ESE 531: Digital Signal Processing

Lec 14: March 3, 2020

All-Pass Systems and Min Phase Decomposition

Penn ESE 531 Spring 2020 – Khanna
Adapted from M. Lustig, EECS Berkeley

Lecture Outline

- Frequency Response of LTI Systems
  - Examples:
    - Zero on Real Axis
    - 2nd order IIR
    - 3rd order Low Pass
- Stability and Causality
- All Pass Systems
- Minimum Phase Systems (If time)

Review: Frequency Response of LTI System

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- We can define a magnitude response…

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right|\left|X\left(e^{j\omega}\right)\right|$$

- a phase response…

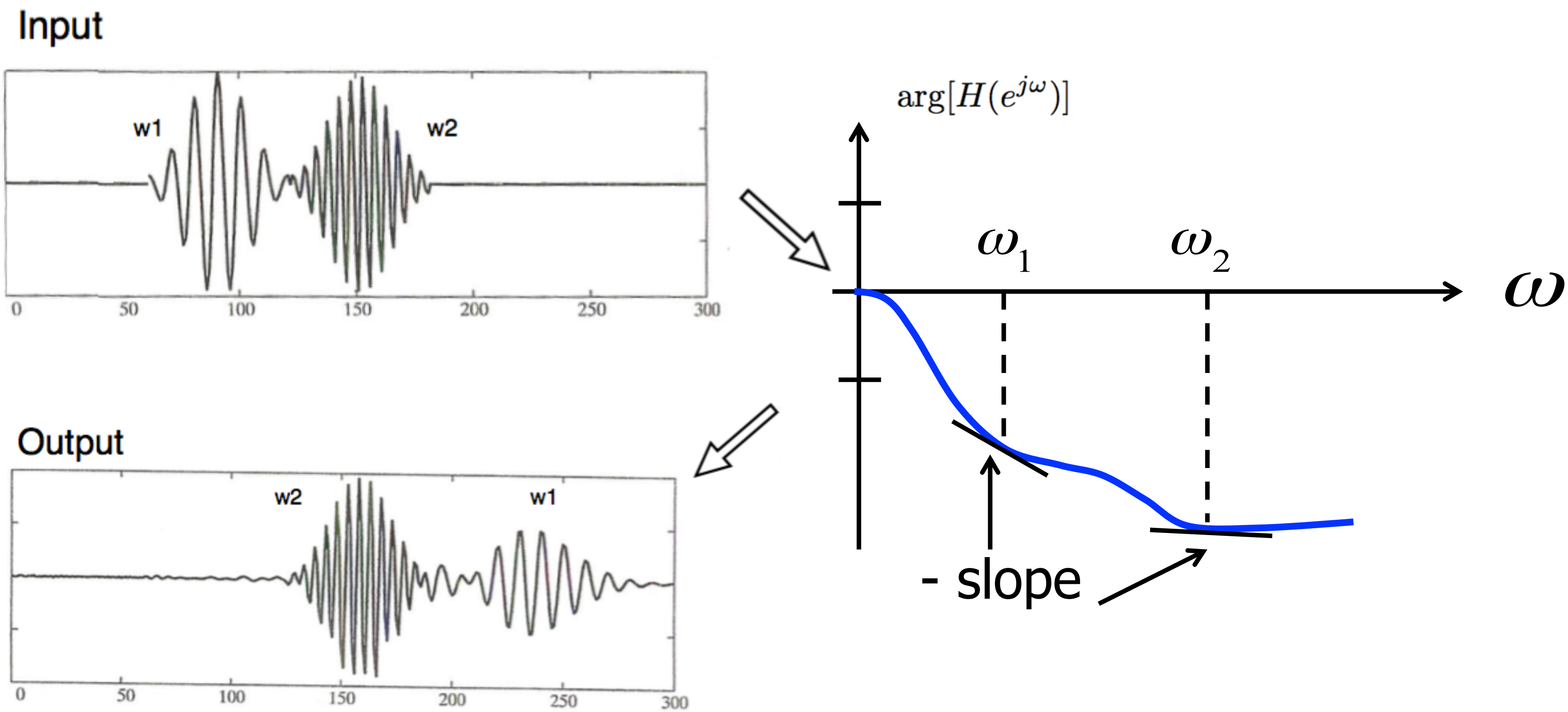
$$\angle Y\left(e^{j\omega}\right) = \angle H\left(e^{j\omega}\right) + \angle X\left(e^{j\omega}\right)$$

- and group delay

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Group Delay

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



Group Delay Math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$$H(e^{j\omega}) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

Group Delay Math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} \qquad H(e^{j\omega}) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}$$

arg of products is sum of args

$$\arg[H(e^{j\omega})] = \sum_{k=1}^{M} \arg[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \arg[1 - d_k e^{-j\omega}]$$

$$\mathrm{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \mathrm{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \mathrm{grd}[1 - d_k e^{-j\omega}]$$

Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \text{grd}[1 - d_k e^{-j\omega}]$$

- Look at each factor:

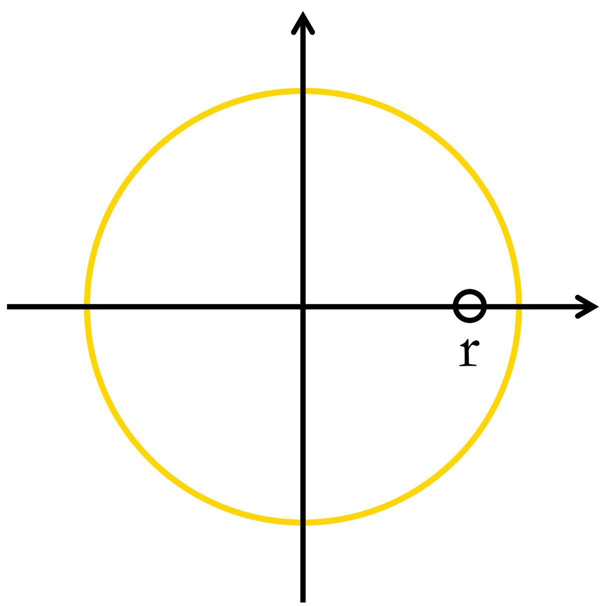
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega - \theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

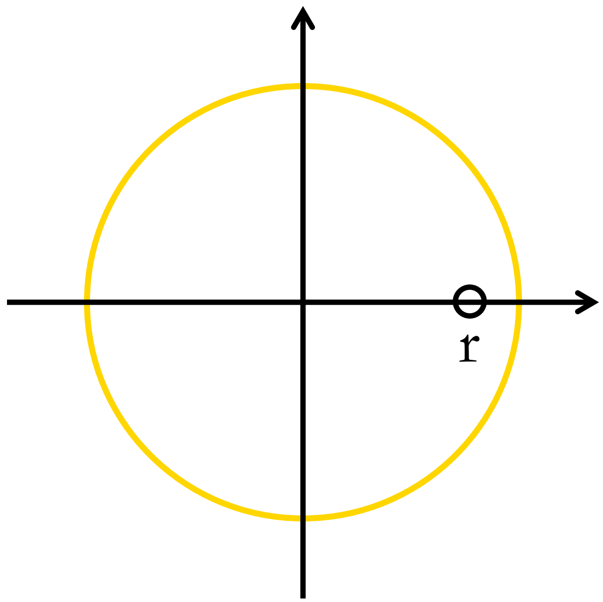
$$\arg[1 - re^{-j\omega}]$$



Example: Zero on Real Axis

- Geometric Interpretation for ($\theta = 0$)

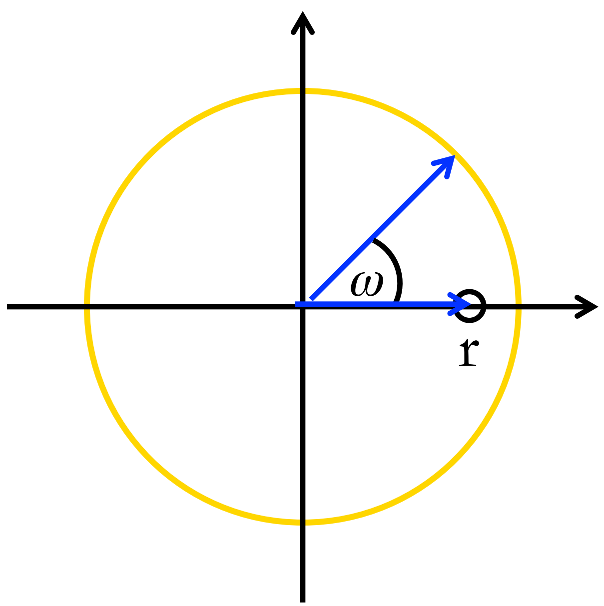
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

- Geometric Interpretation for ($\theta = 0$)

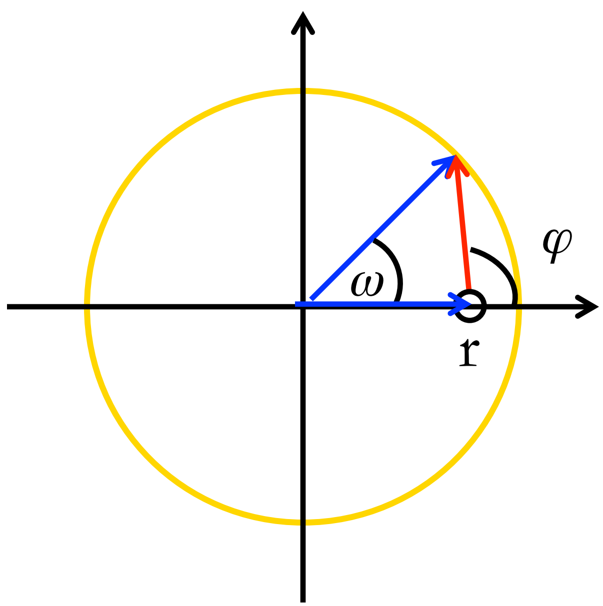
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

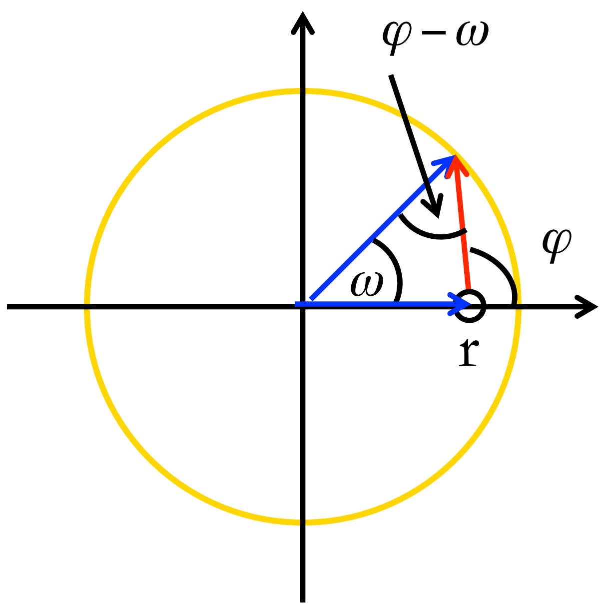
$$\arg[1-re^{-j\omega}] = \arg[(e^{j\omega}-r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega}-r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

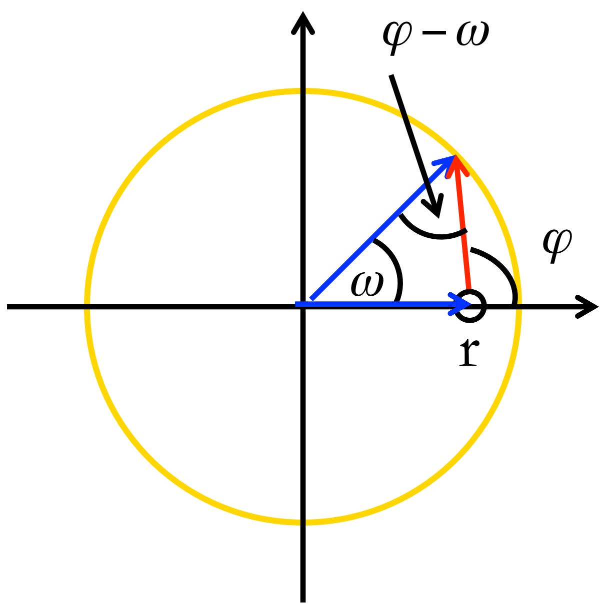
$$\arg[1-re^{-j\omega}]=\arg[(e^{j\omega}-r)e^{-j\omega}]=\underbrace{\arg[e^{j\omega}-r]}_{\varphi}-\underbrace{\arg[e^{j\omega}]}_{\omega}$$

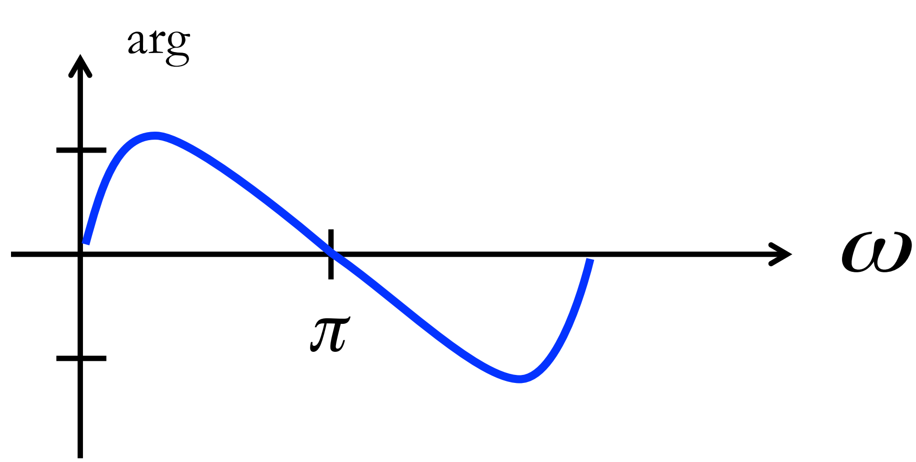


Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



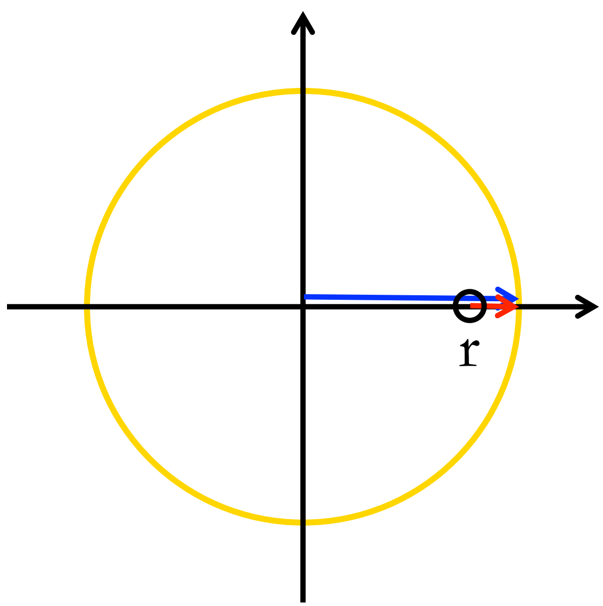


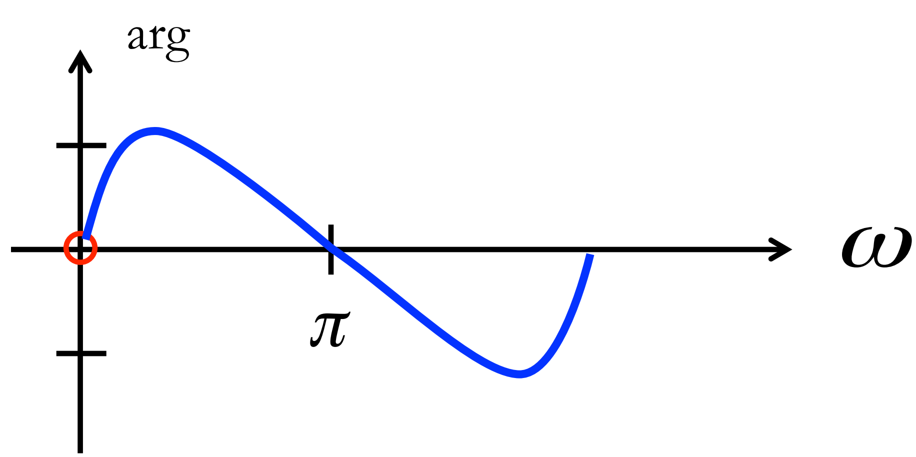
Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

$\omega = 0$



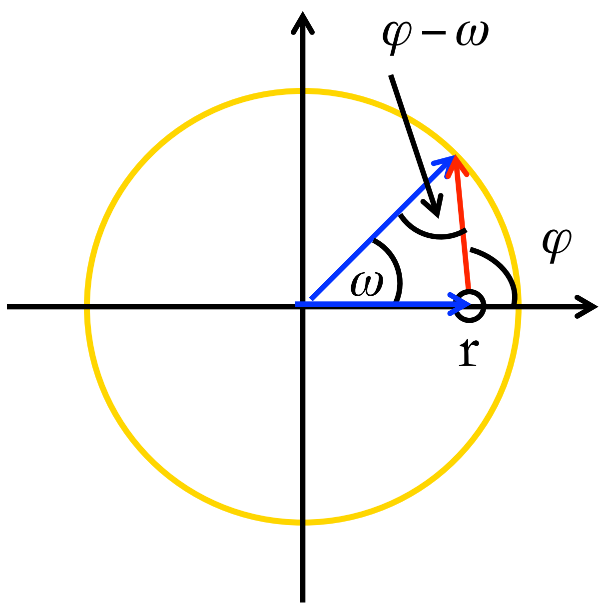


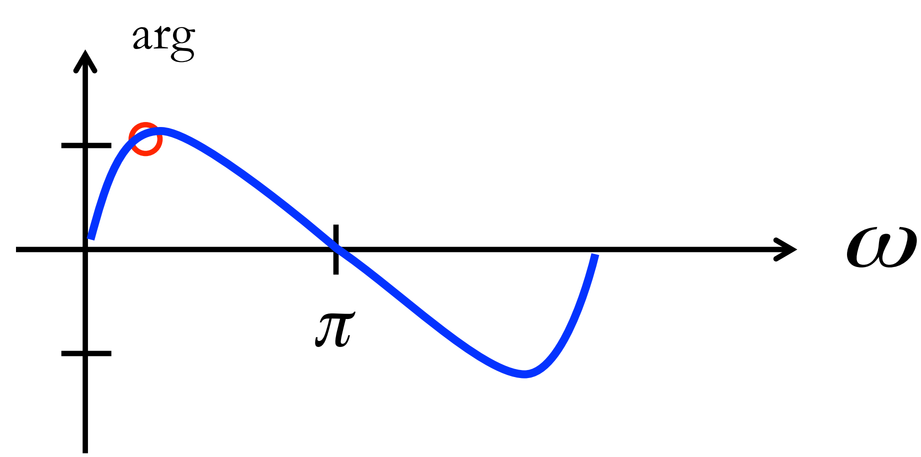
Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \arg[e^{j\omega} - r] - \arg[e^{j\omega}]$$

$$\underbrace{\arg[e^{j\omega} - r]}_{\varphi} \quad \underbrace{\arg[e^{j\omega}]}_{\omega}$$

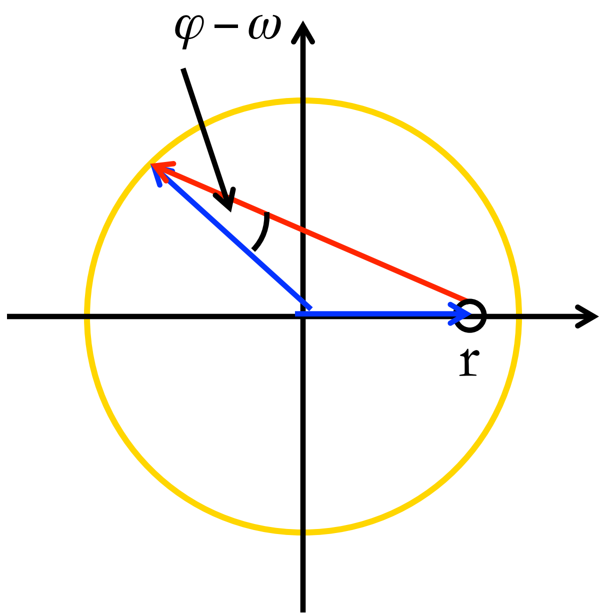


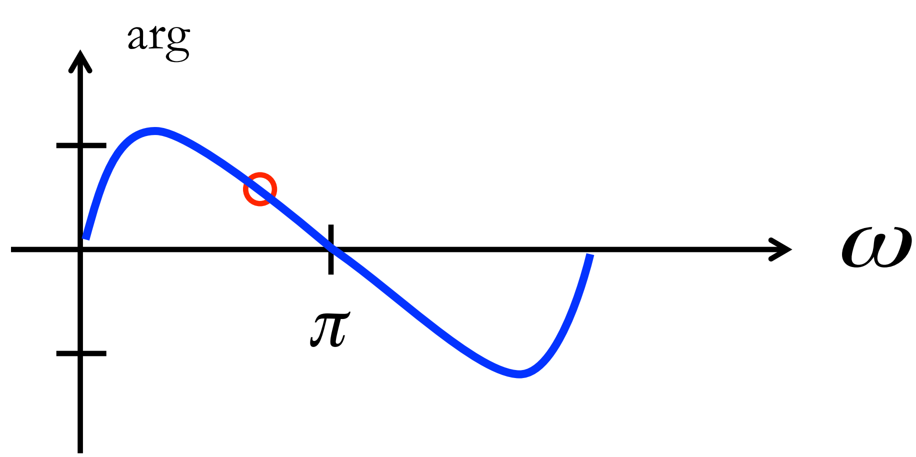


Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



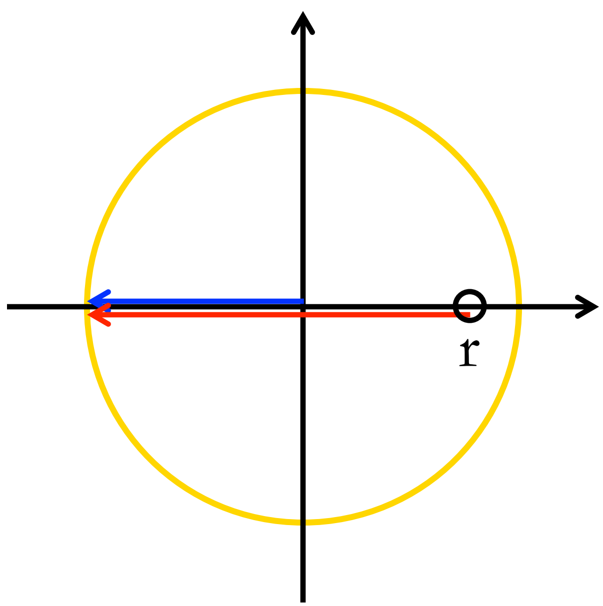


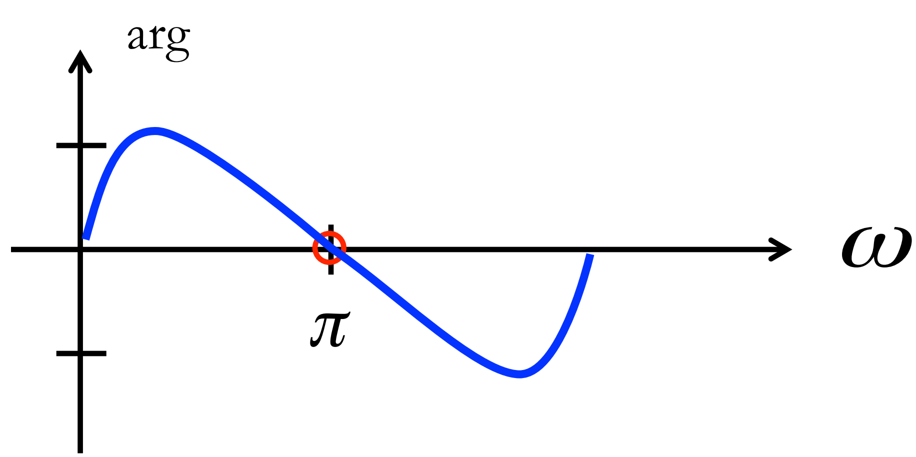
Example: Zero on Real Axis

- Geometric Interpretation for ($\theta = 0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

$\omega = \pi$

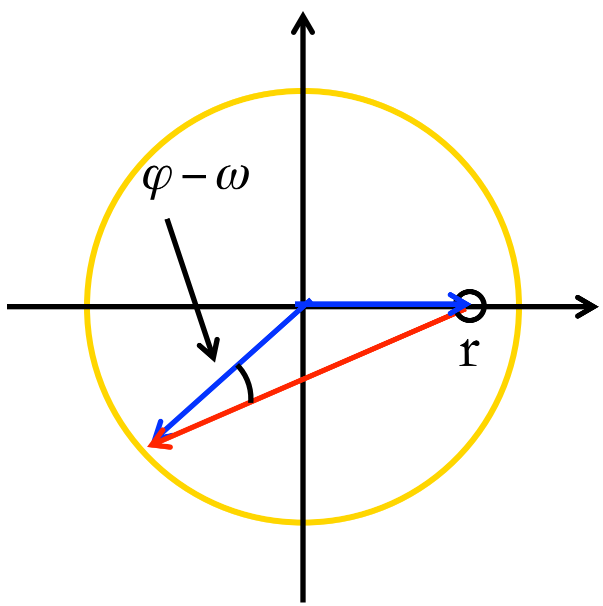


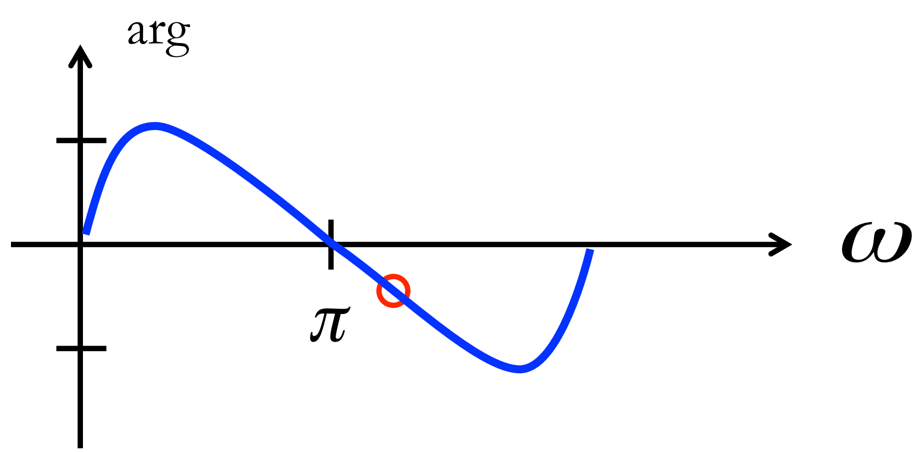


Example: Zero on Real Axis

- Geometric Interpretation for ($\theta = 0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

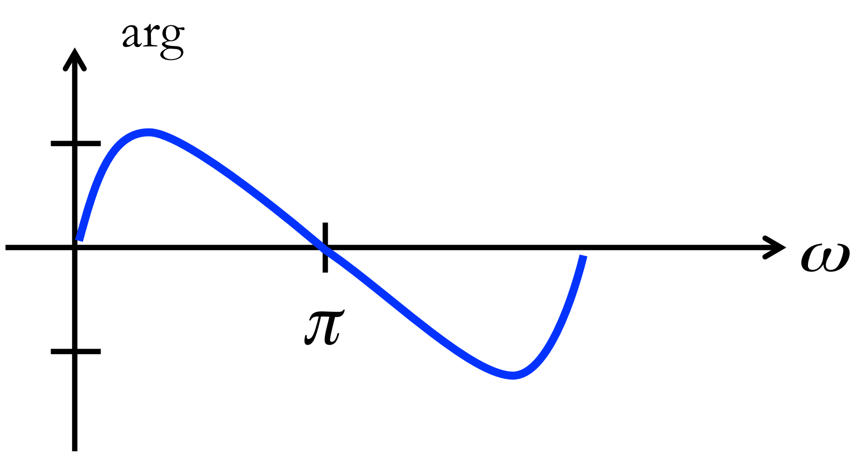


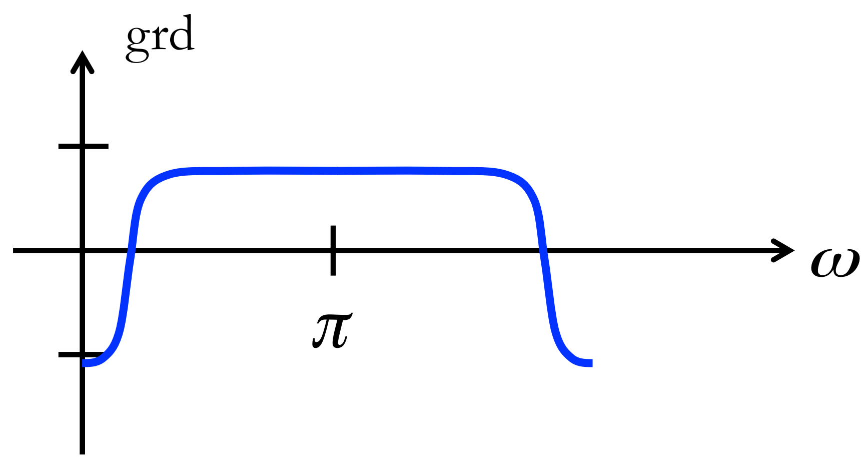


Example: Zero on Real Axis

- Geometric Interpretation for (θ =0)

$$\arg[1-re^{-j\omega}] = \arg[(e^{j\omega}-r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega}-r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

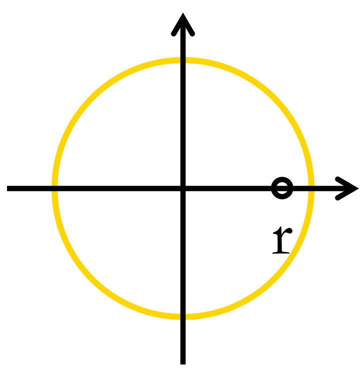


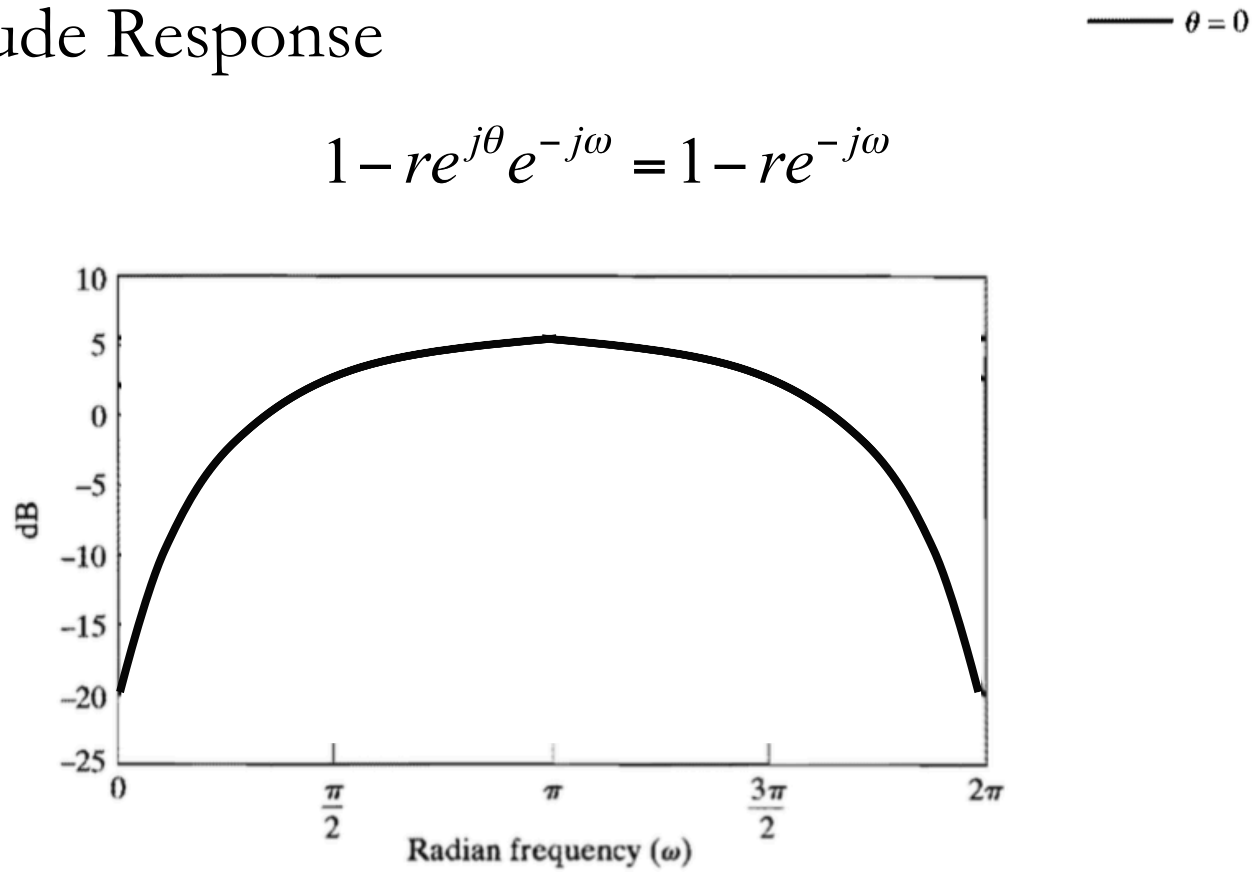


Example: Zero on Real Axis

- Magnitude Response

$$1 - re^{j\theta}e^{-j\omega} = 1 - re^{-j\omega}$$





Group Delay Math

$$\mathrm{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \mathrm{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \mathrm{grd}[1 - d_k e^{-j\omega}]$$

- Look at each factor:

θ≠0?

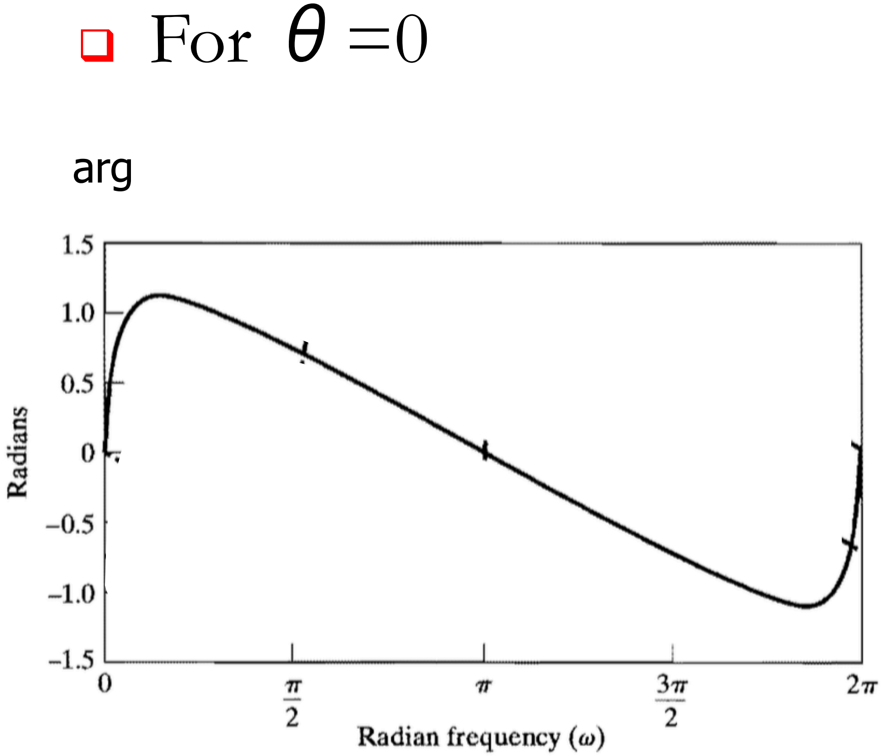
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

$$\mathrm{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

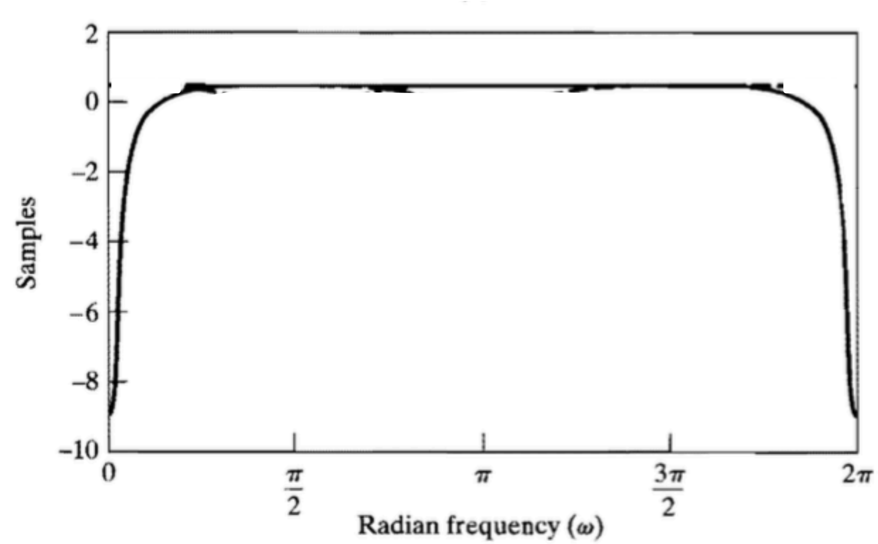
Example: Zero on Real Axis

- For $\theta = 0$

arg



grd

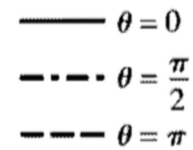


$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

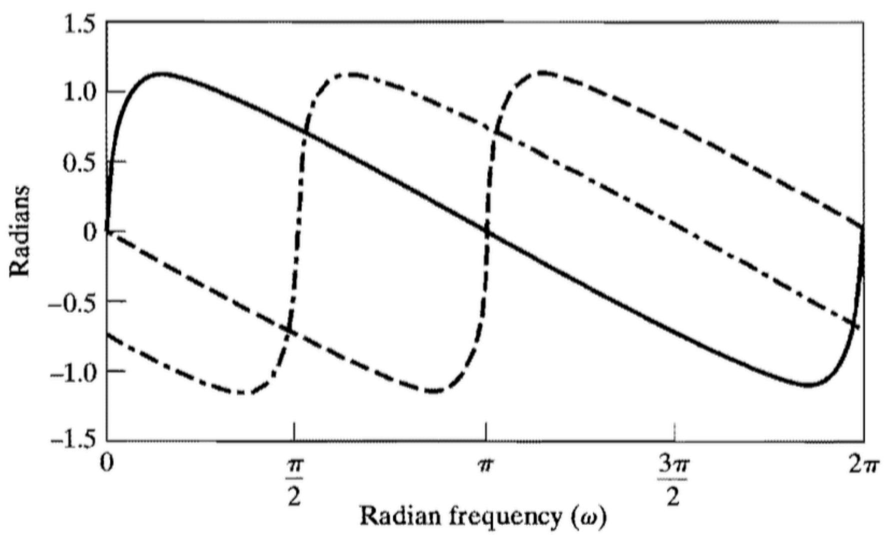
$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

Example: Zero on Real Axis

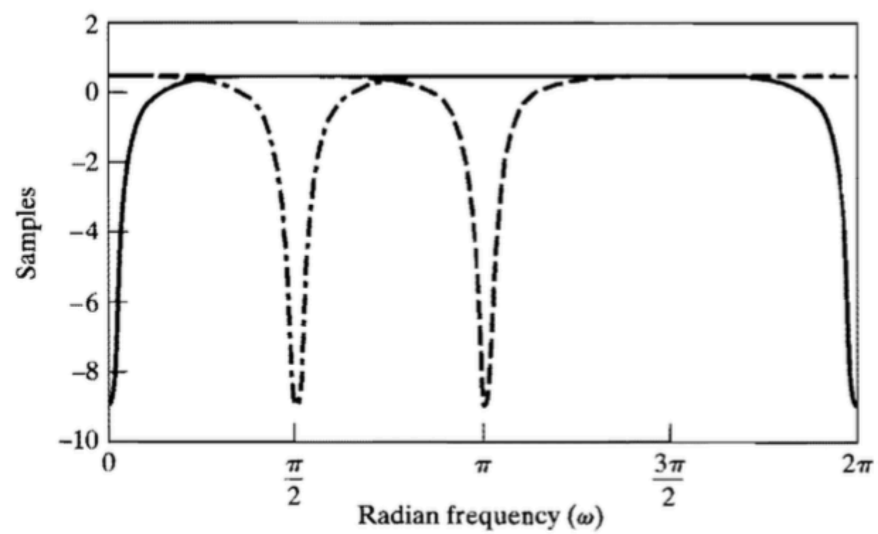
- For $\theta \neq 0$



arg



grd



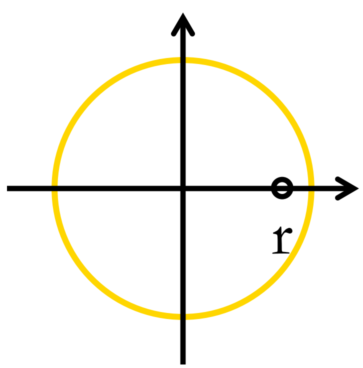
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)$$

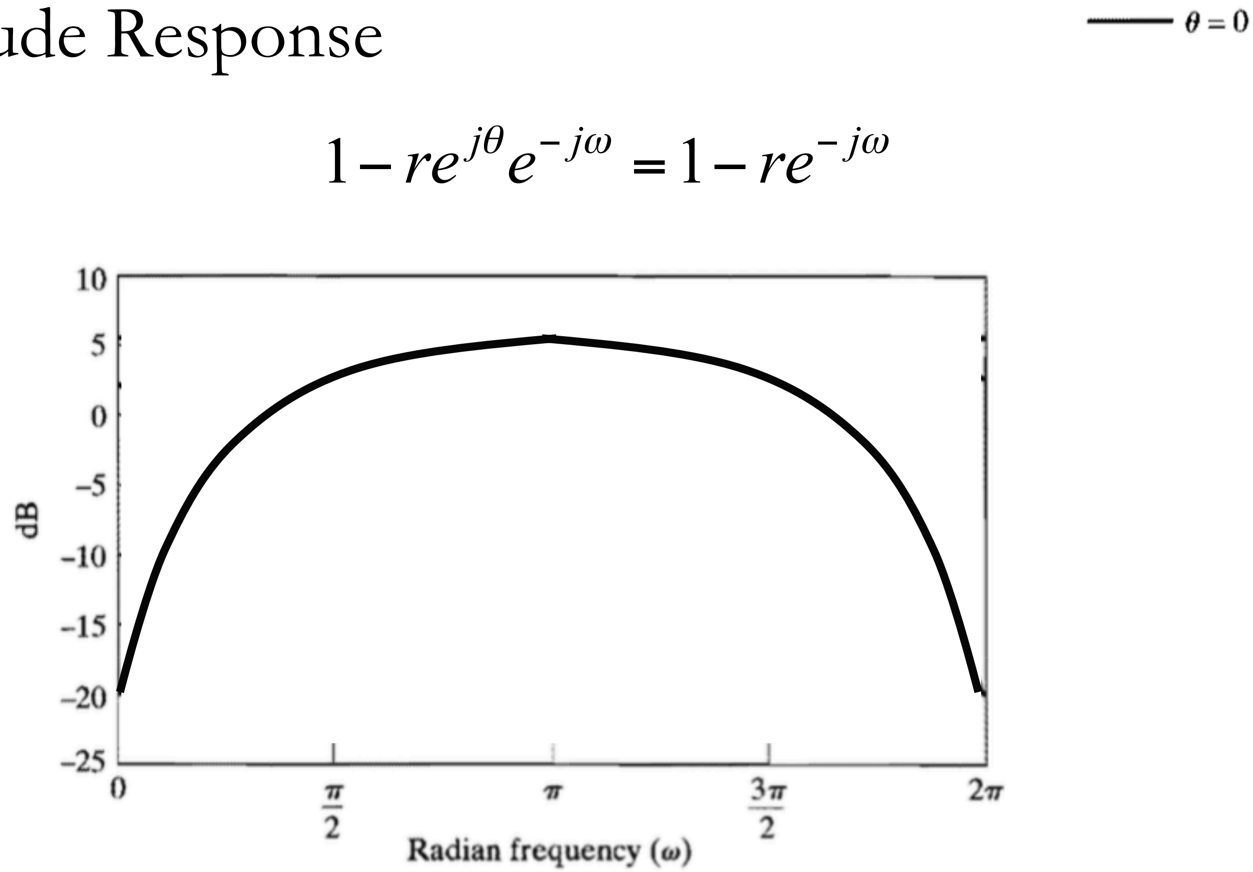
$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

Example: Zero on Real Axis

- Magnitude Response

$$1 - re^{j\theta}e^{-j\omega} = 1 - re^{-j\omega}$$

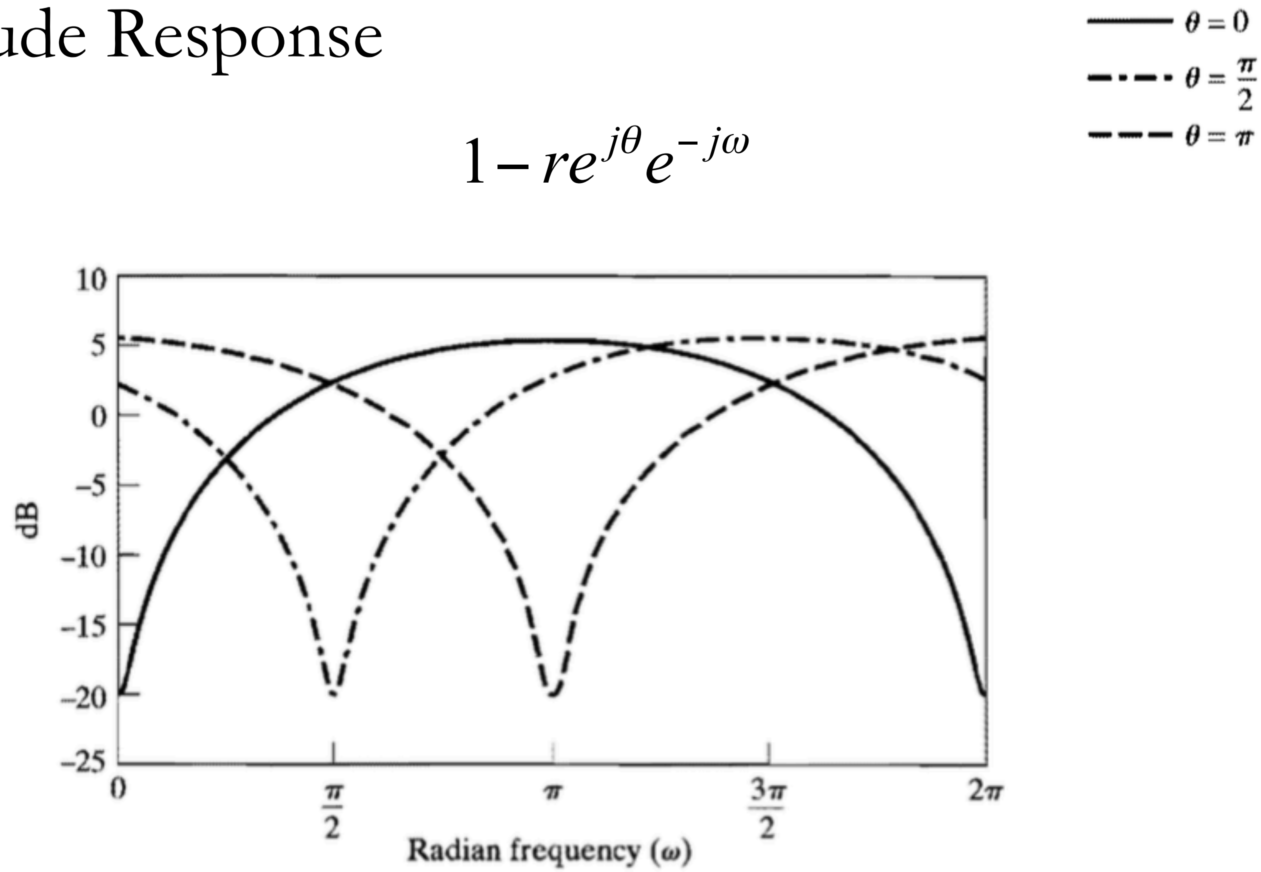


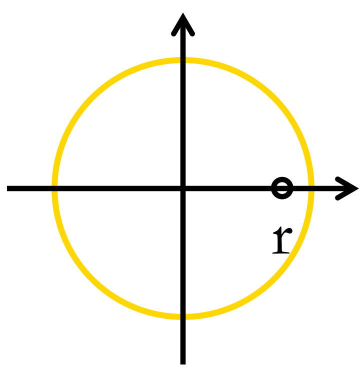


Example: Zero on Real Axis

- Magnitude Response

$$1 - re^{j\theta}e^{-j\omega}$$

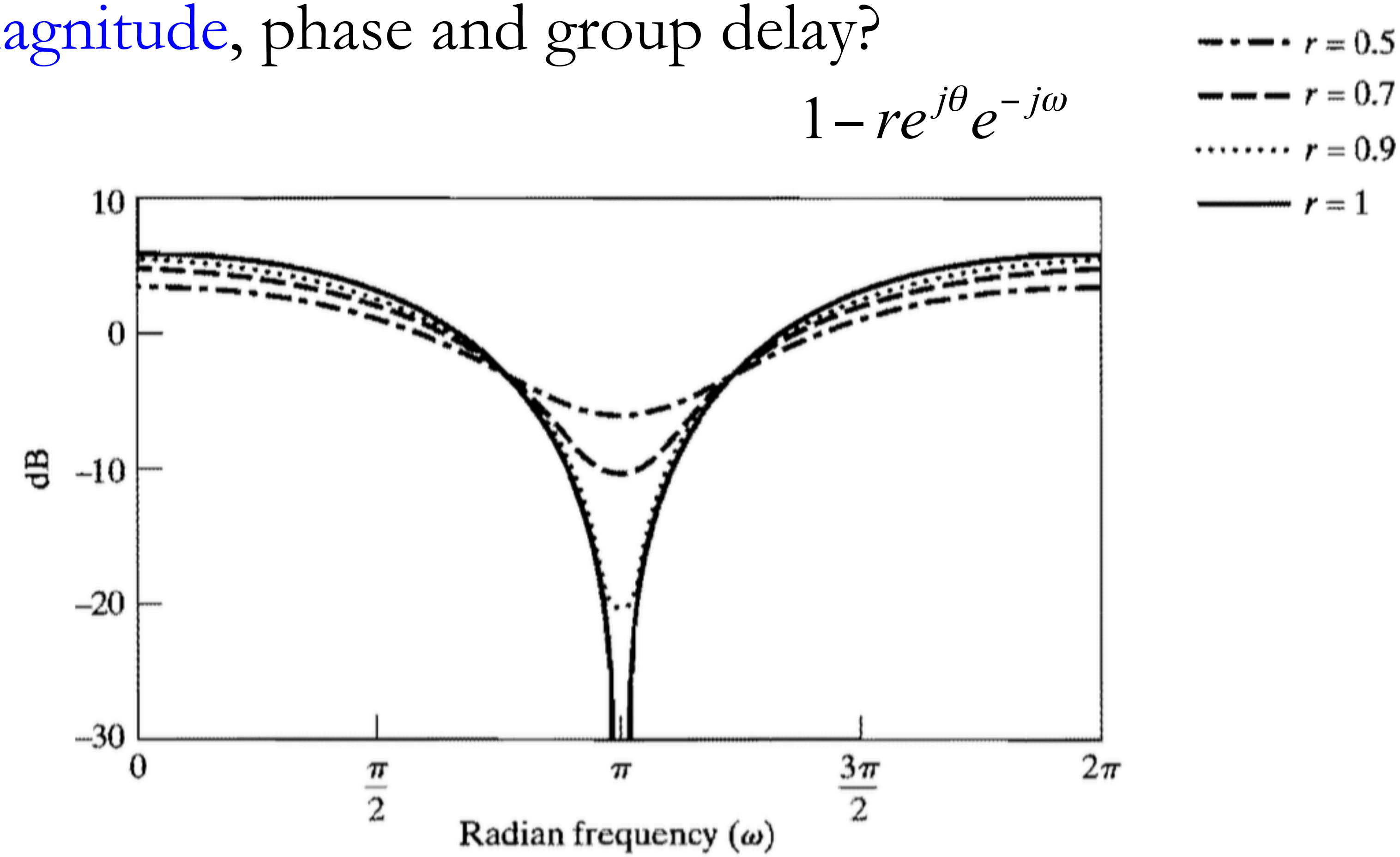




Example: Zero on Real Axis

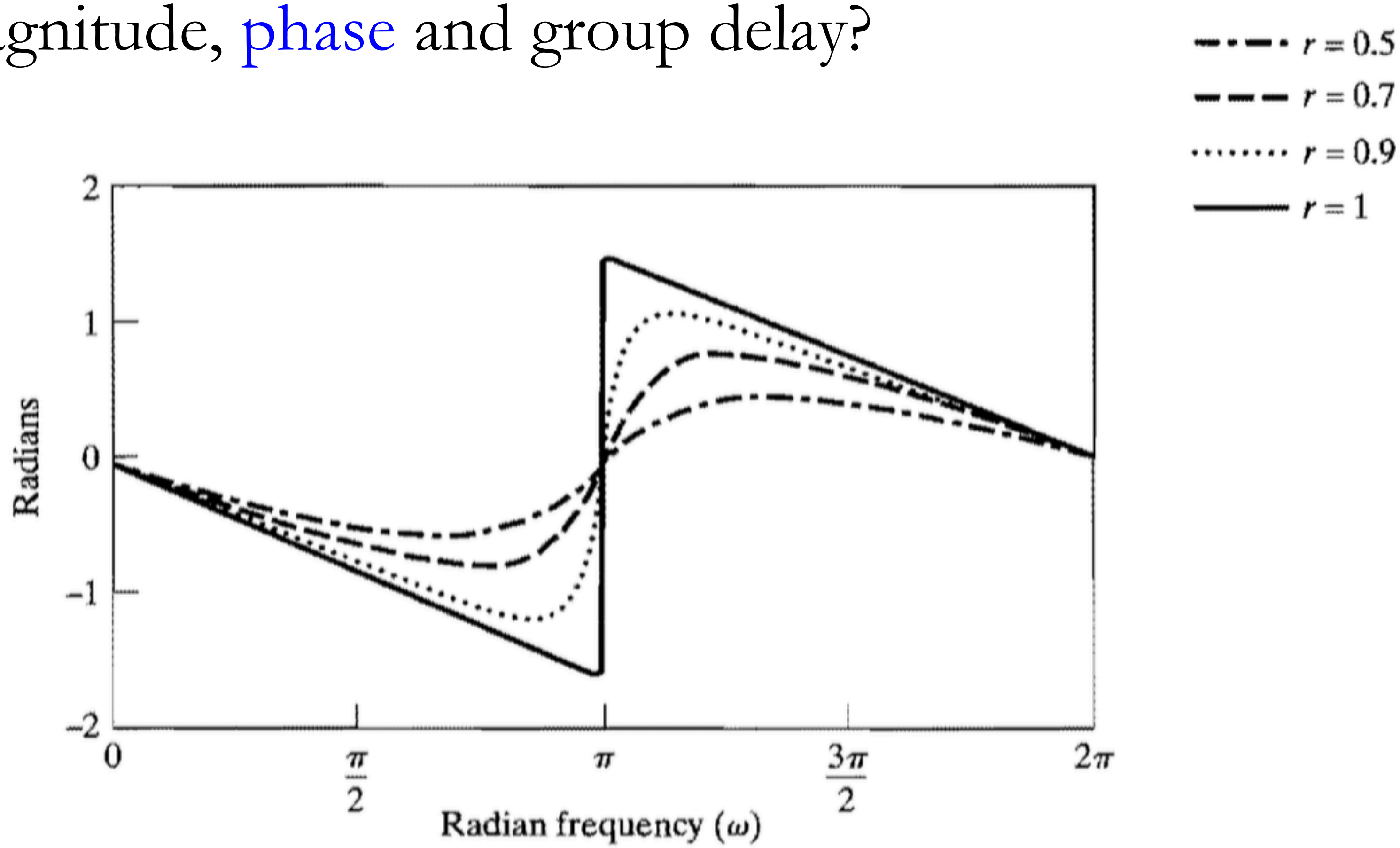
- For $\theta = \pi$, how does zero location effect magnitude, phase and group delay?

$$1 - re^{j\theta}e^{-j\omega}$$



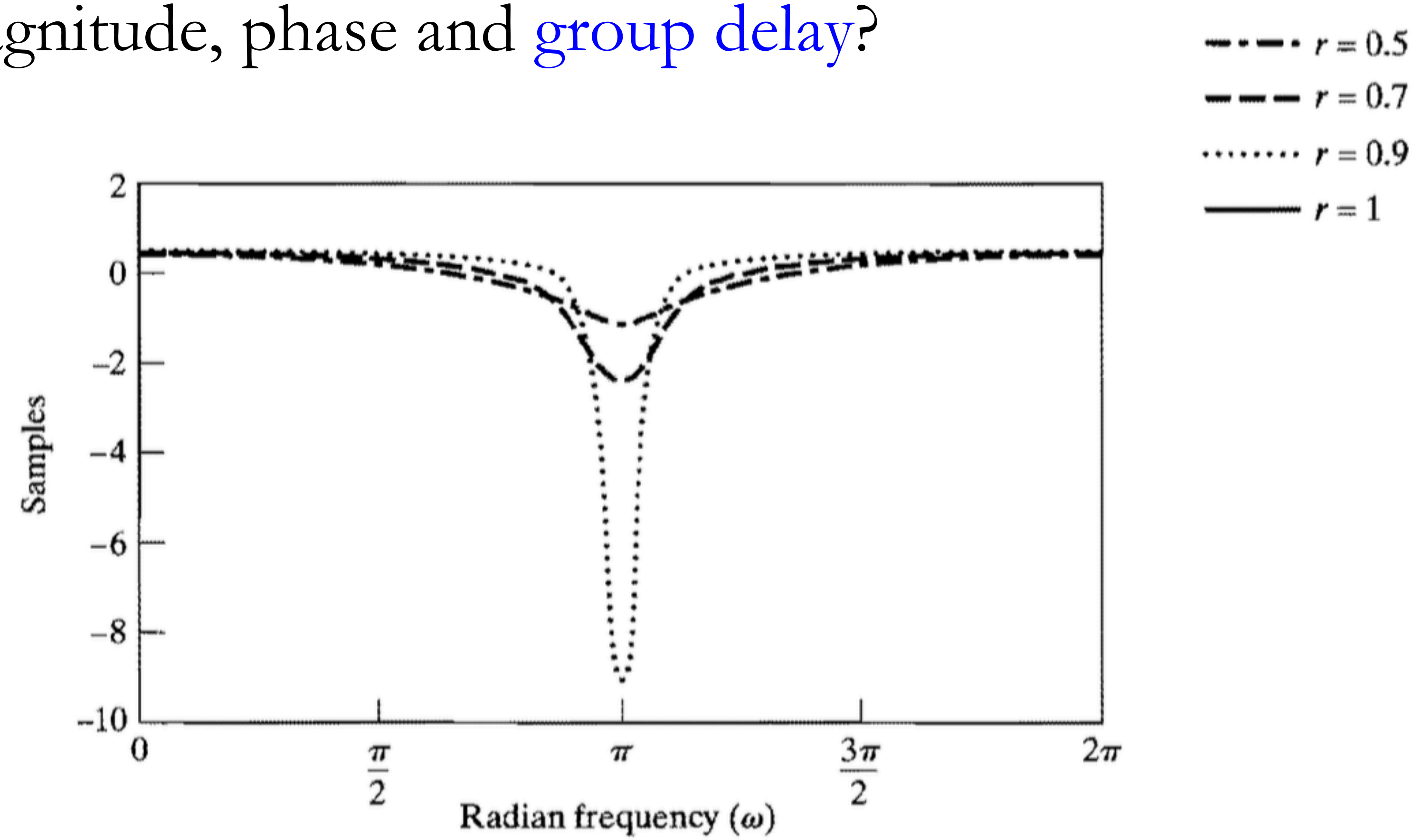
Example: Zero on Real Axis

- For $\theta = \pi$, how does zero location effect magnitude, phase and group delay?



Example: Zero on Real Axis

- For $\theta = \pi$, how does zero location effect magnitude, phase and group delay?

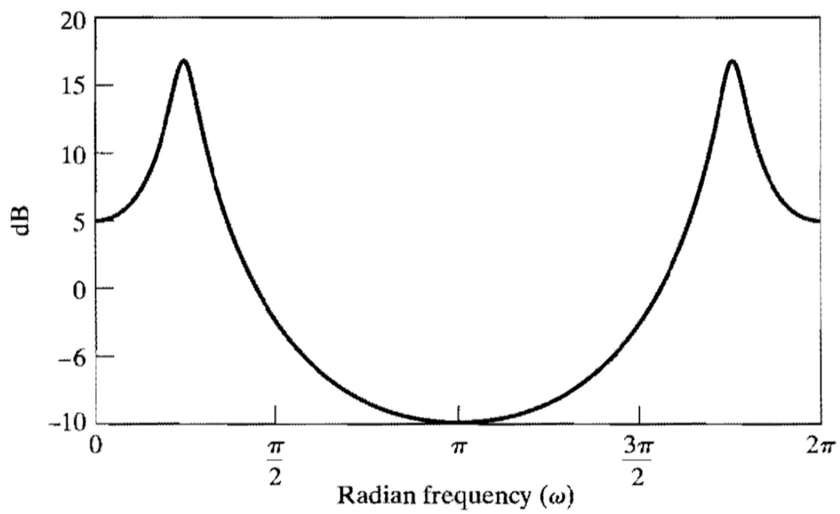


2^{nd} Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

$r = 0.9, \ \theta = \pi/4$

magnitude

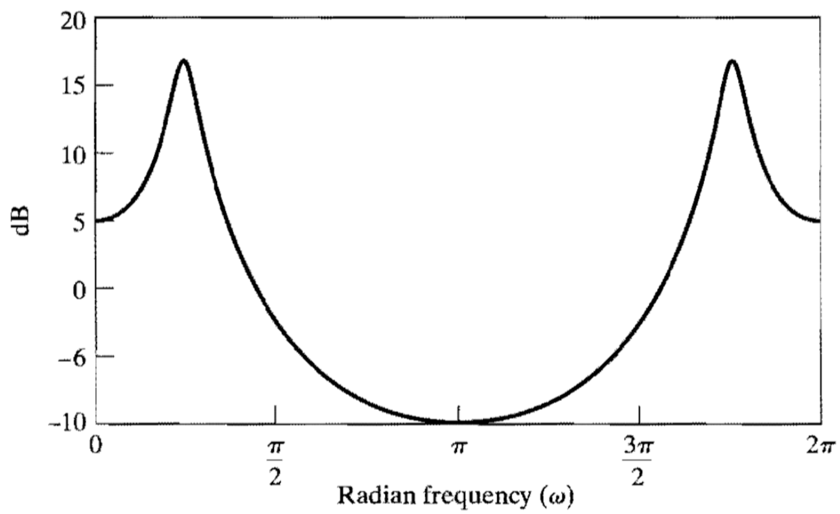


2nd Order IIR with Complex Poles

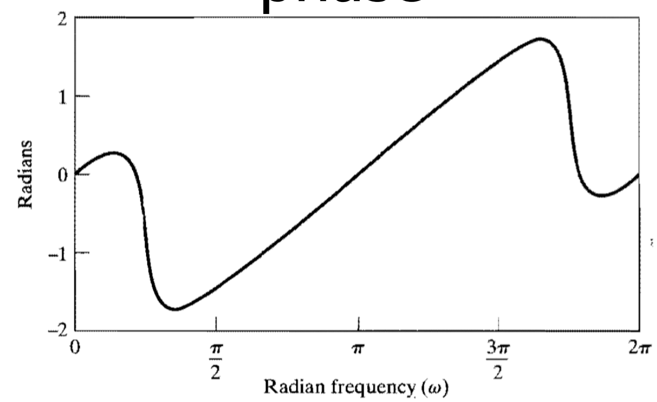
$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

r=0.9, $\theta = \pi / 4$

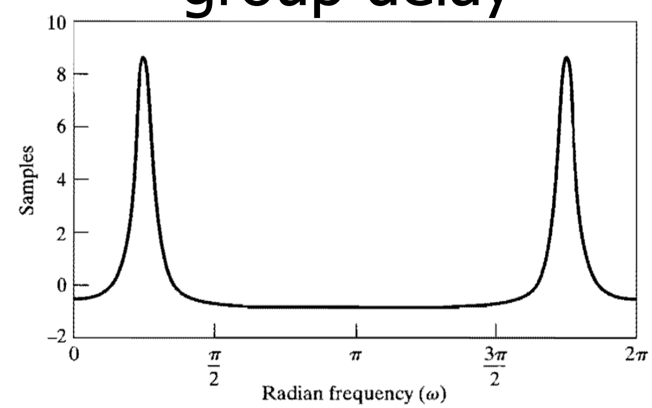
magnitude



phase

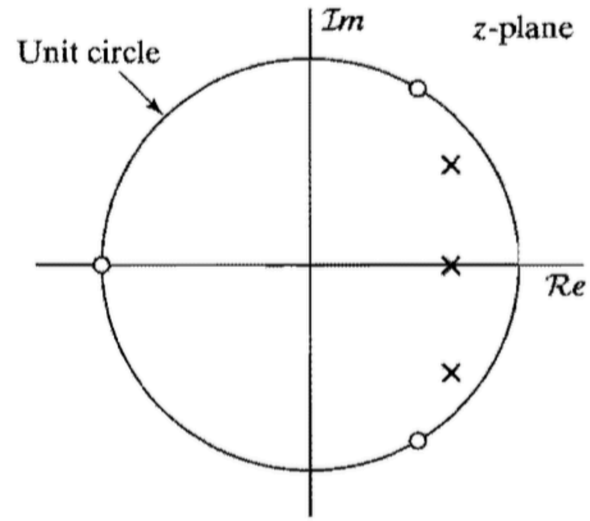


group delay



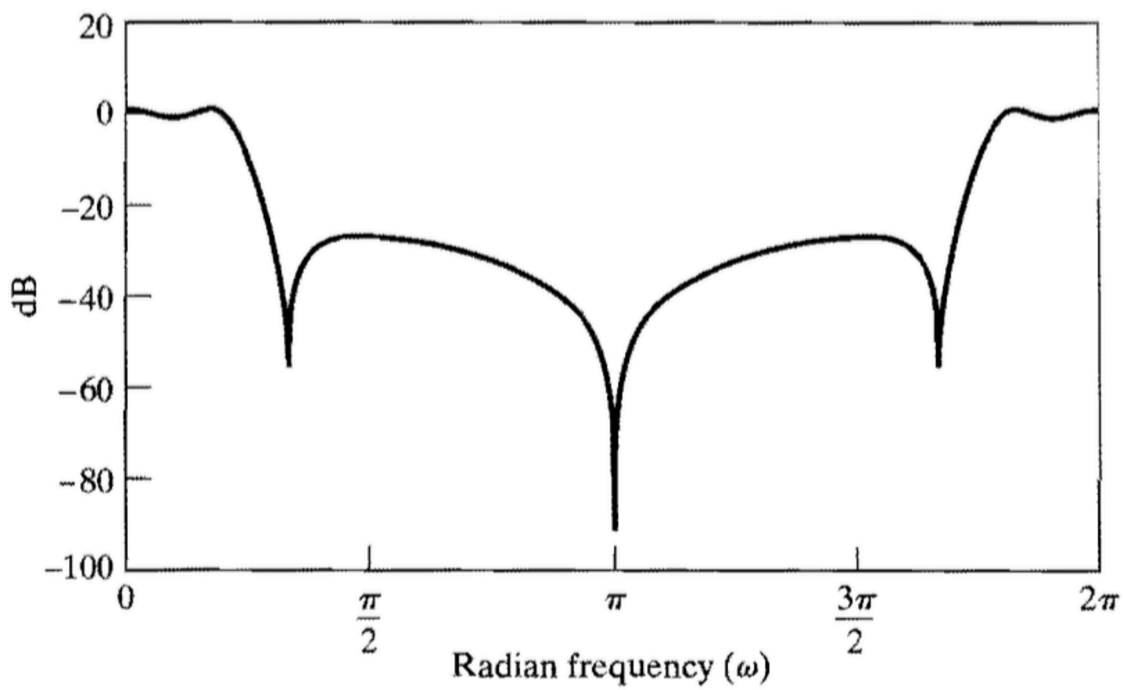
3rd Order IIR Example

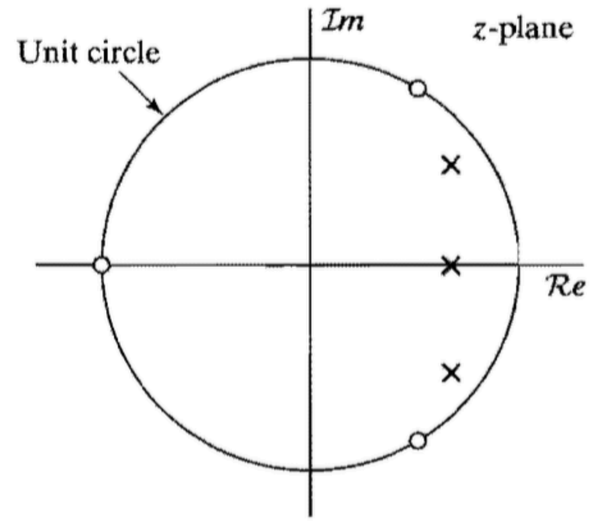
$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



3rd Order IIR Example

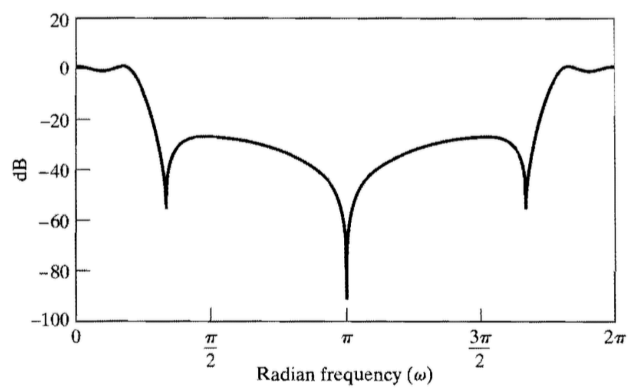
$$H(z) = \frac{0.05634(1+z^{-1})(1-1.0166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$$

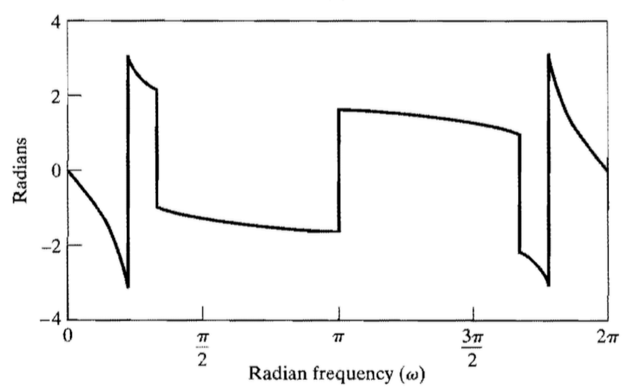


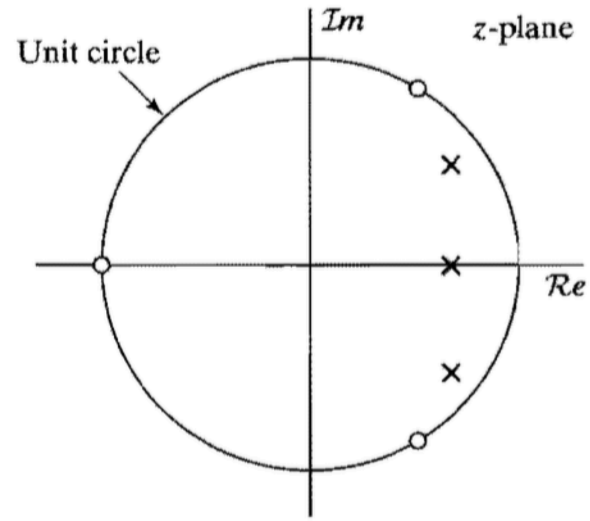


3rd Order IIR Example

$$H(z) = \frac{0.05634(1+z^{-1})(1-1.0166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$$

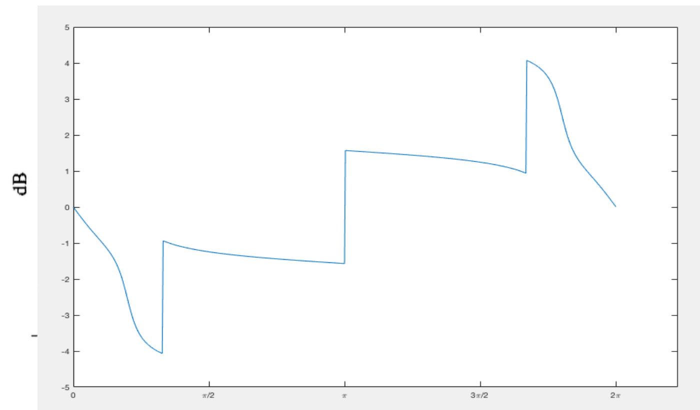


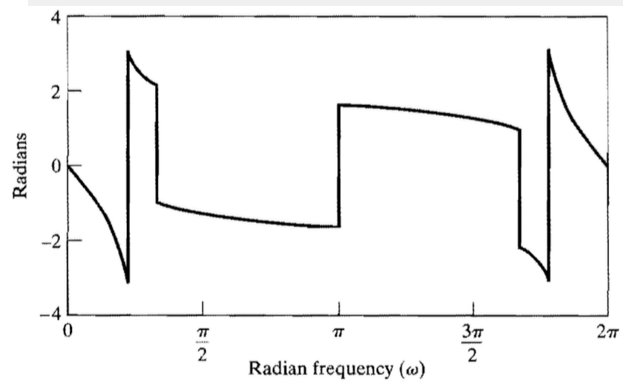


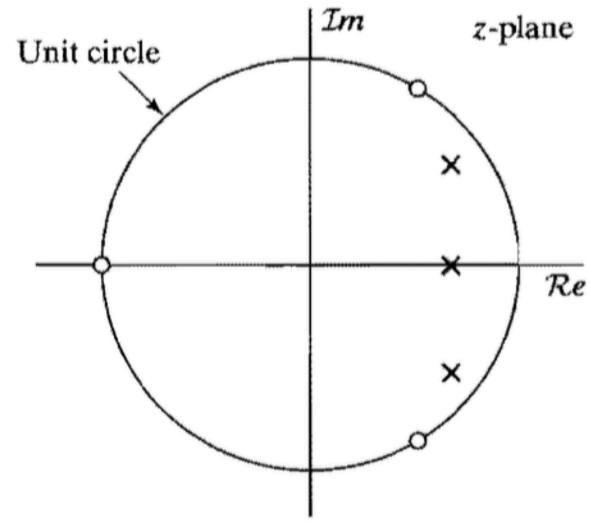


3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$

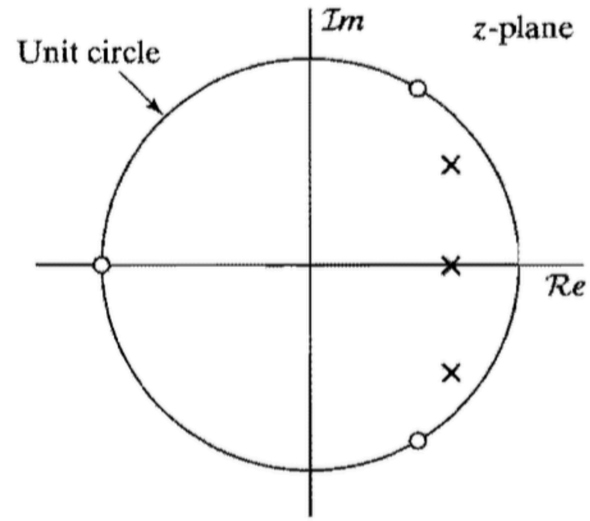


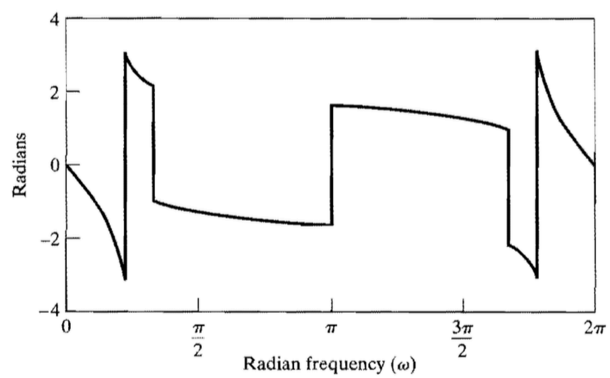


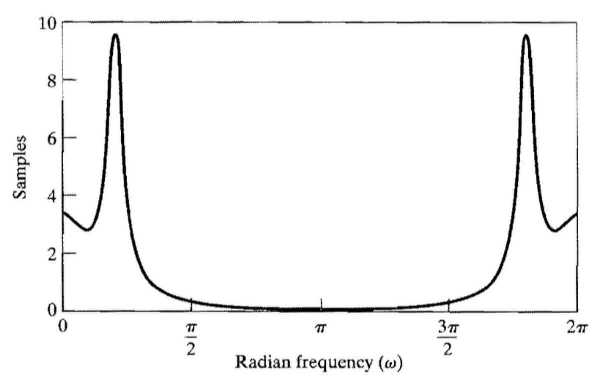


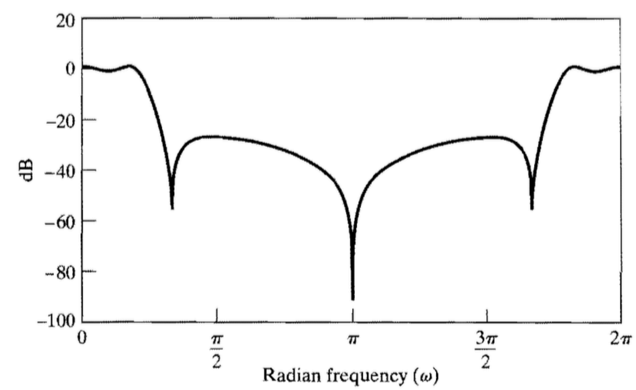
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$









Stability and Causality

LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

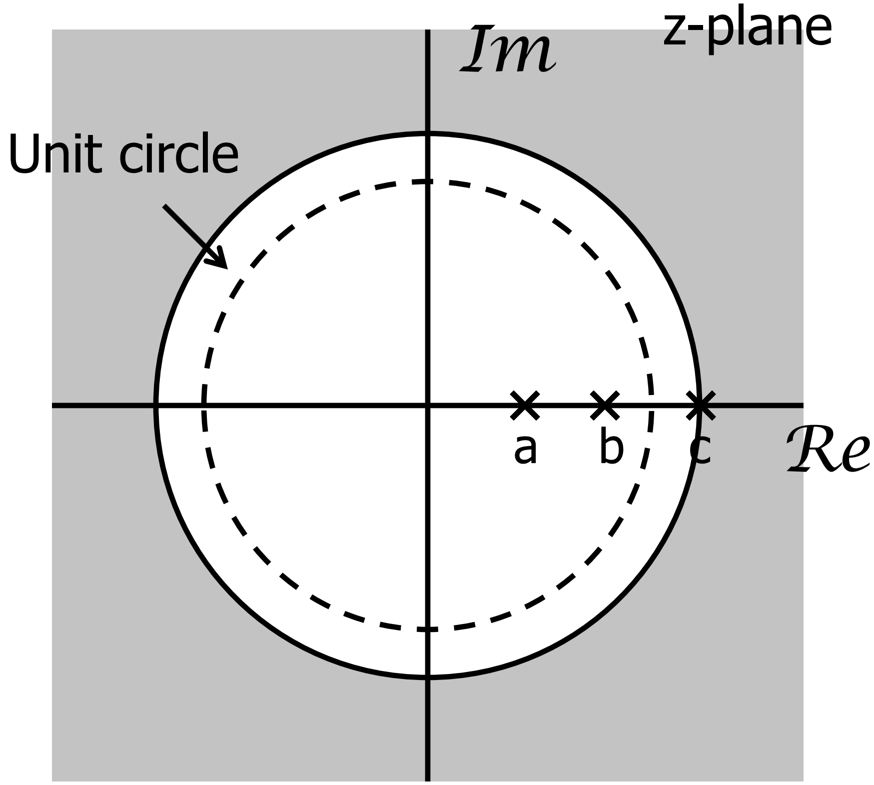
Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

Example: ROC from Pole-Zero Plot

ROC 1: right-sided



LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

If stable and causal, all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

All-Pass Systems

All-Pass Filters

- A system is an all-pass system if

$$\left|H(e^{j\omega})\right| = 1, \ \text{all } \omega$$

- Its phase response $\theta(\omega)$ may be non-trivial

First Order All-Pass Filter (*a* real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

First Order All-Pass Filter (*a* real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$\left|H(e^{j\omega})\right| = \frac{\left|e^{-j\omega} - a^*\right|}{\left|1 - ae^{-j\omega}\right|}$$

$$= \frac{\left|e^{-j\omega}(1 - a^* e^{j\omega})\right|}{\left|1 - ae^{-j\omega}\right|}$$

$$= \frac{\left|1 - a^* e^{j\omega}\right|}{\left|1 - ae^{-j\omega}\right|} = 1$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\mathrm{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

General All-Pass Filter

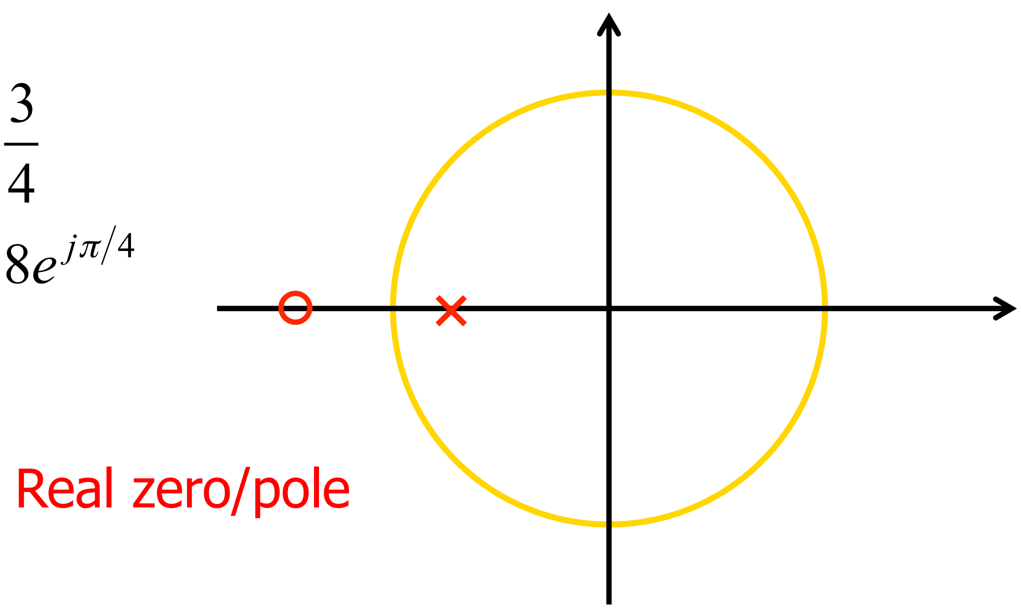
- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$



General All-Pass Filter

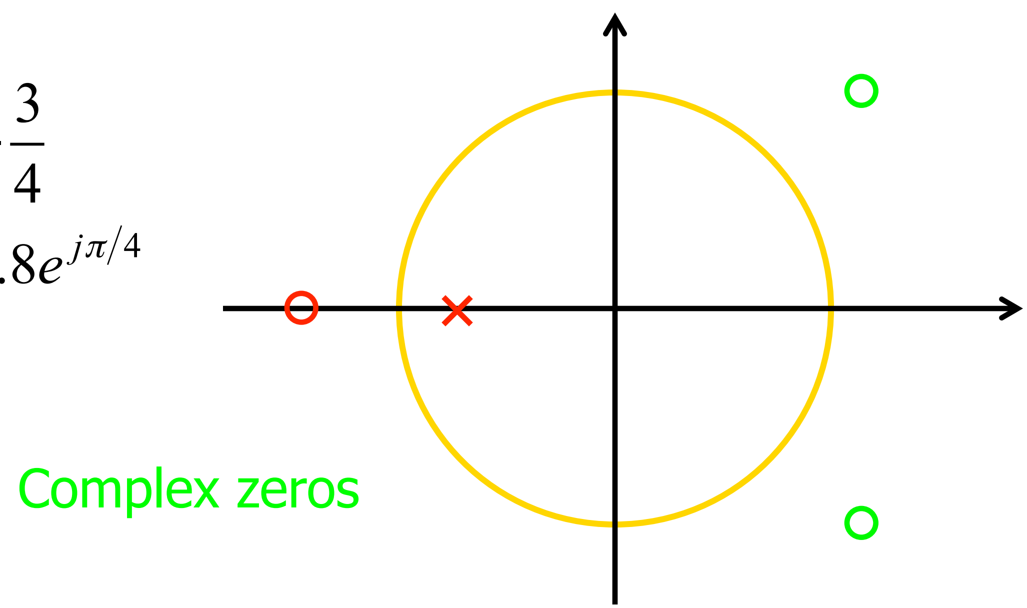
- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\text{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$



General All-Pass Filter

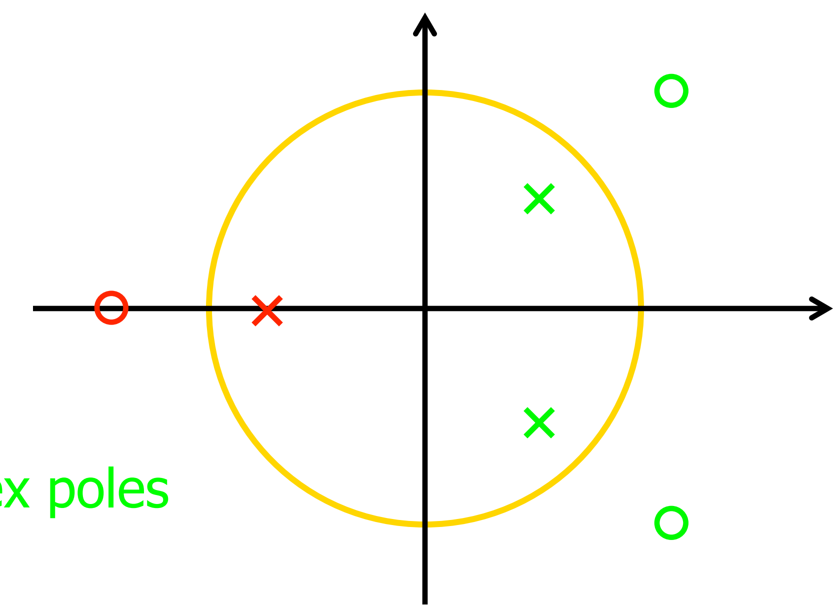
- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$



Complex poles

All Pass Filter Phase Response

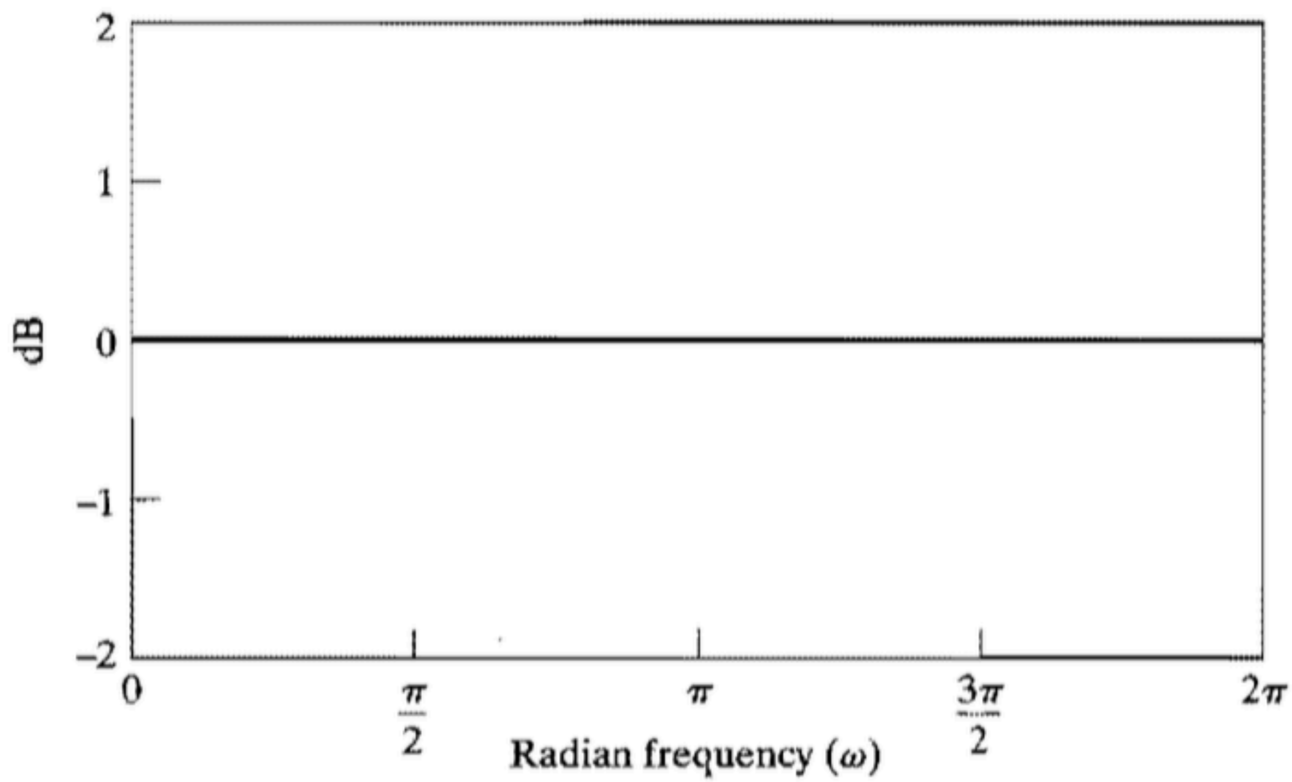
- First order system

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

First Order Example

- Magnitude:



All Pass Filter Phase Response

- First order system $$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$ $$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$
- phase $$\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$$

All Pass Filter Phase Response

- First order system

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

- phase

$$\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$$

$$= \arg\left(\frac{e^{-j\omega}(1 - re^{-j\theta}e^{j\omega})}{1 - re^{j\theta}e^{-j\omega}}\right)$$

$$= \arg(e^{-j\omega}) + \arg(1 - re^{-j\theta}e^{j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$

$$= -\omega - \arg(1 - re^{j\theta}e^{-j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$

$$= -\omega - 2\arg(1 - re^{j\theta}e^{-j\omega})$$

Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \text{grd}[1 - d_k e^{-j\omega}]$$

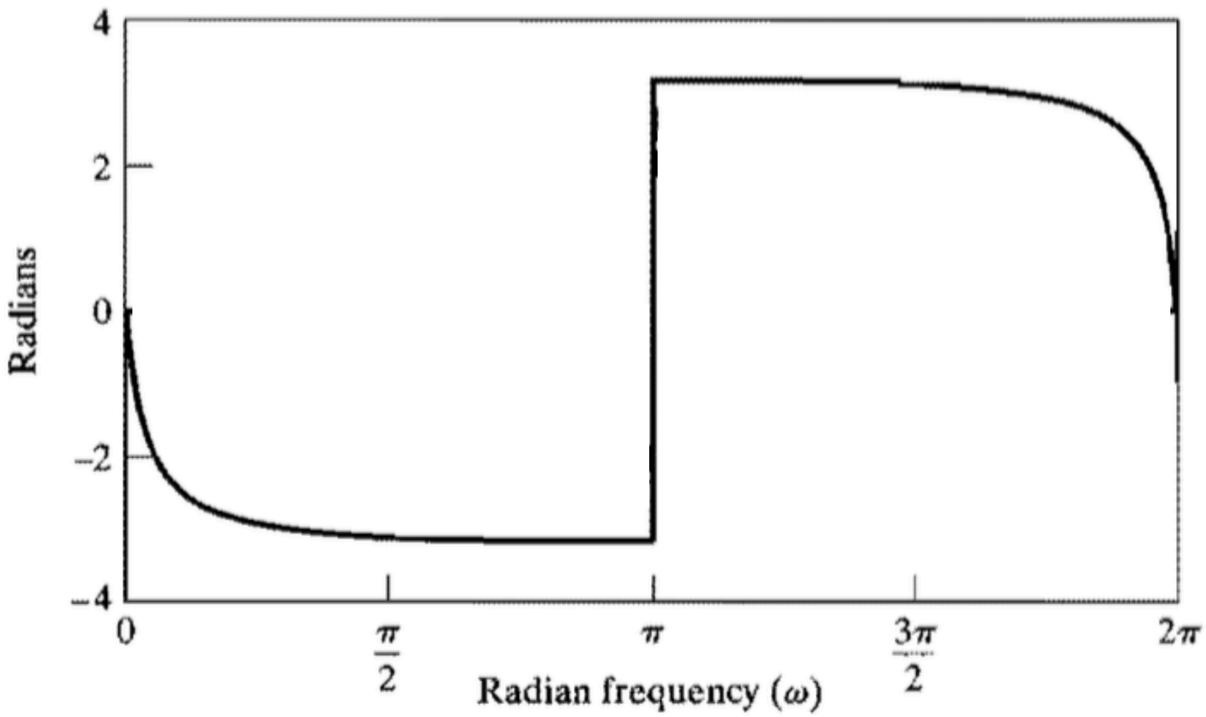
- Look at each factor:

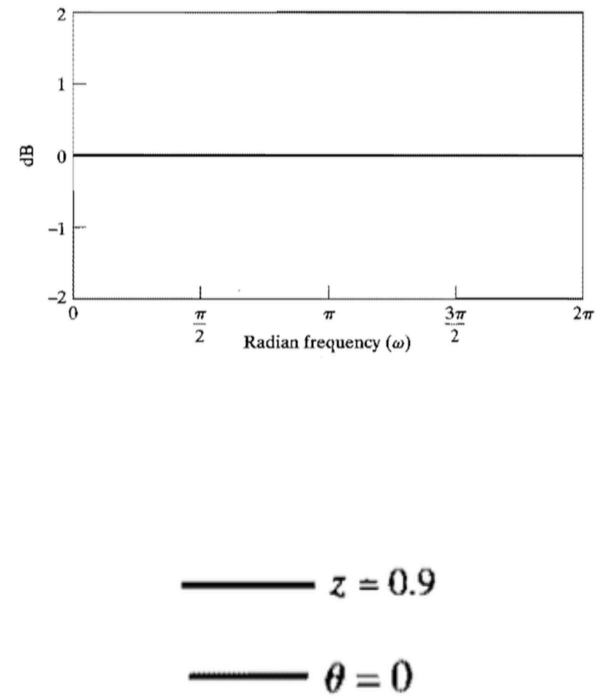
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

First Order Example

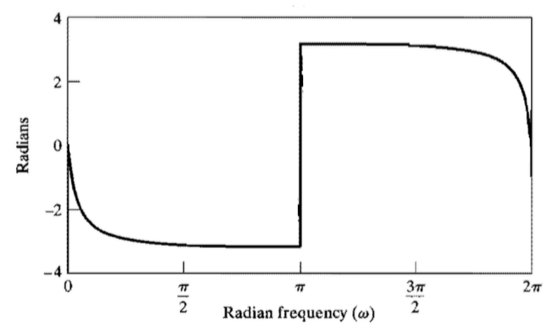
- Phase:

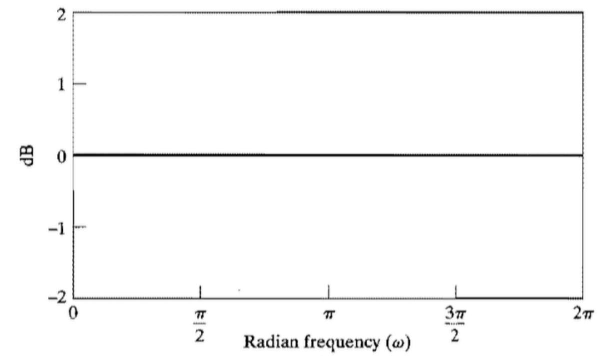


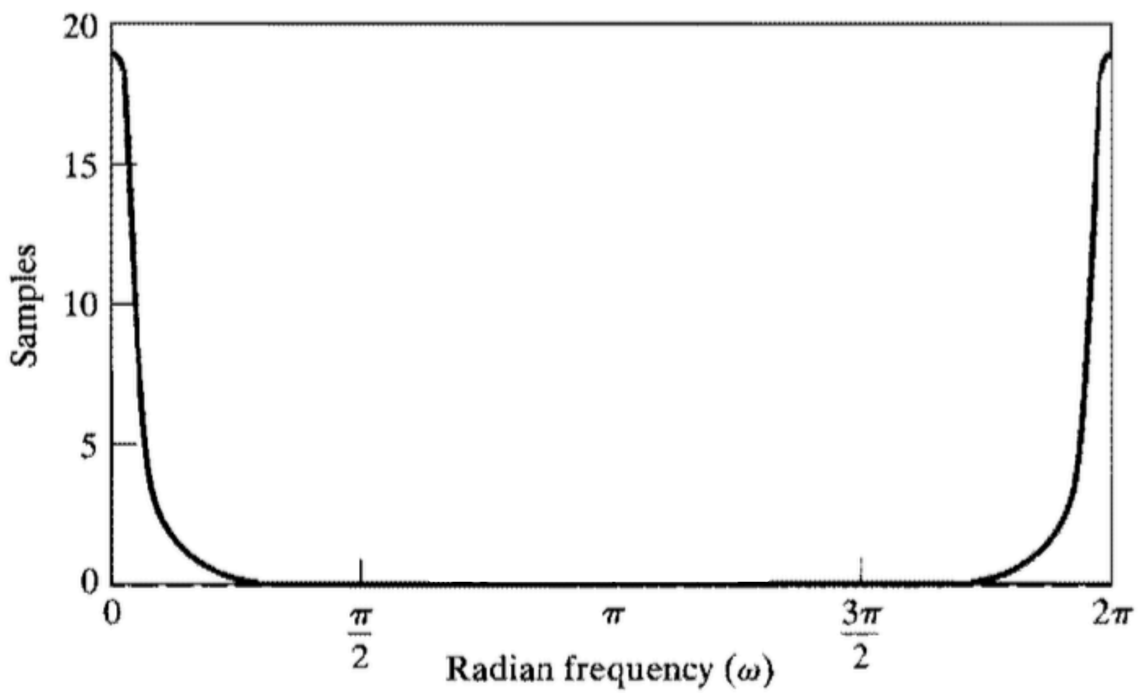


First Order Example

- Group Delay:







All Pass Filter Phase Response

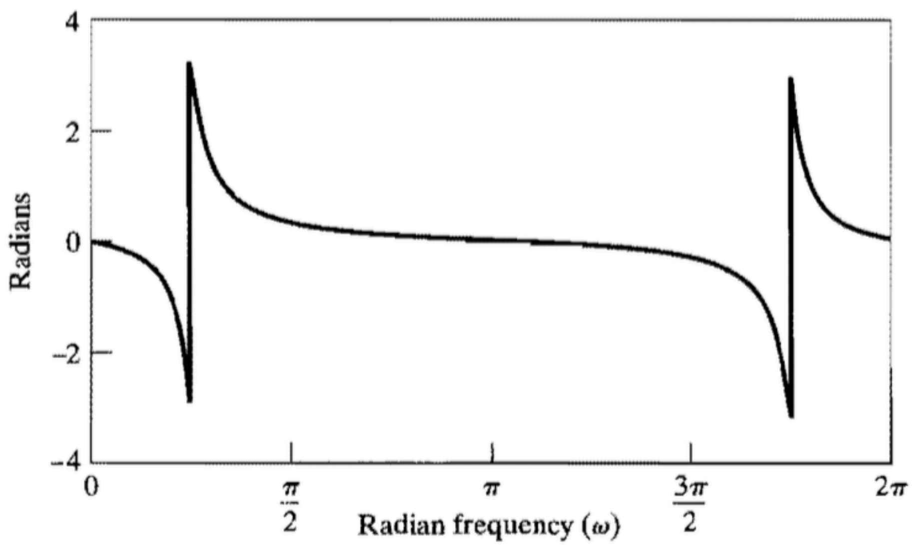
- Second order system with poles at $z = re^{j\theta}, re^{-j\theta}$

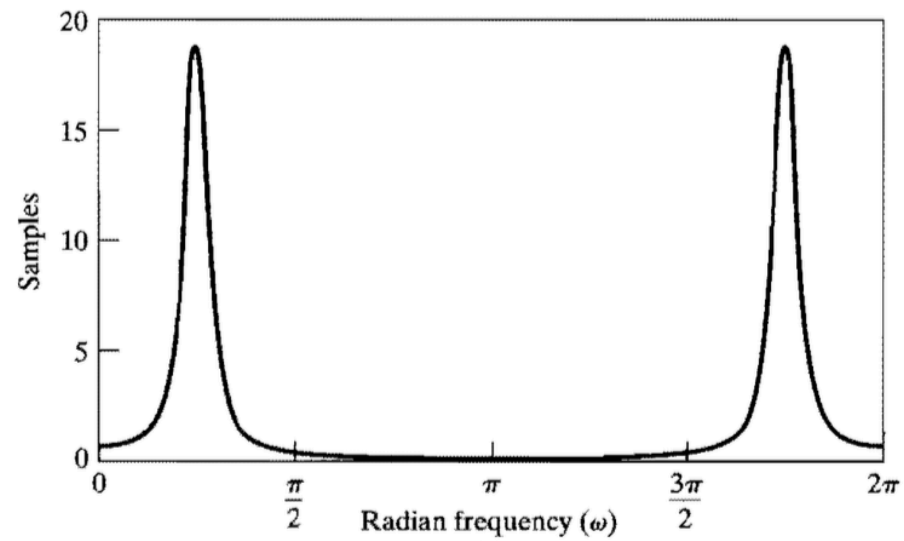
$$\angle\left[\frac{(e^{-j\omega}-re^{-j\theta})(e^{-j\omega}-re^{j\theta})}{(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{-j\omega})}\right] = -2\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

$$-2\arctan\left[\frac{r\sin(\omega+\theta)}{1-r\cos(\omega+\theta)}\right].$$

Second Order Example

- Poles at $z = 0.9e^{\pm j\pi/4}$ (zeros at conjugates)





All-Pass Properties

- Claim: For a stable, causal ($r < 1$) all-pass system:
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
    - Unwrapped phase always non-positive and decreasing
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$
    - Group delay always positive

Pg 309 in text book

  - Intuition
    - delay is positive → system is causal
    - Phase negative → phase lag

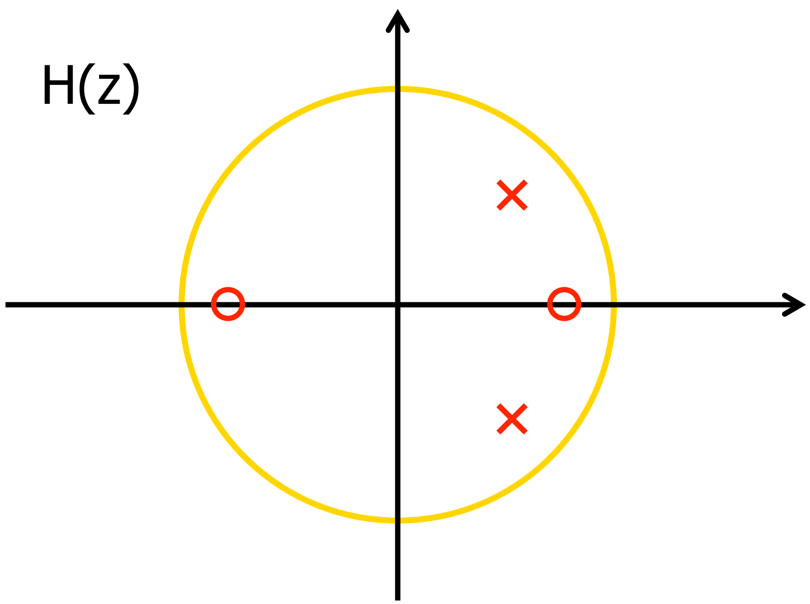
Minimum-Phase Systems

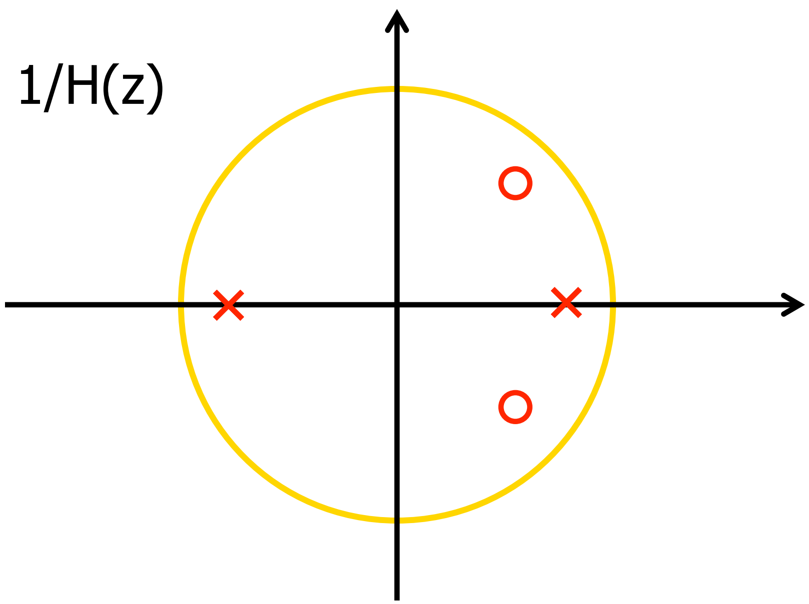
Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)

Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle





All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

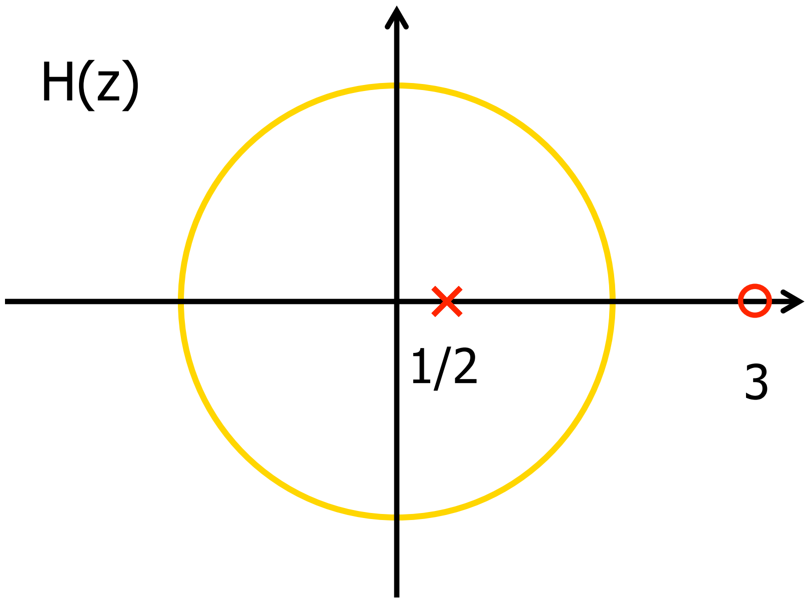
$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

- Approach:
  - (1) First construct H_{ap} with all zeros outside unit circle
  - (2) Compute

$$H_{\min}(z) = \frac{H(z)}{H_{ap}(z)}$$

Min-Phase Decomposition Example

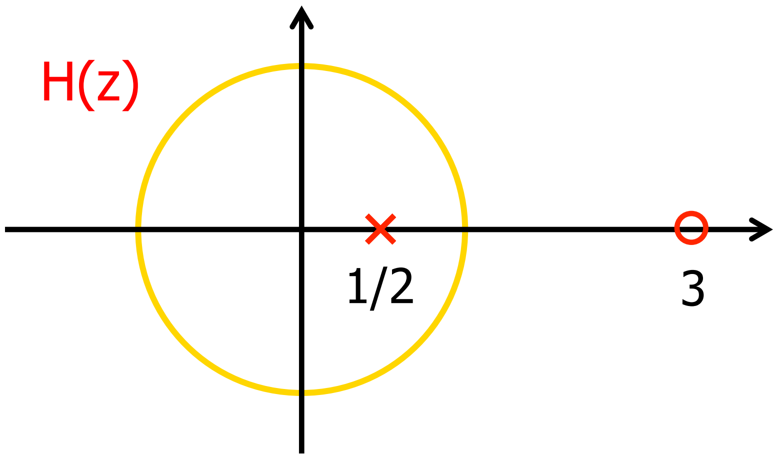
$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

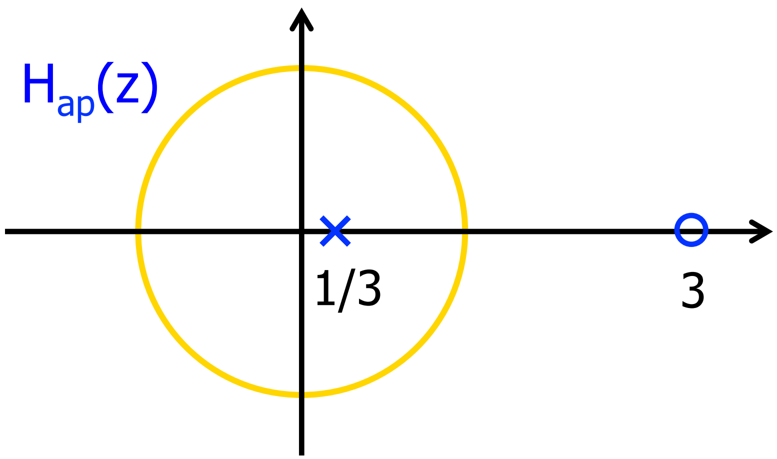


Min-Phase Decomposition Example

$$H(z)=\frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z)=\dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$



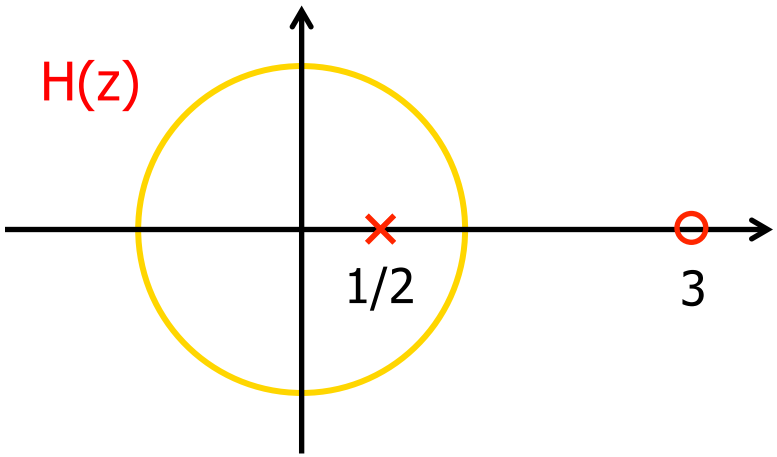


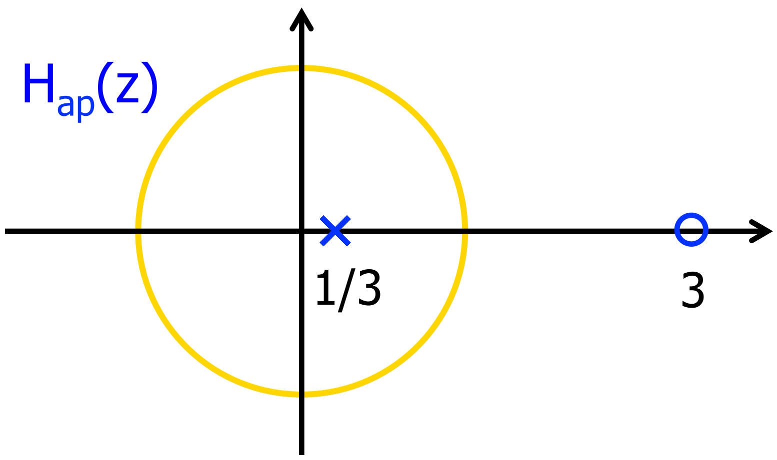
Min-Phase Decomposition Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z) = \dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$

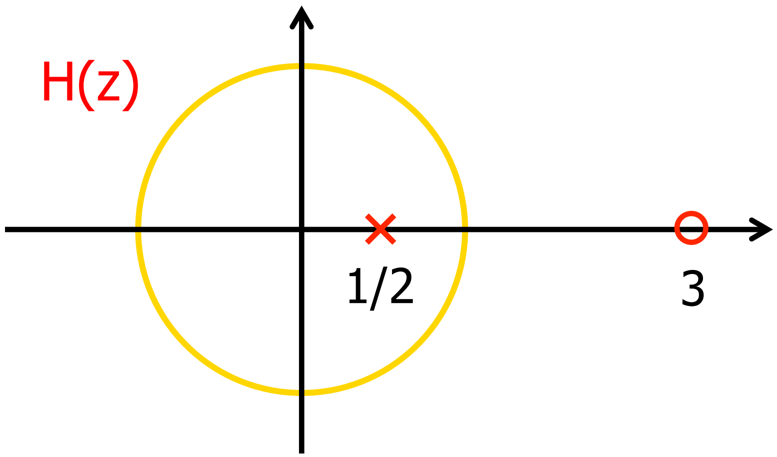
$$H_{\min}(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}\left(\frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}\right)^{-1}$$



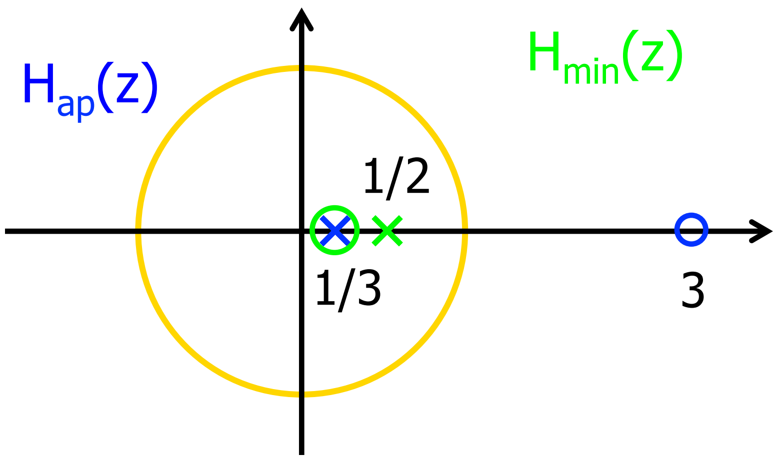


Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$



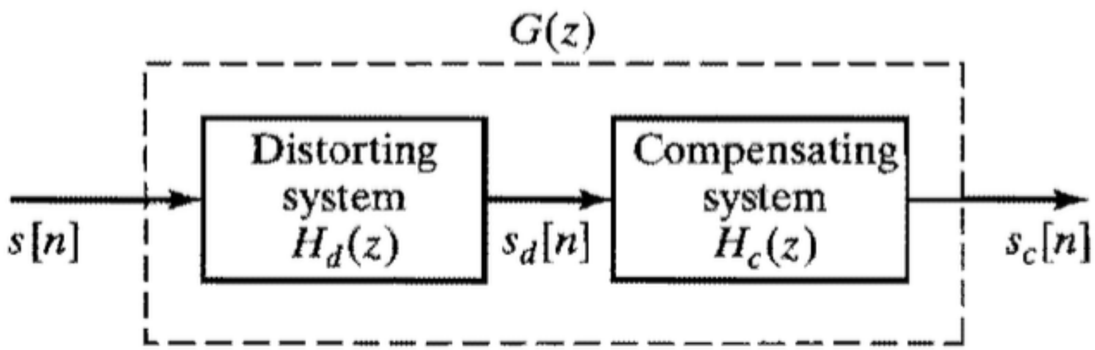
- Set $H_{ap}(z) = \dfrac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$



$$H_{\min}(z) = -3\frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

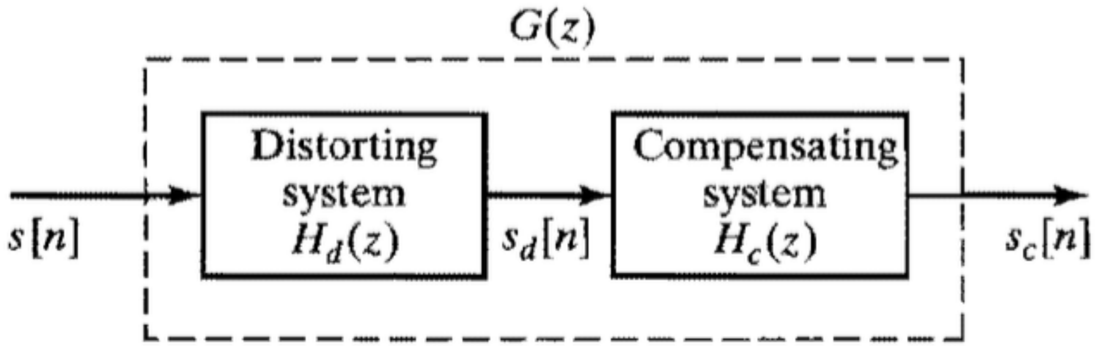
Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



Min-Phase Decomposition Purpose

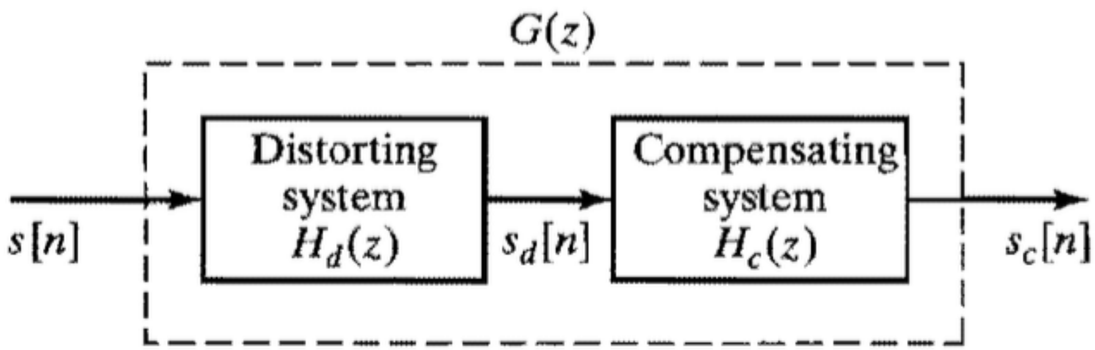
- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal

Min-Phase Decomposition Purpose

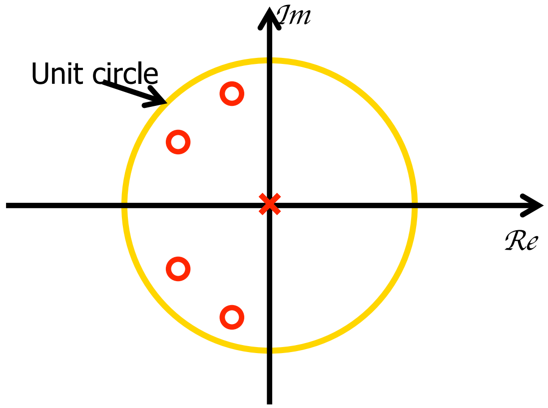
- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z) \rightarrow H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

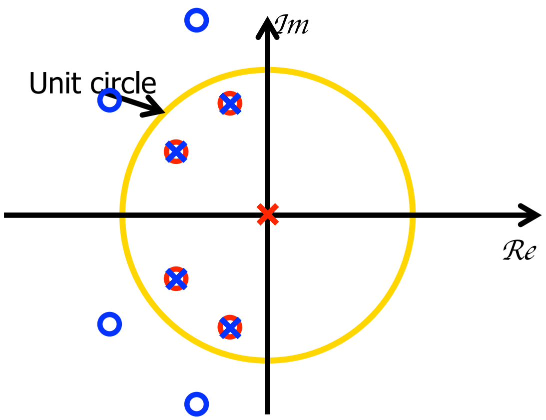
Minimum Phase to Max Phase

Min phase



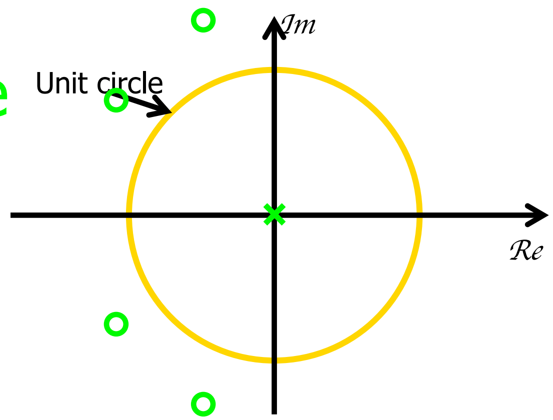
Minimum Phase to Max Phase

Min phase
x All Pass



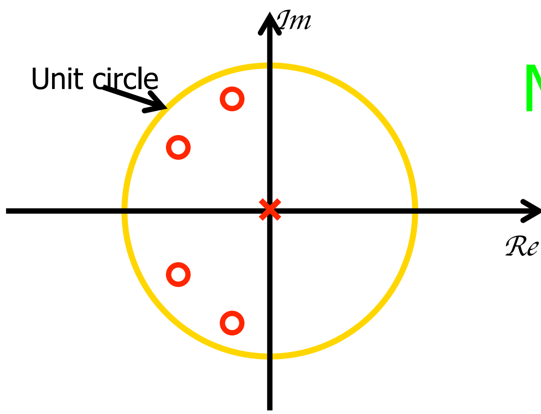
Minimum Phase to Max Phase

Max phase

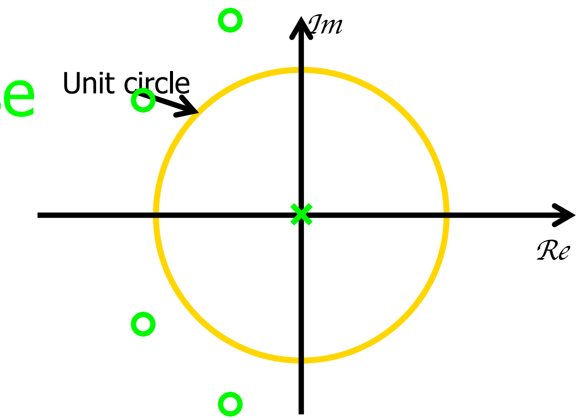


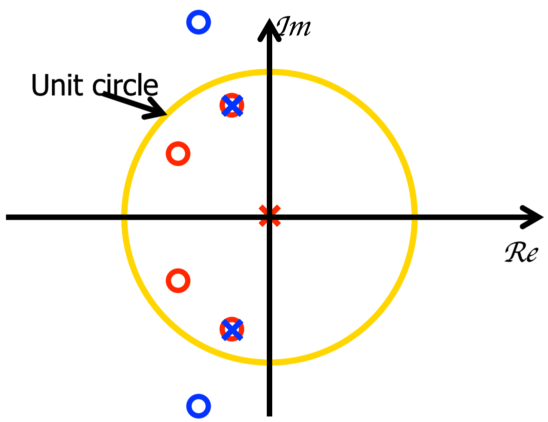
Minimum Phase

Min phase



Max phase

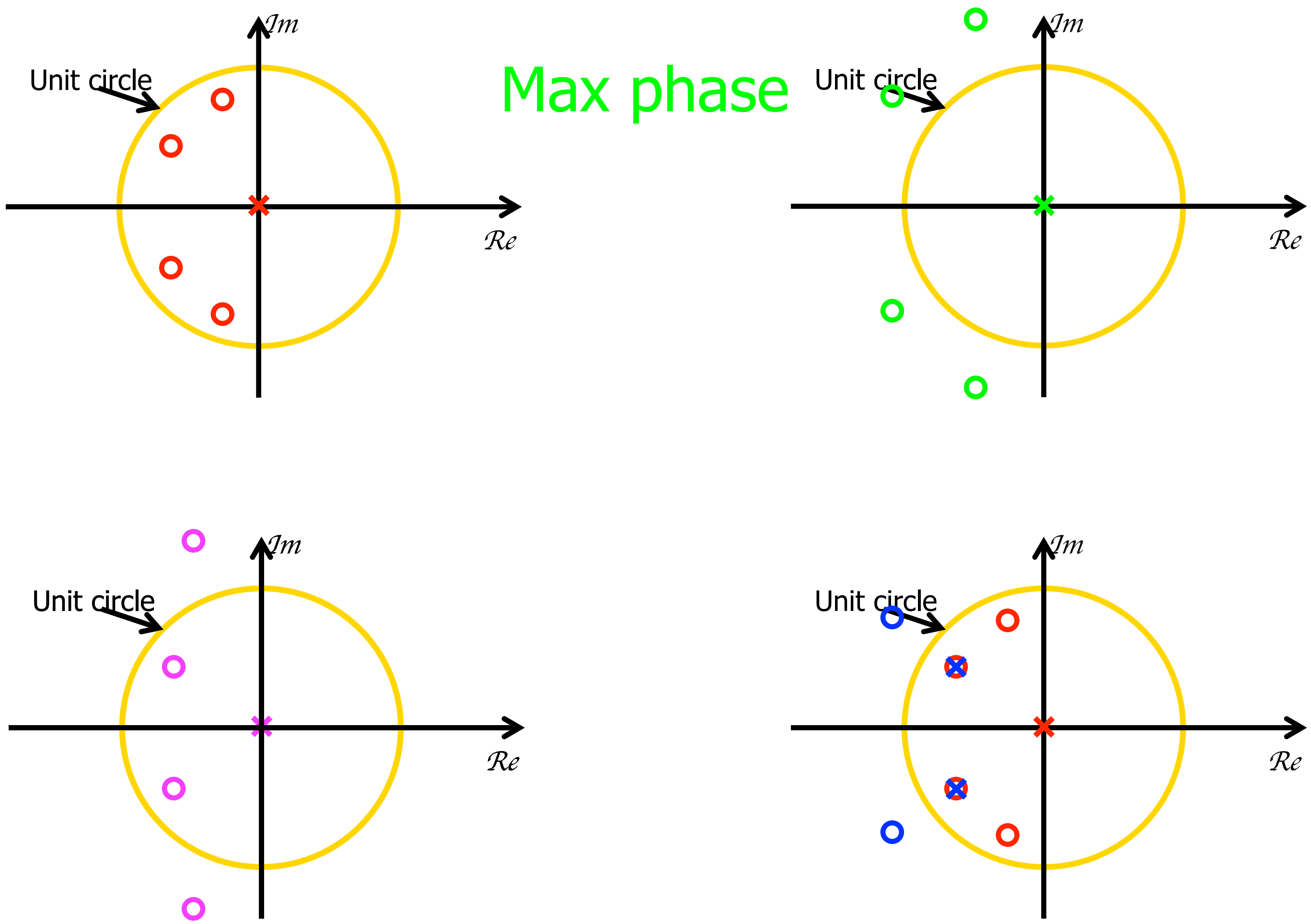




Minimum Phase

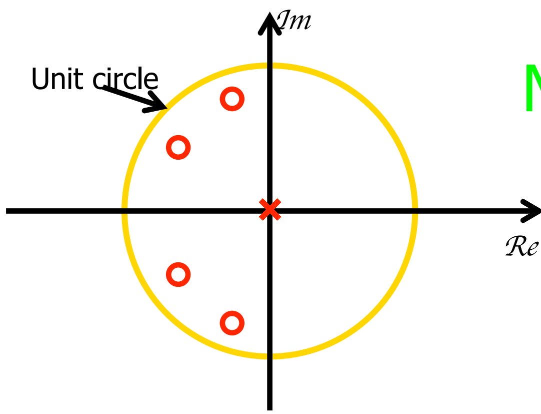
Min phase

Max phase

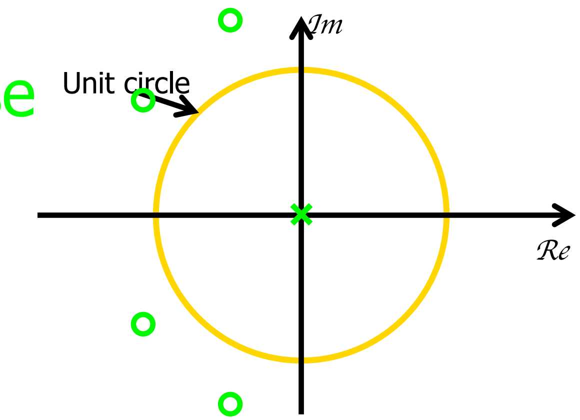


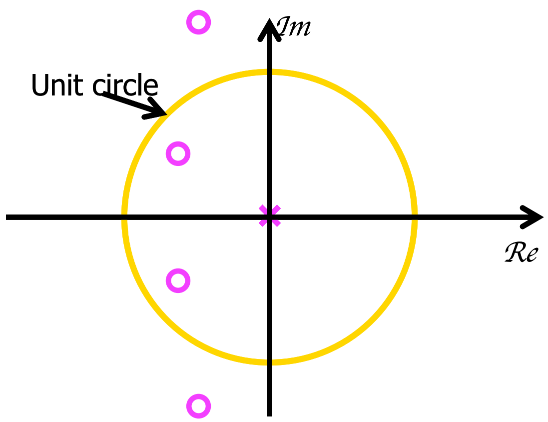
Minimum Phase

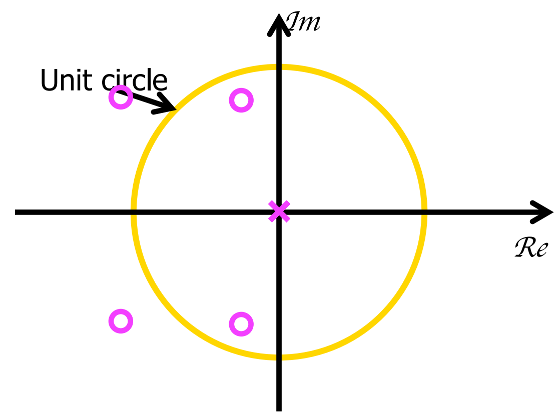
Min phase



Max phase







Minimum Phase Lag Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \le 0$
  - $\mathrm{grd}[H_{ap}(e^{j\omega})] > 0$

- $$\begin{aligned}\arg[H_{max}(e^{j\omega})] &= \arg[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})] \\ &= \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})] \\ &= \quad \le 0 \quad + \quad \le 0\end{aligned}$$

Minimum Group Delay Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

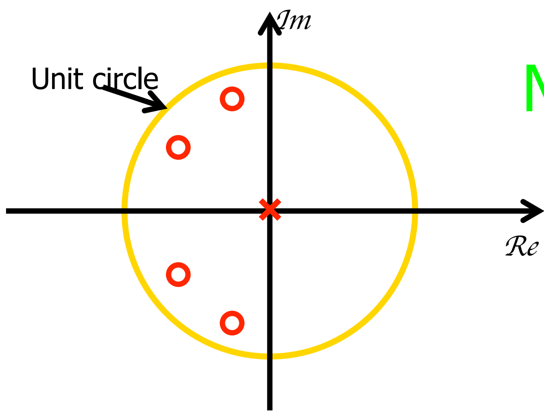
- $\text{grd}[H_{max}(e^{j\omega})] = \text{grd}[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

$$= \text{grd}[H_{min}(e^{j\omega})] + \text{grd}[H_{ap}(e^{j\omega})]$$

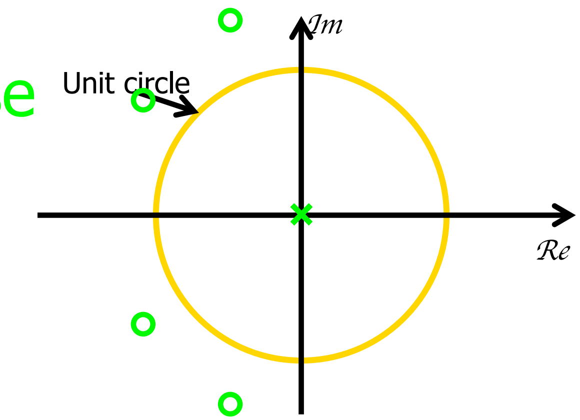
$$= \quad \geq 0 \quad + \quad \geq 0$$

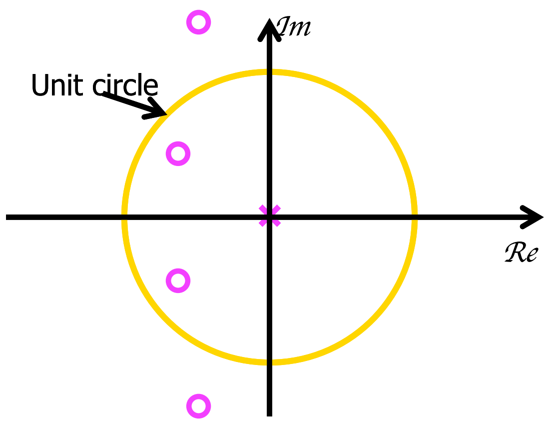
Minimum Phase

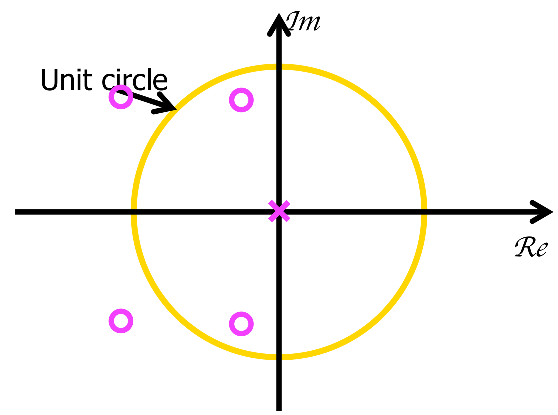
Min phase



Max phase



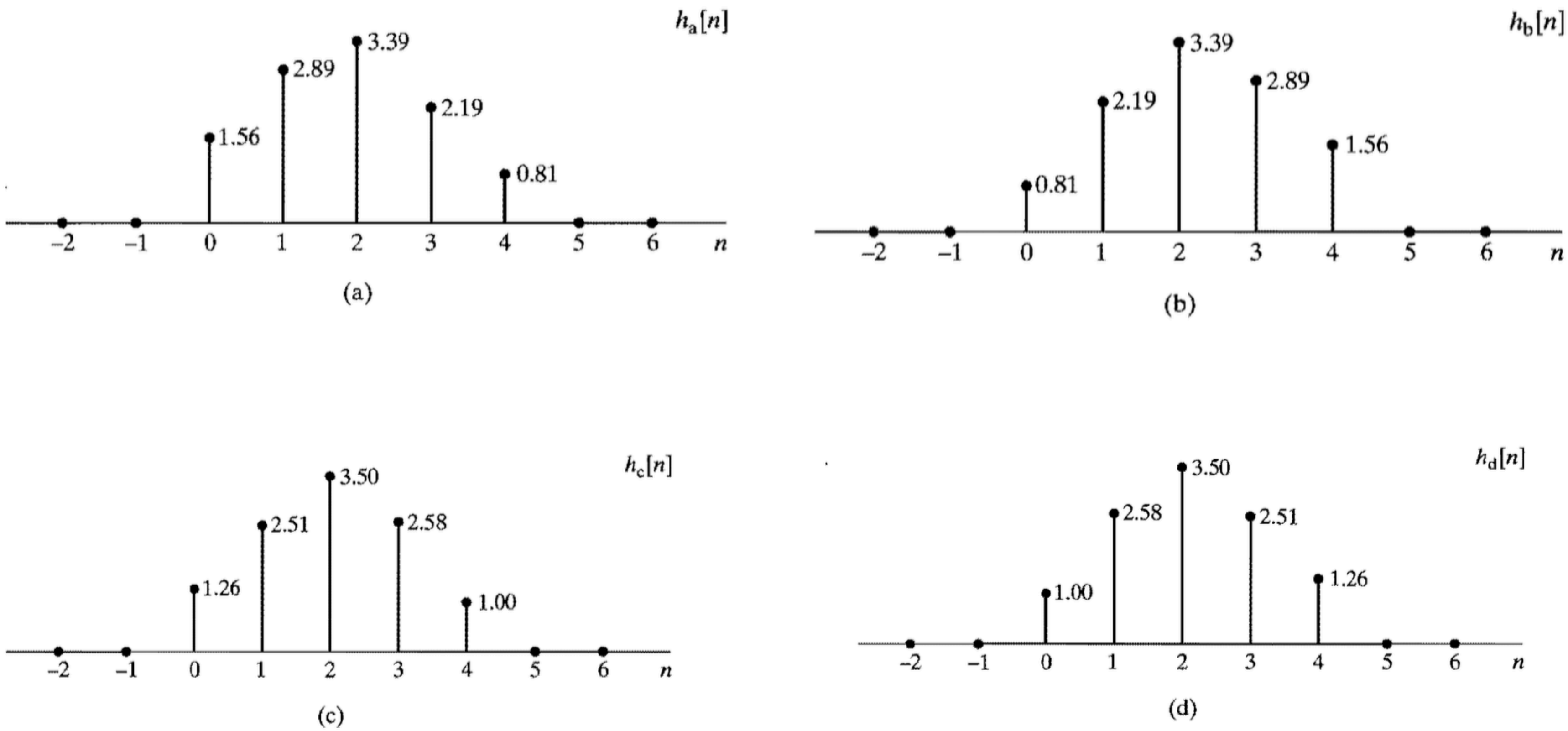




Minimum Energy-Delay Property

Min phase

Max phase



Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
- LTI Stability and Causality
  - If all poles inside unit circle
- All Pass Systems
  - Used for delay compensation
- Minimum Phase Systems
  - Can compensate for magnitude distortion
  - Minimum energy-delay property

Admin

- Dhaval extra now
  - See Piazza for time and location
  - Dhaval TH/F OH cancelled
- Yinghao OH W will be exam review session
- No office hours during spring break
- HW 6 posted after midterm

Admin - Midterm

- Midterm Thursday 3/5
  - During class
    - Starts at **exactly 4:30pm**, ends at exactly 5:50pm (80 minutes)
  - Location DRLB A2
  - Old exams posted on previous years' website
    - Disclaimer: old exams before 2019 covered more material
  - Covers Lec 1 - 11
  - Closed book, one page (8.5x11) cheat sheet allowed
  - Calculators allowed, no smart phones