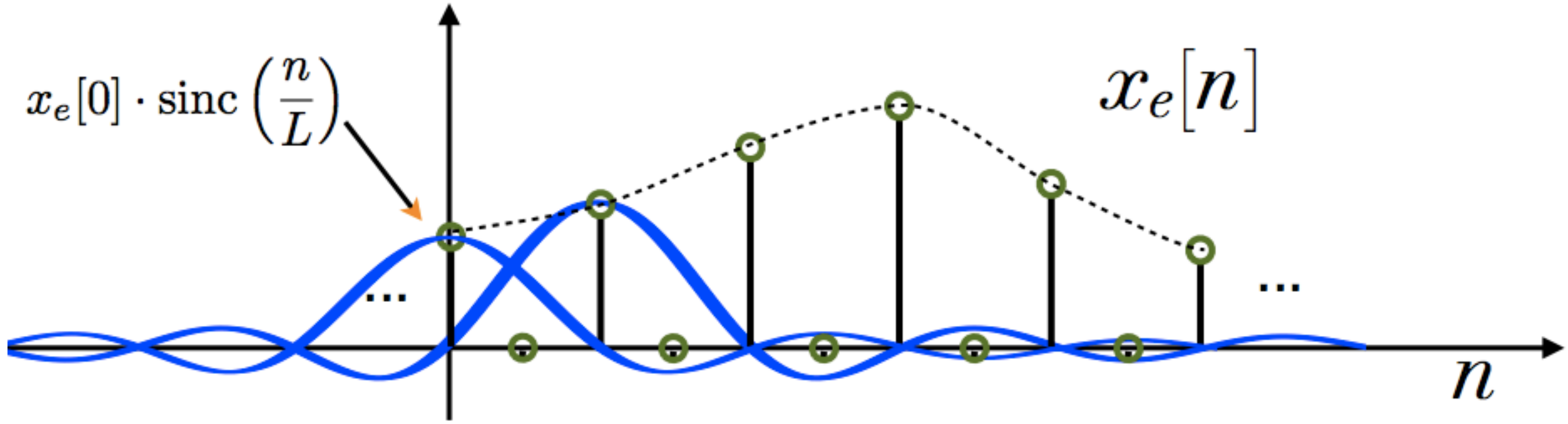
Obtain $x_i[n]$ from $x[n]$ in two steps:

(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

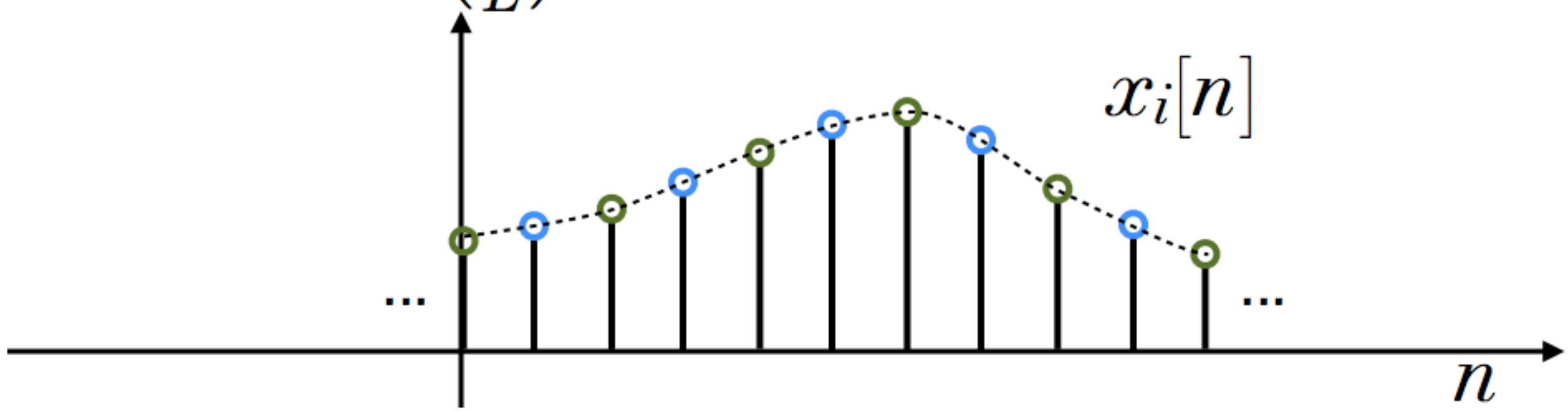


Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:



$$x_i[n] = x_e[n] * \mathrm{sinc}\left(\frac{n}{L}\right)$$



Upsampling

- Much like D/C converter
- Upsample by A LOT → almost continuous

- Intuition:
  - Recall our D/C model: $x[n] \rightarrow x_s(t) \rightarrow x_c(t)$
  - Approximate "$x_s(t)$" by placing zeros between samples
  - Convolve with a sinc to obtain "$x_c(t)$"

Upsampling

$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

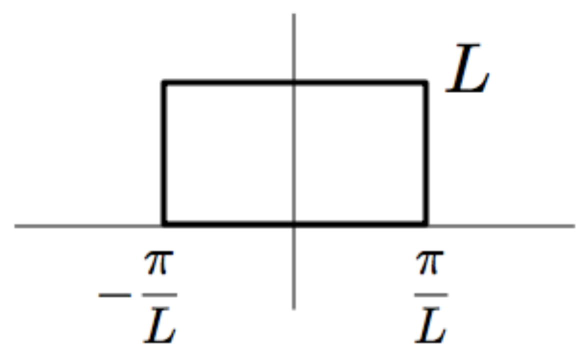
$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\text{sinc}(\frac{n-kL}{L})$$

Frequency Domain Interpretation

$$x_i[n] = x_e[n] * \mathrm{sinc}(n/L)$$

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{x_e[n]} \boxed{\begin{matrix}\text{LPF} \\ \text{gain=L} \\ \pi/L\end{matrix}} \longrightarrow x_i[n]$$

$\mathrm{sinc}(n/L)$ **DTFT** $\Rightarrow$



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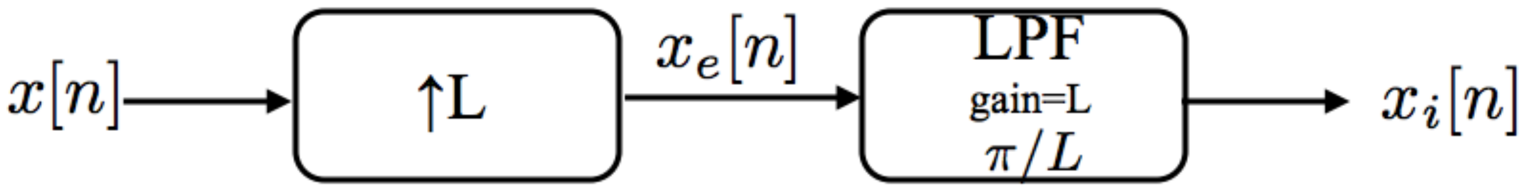
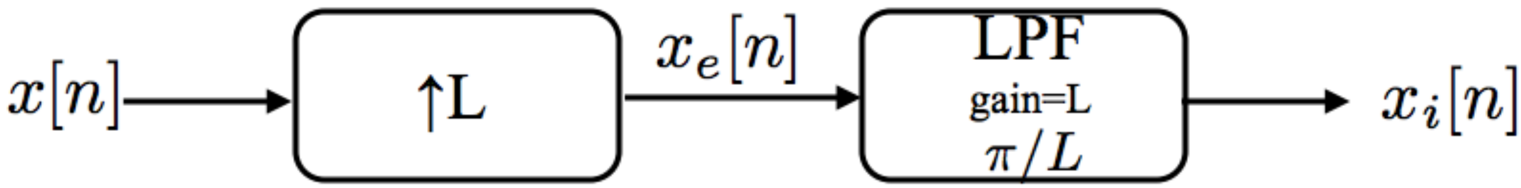


$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

Frequency Domain Interpretation



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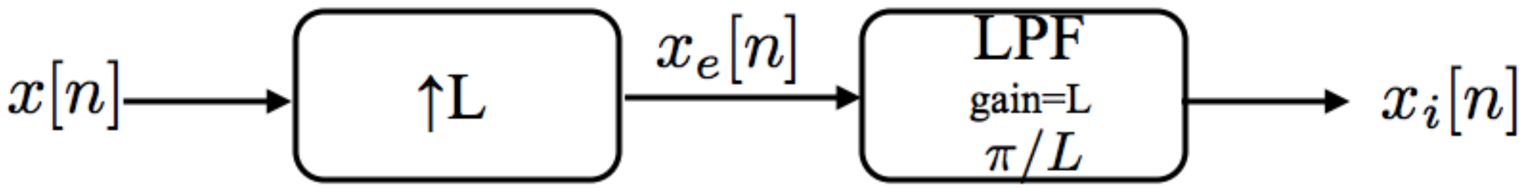

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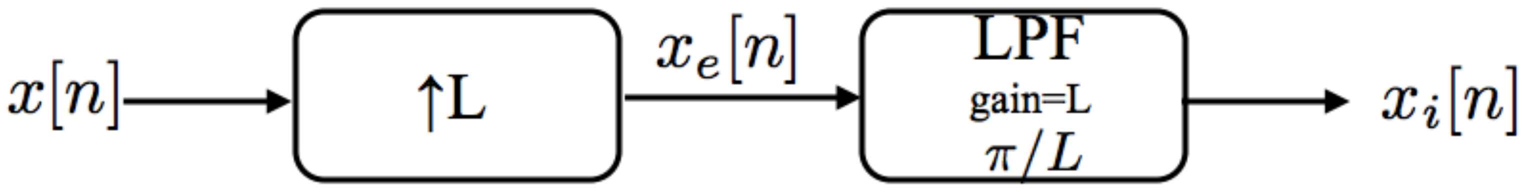

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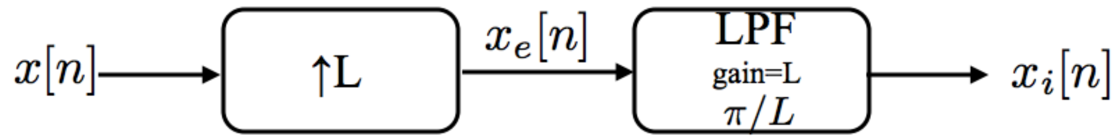
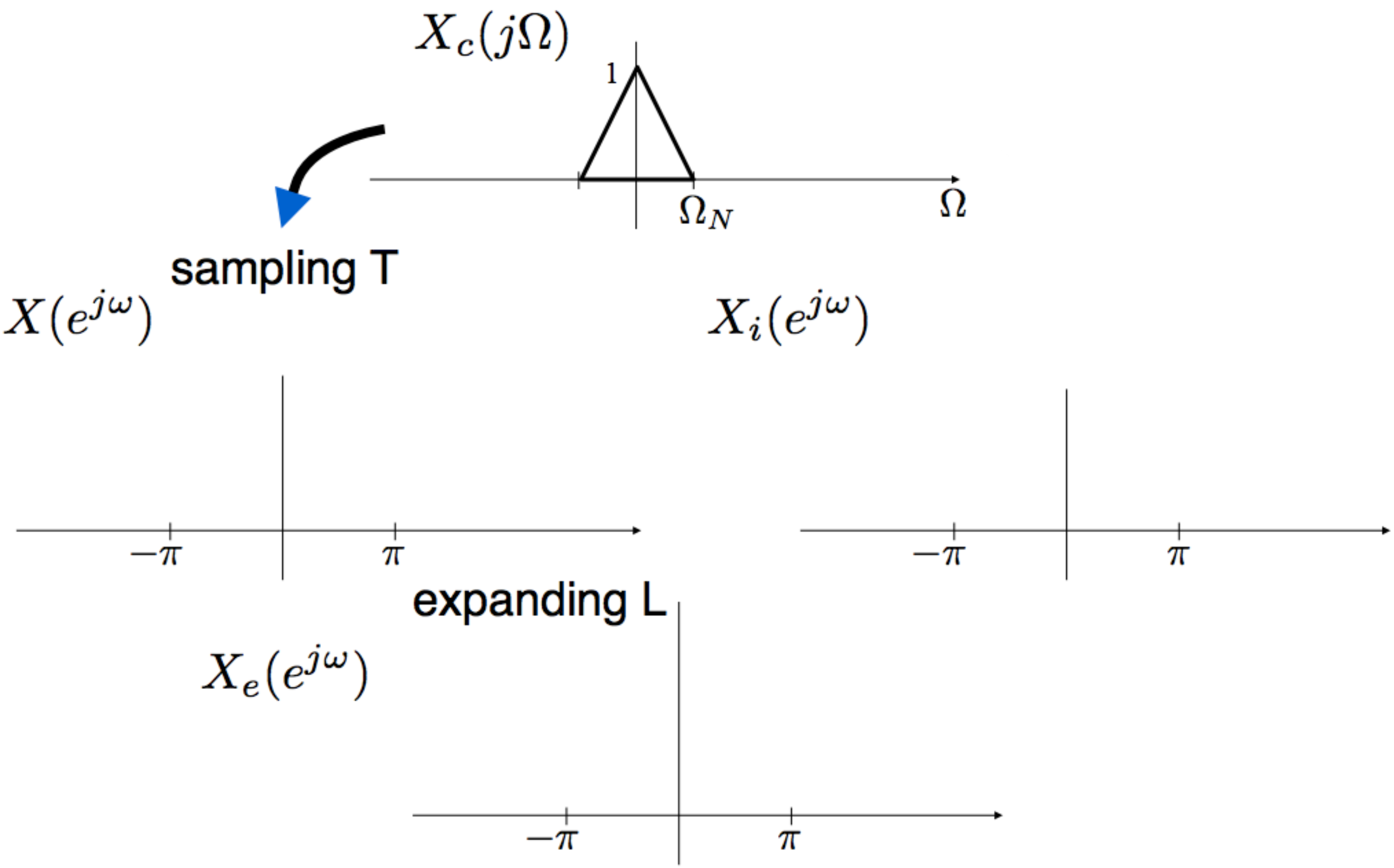
$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} \qquad = X(e^{j\omega L})$$

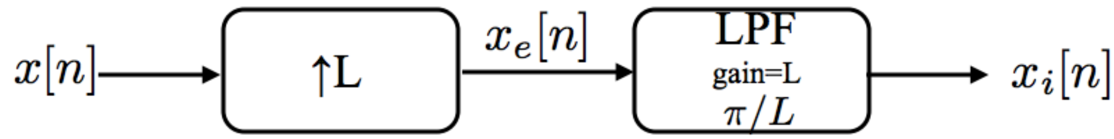
Shrink DTFT by a factor of L!

Example


$x[n]$
$\uparrow$L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example



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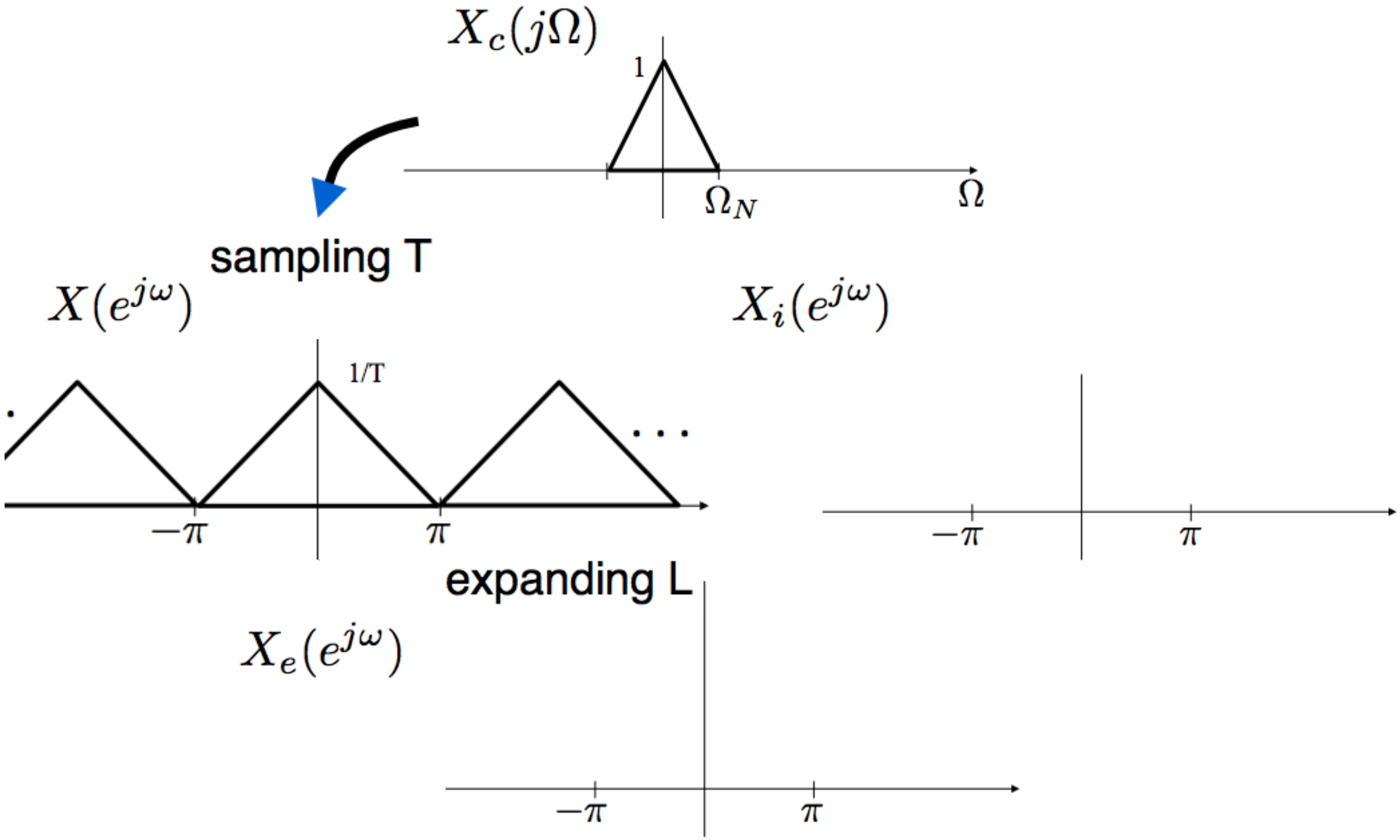

$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
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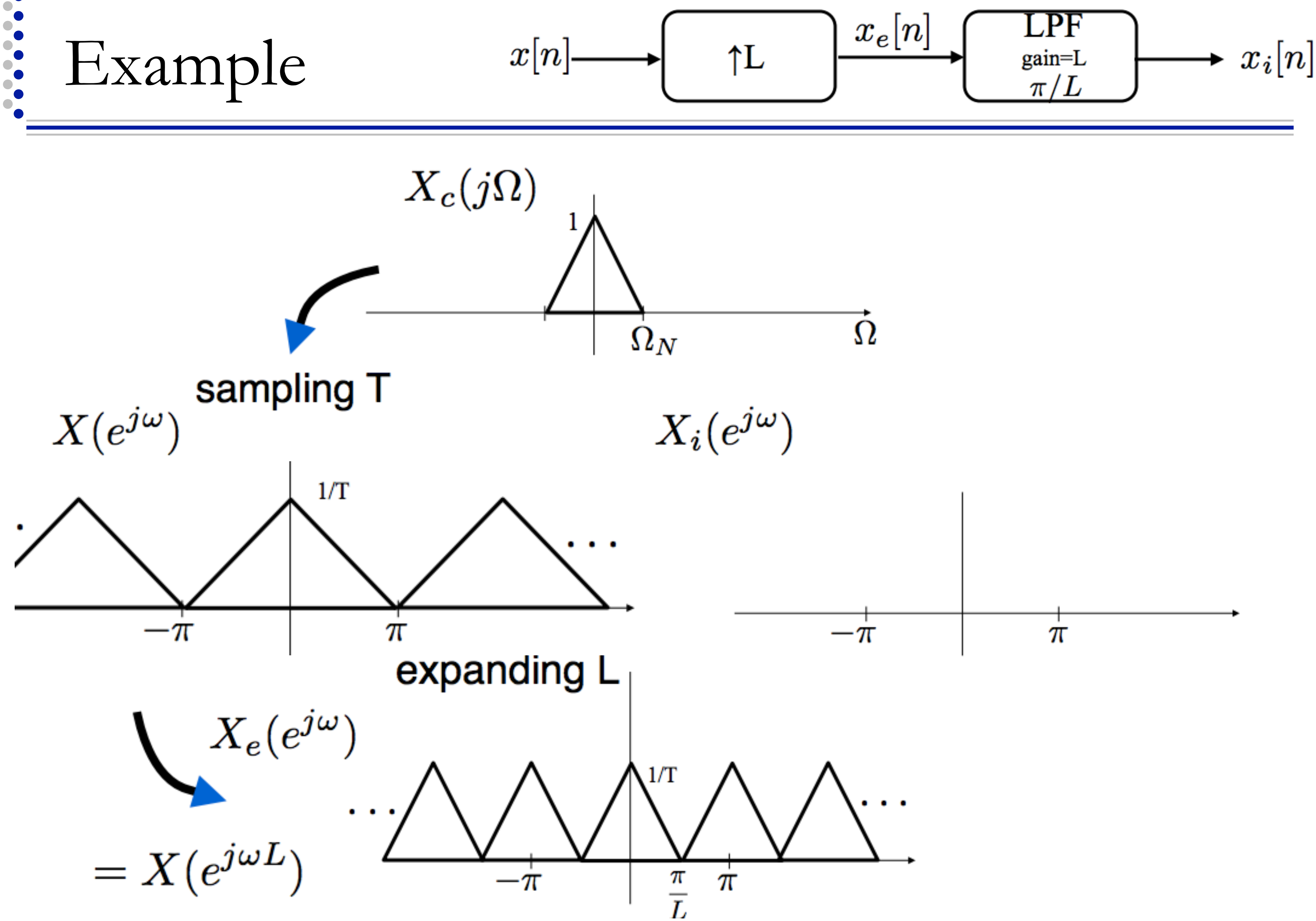

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LPF
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$X_c(j\Omega)$
1
Ω_N
Ω
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1/T
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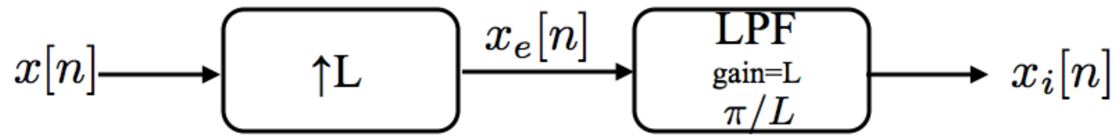
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$= X(e^{j\omega L})$
1/T
$-\pi$
$\frac{\pi}{L}$
π

Example



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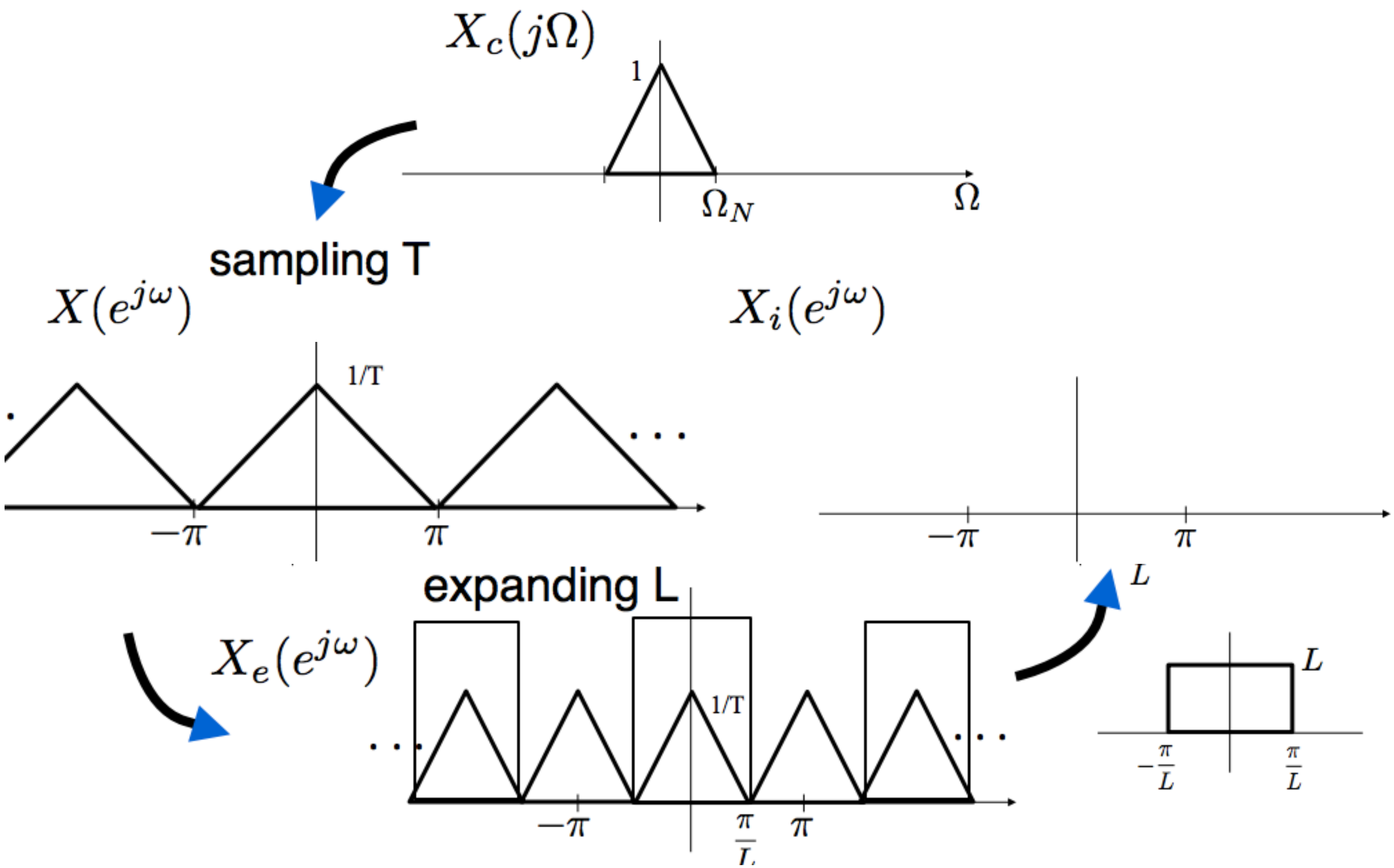

$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
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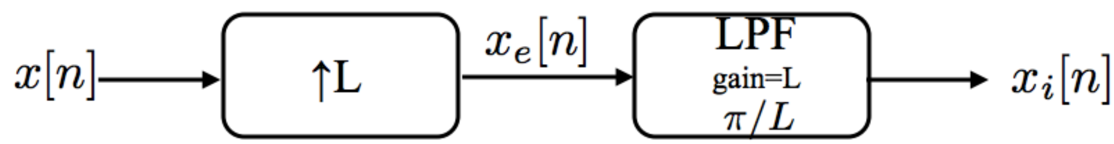
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Example



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gain=L
π/L
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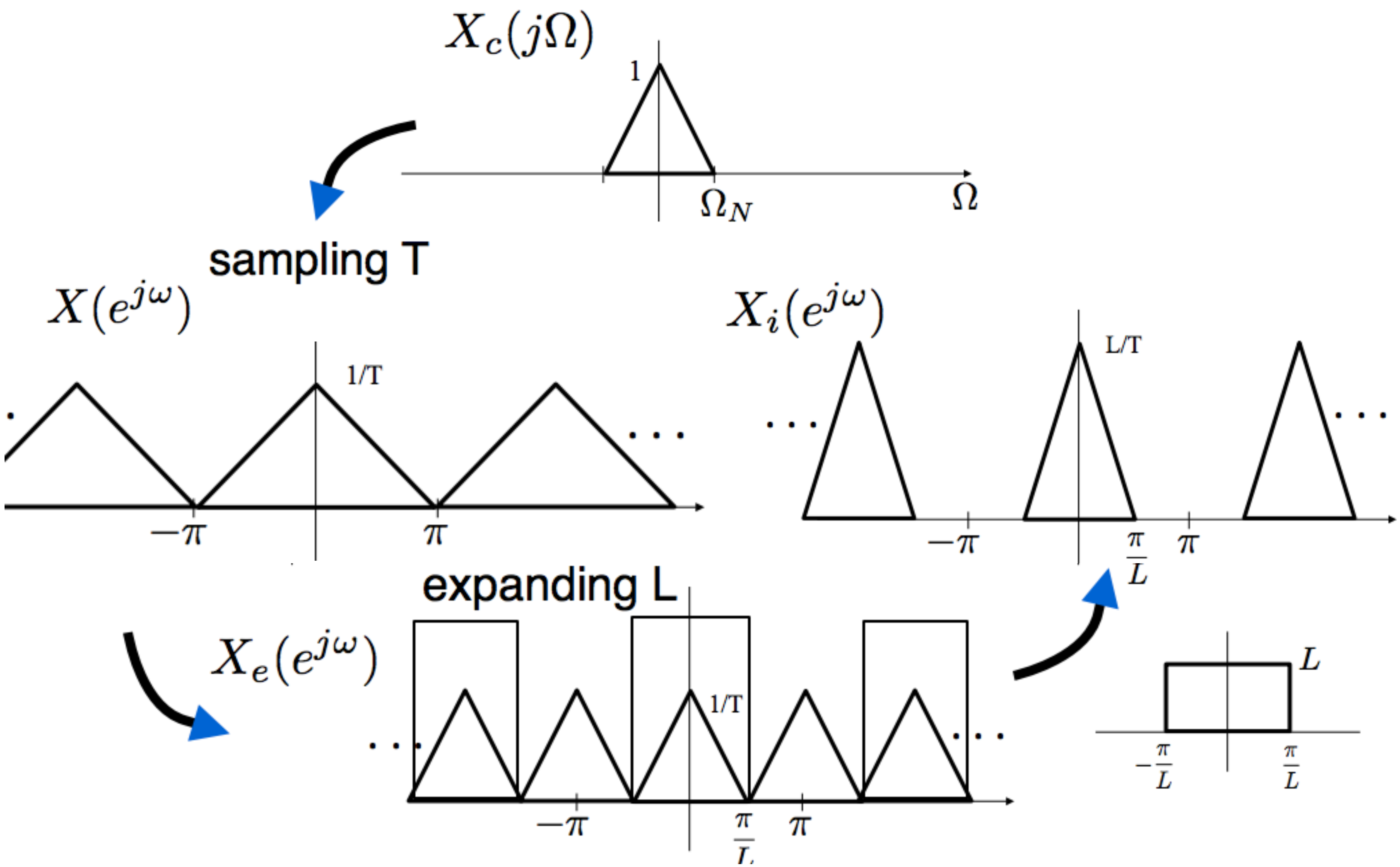

$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
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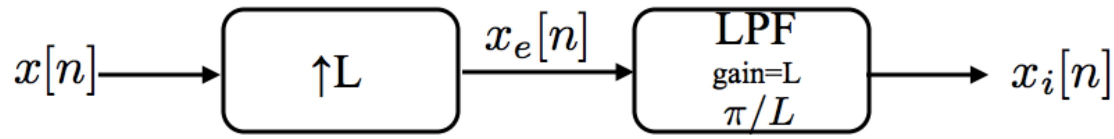
Example



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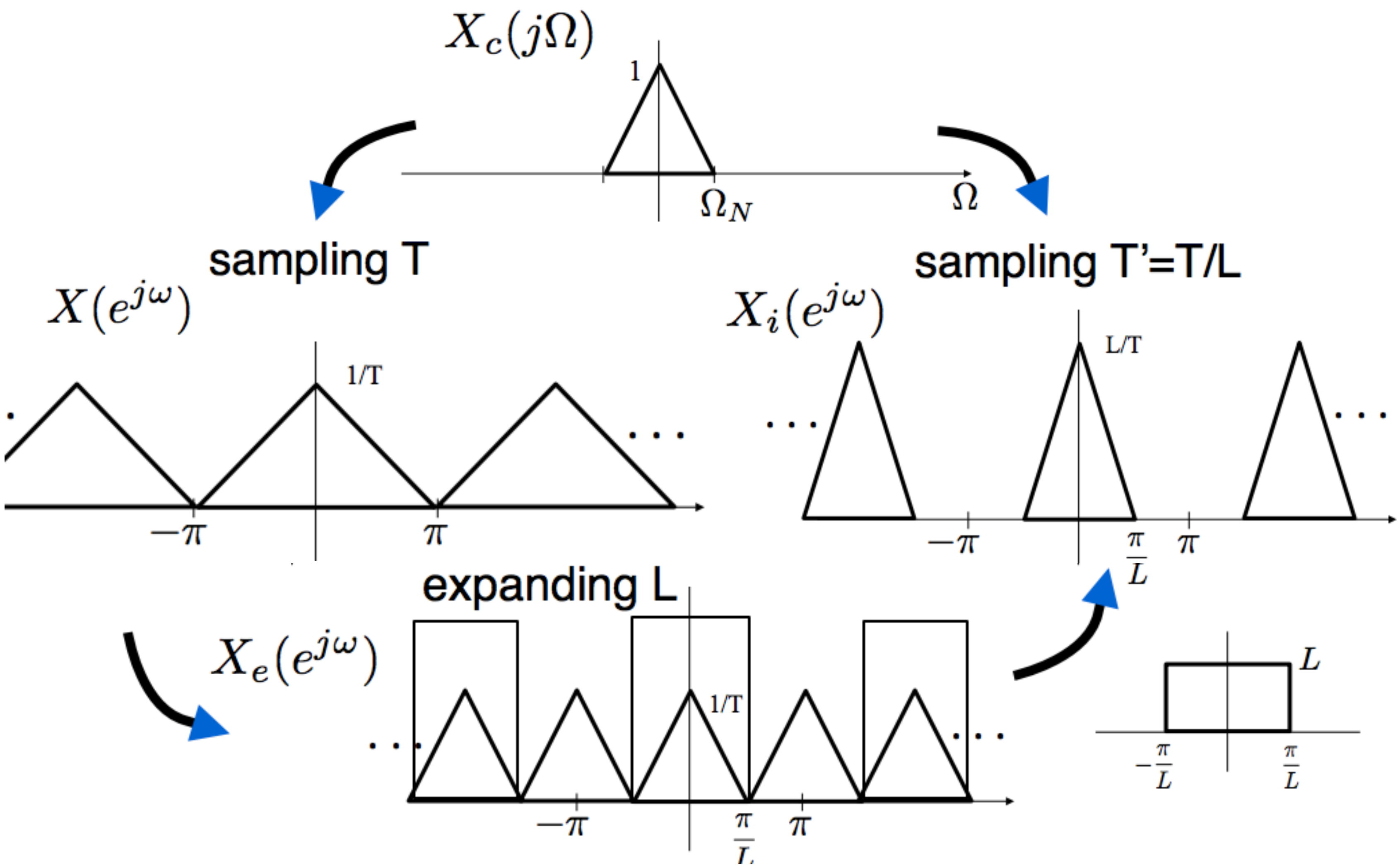
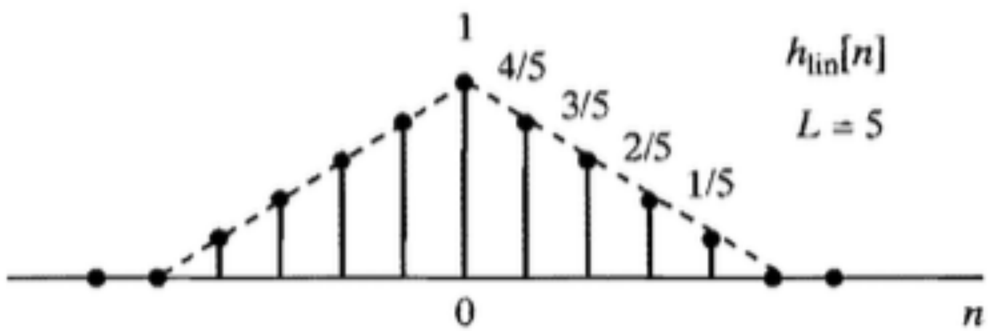


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Practical Interpolation

- Interpolate with simple, practical filters
  - Linear interpolation – samples between original samples fall on a straight line connecting the samples
    - Convolve with triangle instead of sinc

$$h_{\text{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \le L, \\ 0, & \text{otherwise}, \end{cases}$$



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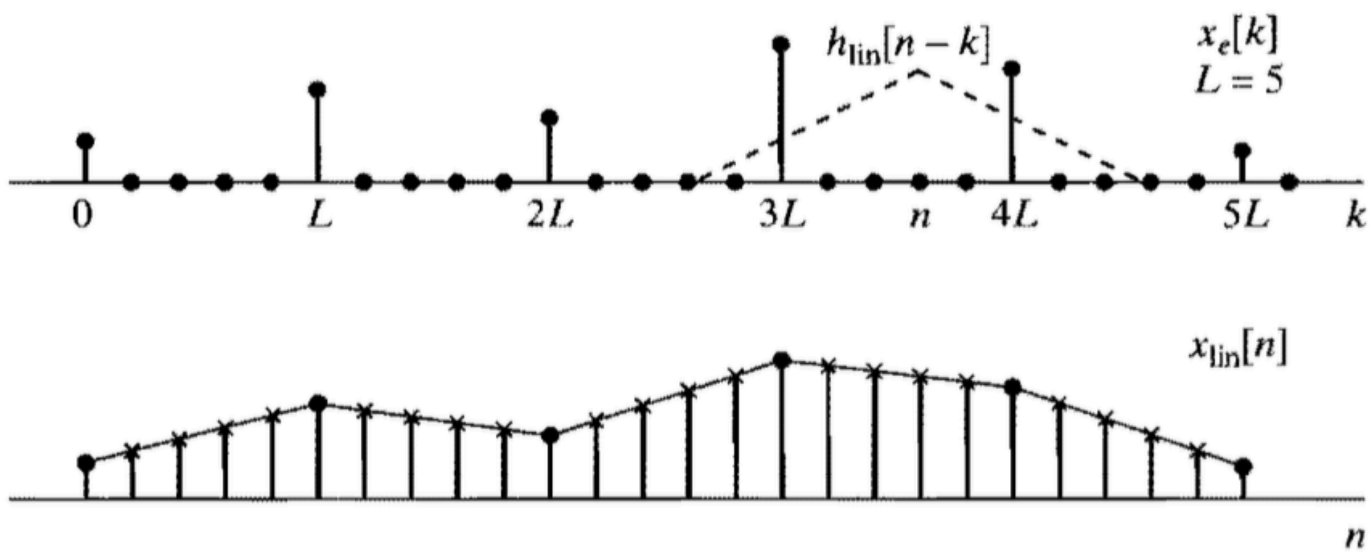
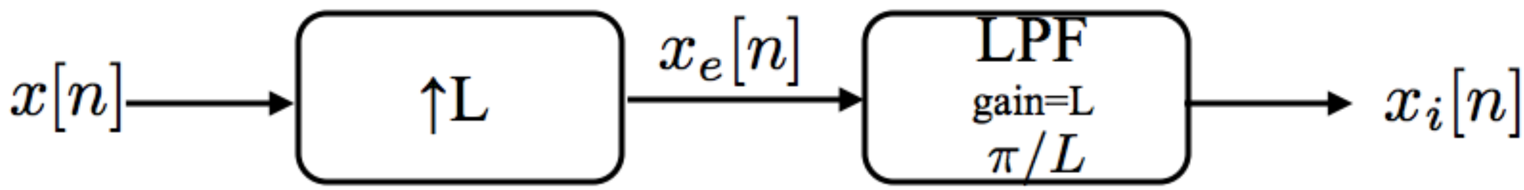
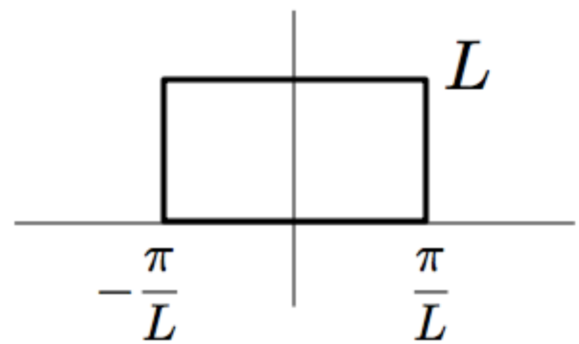


Frequency Domain Interpretation

$$x_i[n] = x_e[n] * \mathrm{sinc}(n/L)$$

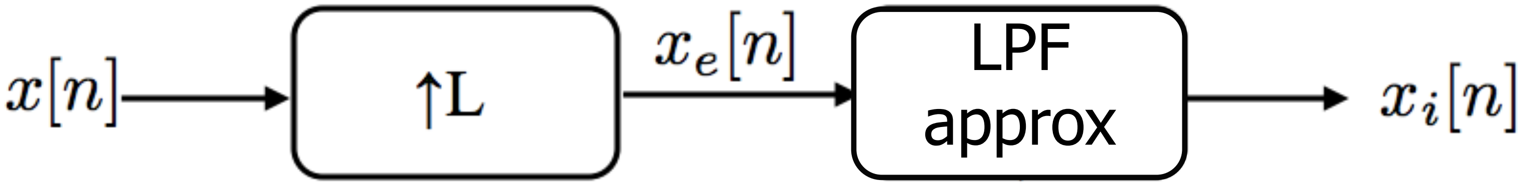


$\mathrm{sinc}(n/L)$ **DTFT ⇒**



Linear Interpolation -- Frequency Domain

$$x_i[n] = x_e[n] * h_{lin}[n]$$



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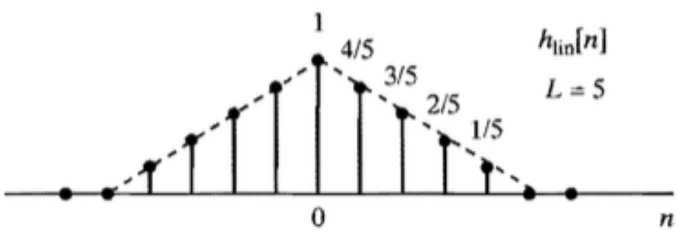

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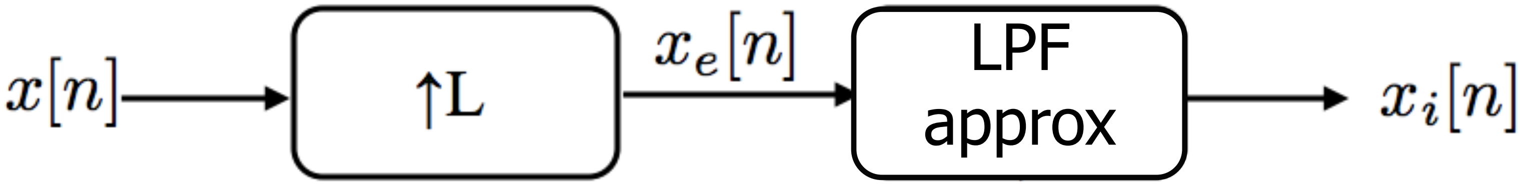

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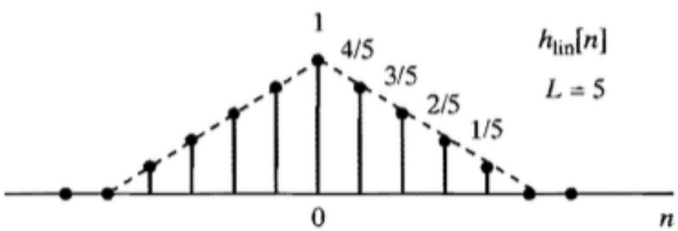
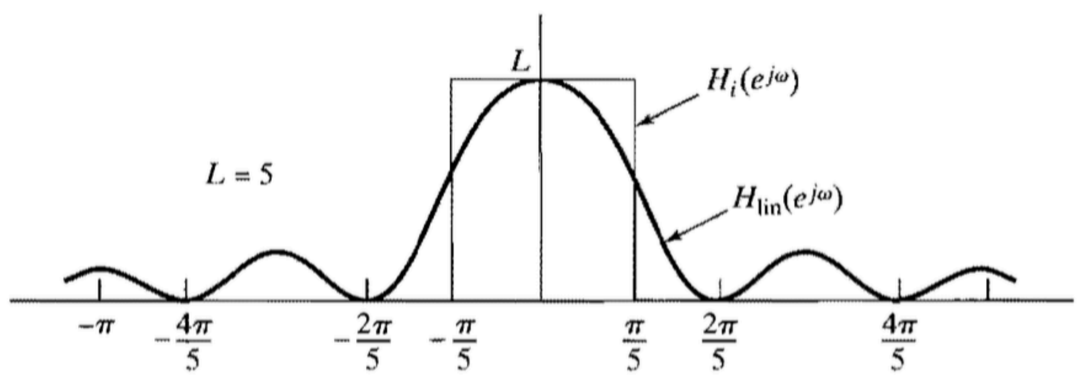


DTFT ⇒



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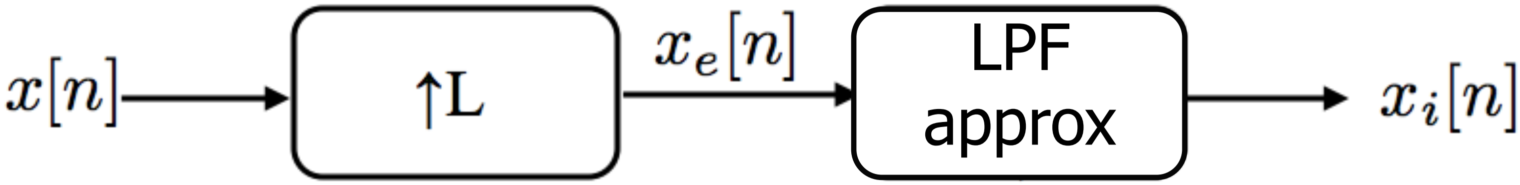

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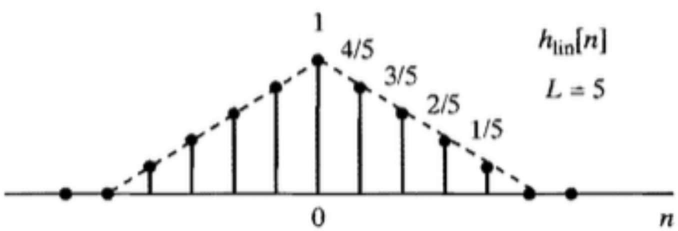

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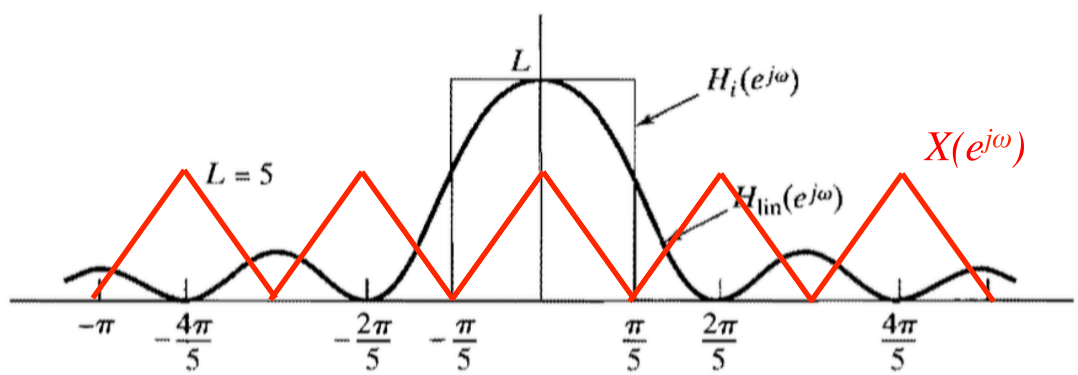

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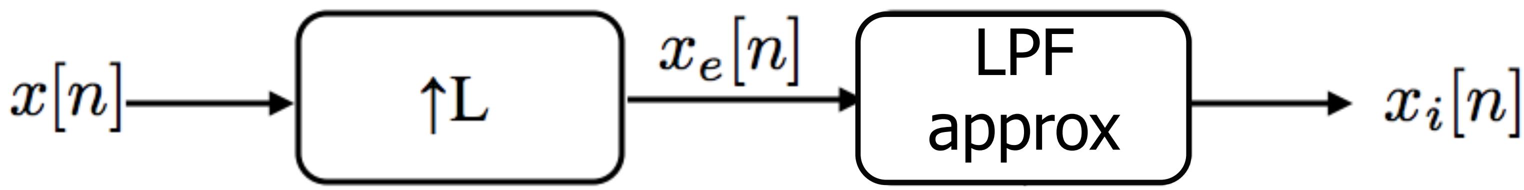

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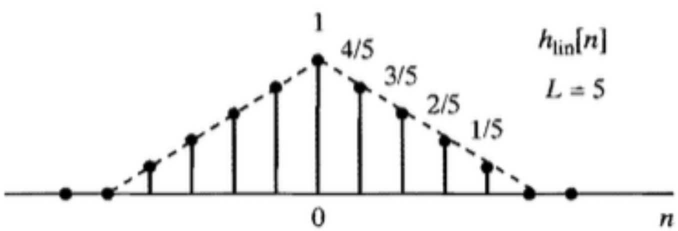

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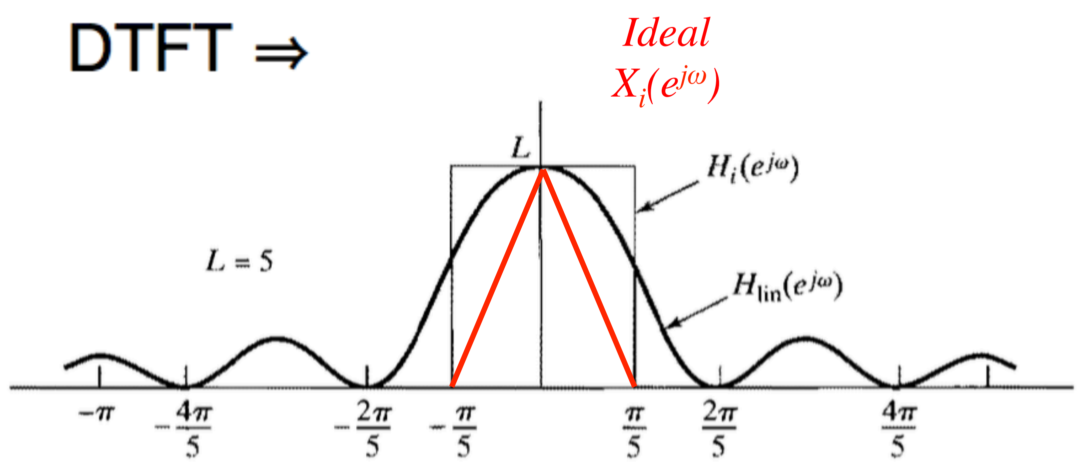

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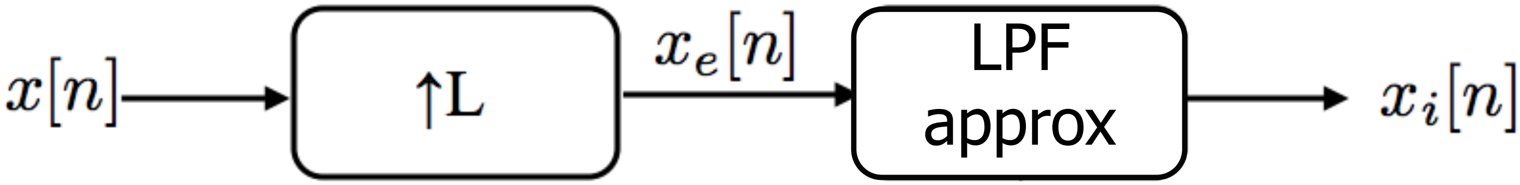

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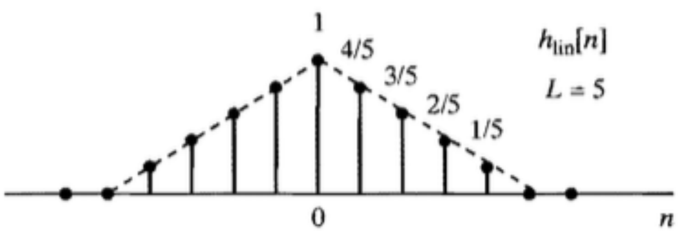

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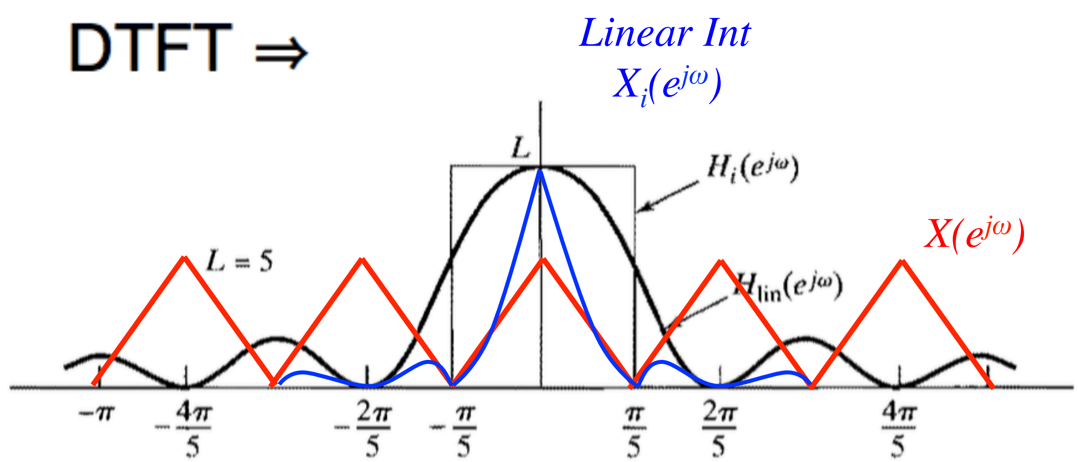

Big Ideas

- CT processing of DT signals
  - Allows for interpretation of DT systems
- Downsampling
  - Like a C/D converter
- Upsampling
  - Like a D/C converter
- Practical Interpolation
  - Linear interpolation
  - Approximate sinc function with triangle

Admin

- HW 4 due Sunday