

ESE 531: Digital Signal Processing

Lec 14: March 3, 2020
All-Pass Systems and Min Phase Decomposition



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Adapted from M. Lustig, EECS Berkeley

Lecture Outline

- Frequency Response of LTI Systems
 - Examples:
 - Zero on Real Axis
 - 2nd order IIR
 - 3rd order Low Pass
- Stability and Causality
- All Pass Systems
- Minimum Phase Systems (If time)

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3

Review: Frequency Response of LTI System

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- We can define a magnitude response...

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

- a phase response...

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

- and group delay

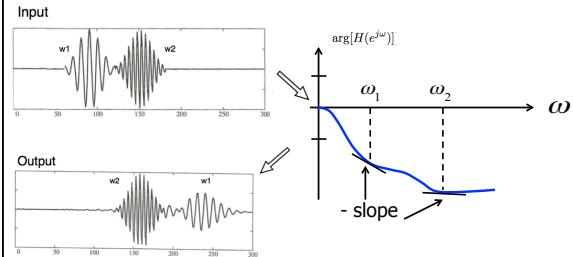
$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

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4

Group Delay

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5

Group Delay Math

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})} \quad H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^M (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^N (1 - d_k e^{-j\omega})}$$

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6

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arg of products is sum of args

$$\arg[H(e^{j\omega})] = \sum_{k=1}^M \arg[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \arg[1 - d_k e^{-j\omega}]$$

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

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7

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□ Look at each factor:

$$\arg[1 - re^{j\theta} e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)}\right)$$

$$\text{grd}[1 - re^{j\theta} e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta} e^{-j\omega}|^2}$$

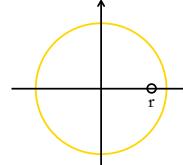
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8

Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta = 0$)

$$\arg[1 - re^{-j\omega}]$$



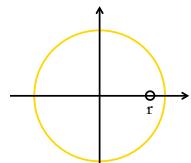
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9

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$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



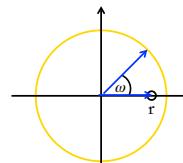
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10

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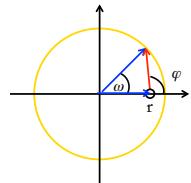
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11

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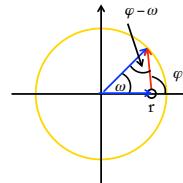
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12

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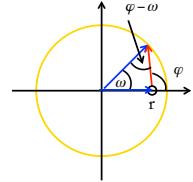
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13

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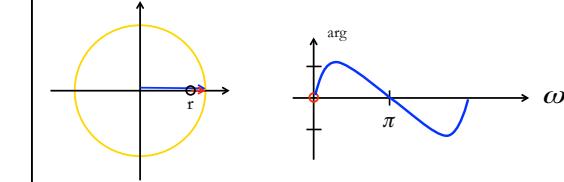
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14

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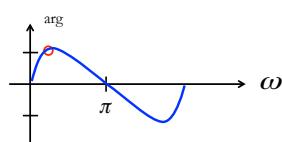
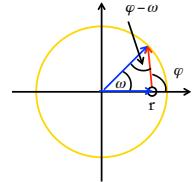
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15

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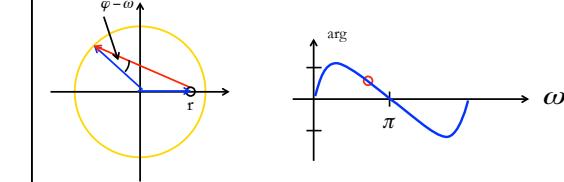
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16

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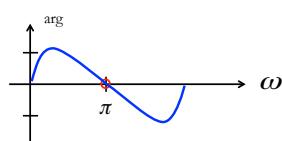
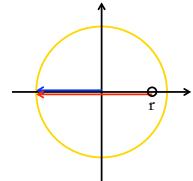
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17

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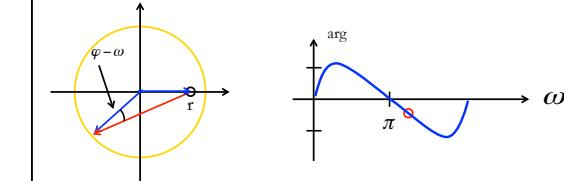
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18

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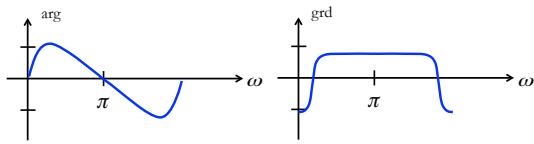
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19

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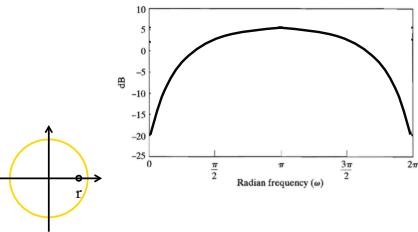
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20

Example: Zero on Real Axis

- Magnitude Response

$$1 - re^{j\theta} e^{-j\omega} = 1 - re^{-j\omega}$$



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21

Group Delay Math

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- Look at each factor:

$$\theta \neq 0?$$

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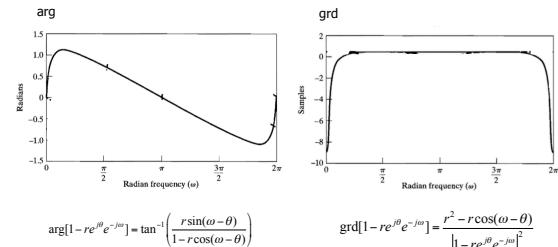
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22

Example: Zero on Real Axis

- For $\theta = 0$

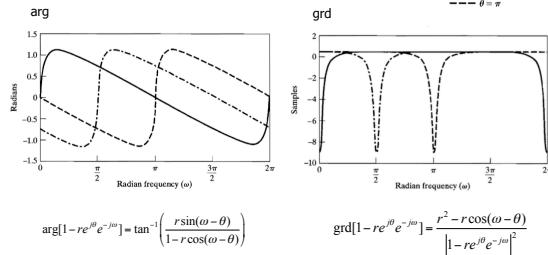


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23

Example: Zero on Real Axis

- For $\theta \neq 0$



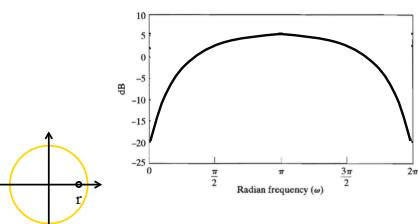
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Example: Zero on Real Axis

- Magnitude Response

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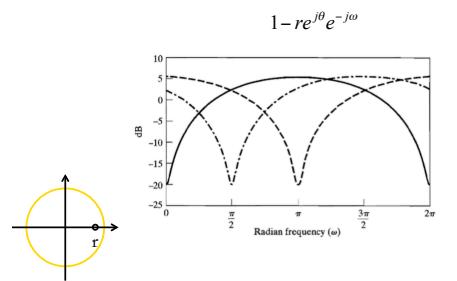


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25

Example: Zero on Real Axis

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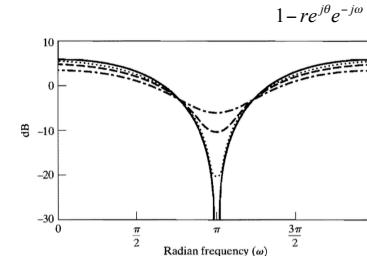


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26

Example: Zero on Real Axis

- For $\theta = \pi$, how does zero location effect magnitude, phase and group delay?

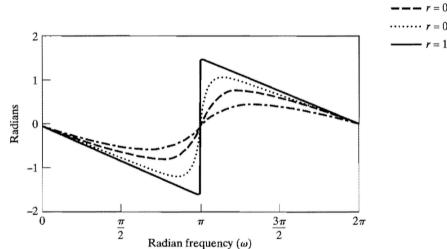


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27

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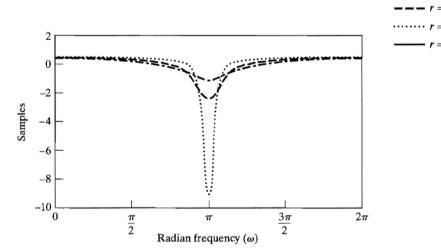


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28

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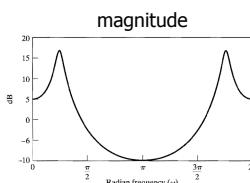


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29

2nd Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} \quad r=0.9, \theta=\pi/4$$

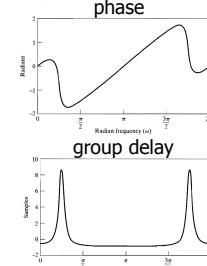
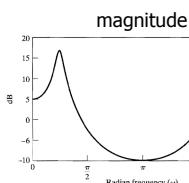


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30

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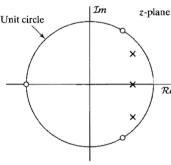


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31

3rd Order IIR Example

$$H(z) = \frac{0.05634(1+z^{-1})(1-1.0166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$$

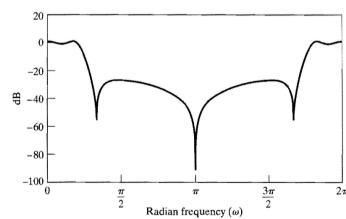


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32

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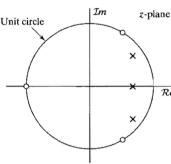
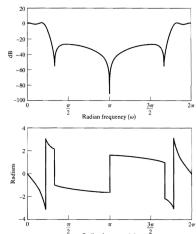


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33

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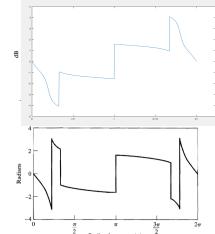


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34

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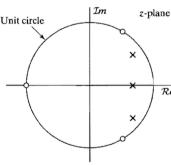
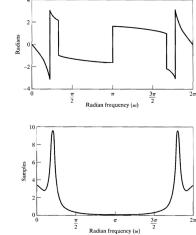


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35

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36

Stability and Causality



LTI System

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

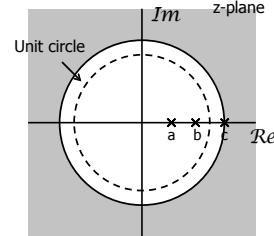
- ❑ Transfer function is not unique without ROC
 - If diff. eq represents LTI and causal system, ROC is region outside all singularities
 - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

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38

Example: ROC from Pole-Zero Plot

ROC 1: right-sided



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39

LTI System

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

If stable and causal, all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

- ❑ Transfer function is not unique without ROC
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40

All-Pass Systems



All-Pass Filters

- ❑ A system is an all-pass system if

$$|H(e^{j\omega})| = 1, \text{ all } \omega$$
- ❑ Its phase response $\theta(\omega)$ may be non-trivial

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42

First Order All-Pass Filter (a real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

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43

First Order All-Pass Filter (a real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$\begin{aligned}|H(e^{j\omega})| &= \left| \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \right| \\&= \frac{\left| e^{-j\omega}(1 - a^* e^{j\omega}) \right|}{\left| 1 - ae^{-j\omega} \right|} \\&= \frac{\left| 1 - a^* e^{j\omega} \right|}{\left| 1 - ae^{-j\omega} \right|} = 1\end{aligned}$$

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44

General All-Pass Filter

- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

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45

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46

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47

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- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

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48

General All-Pass Filter

- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$\begin{aligned}d_k &= -\frac{3}{4} \\e_k &= 0.8e^{j\pi/4}\end{aligned}$$

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49

General All-Pass Filter

- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

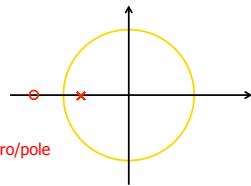
$$H_{\text{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

Real zero/pole



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50

General All-Pass Filter

- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

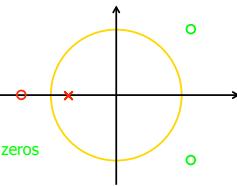
$$H_{\text{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

Complex zeros



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51

General All-Pass Filter

- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

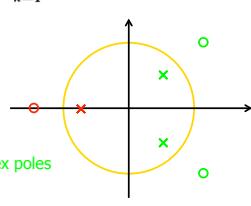
$$H_{\text{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

Complex poles



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52

All Pass Filter Phase Response

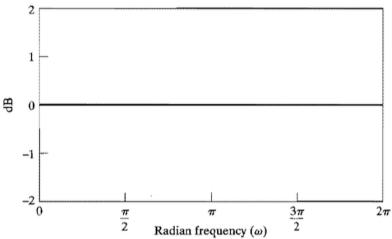
$$\begin{aligned} \text{First order system } H(e^{j\omega}) &= \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \\ &= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}} \end{aligned}$$

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53

First Order Example

- Magnitude:



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54

All Pass Filter Phase Response

$$\begin{aligned} \text{First order system } H(e^{j\omega}) &= \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \\ &= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}} \end{aligned}$$

$$\text{phase } \arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$$

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55

All Pass Filter Phase Response

First order system $H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$

$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

phase $\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$

$$= \arg\left(\frac{e^{-j\omega}(1 - re^{-j\theta}e^{j\omega})}{1 - re^{j\theta}e^{-j\omega}}\right)$$

$$= \arg(e^{-j\omega}) + \arg(1 - re^{-j\theta}e^{j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$

$$= -\omega - \arg(1 - re^{j\theta}e^{-j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$

$$= -\omega - 2\arg(1 - re^{j\theta}e^{-j\omega})$$

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56

Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

Look at each factor:

$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)}\right)$$

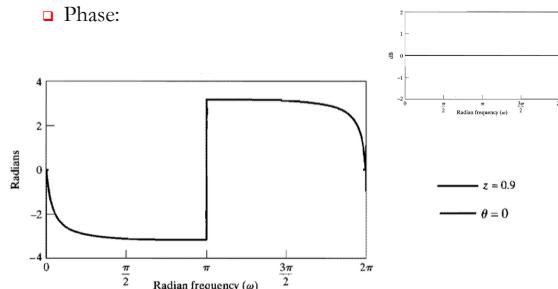
$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

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Adapted from M. Lustig, EECS Berkeley

57

First Order Example

Phase:

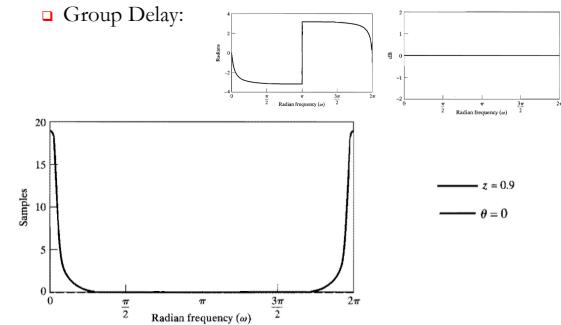


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58

First Order Example

Group Delay:



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59

All Pass Filter Phase Response

Second order system with poles at $z = re^{j\theta}, re^{-j\theta}$

$$\angle \left[\frac{(e^{-j\omega} - re^{-j\theta})(e^{-j\omega} - re^{j\theta})}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})} \right] = -2\omega - 2 \arctan \left[\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right]$$

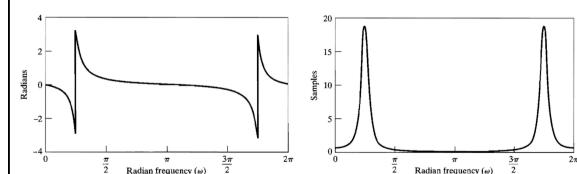
$$-2 \arctan \left[\frac{r \sin(\omega + \theta)}{1 - r \cos(\omega + \theta)} \right].$$

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60

Second Order Example

Poles at $z = 0.9e^{\pm j\pi/4}$ (zeros at conjugates)



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61

All-Pass Properties

- Claim: For a stable, causal ($r < 1$) all-pass system:
 - $\arg[H_{ap}(e^{j\omega})] \leq 0$
 - Unwrapped phase always non-positive and decreasing
 - $\text{grd}[H_{ap}(e^{j\omega})] > 0$
 - Group delay always positive
- Intuition
 - delay is positive \Rightarrow system is causal
 - Phase negative \Rightarrow phase lag

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62

Minimum-Phase Systems



Minimum-Phase Systems

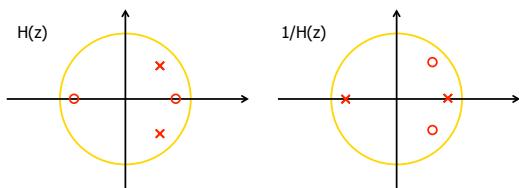
- Definition: A stable and causal system $H(z)$ (i.e. poles inside unit circle) whose inverse $1/H(z)$ is also stable and causal (i.e. zeros inside unit circle)

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64

Minimum-Phase Systems

- Definition: A stable and causal system $H(z)$ (i.e. poles inside unit circle) whose inverse $1/H(z)$ is also stable and causal (i.e. zeros inside unit circle)
 - All poles and zeros inside unit circle



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65

All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

- Approach:
 - (1) First construct H_{ap} with all zeros outside unit circle
 - (2) Compute

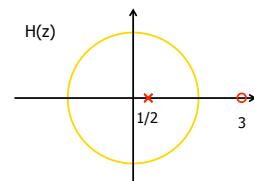
$$H_{\min}(z) = \frac{H(z)}{H_{ap}(z)}$$

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66

Min-Phase Decomposition Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$



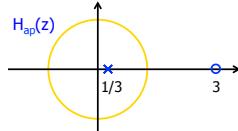
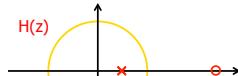
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67

Min-Phase Decomposition Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Set $H_{ap}(z) = \frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$



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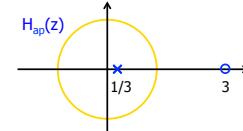
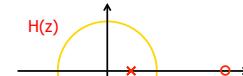
68

Min-Phase Decomposition Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Set $H_{ap}(z) = \frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$

$$H_{min}(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}} \left(\frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}} \right)^{-1}$$



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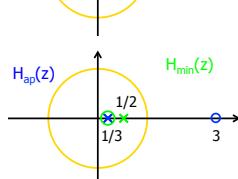
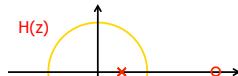
69

Min-Phase Decomposition Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Set $H_{ap}(z) = \frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$

$$H_{min}(z) = -3 \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}}$$

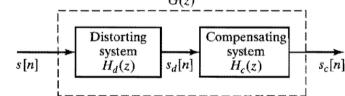


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70

Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:

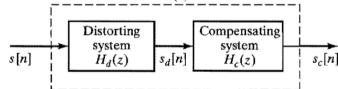


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71

Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:

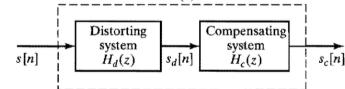
- $H_c(z) = 1/H_d(z)$ ← also stable and causal

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72

Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:

- $H_c(z) = 1/H_d(z)$ ← also stable and causal

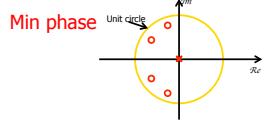
- Else, decompose $H_d(z) = H_{d,min}(z) H_{d,ap}(z)$

- $H_c(z) = 1/H_{d,min}(z) \rightarrow H_d(z) H_c(z) = H_{d,ap}(z)$
- Compensate for magnitude distortion

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73

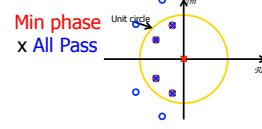
Minimum Phase to Max Phase



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74

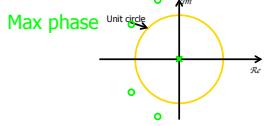
Minimum Phase to Max Phase



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75

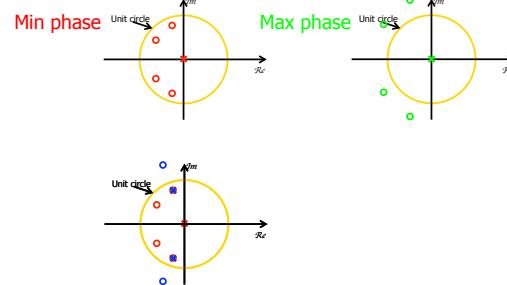
Minimum Phase to Max Phase



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76

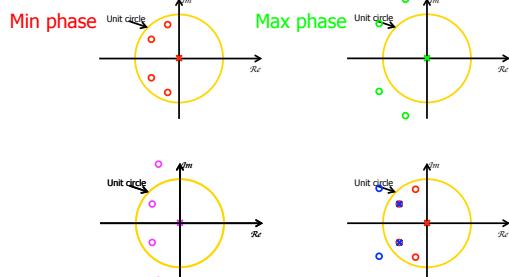
Minimum Phase



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77

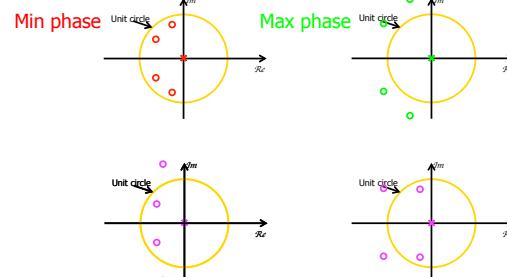
Minimum Phase



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78

Minimum Phase



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79

Minimum Phase Lag Property

- All pass properties

- $\arg[H_{ap}(e^{j\omega})] \leq 0$
- $\text{grd}[H_{ap}(e^{j\omega})] > 0$

$$\begin{aligned} \square \quad \arg[H_{\max}(e^{j\omega})] &= \arg[H_{\min}(e^{j\omega}) * H_{ap}(e^{j\omega})] \\ &= \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})] \\ &= \leq 0 \quad + \quad \leq 0 \end{aligned}$$

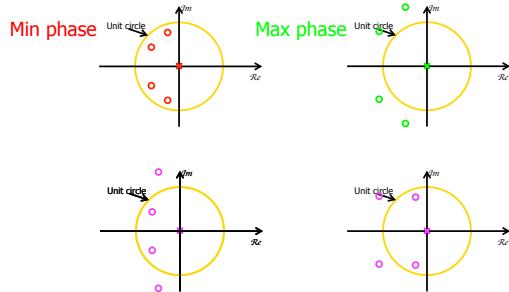
Minimum Group Delay Property

- All pass properties

- $\arg[H_{ap}(e^{j\omega})] \leq 0$
- $\text{grd}[H_{ap}(e^{j\omega})] > 0$

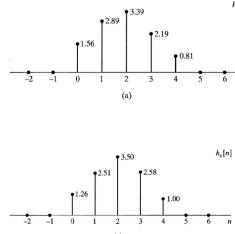
$$\begin{aligned} \square \quad \text{grd}[H_{\max}(e^{j\omega})] &= \text{grd}[H_{\min}(e^{j\omega}) * H_{ap}(e^{j\omega})] \\ &= \text{grd}[H_{\min}(e^{j\omega})] + \text{grd}[H_{ap}(e^{j\omega})] \\ &= \geq 0 \quad + \quad \geq 0 \end{aligned}$$

Minimum Phase

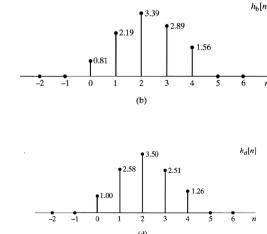


Minimum Energy-Delay Property

Min phase



Max phase



Big Ideas

- Frequency Response of LTI Systems
 - Magnitude Response, Phase Response, Group Delay
- LTI Stability and Causality
 - If all poles inside unit circle
- All Pass Systems
 - Used for delay compensation
- Minimum Phase Systems
 - Can compensate for magnitude distortion
 - Minimum energy-delay property

Admin

- Dhaval extra now
 - See Piazza for time and location
 - Dhaval TH/F OH cancelled
- Yinghao OH W will be exam review session
- No office hours during spring break
- HW 6 posted after midterm

Admin - Midterm

- Midterm Thursday 3/5
 - During class
 - Starts at **exactly 4:30pm**, ends at exactly 5:50pm (80 minutes)
 - Location DRLB A2
 - Old exams posted on previous years' website
 - Disclaimer: old exams before 2019 covered more material
 - Covers Lec 1 - 11
 - Closed book, one page (8.5x11) cheat sheet allowed
 - Calculators allowed, no smart phones