ESE531

Spring 2019

University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2019 — Midterm — Tuesday, March 12

- 5 Problems with point weightings shown. All ~~4~~ 5 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.
- Final answers here.
- Additional workspace in "blue" book. **Note where to find work in "blue" book if relevant.**

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Q5	
Total	**Mean: 63.3, Stdev: 13.8**

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\vert a\vert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\vert a\vert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\vert r\vert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \vert\omega\vert < \omega_c, \\ 0, & \omega_c < \vert\omega\vert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad (n_d \text{ an integer})$	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega}) \quad \text{if } x[n] \text{ real.}$
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} \vert x[n]\vert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \vert X(e^{j\omega})\vert^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\vert z\vert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\vert z\vert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\vert z\vert > \vert a\vert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\vert z\vert < \vert a\vert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\vert z\vert > \vert a\vert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\vert z\vert < \vert a\vert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\vert z\vert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\vert z\vert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\vert z\vert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\vert z\vert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\vert z\vert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\vert z_0\vert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \vert r\vert < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$
Downsampling by M (↓M): $X_{down} = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

Interchange Identities:

$$x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow \mathrm{L}} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{\uparrow \mathrm{L}} \rightarrow \boxed{H(z^L)} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{\downarrow \mathrm{M}} \rightarrow \boxed{H(z)} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow \mathrm{M}} \rightarrow y[n]$$

1. (10 points) Consider a discrete-time system with input $x[n]$ and output $y[n]$ that satisfy the difference equation:

$$y[n] = n \cdot y[n-1] + x[n]. \tag{1}$$

The system is causal and satisfies initial-rest conditions; i.e. if $x[n] = 0$ for $n < 0$, then $y[n] = 0$ for $n < 0$. If $x[n] = \delta[n]$, determine $y[n]$ for all n.

$y[0] = 0 \cdot y[-1] + x[0] = 1$

$y[1] = 1 \cdot y[0] + x[1] = 1$

$y[2] = 2 \cdot y[1] + x[2] = 2 \cdot 1$

$y[3] = 3 \cdot y[2] + x[3] = 3 \cdot 2 \cdot 1$

$\vdots$

$y[n] = n \cdot y[n-1] + x[n] = n \cdot (n-1) \cdot (n-2) \cdots (1)$

$y[n] = n! \quad \forall n$

2. (20 points) Suppose you have a discrete-time system with impulse response

$$h[n] = 0.9^n u[n] \tag{2}$$

where $u[n]$ is the unit step function. If an input

$$x[n] = cos\left(\frac{\pi}{6}n\right) \tag{3}$$

is applied to the system, determine the output $y[n]$.

$$H(z) = \frac{1}{1-0.9z^{-1}} \qquad \text{ROC: } |z| > 0.9$$

causal and stable $\Rightarrow$ $H(e^{jw})$ exists

$$\longrightarrow \quad H(e^{jw}) = H(z)\Big|_{z=e^{jw}} = \frac{1}{1-0.9e^{-jw}}$$

$$\longrightarrow \quad x[n] = \cos\left(\tfrac{\pi}{6}n\right) \longrightarrow \boxed{H(z)} \longrightarrow y[n] = \underbrace{a}_{\substack{\text{mag.}\\\text{scaling}}} \cos\left(\tfrac{\pi}{6}n + \underbrace{\theta}_{\text{phase shift}}\right)$$

$$a = |H(e^{jw})|\Big|_{w=\frac{\pi}{6}} \qquad \theta = \angle H(e^{jw})\Big|_{w=\frac{\pi}{6}}$$

$$H(e^{jw})\Big|_{w=\frac{\pi}{6}} = \frac{1}{1-0.9e^{-j\frac{\pi}{6}}} = 0.8783 - 1.7917j$$

$$= \underbrace{1.9954}_{a} \measuredangle \underbrace{-1.1150}_{\theta}$$

$$y[n] = 1.9954\cos\left(\tfrac{\pi}{6}n - 1.1150\right)$$

2. (20 points) Suppose you have a discrete-time system with impulse response

$$h[n] = 0.9^n u[n] \tag{2}$$

where $u[n]$ is the unit step function. If an input

$$x[n] = cos\left(\frac{\pi}{6}n\right) = \frac{1}{2}\left(e^{j\frac{\pi}{6}n} + e^{-j\frac{\pi}{6}n}\right) \tag{3}$$

is applied to the system, determine the output $y[n]$.

$$H(e^{j\omega}) = \frac{1}{1-0.9e^{-j\omega}}$$

$$H(e^{j\omega})\Big|_{\frac{\pi}{6}=\omega} = 1.9954e^{-1.1150j}$$

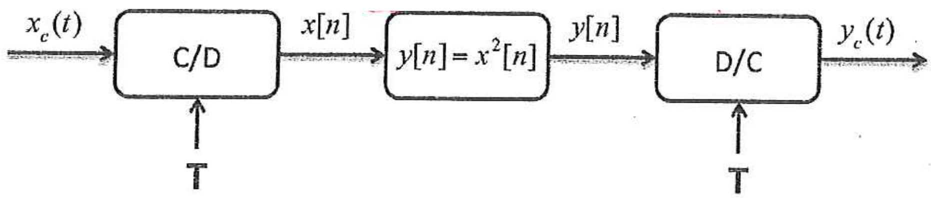
$$H(e^{j\omega})\Big|_{-\frac{\pi}{6}=\omega} = 1.9954\,e^{1.1150j}$$

$$y[n] = \frac{1}{2}\left(1.9954\,e^{-1.1150j}\,e^{j\frac{\pi}{6}n} + 1.9954\,e^{1.1150j}\,e^{-j\frac{\pi}{6}n}\right)$$

$$= \frac{1}{2}\cdot 1.9954\left(e^{j(\frac{\pi}{6}n-1.1150)} + e^{-j(\frac{\pi}{6}-1.1150)}\right)$$

$$= 1.9954\cos\left(\frac{\pi}{6}n-1.1150\right)$$

3. (20 points) Consider a standard system (shown below) consisting of a C/D converter, a discrete-time system, and a D/C converter. Suppose that the Fourier transform $X_c(j\Omega)$ of $x_c(t)$ obeys $X_c(j\Omega) = 0$ for $|\Omega| \geq 2\pi 1000$, and that the discrete-time system is a squarer, i.e. $y[n] = x^2[n]$.



(a) What is the largest value of T such that $y_c(t) = x_c^2(t)$? Justify your answer.

(b) Is the discrete-time system (the squarer) an LTI system? Justify your answer.

a) $y[n] = x^2[n] \Rightarrow$

$$Y(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) \circledast X(e^{j\omega})$$

therefore $Y(e^{j\omega})$ will occupy twice the bandwidth of $X(e^{j\omega})$ if no aliasing occurs

If $Y(e^{j\omega}) \neq 0$ for $-\pi < \omega < \pi$, then $X(e^{j\omega}) \neq 0$ for $-\frac{\pi}{2} < \omega < \frac{\pi}{2}$, and $X(e^{j\omega}) = 0$ for $\frac{\pi}{2} \leq |\omega| \leq \pi$

In other words the max signal frequency of $x_c(t)$ must fall on $\omega = \frac{\pi}{2}$ after sampling for $x_c^2(t) = y_c(t)$.

$$X_c(j\Omega) = 0 \quad \forall \quad \underbrace{\Omega \geq 2\pi \cdot 1000}_{\Omega_{max}}$$

$\omega = \Omega T$

$\Rightarrow \frac{\pi}{2} = \Omega_{max} \cdot T = 2\pi \cdot 1000\, T$

$\Rightarrow T \leq \frac{1}{4000}$

b) Not linear. $\Rightarrow$ not LTI

$x'[n] = x_1[n] + x_2[n]$

$y'[n] = (x'[n])^2 = x_1^2[n] + 2x_1[n]x_2[n] + x_2^2[n]$

$\neq y_1[n] + y_2[n]$

also output frequencies different from input. frequency spreading

4. (25 points) Consider an LTI system whose response to the input signal

$$x[n] = 2\delta[n-2] + \delta[n-3] + \delta[n-4] \quad (4)$$

is the output signal

$$y[n] = 6\delta[n] - \delta[n-1] + 3\delta[n-2] - \delta[n-3] + \delta[n-4]. \quad (5)$$

(a) Find the transfer function, $H(z)$ and the impulse response, $h[n]$, of this system.

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{6 - z^{-1} + 3z^{-2} - z^{-3} + z^{-4}}{2z^{-2} + z^{-3} + z^{-4}}$$

$$= \frac{(2z^{-2} + z^{-3} + z^{-4})(3z^{2} - 2z + 1)}{2z^{-2} + z^{-3} + z^{-4}}$$

$$= 3z^{2} - 2z + 1$$

$$h[n] = 3\delta[n+2] - 2\delta[n+1] + \delta[n]$$

(You may continue problem 1 on this almost blank page.)

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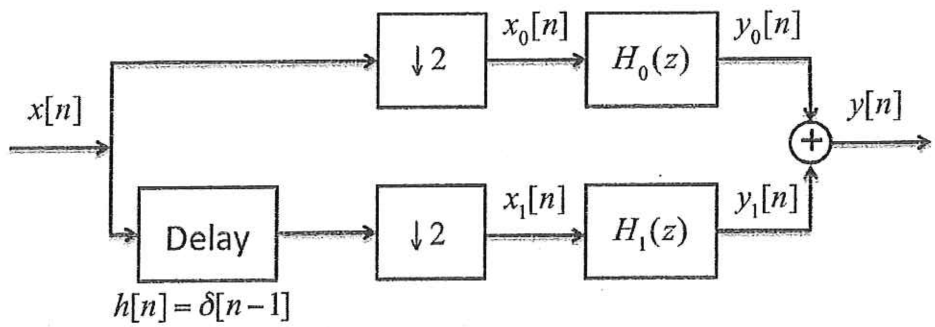
5. (25 pts) Consider the decimation filter structure shown below:



where $y_0[n]$ and $y_1[n]$ are generated according to the following difference equations:

$$y_0[n] = \frac{1}{4}y_0[n-1] - \frac{1}{3}x_0[n] + \frac{1}{8}x_0[n-1] \quad (6)$$

$$y_1[n] = \frac{1}{4}y_1[n-1] + \frac{1}{12}x_1[n] \quad (7)$$

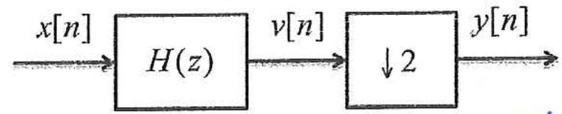
(a) How many multiplies per output sample does the implementation of the filter structure require? (I.e How many multiplications are necessary to evaluate the current output sample assuming you have all previous inputs and outputs available.) Consider a divide to be equivalent to a multiply.

a) $y_0[n]$ requires 3 multiplies
$y_1[n]$ requires 2 multiplies

delay, downsampling and summing requires no multiplies

↳ 5 total multiplies per sample.

The decimation filter can also be implemented as shown below:

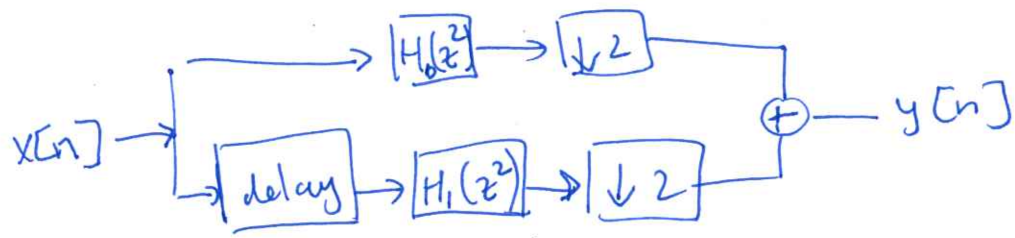


where $v[n] = a \cdot v[n-1] + b \cdot x[n] + c \cdot x[n-1]$. $\Rightarrow$ $\dfrac{V(z)}{X(z)} = \dfrac{b + cz^{-1}}{1 - az^{-1}}$

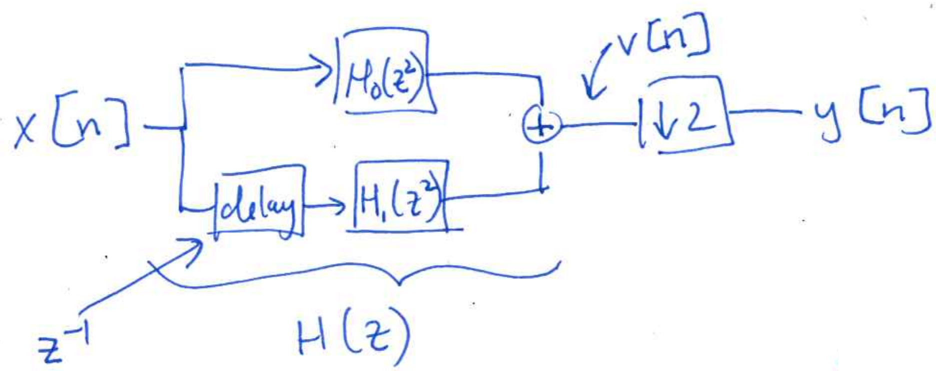
(b) Determine a, b, and c.

(c) How many multiplies per output sample does this second implementation require?

b) using interchange identity



and moving downsampling



$$H(z) = H_0(z^2) + z^{-1} H_1(z^2)$$

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8} z^{-1}}{1 - \frac{1}{4} z^{-1}} \qquad H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4} z^{-1}}$$

$$\frac{V(z)}{X(z)} = H(z) = \frac{-\frac{1}{3} + \frac{1}{8} z^{-2}}{1 - \frac{1}{4} z^{-2}} + \frac{\frac{1}{12} z^{-1}}{1 - \frac{1}{4} z^{-2}} = \frac{-\frac{1}{3} + \frac{1}{4} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$\Rightarrow a = \frac{1}{2}, \ b = -\frac{1}{3}, \ c = \frac{1}{4}$

c) each $v[n]$ requires 3 multiplies, for every $y[n]$ we need 2 $v[n]$ samples. Therefore, each output sample requires 6 multiplies