

ESE 531: Digital Signal Processing

Lec 13: February 27, 2020
Frequency Response of LTI Systems

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Lecture Outline

- Frequency Response of LTI Systems
 - Magnitude Response
 - Simple Filters
 - Phase Response
 - Group Delay
 - Example: Zero on Real Axis

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Frequency Response of LTI Systems

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Frequency Response of LTI System

- LTI Systems are uniquely determined by their impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[k] * h[k]$$

- We can write the input-output relation also in the z-domain

$$Y(z) = H(z)X(z)$$

- Or we can define an LTI system with its frequency response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- $H(e^{j\omega})$ defines magnitude and phase change at each frequency

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Frequency Response of LTI System

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- We can define a magnitude response

$$|Y(e^{j\omega})| = |H(e^{j\omega})| |X(e^{j\omega})|$$

- And a phase response

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

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Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$

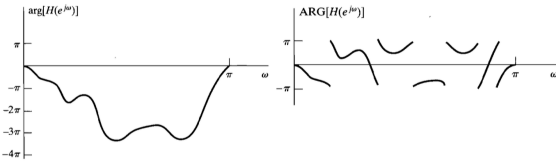
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Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

- More later...

Unwrapped phase

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Linear Difference Equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

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Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

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Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$

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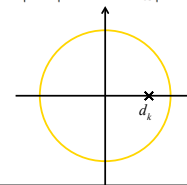
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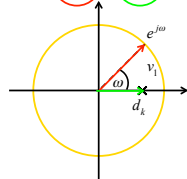
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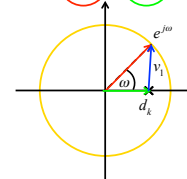
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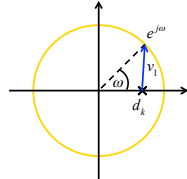
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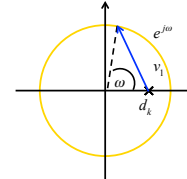
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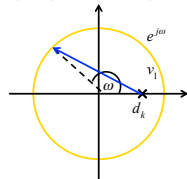
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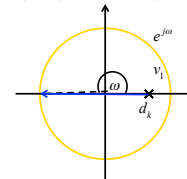
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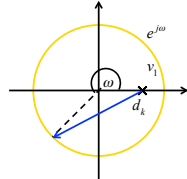
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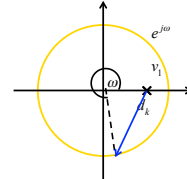
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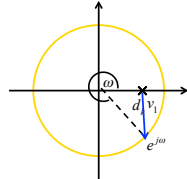
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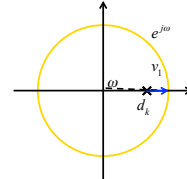
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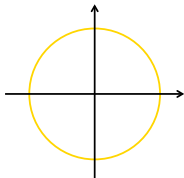
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Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



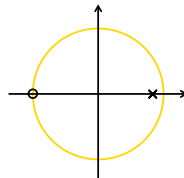
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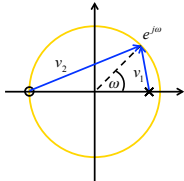
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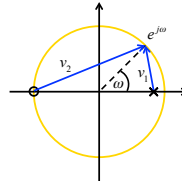
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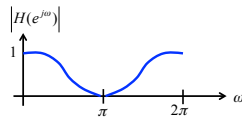
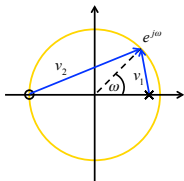
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Simple Low Pass Filter

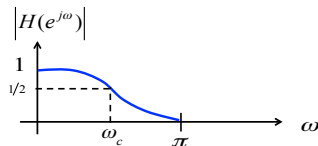
$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$

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Simple Low Pass Filter

$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$



$$\omega_c \text{ is the 3dB cutoff frequency} \quad \alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$$

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Simple High Pass Filter

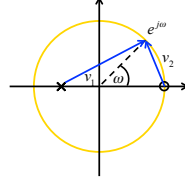
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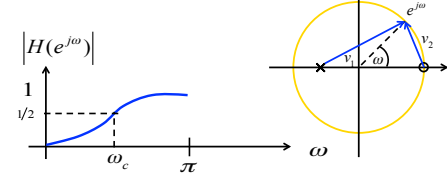


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ω_c is the 3dB cutoff frequency $\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$

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Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$

Note: $1-2\beta z^{-1}+z^{-2} = (1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})$
 $\cos(\omega_0) = \beta$

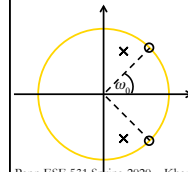
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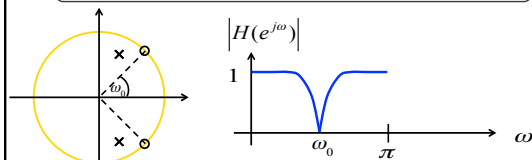
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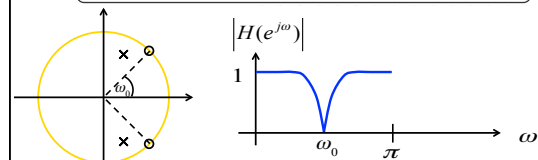
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Note:

$$H_{BS}(\mp 1) = \frac{1+\alpha}{2} \frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$$



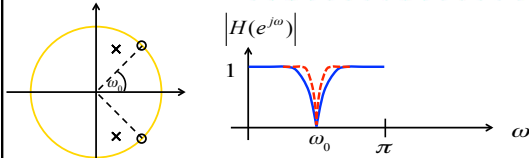
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Note: As $\alpha \rightarrow 1$ poles approach zeros
 $H_{BS}(\mp 1) = \frac{1+\alpha}{2} \frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$



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Simple Band-Pass Filter

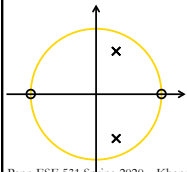
$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad \begin{matrix} |\alpha| < 1 \\ |\beta| < 1 \end{matrix}$$

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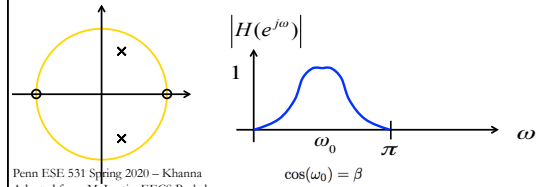


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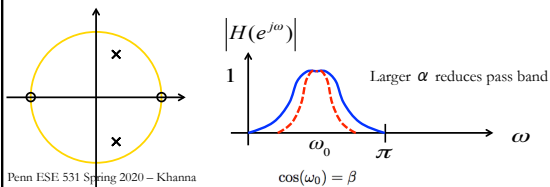


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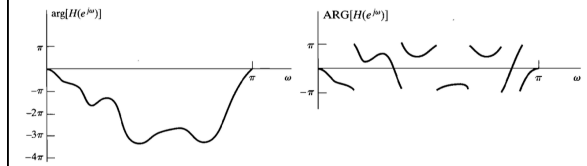
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Phase Response

Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$



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Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

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Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

ARG is the wrapped phase
arg is the unwrapped phase

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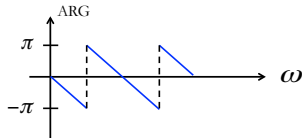
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arg is the unwrapped phase



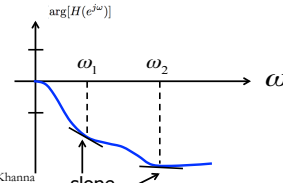
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Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$



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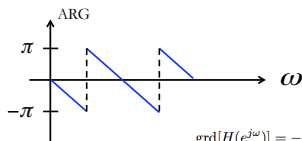
Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

ARG is the wrapped phase
arg is the unwrapped phase



$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

For linear phase system, group delay is n_d

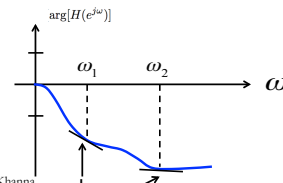
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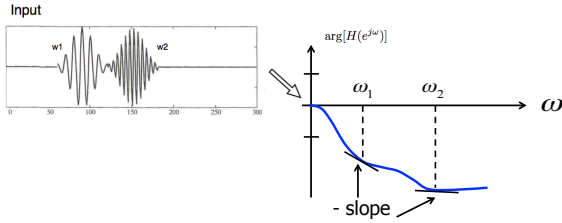


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Group Delay

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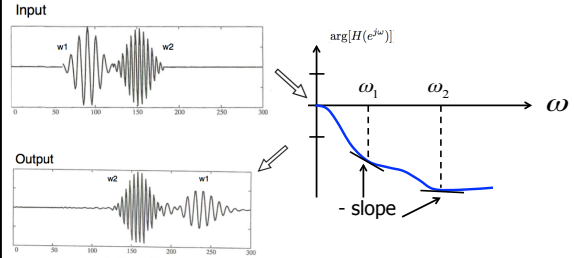


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Group Delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



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Group Delay Math

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})} \quad H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^M (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^N (1 - d_k e^{-j\omega})}$$

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Group Delay Math

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arg of products is sum of args

$$\arg[H(e^{j\omega})] = \sum_{k=1}^M \arg[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \arg[1 - d_k e^{-j\omega}]$$

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

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Look at each factor:

$$\arg[1 - r e^{j\theta} e^{-j\omega}] = \tan^{-1} \left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right)$$

$$\text{grd}[1 - r e^{j\theta} e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - r e^{j\theta} e^{-j\omega}|^2}$$

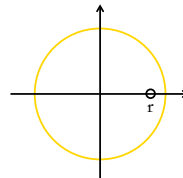
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Example: Zero on Real Axis

Geometric Interpretation for ($\theta = 0$)

$$\arg[1 - r e^{-j\omega}]$$



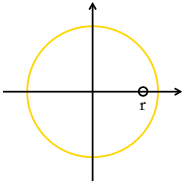
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Example: Zero on Real Axis

- Geometric Interpretation for ($\theta = 0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{-j\omega}]}_{\omega}$$



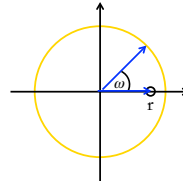
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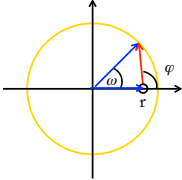
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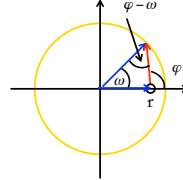
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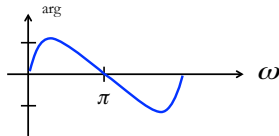
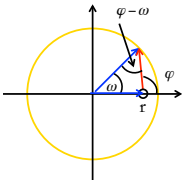
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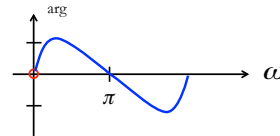
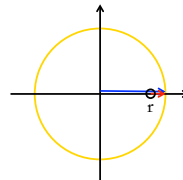
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$\omega = 0$



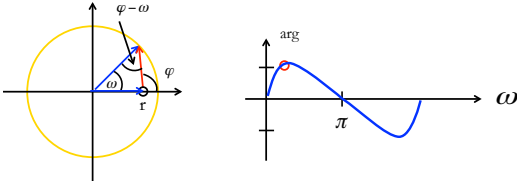
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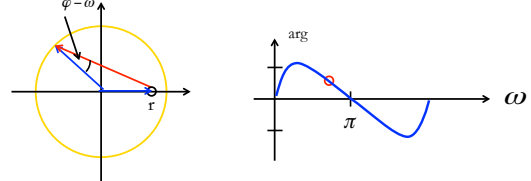
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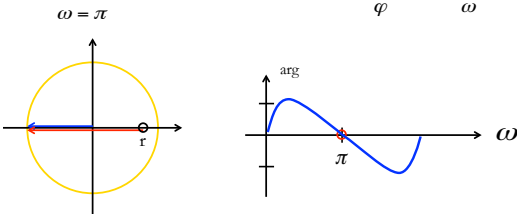
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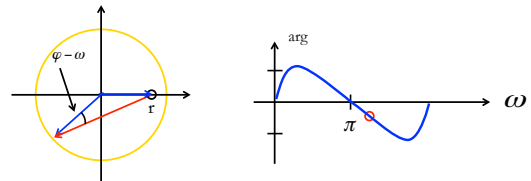
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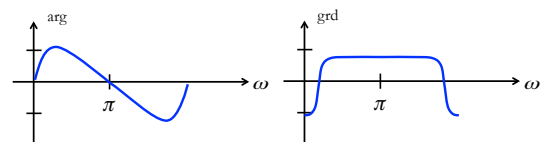
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Group Delay Math

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- Look at each factor:

$\theta \neq 0$?

$$\arg[1 - re^{j\theta} e^{-j\omega}] = \tan^{-1} \left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right)$$

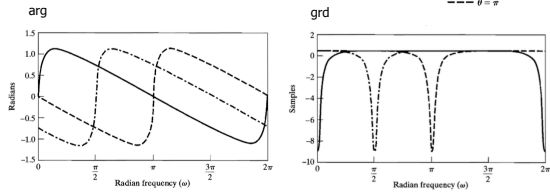
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Example: Zero on Real Axis

□ For $\theta \neq 0$



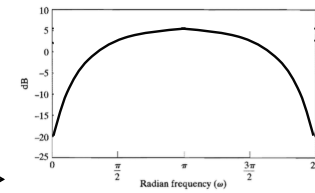
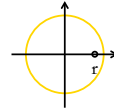
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Example: Zero on Real Axis

□ Magnitude Response

$$1 - re^{j\theta}e^{-j\omega} = 1 - re^{-j\omega}$$



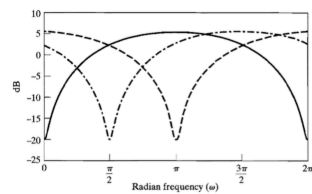
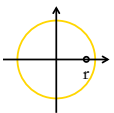
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Example: Zero on Real Axis

□ Magnitude Response

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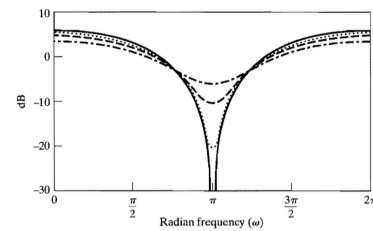
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Example: Zero on Real Axis

□ For $\theta = \pi$, how does zero location effect magnitude, phase and group delay?

$$1 - re^{j\theta}e^{-j\omega}$$

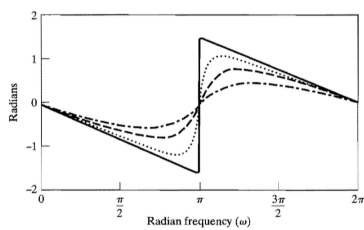


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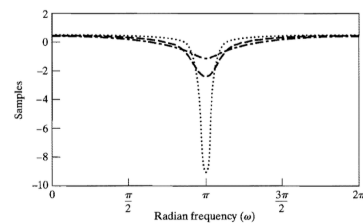


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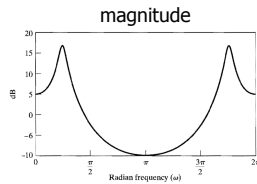


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2nd Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} \quad r=0.9, \theta = \pi/4$$

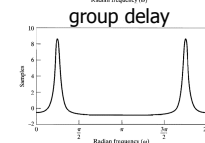
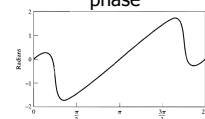
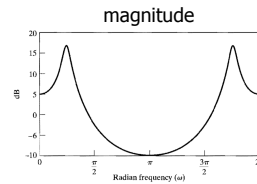


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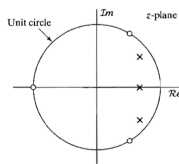


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3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$

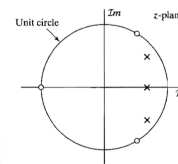
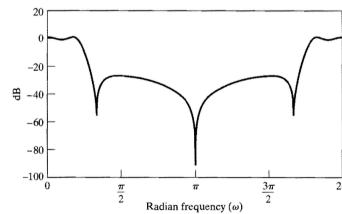


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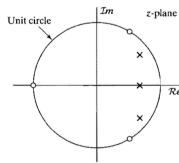
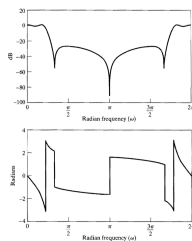


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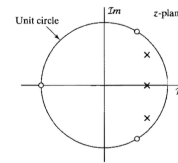
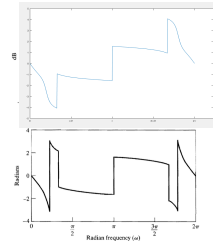


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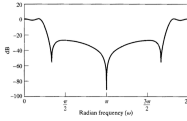
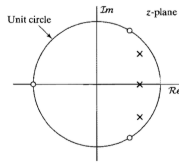
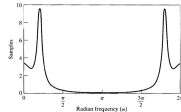
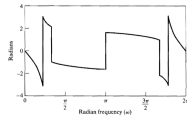


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Big Ideas

Frequency Response of LTI Systems

- Magnitude Response
 - Simple Filters
- Phase Response
 - Group Delay
- Example: Zero on Real Axis

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Admin

- HW 5 due Sunday
- HW 1-4 grading backlog
 - Will have all grades by Sunday
- Dhaval extra OH on Tu evening for exam prep
 - See Piazza for time and location
 - Dhaval TH/F OH cancelled
- Yinghao OH W will be exam review session
- No office hours during spring break

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Admin - Midterm

- Midterm next week Thursday 3/5
 - During class
 - Starts at **exactly 4:30pm**, ends at exactly 5:50pm (80 minutes)
 - Location DRLB A2
 - Old exams posted on previous years' website
 - Disclaimer: old exams before 2019 covered more material
 - Covers Lec 1 - 11
 - Closed book, one page (8.5x11) cheat sheet allowed
 - Calculators allowed, no smart phones

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