ESE 531: Digital Signal Processing

Lec 12: February 25, 2020

Data Converters, Noise Shaping

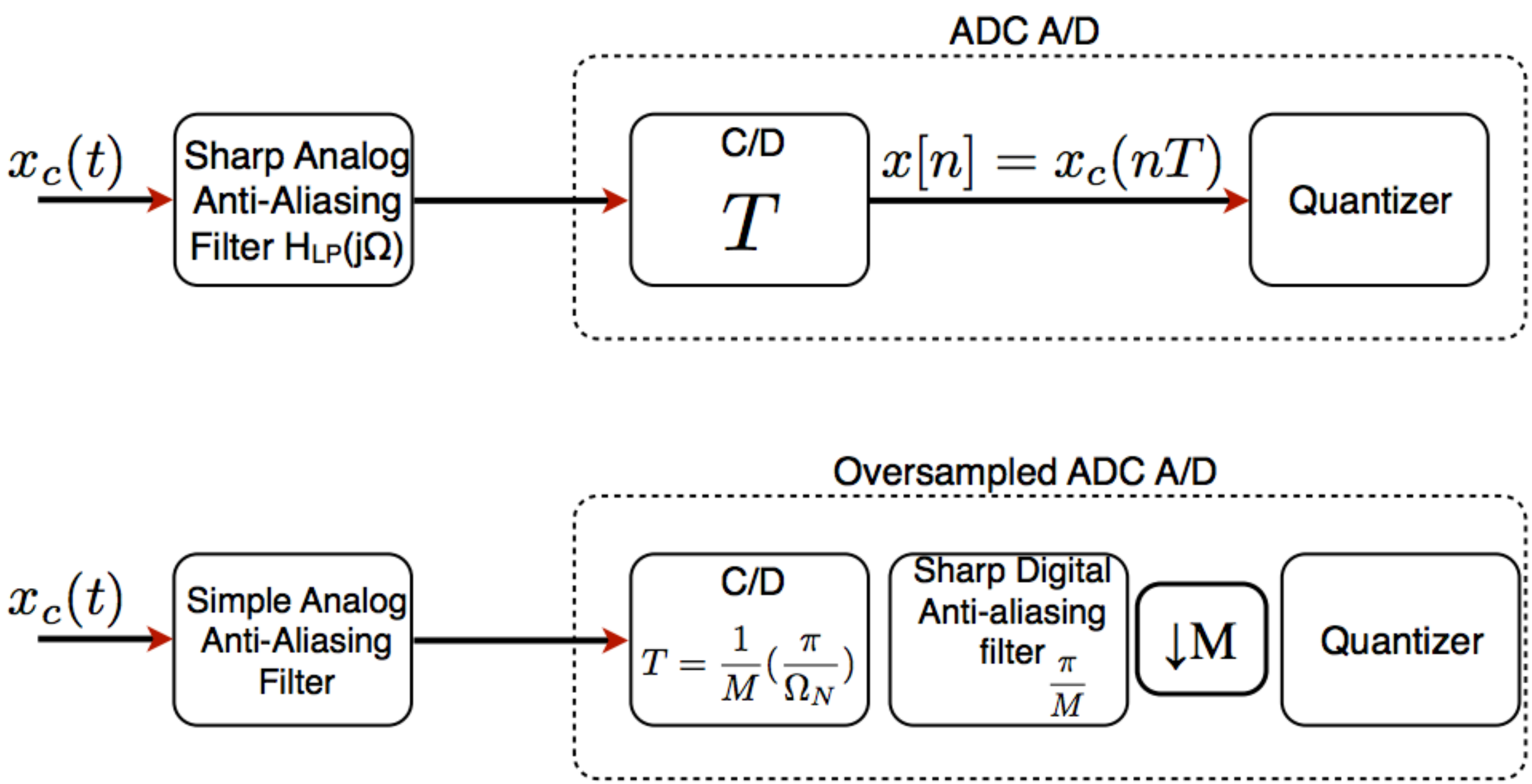
Lecture Outline

- Data Converters
  - Anti-aliasing
  - ADC
    - Quantization
  - Practical DAC
- Noise Shaping

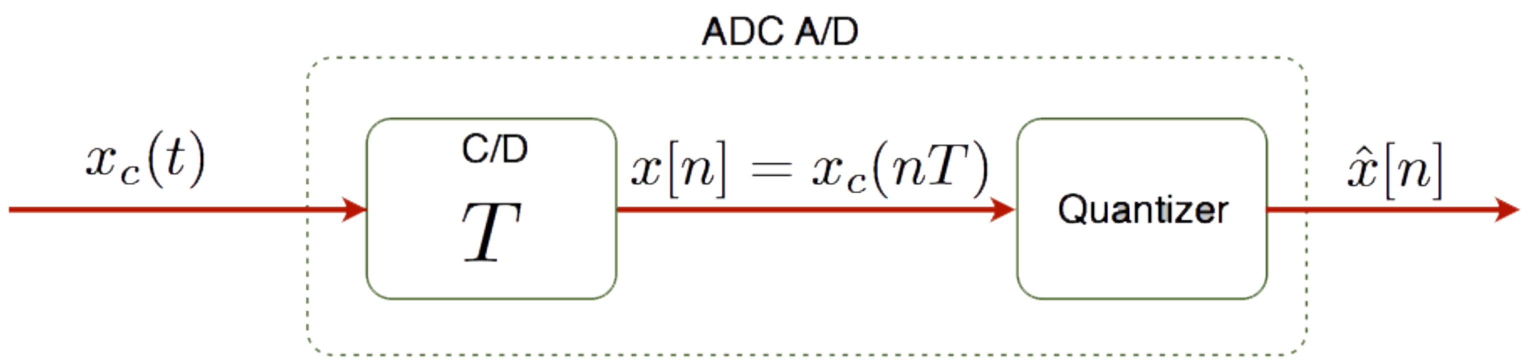
ADC

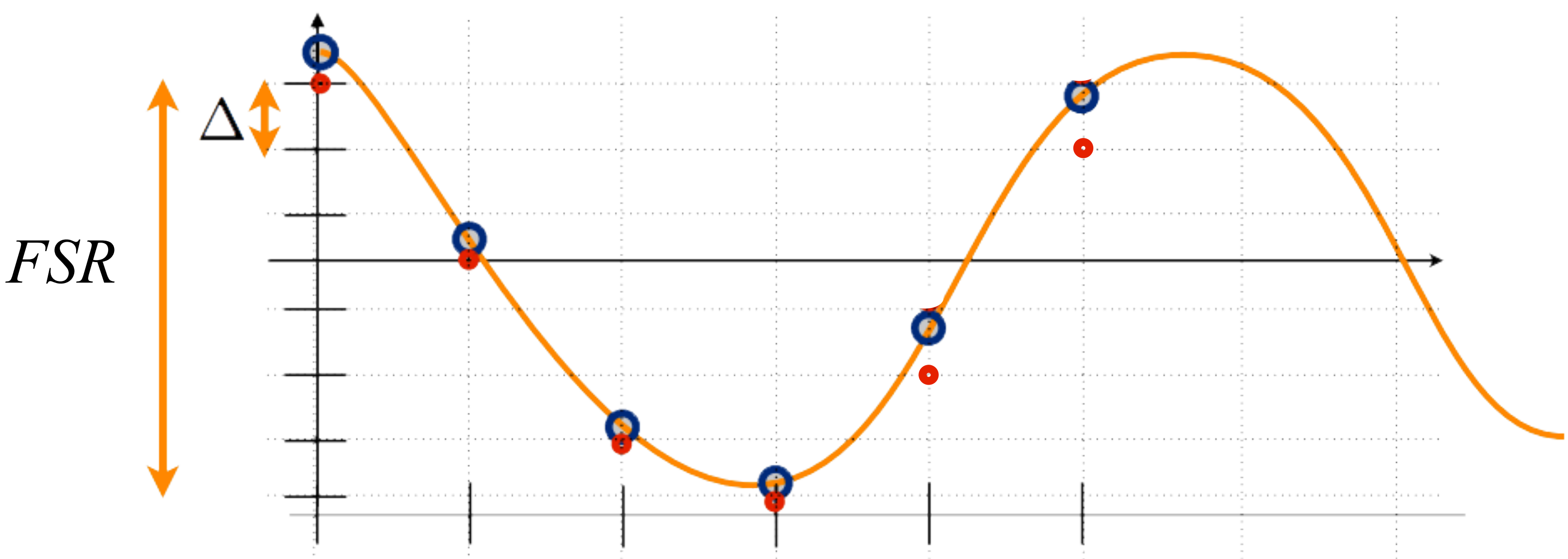
Analog to Digital Converter

Oversampled ADC



Sampling and Quantization

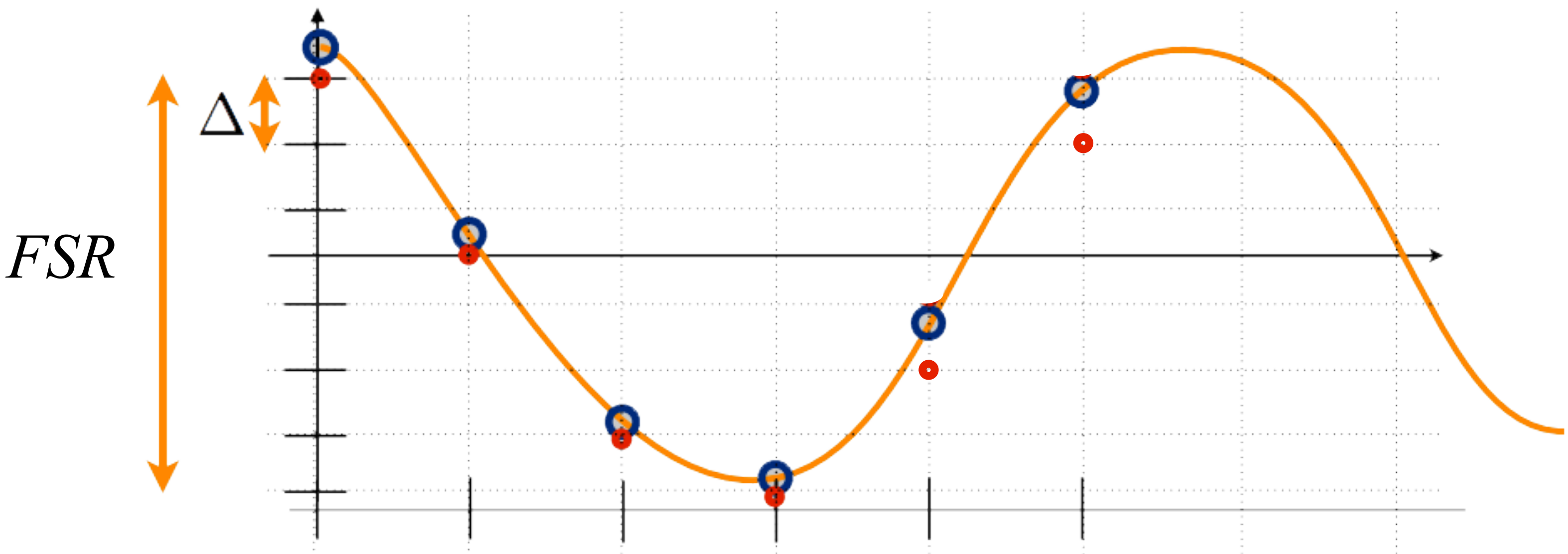




Sampling and Quantization

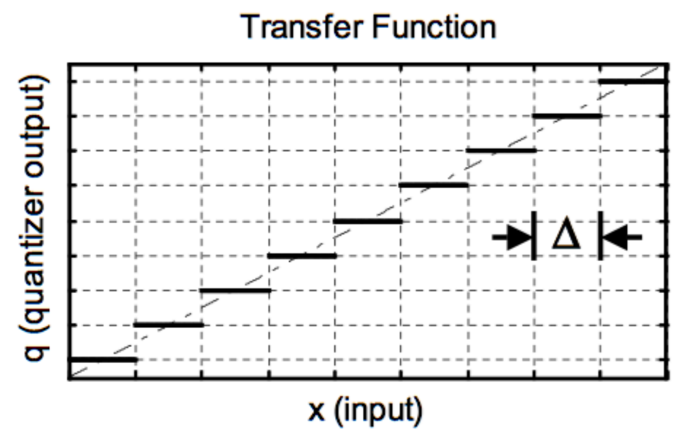
- For an input signal with V_{pp}=FSR with B bits

$$\Delta = \frac{FSR}{2^B}$$



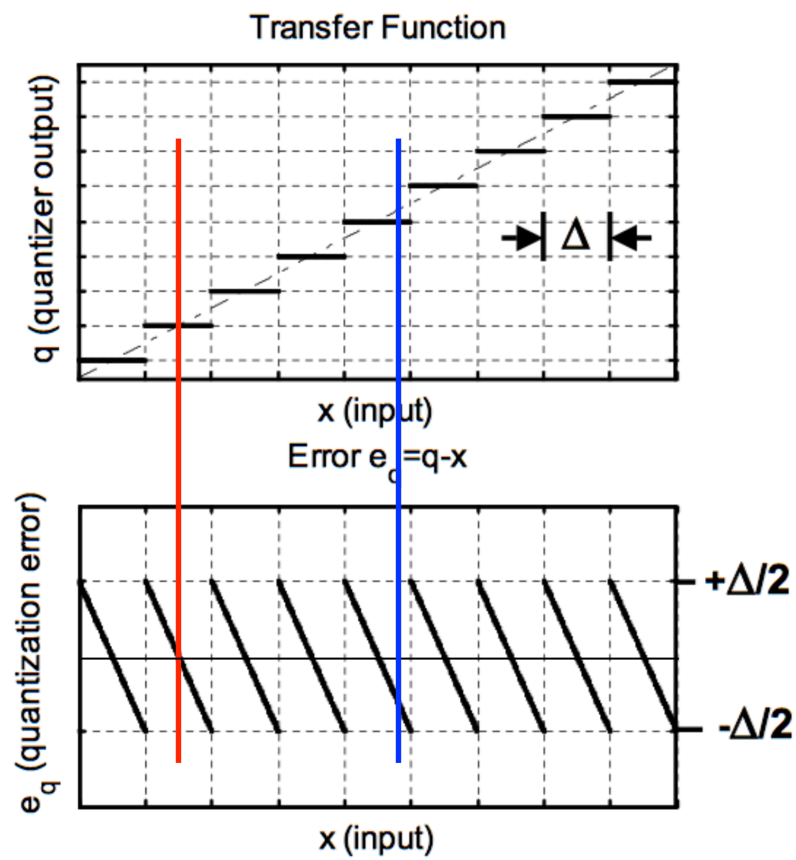
Ideal Quantizer

- Quantization step Δ



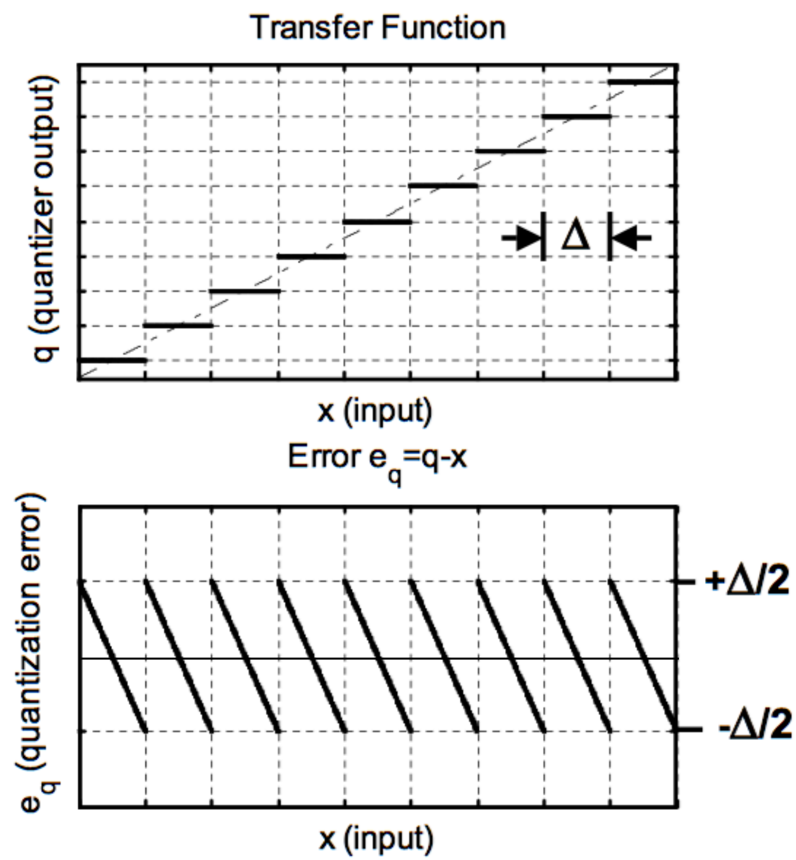
Ideal Quantizer

- Quantization step Δ
- Quantization error has sawtooth shape
  - Bounded by $-\Delta/2$, $+\Delta/2$



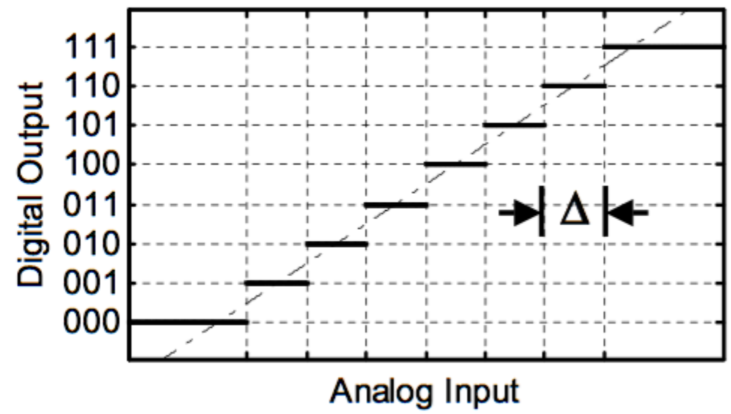
Ideal Quantizer

- Quantization step Δ
- Quantization error has sawtooth shape
  - Bounded by $-\Delta/2$, $+\Delta/2$
- Ideally infinite input range and infinite number of quantization levels



Ideal B-bit Quantizer

- Practical quantizers have a limited input range and a finite set of output codes
- E.g. a 3-bit quantizer can map onto $2^3=8$ distinct output codes



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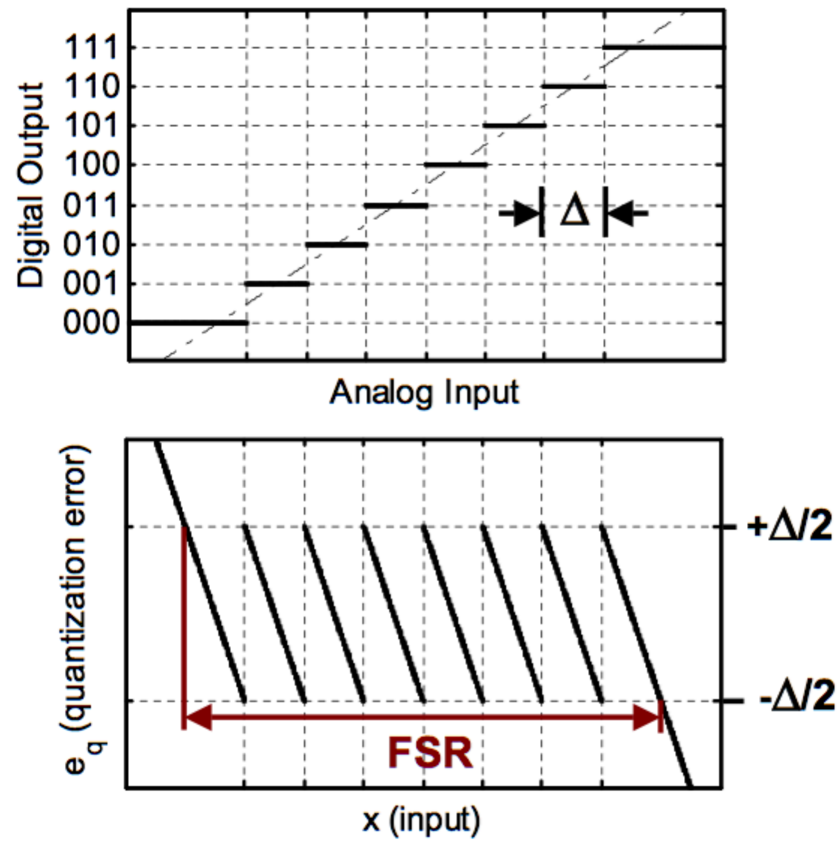
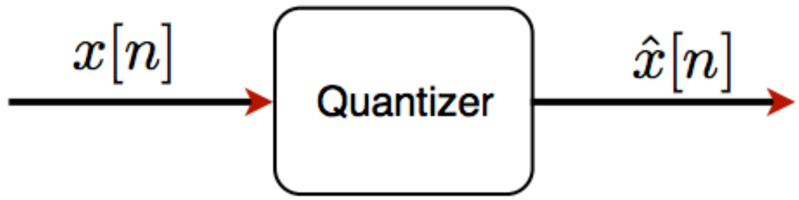


- Quantization error grows out of bounds beyond code boundaries
- We define the full scale range (FSR) as the maximum input range that satisfies $|e_q| \leq \Delta/2$
  - Implies that $FSR = 2^B \cdot \Delta$

Effect of Quantization Error on Signal

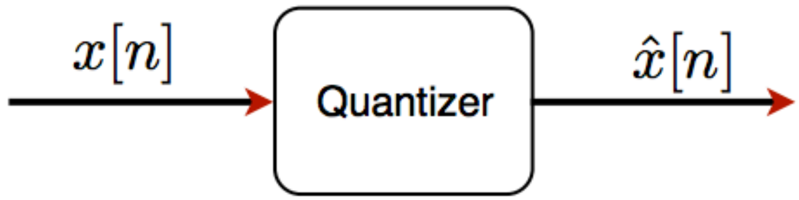
- Quantization error is a deterministic function of the signal
  - Consequently, the effect of quantization strongly depends on the signal itself
- Unless, we consider fairly trivial signals, a deterministic analysis is usually impractical
  - More common to look at errors from a statistical perspective
  - "Quantization noise"
- Two aspects
  - How much noise power (variance) does quantization add to our samples?
  - How is this noise distributed in frequency?

Quantization Error



- Model quantization error as noise:

Quantization Error



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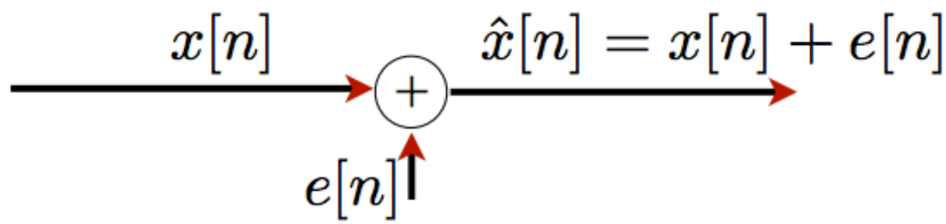
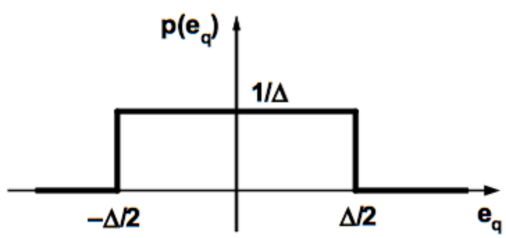


- In that case:

$$-\Delta/2 \leq e[n] < \Delta/2$$

Quantization Error Statistics

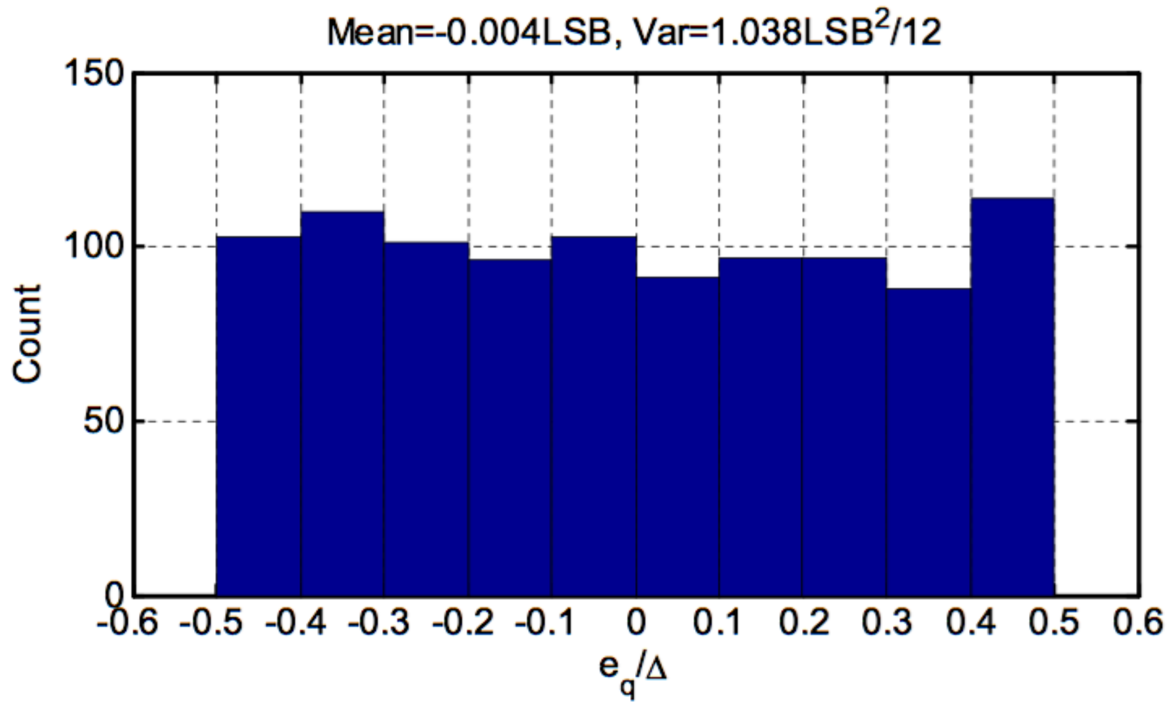
- Crude assumption: $e_q(x)$ has uniform probability density
- This approximation holds reasonably well in practice when
  - Signal spans large number of quantization steps
  - Signal is "sufficiently active"
  - Quantizer does not overload



Mean $$\bar{e} = \int_{-\Delta/2}^{+\Delta/2} \frac{e}{\Delta} de = 0$$

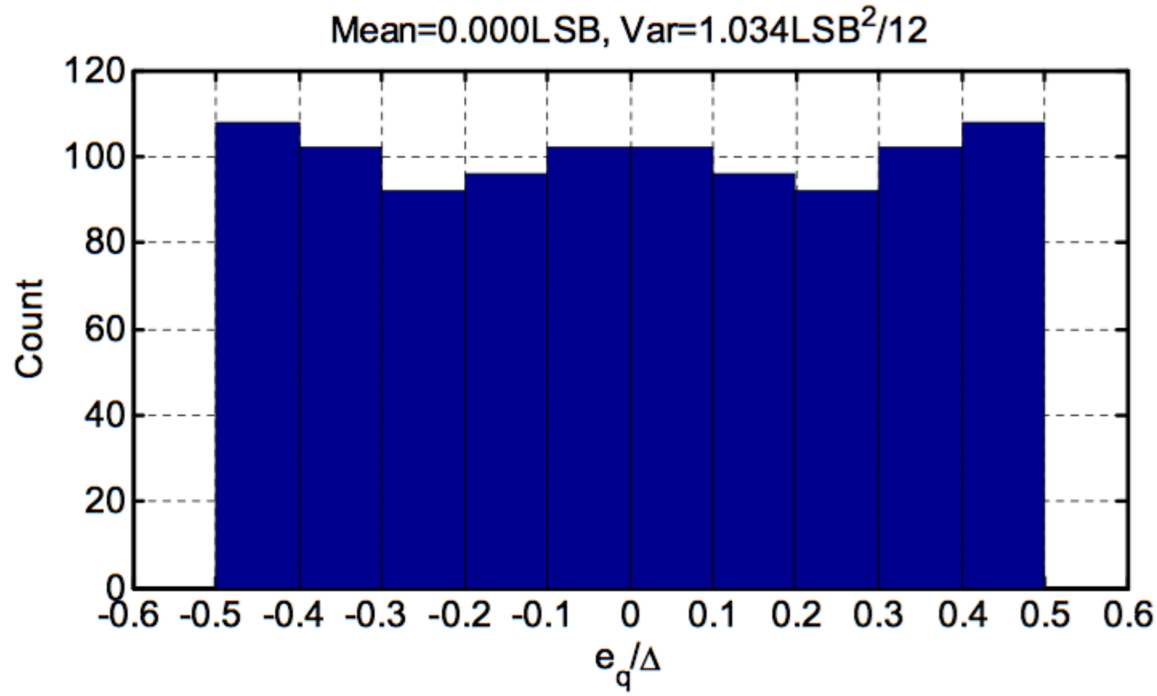
Variance $$\overline{e^2} = \int_{-\Delta/2}^{+\Delta/2} \frac{e^2}{\Delta} de = \frac{\Delta^2}{12}$$

Reality Check



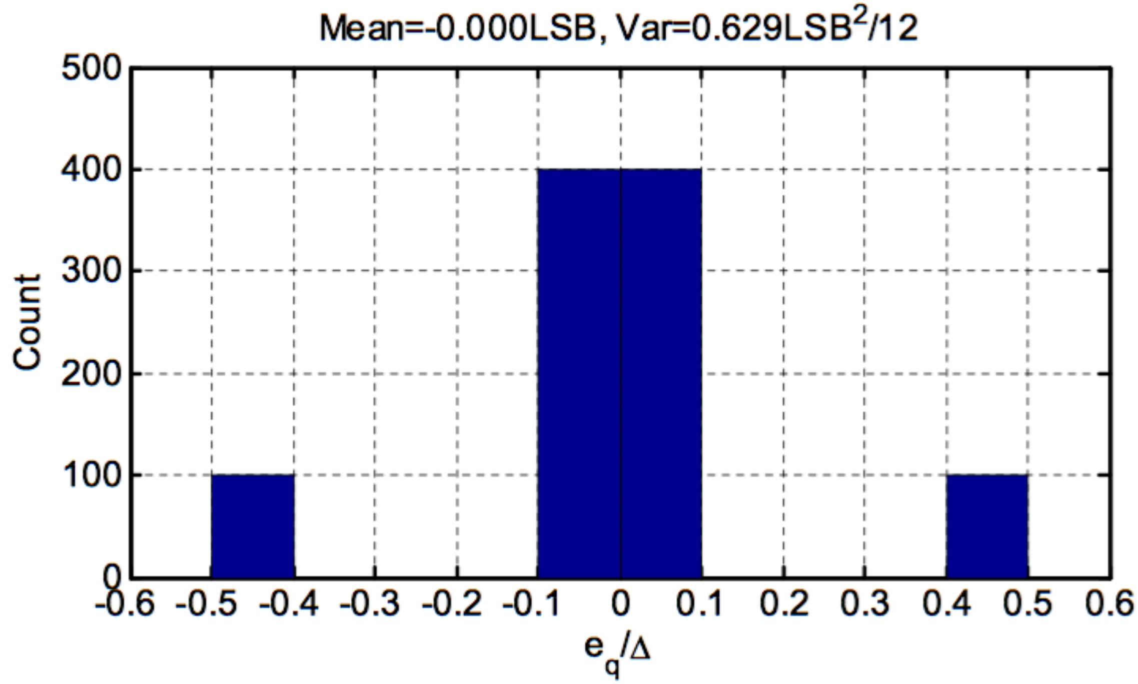
- Shown is a histogram of e_q in an 8-bit quantizer
  - Input sequence consists of 1000 samples with Gaussian distribution, 4σ =FSR

Reality Check



- Same as before, but now using a sinusoidal input signal with f_{sig}/f_s=101/1000

Reality Check



- Same as before, but now using a sinusoidal input signal with f_{sig}/f_s=100/1000
- What went wrong?

Analysis

$$v_{sig}[n] = \cos\left(2\pi \cdot \frac{f_{sig}}{f_S} \cdot n\right)$$

- Signal repeats every m samples, where m is the smallest integer that satisfies $$m \cdot \frac{f_{sig}}{f_S} = \text{integer}$$

Analysis

$$v_{sig}(n) = \cos\left(2\pi \cdot \frac{f_{sig}}{f_S} \cdot n\right)$$

- Signal repeats every m samples, where m is the smallest integer that satisfies

$$m \cdot \frac{f_{sig}}{f_S} = \text{integer}$$

$$m \cdot \frac{101}{1000} = \text{integer} \Rightarrow \text{m=1000}$$

$$m \cdot \frac{100}{1000} = \text{integer} \Rightarrow \text{m=10}$$

- This means that in the last case $e_q(n)$ consists at best of 10 different values, even though we took 1000 samples

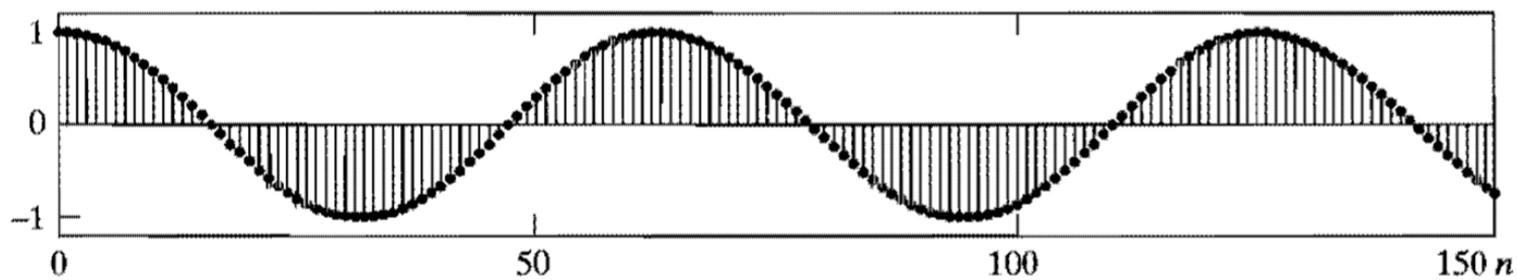
Noise Model for Quantization Error

- Assumptions:
  - Model e[n] as a sample sequence of a stationary random process
  - e[n] is not correlated with x[n]
  - e[n] not correlated with e[m] where $m \neq n$
  - e[n] ~ $U[-\Delta/2, \Delta/2]$ (uniform pdf)

- Result:
- Variance is: $\sigma_e^2 = \frac{\Delta^2}{12}$
- Assumptions work well for signals that change rapidly, are not clipped, and for small Δ

Quantization Noise

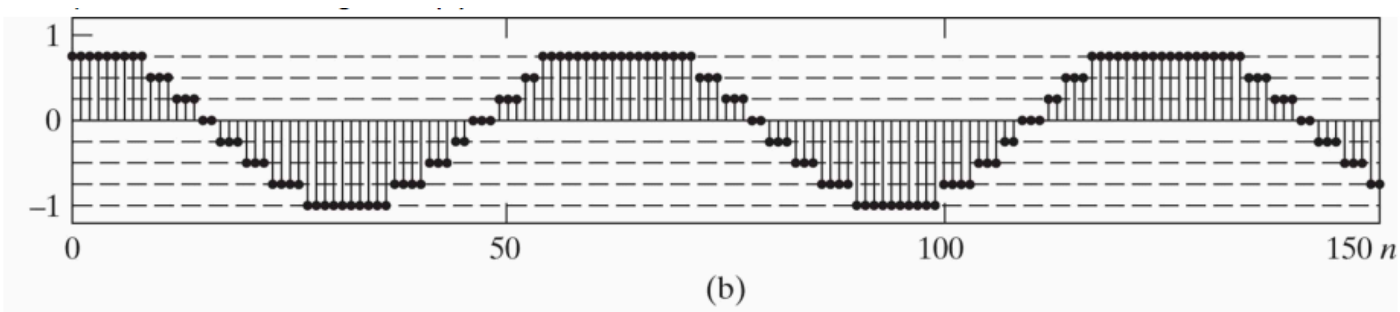
- **Figure 4.57** Example of quantization noise. (a) Unquantized samples of the signal x[n] = 0.99cos(n/10).



(a)

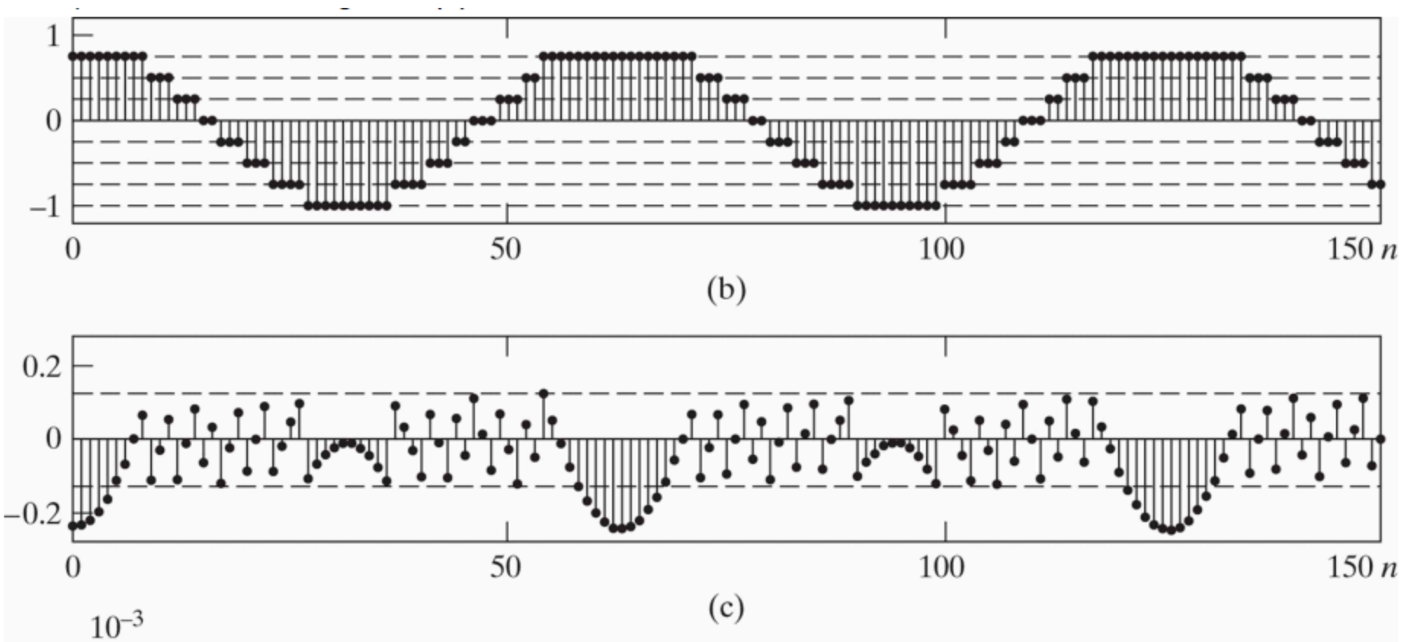
Quantization Noise

- **Figure 4.57**(continued) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer.



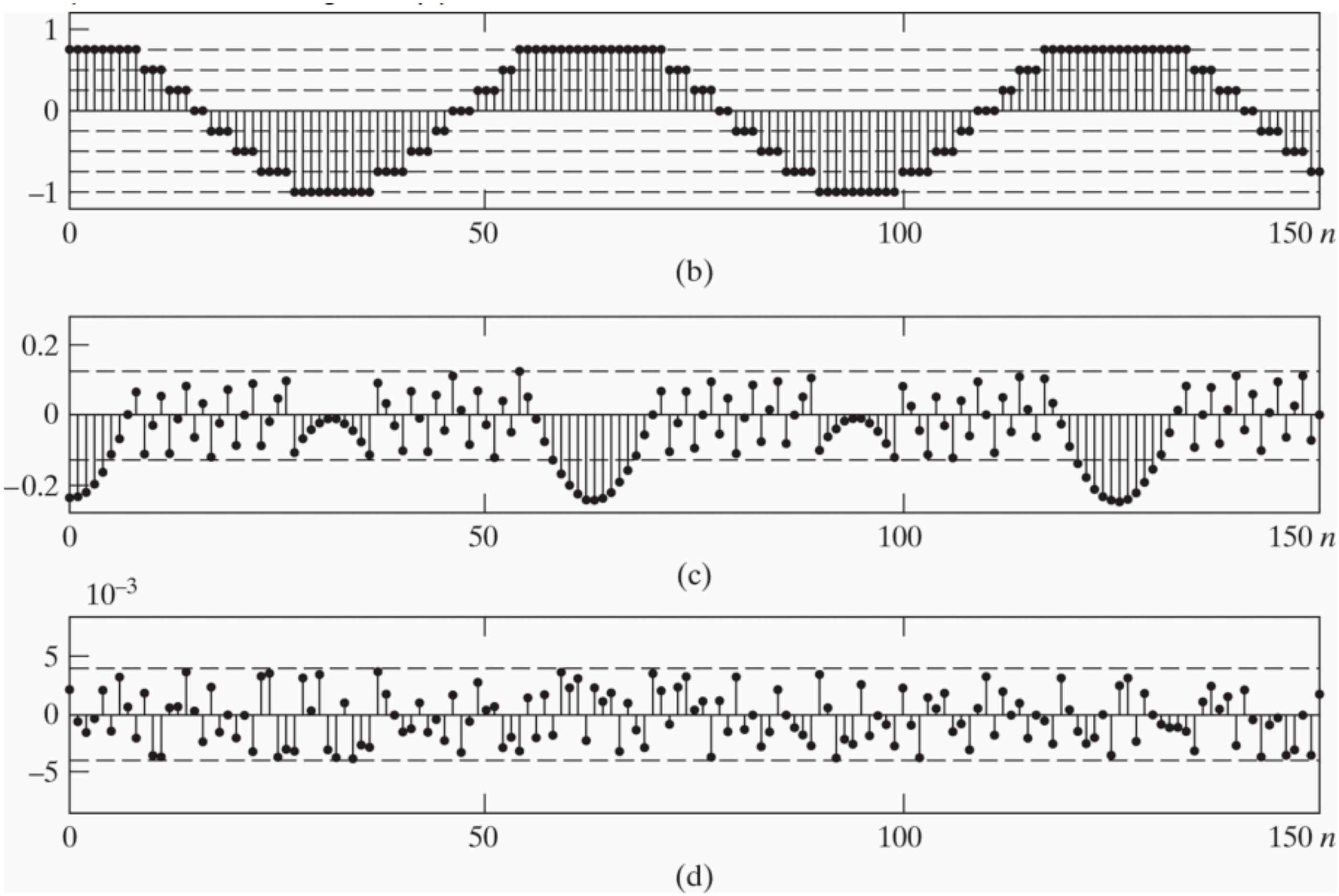
Quantization Noise

- **Figure 4.57**(continued) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer. (c) Quantization error sequence for 3-bit quantization of the signal in (a).



Quantization Noise

- **Figure 4.57**(continued) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer. (c) Quantization error sequence for 3-bit quantization of the signal in (a). (d) Quantization error sequence for 8-bit quantization of the signal in (a).



Signal-to-Quantization-Noise Ratio

- For uniform B bits quantizer

$$SNR_Q = 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right)$$

Signal-to-Quantization-Noise Ratio

- For uniform B bits quantizer

$$
\begin{aligned}
SNR_Q &= 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right) \\
&= 10\log_{10}\left(\frac{12\cdot 2^{2B}\sigma_x^2}{FSR^2}\right)
\end{aligned}
$$

$$
\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{FSR}{\sigma_x}\right)
$$

Quantizer range

rms of amp

Signal-to-Quantization-Noise Ratio

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{FSR}{\sigma_x}\right)$$

Quantizer range

rms of amp

- Improvement of 6dB with every bit
- The range of the quantization must be adapted to the rms amplitude of the signal
  - Tradeoff between clipping and noise!
  - Often use pre-amp
  - Sometimes use analog auto gain controller (AGC)

Signal-to-Quantization-Noise Ratio

- Assuming full-scale sinusoidal input, we have

$$\mathrm{SQNR} = \frac{P_{sig}}{P_{qnoise}} =$$

Signal-to-Quantization-Noise Ratio

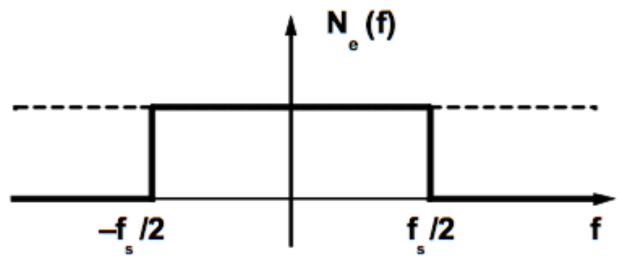
- Assuming full-scale sinusoidal input, we have

$$\mathrm{SQNR} = \frac{P_{sig}}{P_{qnoise}} = 6.02B + 1.76 \ \ \mathrm{dB}$$

B (Number of Bits)	SQNR
8	50dB
12	74dB
16	98dB
20	122dB

Quantization Noise Spectrum

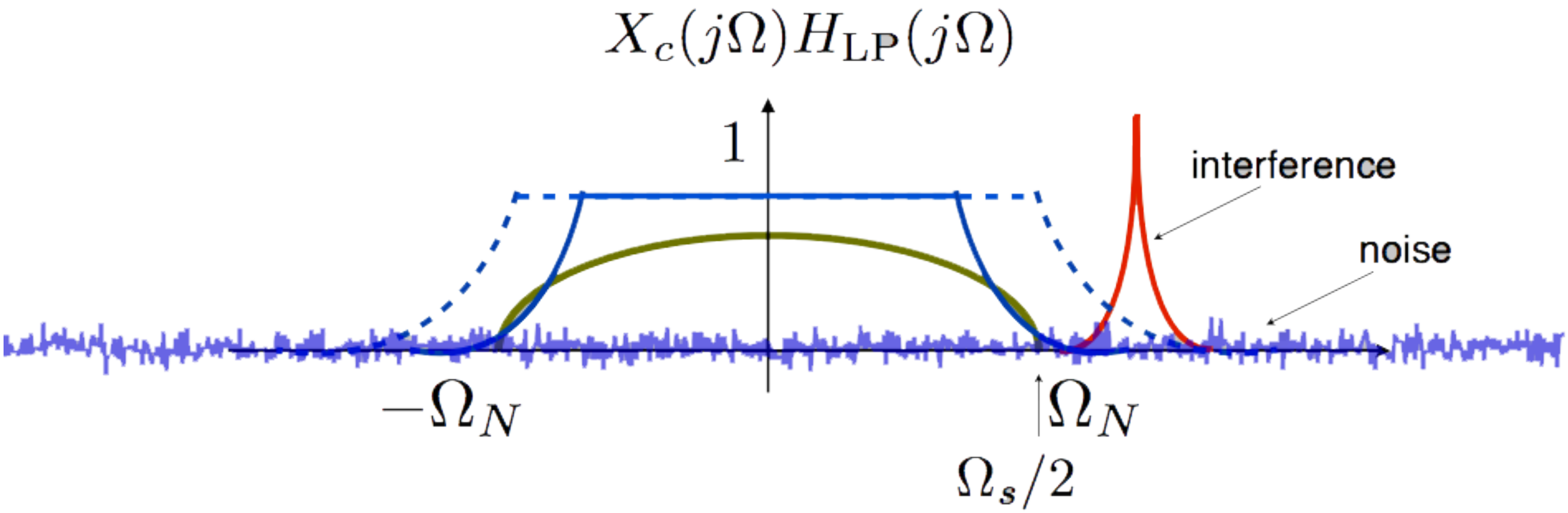
- If the quantization error is "sufficiently random", it also follows that the noise power is uniformly distributed in frequency



$$\frac{\Delta^2}{12} \cdot \frac{1}{f_s}$$

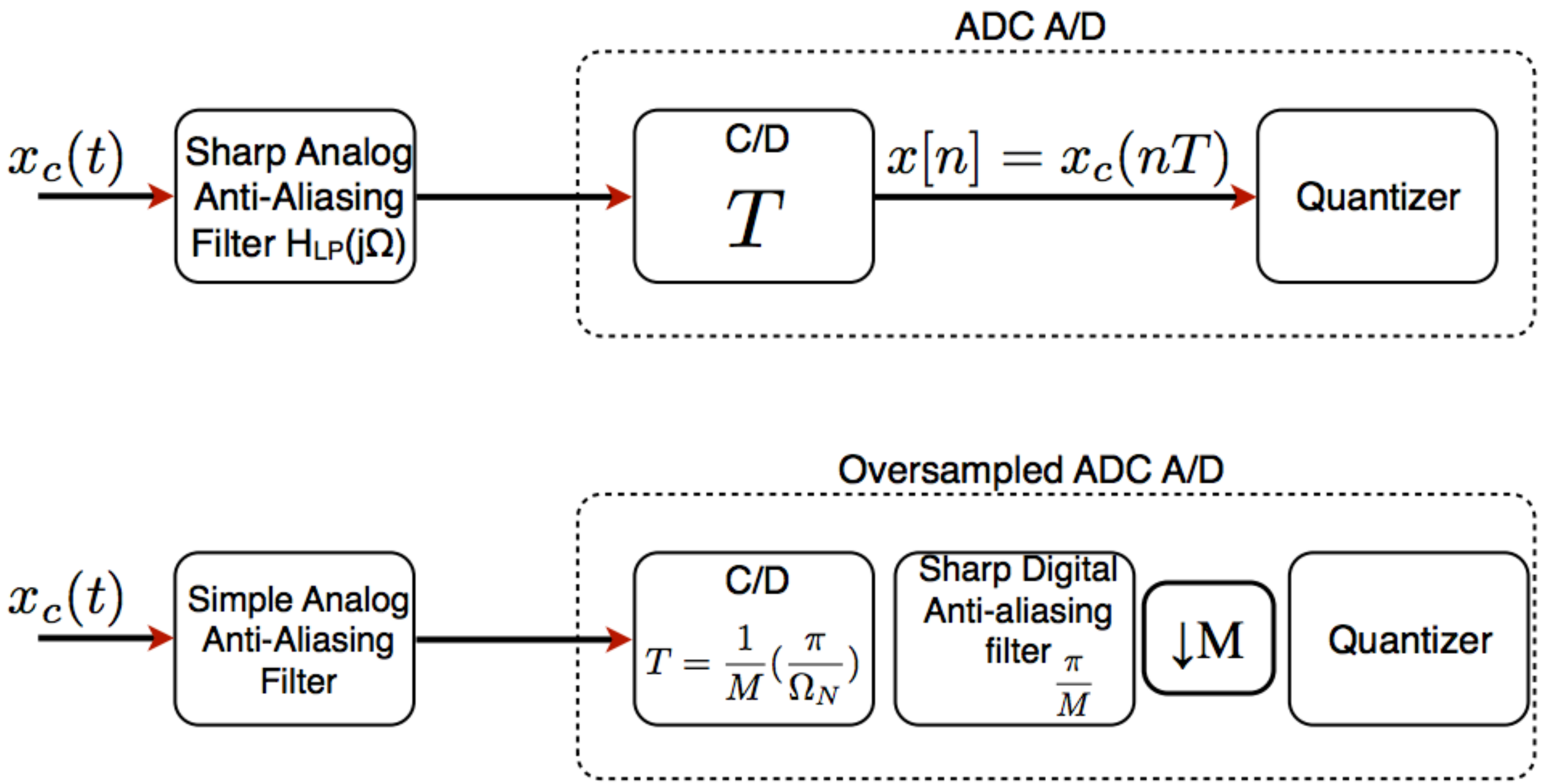
- References
  - W. R. Bennett, "Spectra of quantized signals," Bell Syst. Tech. J., pp. 446-72, July 1988.
  - B. Widrow, "A study of rough amplitude quantization by means of Nyquist sampling theory," IRE Trans. Circuit Theory, vol. CT-3, pp. 266-76, 1956.

Non-Ideal Anti-Aliasing Filter

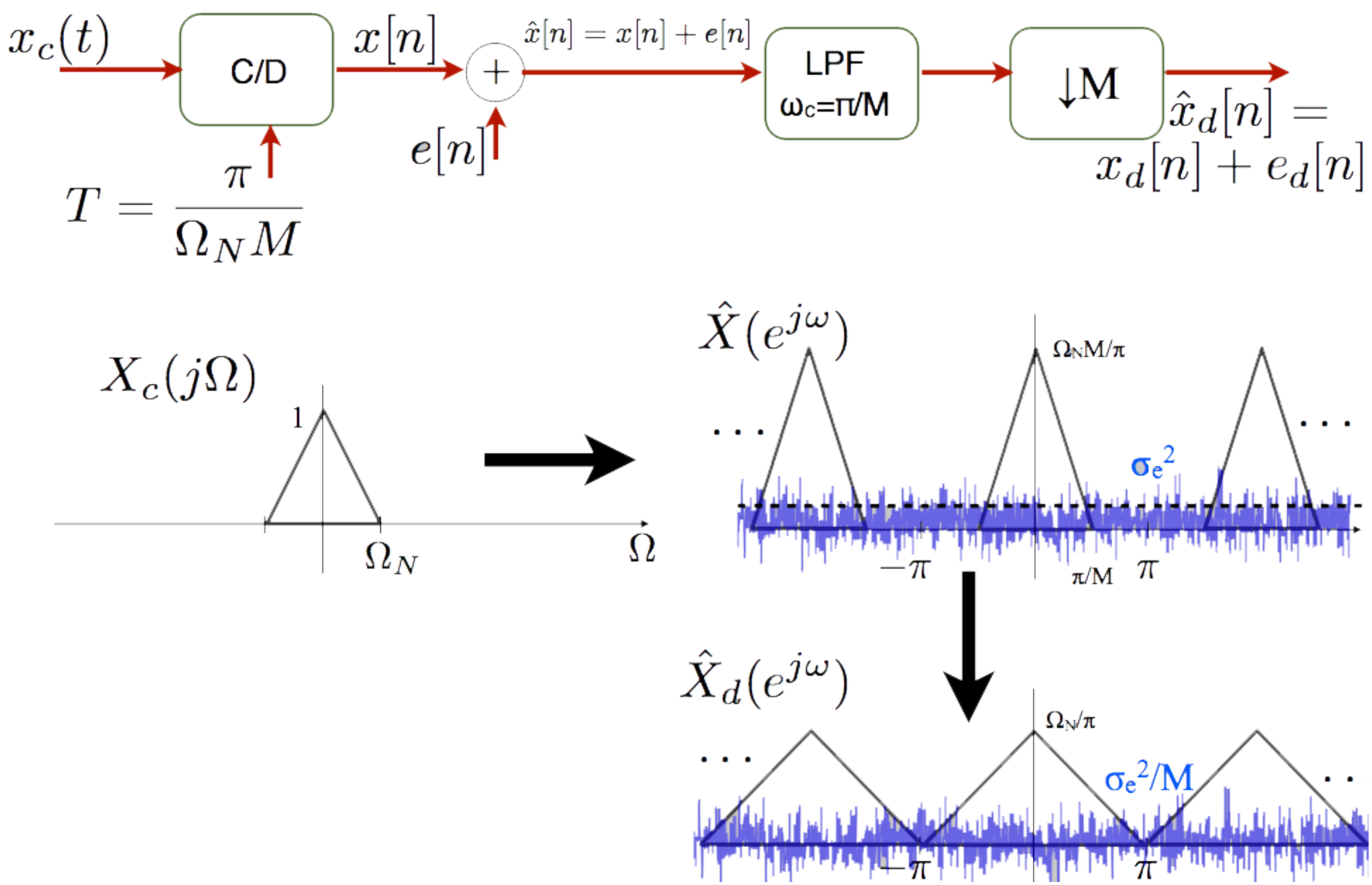


- Problem: Hard to implement sharp analog filter
- Consequence: Crop part of the signal and suffer from noise and interference

Oversampled ADC



Quantization Noise with Oversampling



Quantization Noise with Oversampling

- Energy of $x_d[n]$ equals energy of x[n]
  - No filtering of signal!
- Noise variance is reduced by factor of M

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{FSR}{\sigma_x}\right) + 10\log_{10} M$$

- For doubling of M we get 3dB improvement, which is the same as 1/2 a bit of accuracy
  - With oversampling of 16 with 8bit ADC we get the same quantization noise as 10bit ADC!

Practical DAC

Practical DAC

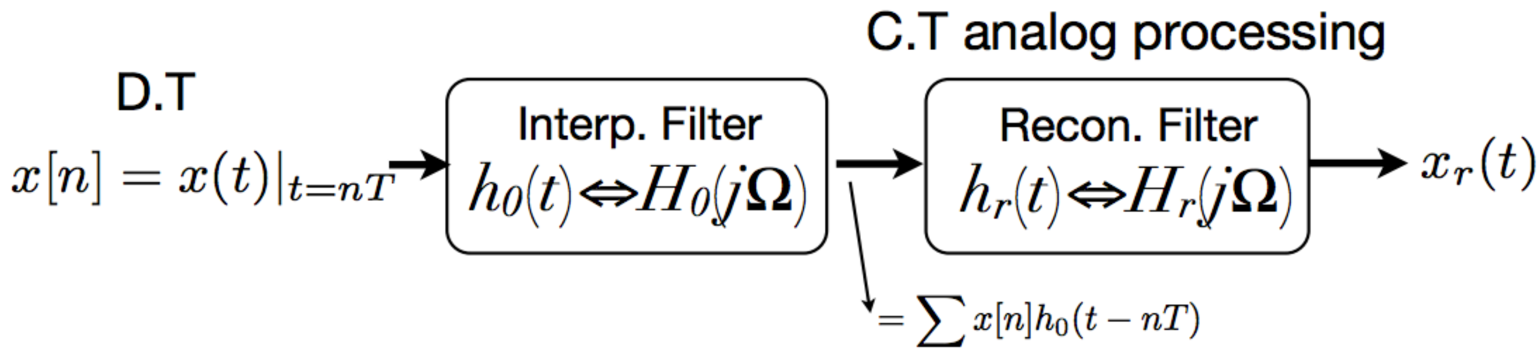
D.T

$$x[n] = x(t)|_{t=nT} \rightarrow \boxed{\text{sinc pulse generator}} \rightarrow x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

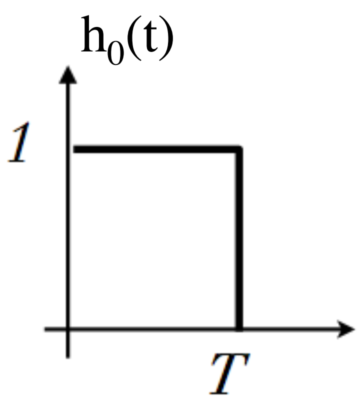
C.T

- Scaled train of sinc pulses
- Difficult to generate sinc → Too long!

Practical DAC



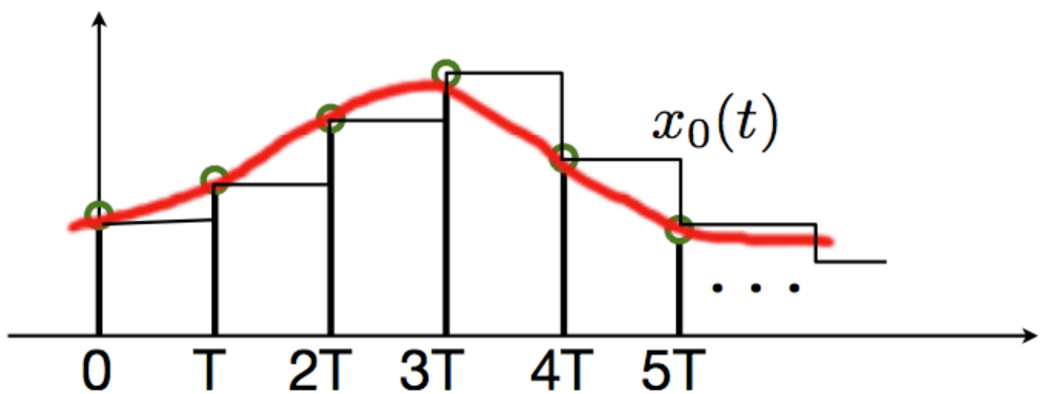
- $h_0(t)$ is finite length pulse → easy to implement
- For example: zero-order hold



$$H_0(j\Omega) = Te^{-j\Omega\frac{T}{2}} \mathrm{sinc}(\frac{\Omega}{\Omega_s})$$

Practical DAC

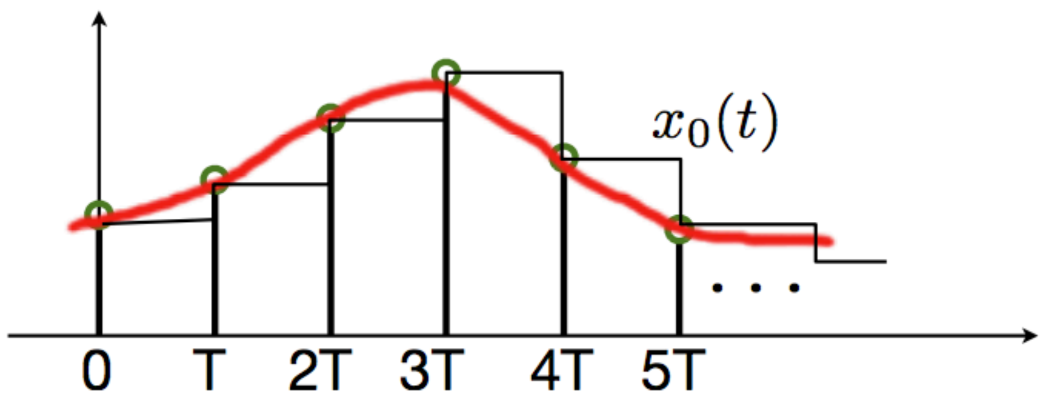
Zero-Order-Hold interpolation



$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t - nT) = h_0(t) * x_s(t)$$

Practical DAC

Zero-Order-Hold interpolation



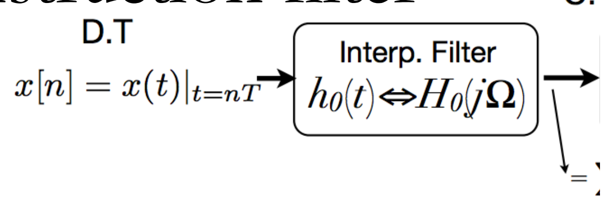
$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n] h_0(t - nT) = h_0(t) * x_s(t)$$

Taking a FT:

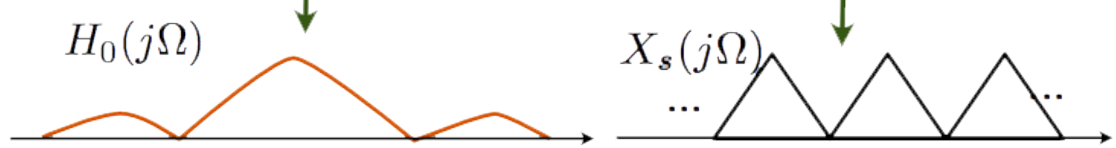
$$\begin{aligned} X_{(}j\Omega) &= H_0(j\Omega) X_s(j\Omega) \\ &= H_0(j\Omega) \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s)) \end{aligned}$$

Practical DAC

- Output of the reconstruction filter



$$
\begin{aligned}
X_r(j\Omega) &= H_0(j\Omega)\cdot X_s(j\Omega) \\
&= \underbrace{Te^{-j\Omega\frac{T}{2}}\,\mathrm{sinc}\left(\frac{\Omega}{\Omega_s}\right)}_{\text{from zero-order hold}} \cdot \underbrace{\frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s))}_{\text{Shifted copies from sampling}}
\end{aligned}
$$



Practical DAC

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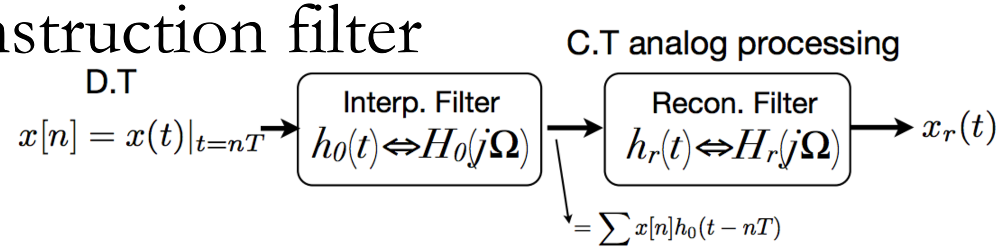
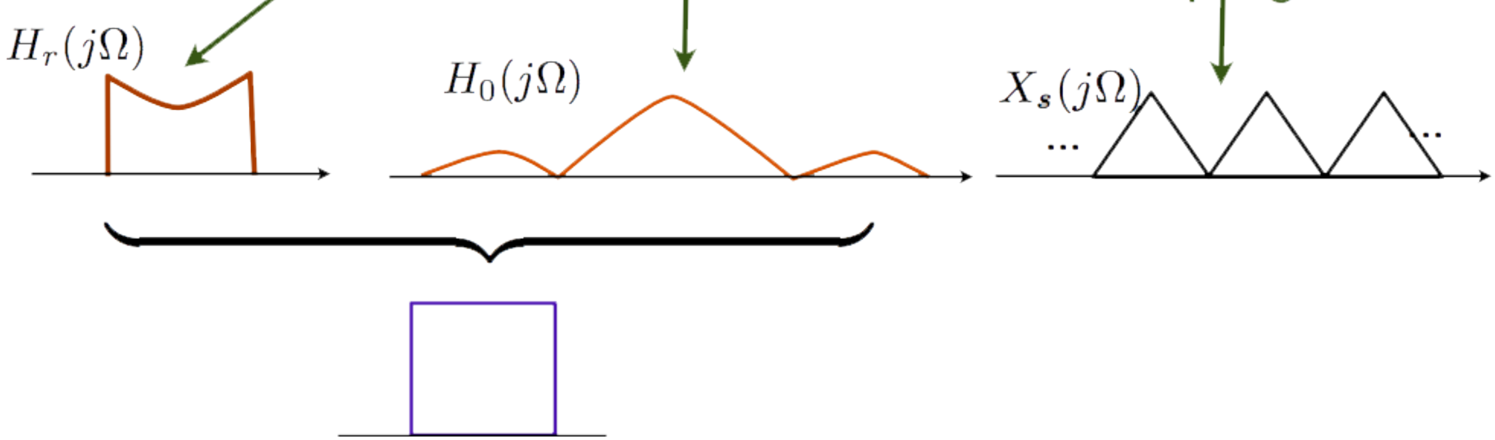
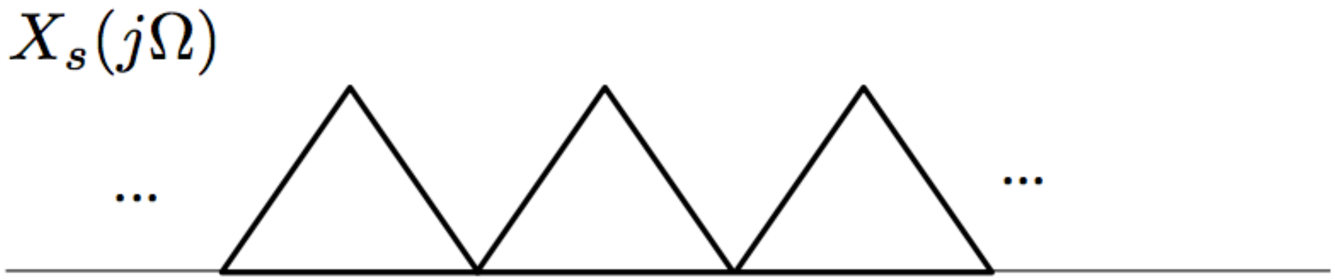


$$X_r(j\Omega) = H_r(j\Omega) \cdot H_0(j\Omega) \cdot X_s(j\Omega)$$

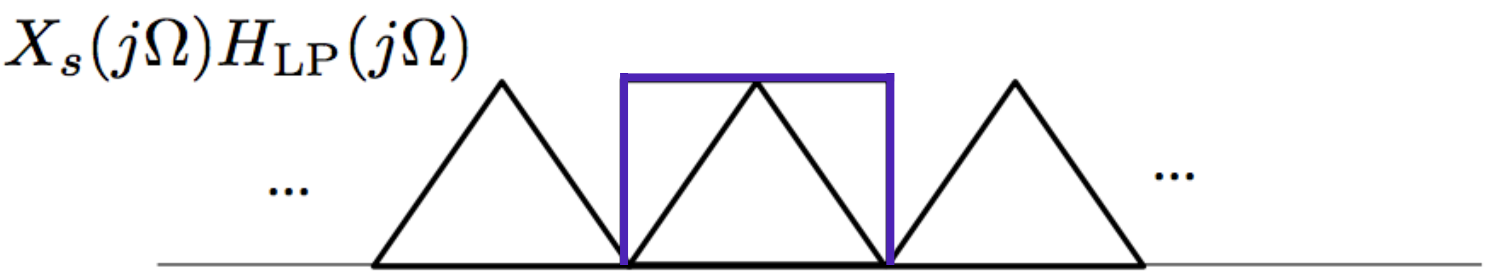
$$= \underbrace{H_r(j\Omega)}_{\text{recon filter}} \cdot \underbrace{T e^{-j\Omega\frac{T}{2}} \mathrm{sinc}(\frac{\Omega}{\Omega_s})}_{\text{from zero-order hold}} \cdot \underbrace{\frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s))}_{\text{Shifted copies from sampling}}$$



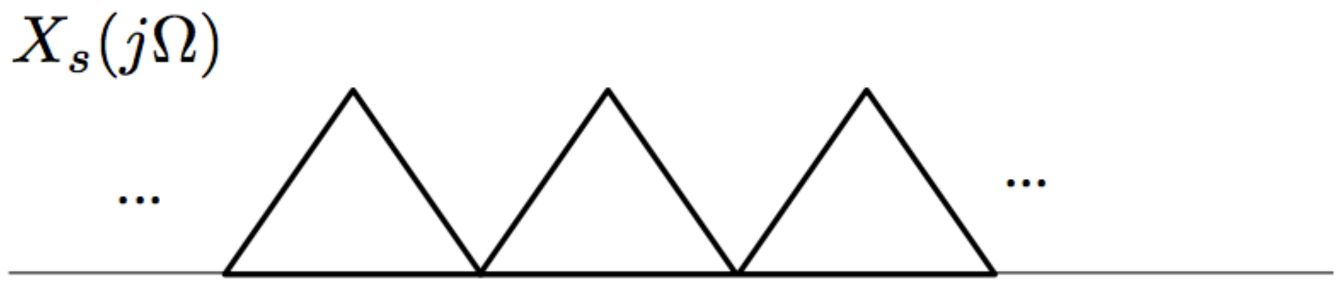
Practical DAC



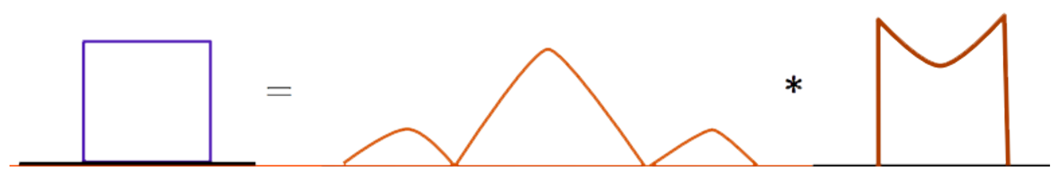
Ideally:

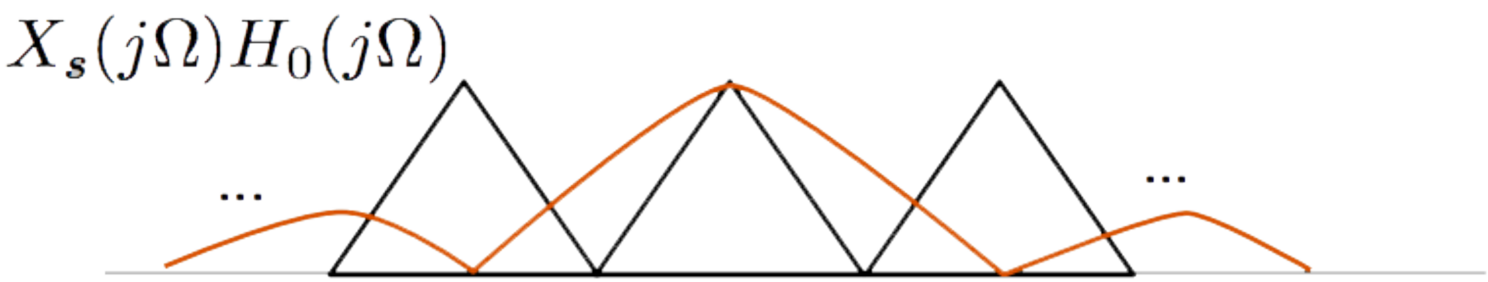


Practical DAC



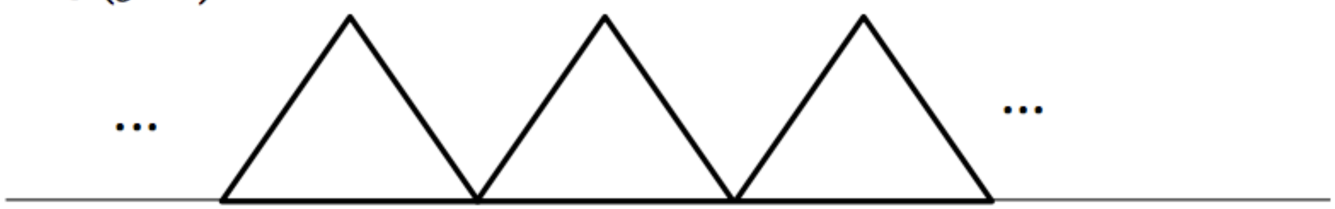
Practically:





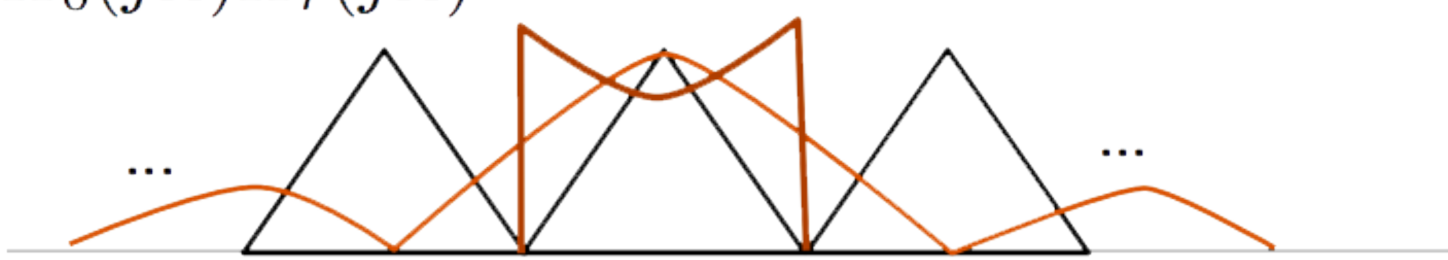
Practical DAC

$X_s(j\Omega)$

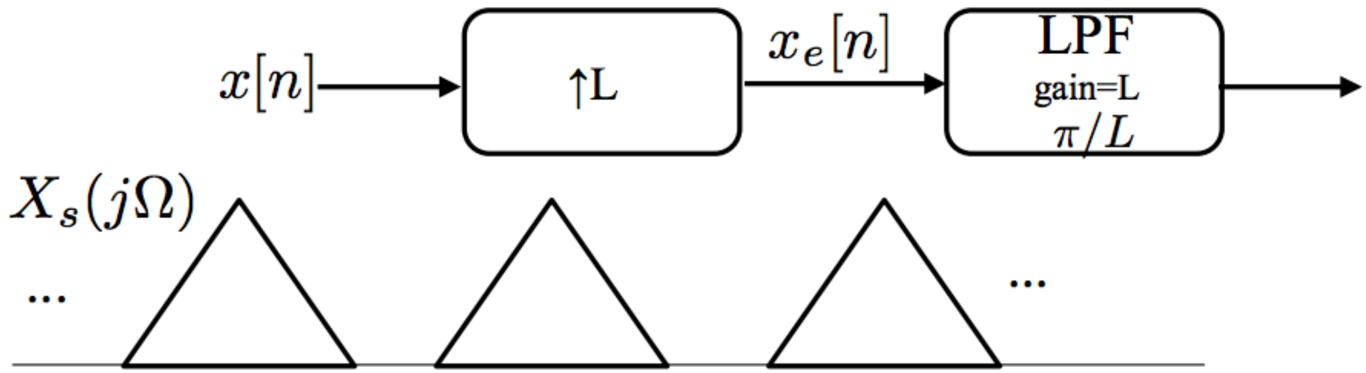


Practically:

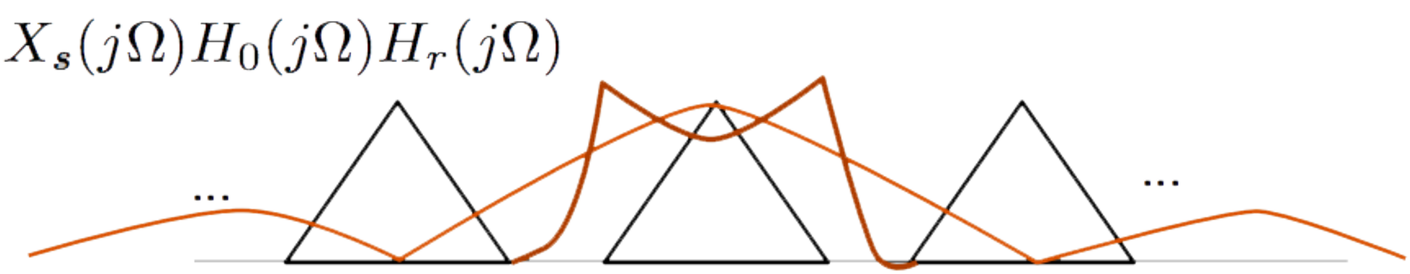
$X_s(j\Omega)H_0(j\Omega)H_r(j\Omega)$



Practical DAC with Upsampling

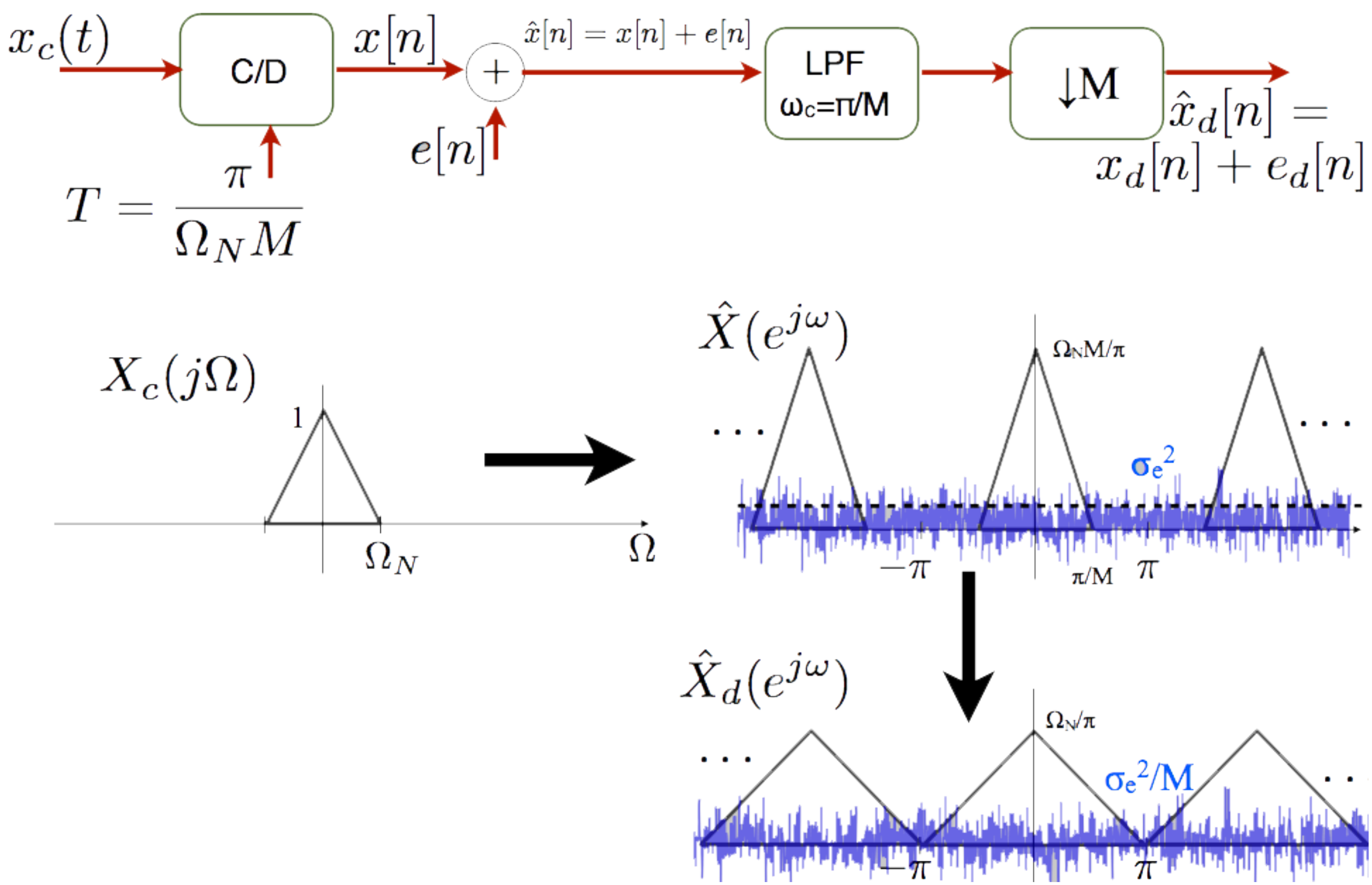


Practically:



Noise Shaping

Quantization Noise with Oversampling



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- Energy of $x_d[n]$ equals energy of x[n]
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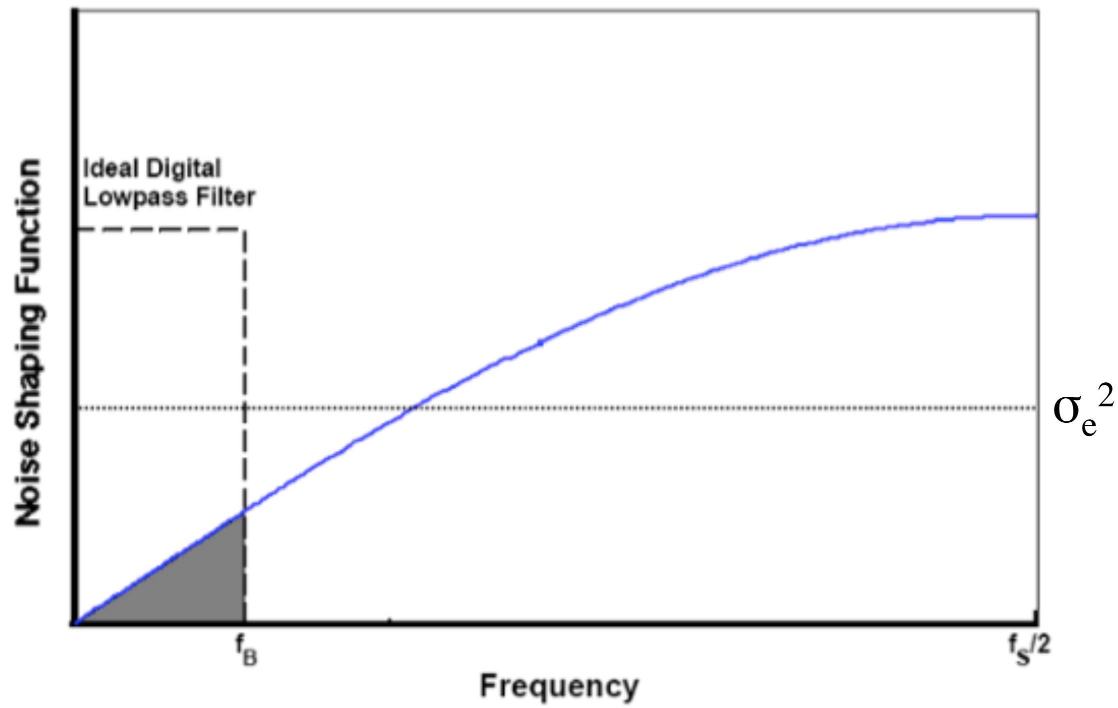
$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right) + 10\log_{10} M$$

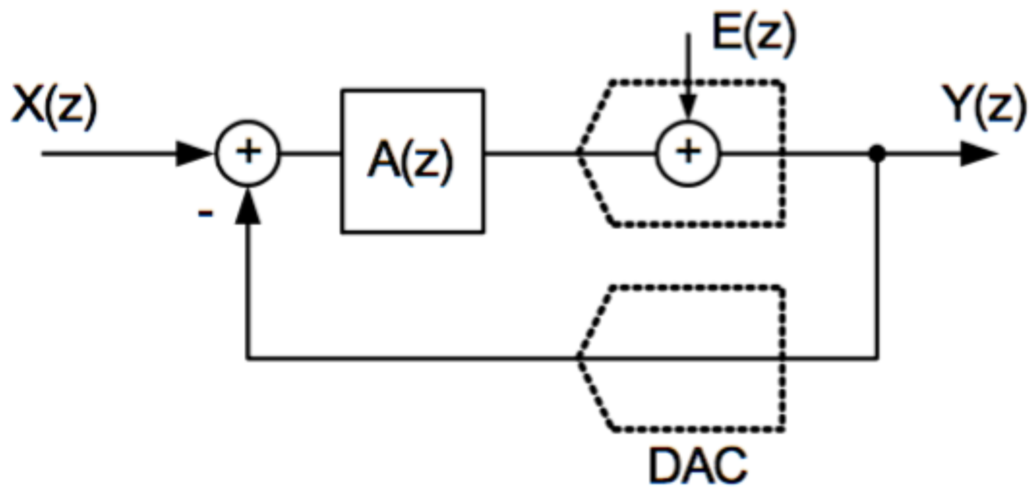
- For doubling of M we get 3dB improvement, which is the same as 1/2 a bit of accuracy
  - With oversampling of 16 with 8bit ADC we get the same quantization noise as 10bit ADC!

Noise Shaping

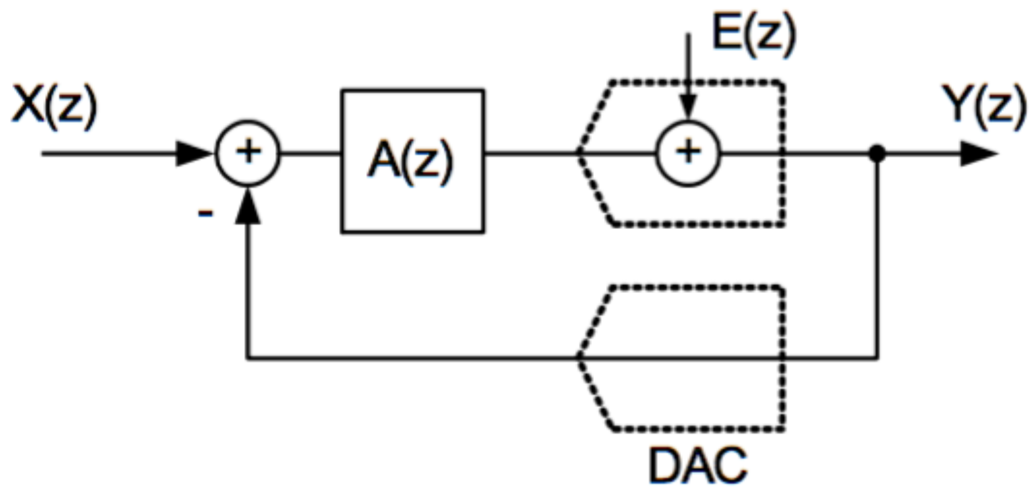


- Idea: "Somehow" build an ADC that has most of its quantization noise at high frequencies
- Key: Feedback

Noise Shaping Using Feedback



Noise Shaping Using Feedback



$$Y(z) = E(z) + A(z)X(z) - A(z)Y(z)$$

$$= E(z)\frac{1}{1+A(z)} + X(z)\frac{A(z)}{1+A(z)}$$

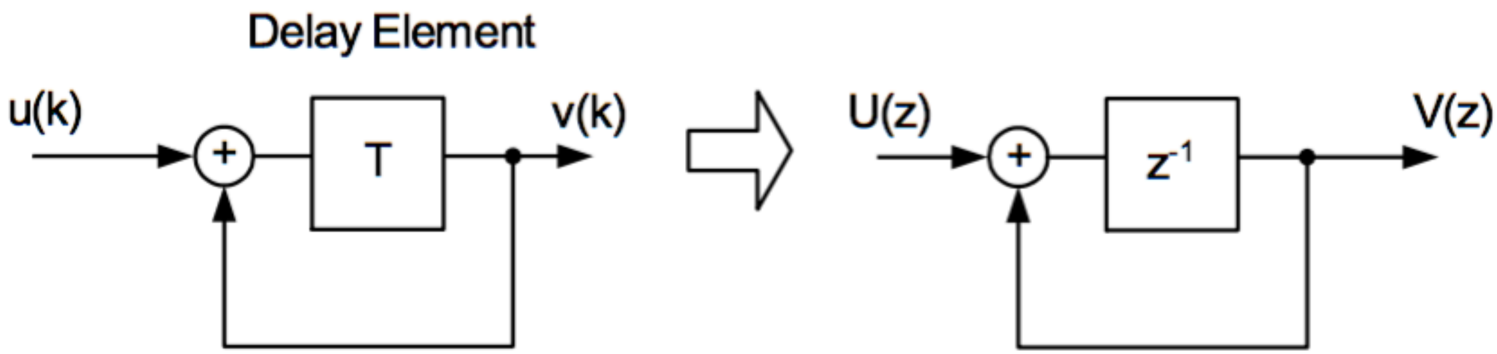
$$= E(z)\underbrace{H_E(z)}_{\substack{\text{Noise}\\\text{Transfer}\\\text{Function}}} + X(z)\underbrace{H_X(z)}_{\substack{\text{Signal}\\\text{Transfer}\\\text{Function}}}$$

Noise Shaping Using Feedback

$$Y(z) = E(z)\underbrace{\frac{1}{1+A(z)}}_{\substack{\text{Noise}\\ \text{Transfer}\\ \text{Function}}} + X(z)\underbrace{\frac{A(z)}{1+A(z)}}_{\substack{\text{Signal}\\ \text{Transfer}\\ \text{Function}}}$$

- Objective
  - Want to make STF unity in the signal frequency band
  - Want to make NTF "small" in the signal frequency band
- If the frequency band of interest is around DC (0...f_B) we achieve this by making $|A(z)| >> 1$ at low frequencies
  - Means that NTF << 1
  - Means that STF ≅ 1

Discrete Time Integrator

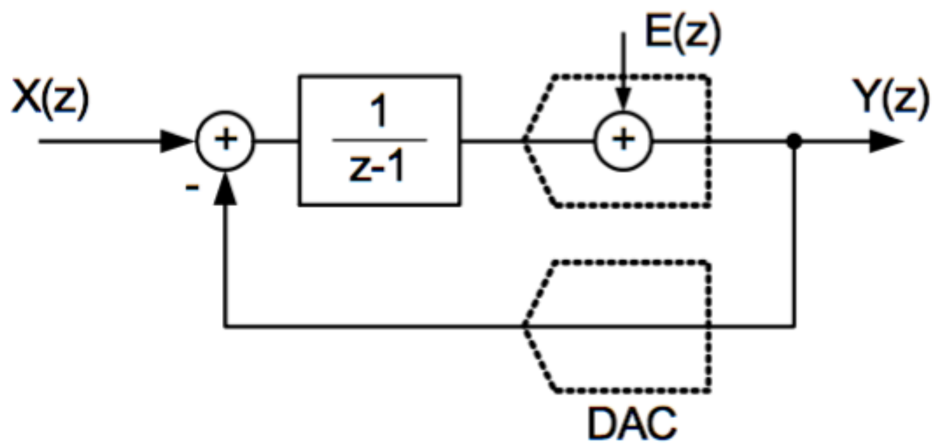


$$v(k) = u(k-1) + v(k-1) \qquad V(z) = z^{-1}U(z) + z^{-1}V(z)$$

$$\frac{V(z)}{U(z)} = \frac{z^{-1}}{1-z^{-1}} = \frac{1}{z-1} \qquad z = e^{j\omega T}$$

- "Infinite gain" at DC ($\omega=0$, z=1)

First Order Sigma-Delta Modulator



$$Y(z) = E(z)\frac{1}{1+\frac{1}{z-1}} + X(z)\frac{\frac{1}{z-1}}{1+\frac{1}{z-1}}$$

$$= E(z)\left(1-z^{-1}\right) + X(z)z^{-1}$$

- Output is equal to delayed input plus filtered quantization noise

NTF Frequency Domain Analysis

$$H_e(z) = 1 - z^{-1}$$

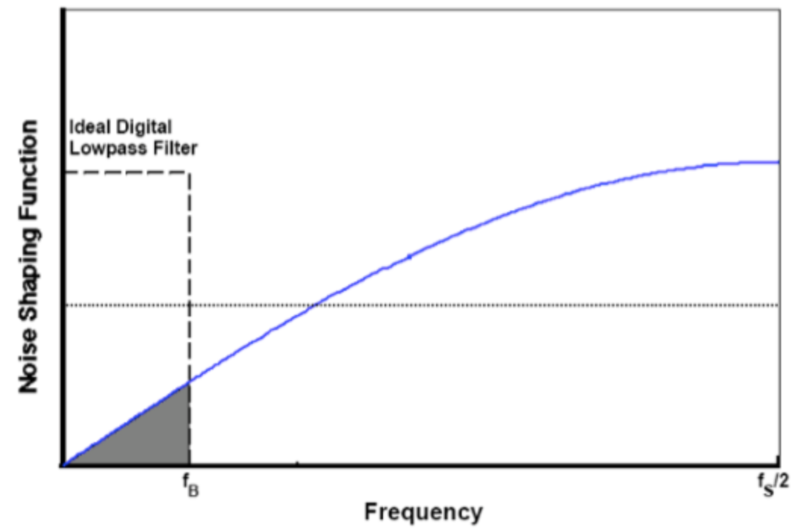
NTF Frequency Domain Analysis

$$H_e(z) = 1 - z^{-1}$$

$$H_e(j\omega) = \left(1 - e^{-j\omega T}\right) = 2e^{-j\omega T/2}\left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2}\right)$$

$$= 2e^{-j\frac{\omega T}{2}}\left(j\,sin\left(\frac{\omega T}{2}\right)\right) = 2\,sin\left(\frac{\omega T}{2}\right)e^{-j\frac{\omega T - \pi}{2}}$$

$$\left|H_e(f)\right| = 2\left|sin(\pi fT)\right| = 2\left|sin\left(\pi\frac{f}{f_s}\right)\right|$$

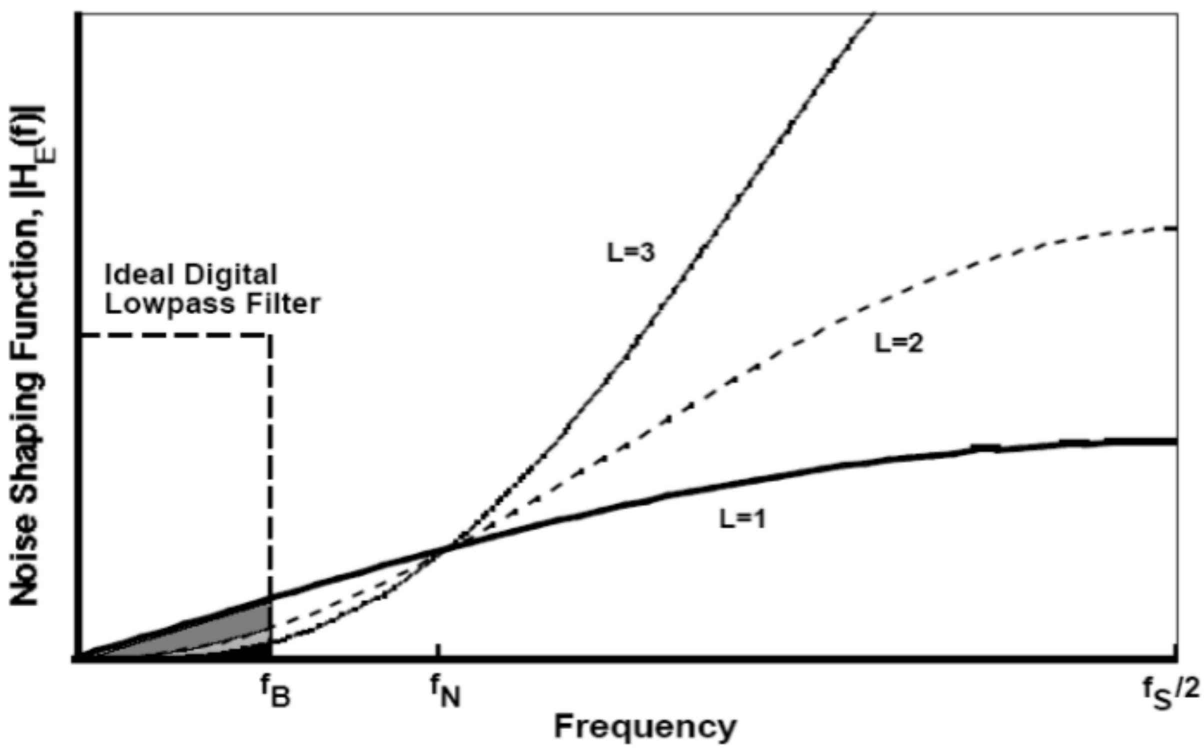


- "First order noise Shaping"
  - Quantization noise is attenuated at low frequencies, amplified at high frequencies

Higher Order Noise Shaping

- L^{th} order noise transfer function

$$H_E(z) = \left(1 - z^{-1}\right)^L$$



In-Band Quantization Noise

- Question: If we had an ideal digital lowpass, what is the achieved SQNR as a function of oversampling ratio?
- Can integrate shaped quantization noise spectrum up to f_B and compare to full-scale signal

$$P_{qnoise} = \int_0^{f_B} \frac{\Delta^2}{12} \cdot \frac{2}{f_s} \cdot \left[2\sin\left(\pi \frac{f}{f_s} \right) \right]^2 df$$

$$\cong \int_0^{f_B} \frac{\Delta^2}{12} \cdot \frac{2}{f_s} \cdot \left[2\pi \frac{f}{f_s} \right]^2 df$$

$$\cong \frac{\Delta^2}{12} \cdot \frac{\pi^2}{3} \left[\frac{2f_B}{f_s} \right]^3 = \frac{\Delta^2}{12} \cdot \frac{\pi^2}{3} \frac{1}{M^3}$$

In-Band Quantization Noise

- Assuming a full-scale sinusoidal signal, we have

$$SQNR \cong \frac{P_{sig}}{P_{qnoise}} = \frac{\frac{1}{2}\left(\frac{(2^B-1)\Delta}{2}\right)^2}{\frac{\Delta^2}{12}\cdot\frac{\pi^2}{3}\frac{1}{M^3}} = 1.5\times\left(2^B-1\right)^2\times\underbrace{\frac{3}{\pi^2}\times M^3}_{\substack{\text{Due to noise}\\ \text{shaping \&}\\ \text{digital filter}}}$$

$$\cong 1.76 + 6.02B - 5.2 + 30\log(M) \quad [dB] \quad (\text{for large } B)$$

- Each 2x increase in M results in 8x SQNR improvement
  - Also added ½ bit resolution

Digital Noise Filter

- Increasing M by 2x, means 3-dB reduction in quantization noise power, and thus 1/2 bit increase in resolution
  - "1/2 bit per octave"
- Is this useful?
- Reality check
  - Want 16-bit ADC, f_B=1MHz
  - Use oversampled 8-bit ADC with digital lowpass filter
  - 8-bit increase in resolution necessitates oversampling by 16 octaves

$$f_s \geq 2 \cdot f_B \cdot M = 2 \cdot 1MHz \cdot 2^{16}$$
$$\geq 131GHz$$

SQNR Improvement

- Example Revisited
  - Want 16-bit ADC, f_B=1MHz
  - Use oversampled 8-bit ADC, first order noise shaping and (ideal) digital lowpass filter
    - SQNR improvement compared to case without oversampling is -5.2dB+30log(M)
  - 8-bit increase in resolution (48 dB SQNR improvement) would necessitate $M \cong 60 \rightarrow f_S$=120MHz
- Not all that bad!

M	SQNR improvement
16	31dB (~5 bits)
256	67dB (~11 bits)
1024	85dB (~14 bits)

Big Ideas

- Quantizers
  - Introduces quantization noise
- Data Converters
  - Oversampling to reduce interference and quantization noise → increase ENOB (effective number of bits)
  - Practical DACs use practical interpolation and reconstruction filters with oversampling
- Noise Shaping
  - Use feedback to reduce oversampling factor

Admin

- HW 5 due Sunday
- Midterm Next week Th 3/5
  - During class
    - Starts at exactly 4:30pm, ends at exactly 5:50pm (80 minutes)
  - Location DRLB A2
  - Old exams posted on previous years' websites
    - Disclaimer: exams before 2020 covered more material
  - Covers Lec 1- 11
  - Closed book, one (8.5x11 front/back) page cheat sheet allowed
  - Calculators allowed, no smart phones