ESE 531: Digital Signal Processing

Lec 11: February 20, 2020

Polyphase Decomposition and Multi-rate Filter Banks

Lecture Outline

- Review: Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks

Expander and Compressor

not LTI!

"expander"

Upsampling
-expanding in time
-compressing in frequency

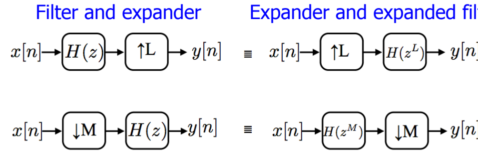
"compressor"

Downsampling
-compressing in time
-expanding in frequency

Interchanging Operations - Summary

Filter and expander

Expander and expanded filter*



Compressor and filter

Expanded filter* and compressor

*Expanded filter = expanded impulse response, compressed freq response

Polyphase Decomposition

- The polyphase decomposition of a sequence is obtained by representing it as a superposition of M subsequences, each consisting of every Mth value of successively delayed versions of the sequence.
- When this decomposition is applied to a filter impulse response, it can lead to efficient implementation structures for linear filters in several contexts.

Polyphase Decomposition

- We can decompose an impulse response (of our filter) to:

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

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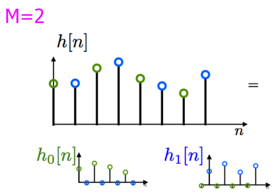

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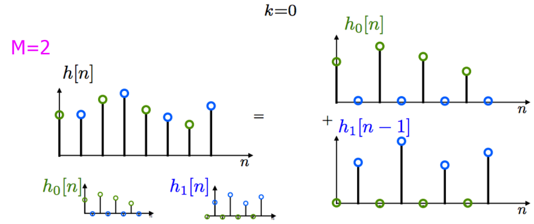
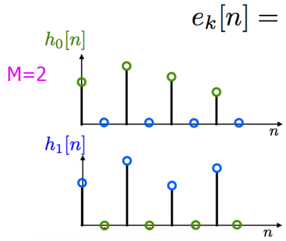


Polyphase Decomposition

$$h_k[n] \rightarrow \boxed{\downarrow M} \rightarrow e_k[n]$$

$$e_k[n] = h_k[nM]$$



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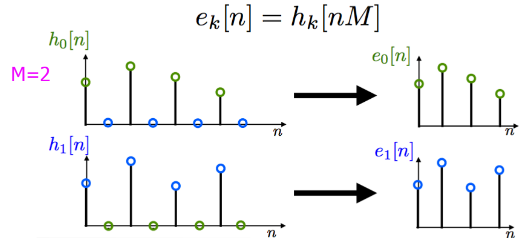

Polyphase Decomposition

$$e_k[n] \rightarrow \boxed{\uparrow M} \rightarrow h_k[n]$$

recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Polyphase Decomposition

$$e_k[n] \rightarrow \boxed{\uparrow M} \rightarrow h_k[n]$$

recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Also, recall:

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

So,

$$\boxed{H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}}$$

Polyphase Decomposition

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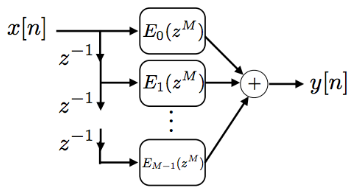
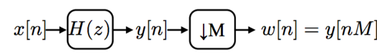
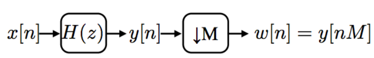


Polyphase Implementation of Decimation



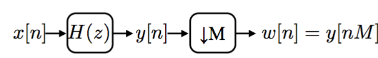
- Problem:
  - Compute all y[n] and then throw away -- wasted computation!

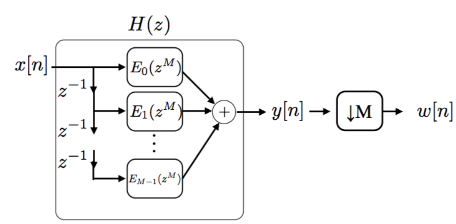
Polyphase Implementation of Decimation



- Problem:
  - Compute all y[n] and then throw away -- wasted computation!
  - For FIR length N → N multiplications/unit time

Polyphase Implementation of Decimation





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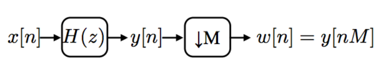

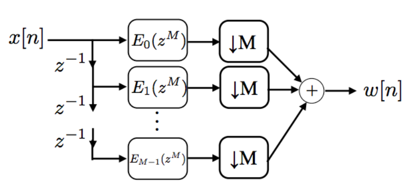

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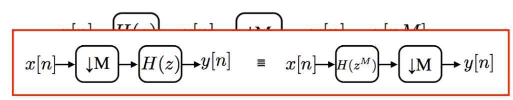

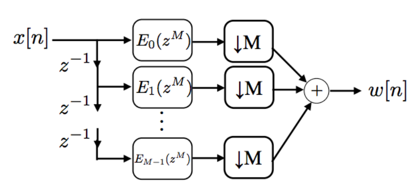

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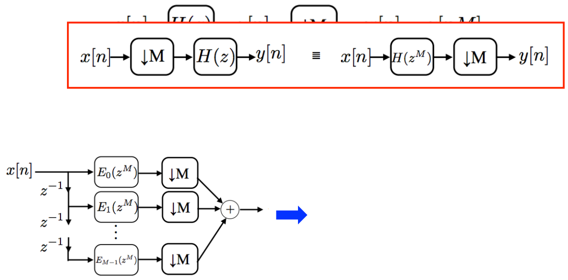

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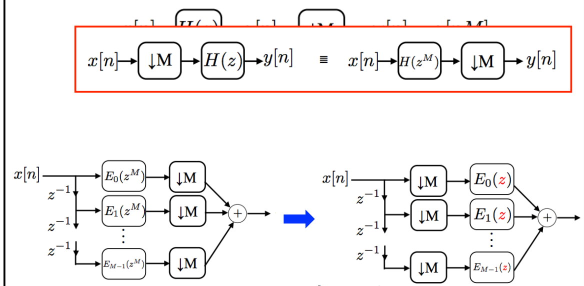

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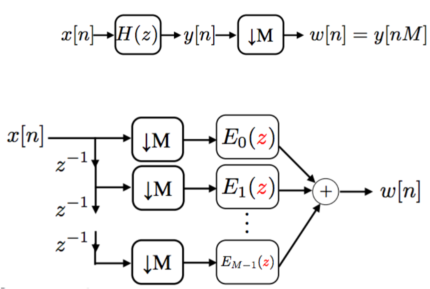

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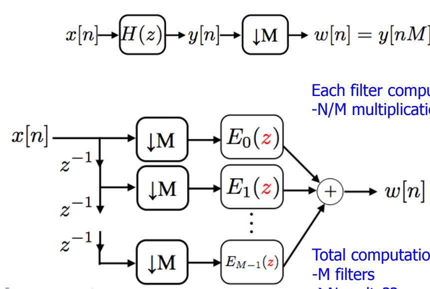
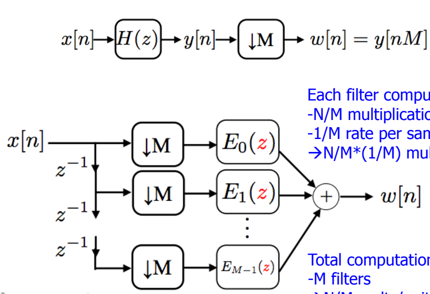


Each filter computation:
-N/M multiplications

Total computation:
-M filters
→N mults??

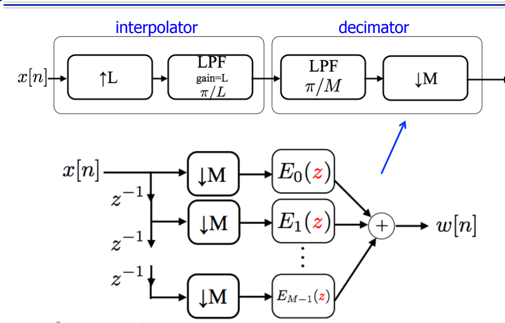
Polyphase Implementation of Decimation



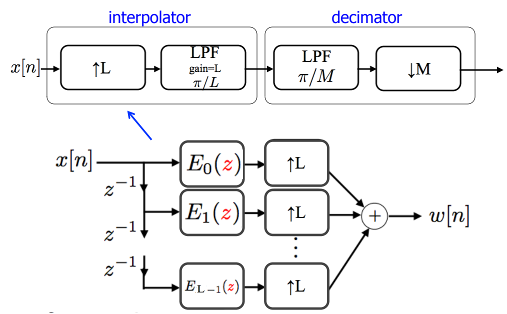
Each filter computation:
-N/M multiplications
-1/M rate per sample
→N/M*(1/M) mults/unit time

Total computation:
-M filters
→N/M mults/unit time

Polyphase Implementation of Decimator



Polyphase Implementation of Interpolation

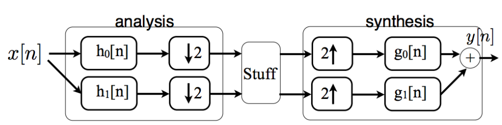


Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

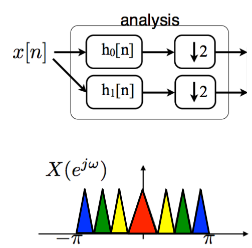
Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering
- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π



Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C=\pi/2$



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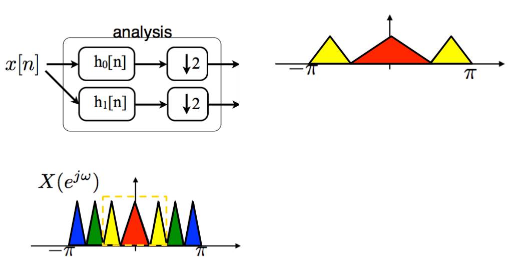

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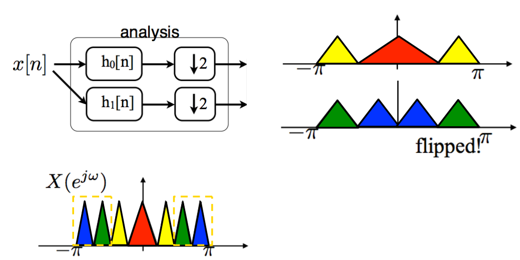

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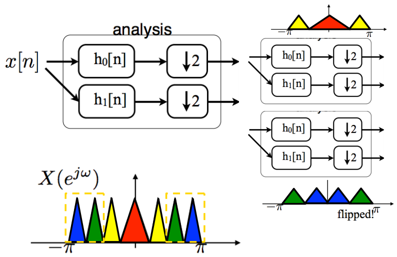

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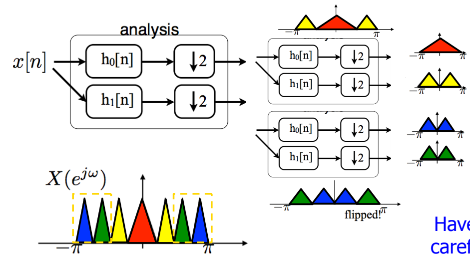

Have to be careful with order!

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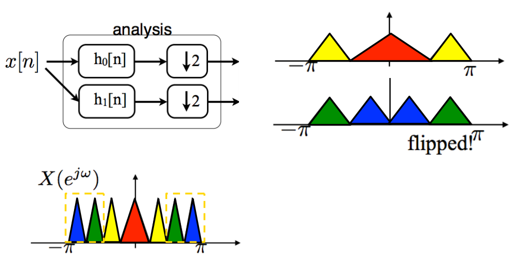

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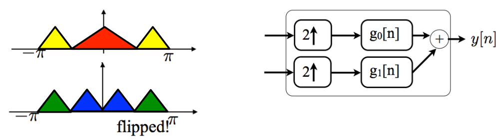

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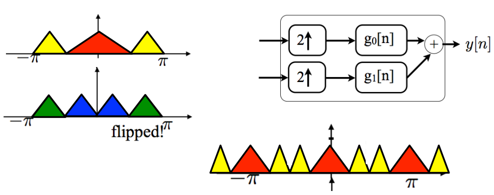

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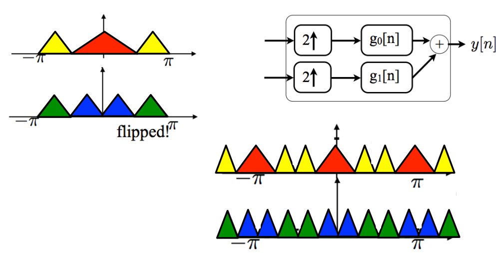

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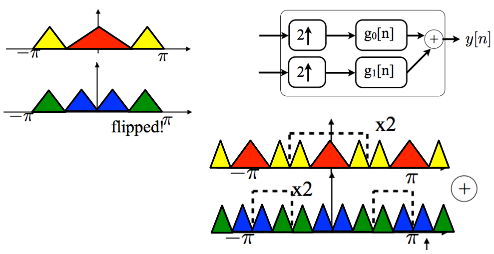

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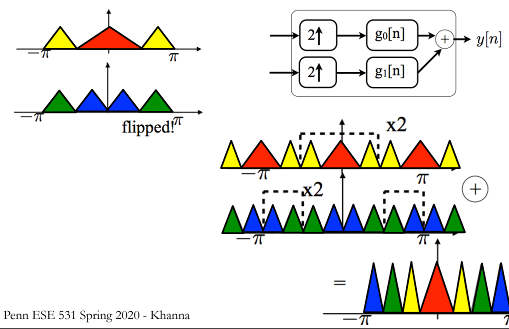

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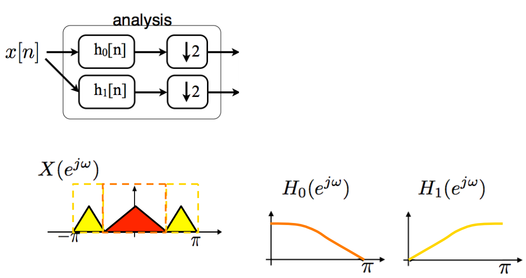

Non Ideal Filters

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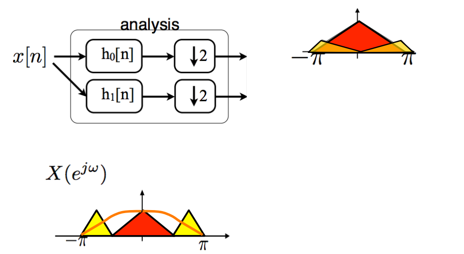

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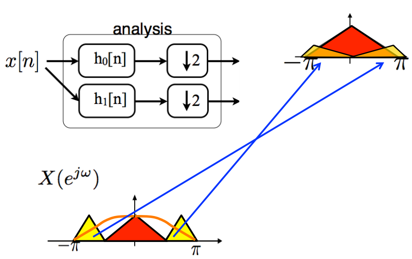

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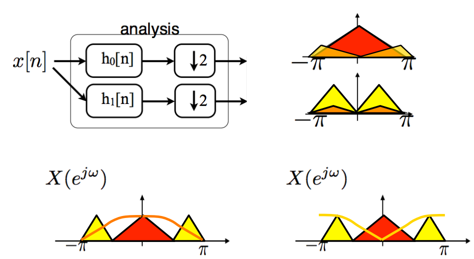

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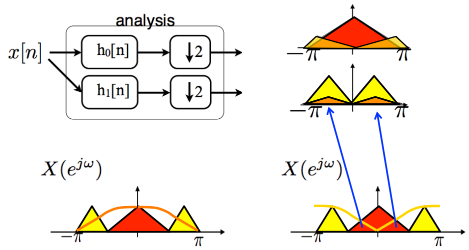

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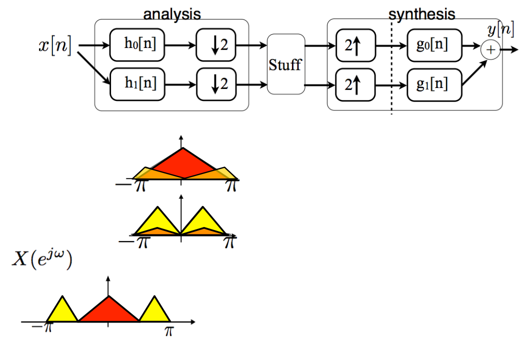

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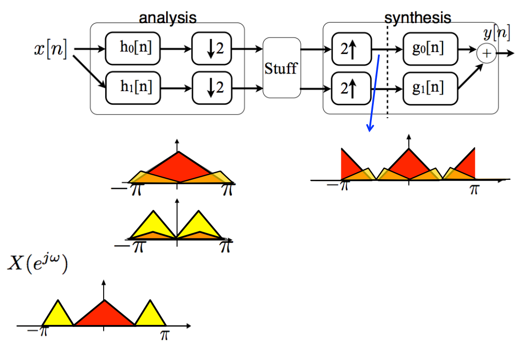

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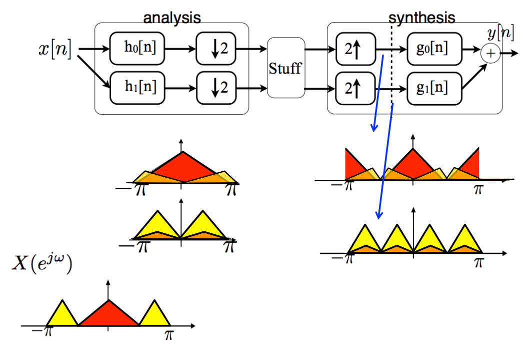
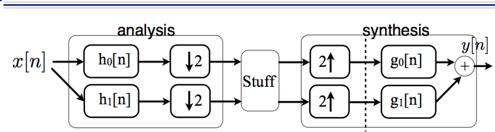


Perfect Reconstruction non-Ideal Filters

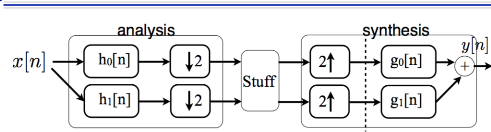


$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]X(e^{j(\omega-\pi)})$$

need to cancel!

aliasing

Quadrature Mirror Filters



Quadrature mirror filters

$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$
$$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$$
$$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$$

Perfect Reconstruction non-Ideal Filters

$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
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need to cancel!

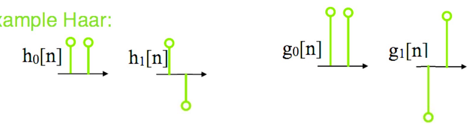
aliasing

$$\begin{aligned} H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\ G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\ G_1(e^{j\omega}) &= -2H_1(e^{j\omega}) \end{aligned}$$

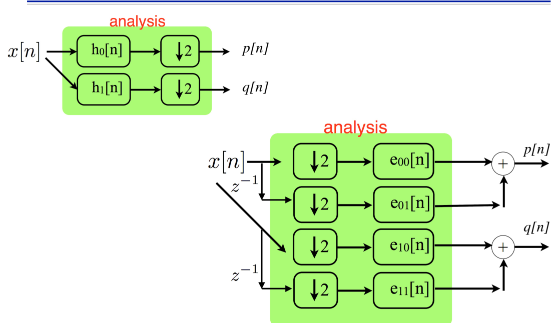
Haar Filter Example

$$\begin{aligned} H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\ G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\ G_1(e^{j\omega}) &= -2H_1(e^{j\omega}) \end{aligned}$$

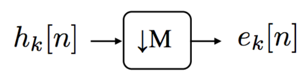
Example Haar:



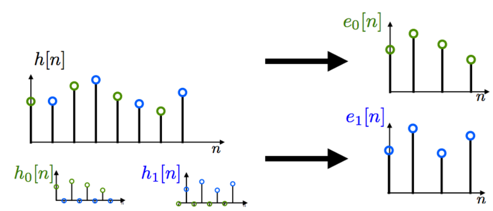
Polyphase Filter Bank



Polyphase Decomposition



$$e_k[n] = h_k[nM]$$



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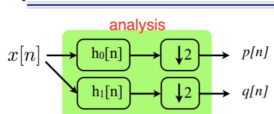
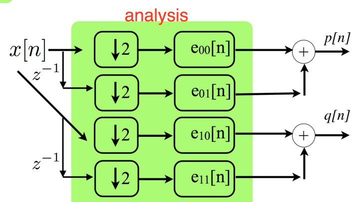
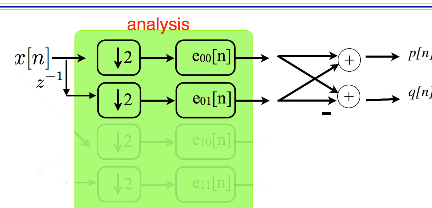




$$\begin{aligned} e_{00} &= h_0[2n] \\ e_{01} &= h_0[2n+1] \\ e_{10} &= e_{00}[n] \\ e_{11} &= -e_{01}[n] \end{aligned}$$

Polyphase Filter Bank

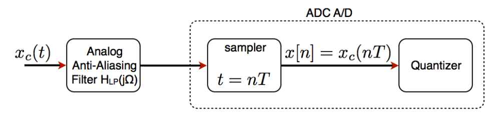


$$\begin{aligned} e_{00} &= h_0[2n] \\ e_{01} &= h_0[2n+1] \\ e_{10} &= e_{00}[n] \\ e_{11} &= -e_{01}[n] \end{aligned}$$

ADC

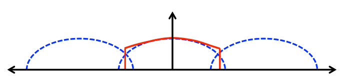
Analog to Digital Converter

Anti-Aliasing Filter with ADC



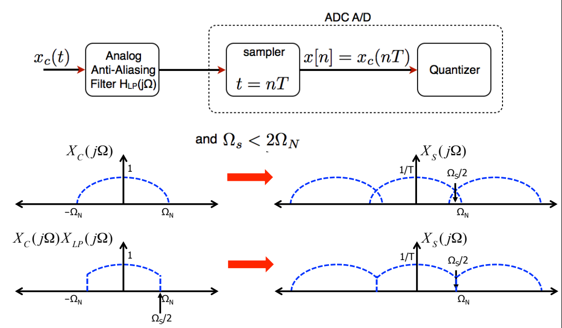
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

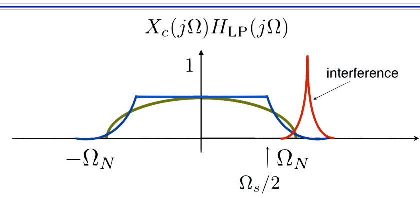


$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } |\Omega| \le \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

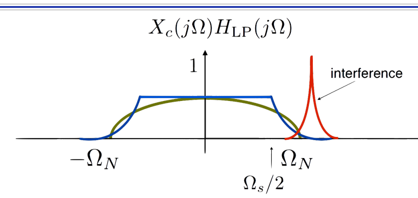
Anti-Aliasing Filter with ADC

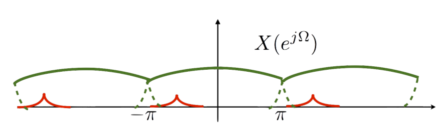


Non-Ideal Anti-Aliasing Filter

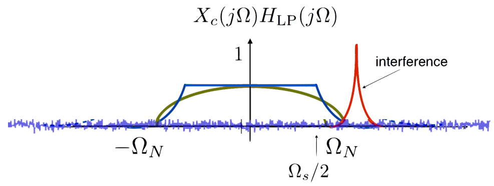


Non-Ideal Anti-Aliasing Filter



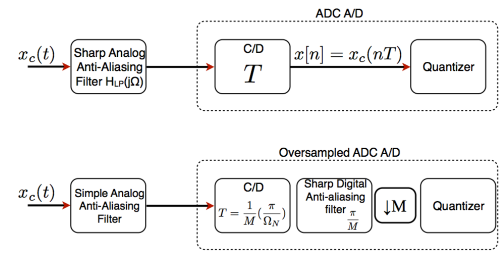


Non-Ideal Anti-Aliasing Filter

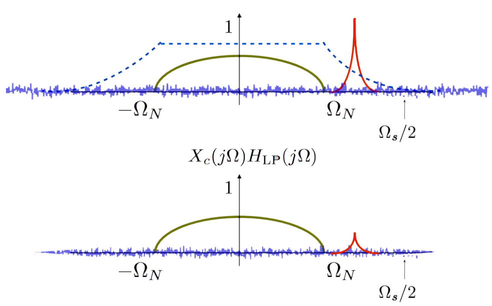


- Problem: Hard to implement sharp analog filter
- Consequence: Crop part of the signal and suffer from noise and interference

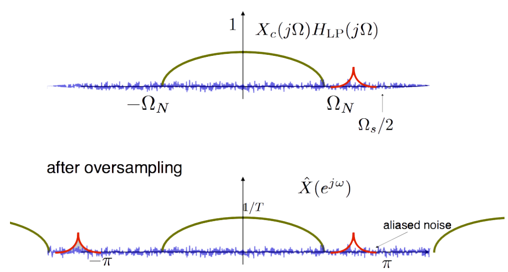
Oversampled ADC



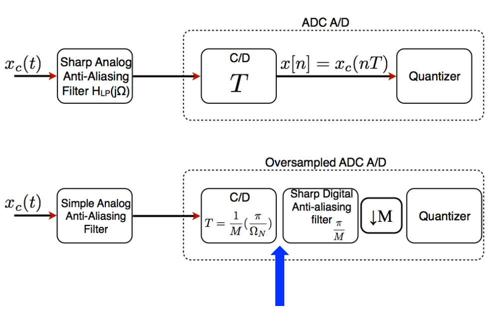
Oversampled ADC – Simple filter



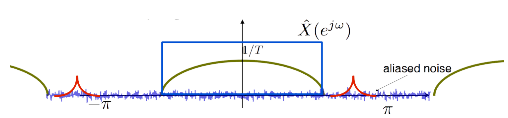
Oversampled ADC – M=2



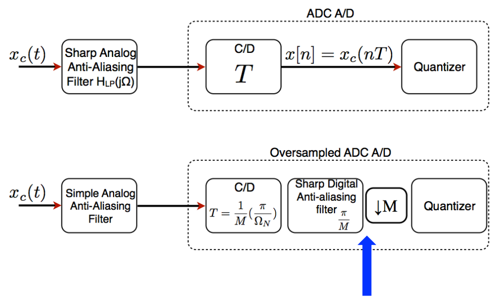
Oversampled ADC



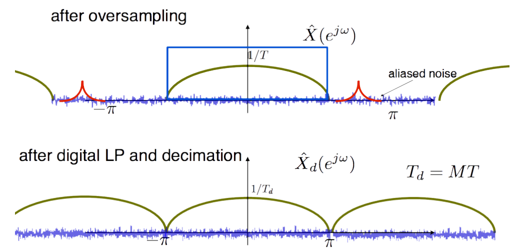
Oversampled ADC – Sharp digital filter/Downsample



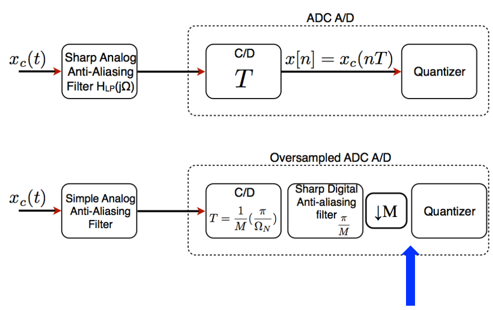
Oversampled ADC



Oversampled ADC – Sharp digital filter/Downsample

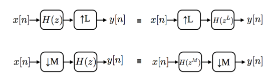


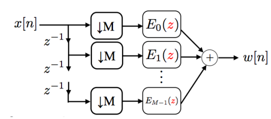
Oversampled ADC

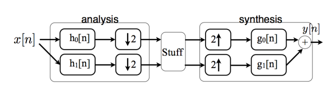


Big Ideas

- Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks







Admin

- HW 4 due Sunday