

ESE 531: Digital Signal Processing

Lec 15: March 24, 2020

Generalized Linear Phase Systems, Design
of IIR Filters



Your First Priority

- ❑ Your first priority is your health
 - You should abide by all health guidelines
 - Wash your hands
 - Cough/Sneeze into your elbow
 - Don't touch your face
 - Maintain ~~social~~ physical distancing
 - Virtual social interaction is encouraged!
 - Stay home except for necessary activities
 - Part of your health is your mental and emotional health
 - See <https://caps.wellness.upenn.edu/selfhelp/> for help



Advice for Remote Learning Success

- ❑ Attend lecture in real time and stay on schedule
 - Rewatch recorded lectures, attend virtual office hours
 - We are here to help, but you need to let us know you need help
- ❑ Start homework early!
- ❑ Form a virtual study group with classmates
 - Use piazza to find groups
- ❑ Useful study habits reference:
 - <https://lsa.umich.edu/content/dam/rll-assets/rll-docs/Study%20Habits.pdf>



Course Grading Changes

- ❑ Homework
 - 20% of grade (same)
 - Only 8 HW instead of 9
 - Still drop lowest homework (same)
- ❑ Final Project
 - 30% of grade (increased)
- ❑ Midterm Exam
 - 25% of grade (same)
- ❑ Final Exam
 - 25% of grade (decreased)



Course Structure Changes

- ❑ Homework Canvas submission, course webpage and Piazza all the same
 - Recommend you turn on Piazza email notifications
- ❑ Virtual lectures via Zoom
 - See Piazza threads for link
- ❑ Virtual Office hours via Google Hangouts/Zoom
 - See Piazza threads for schedule and links
- ❑ Big challenge will be staying on pace with material
 - Requires self-motivation



I want to hear from you...

- ☐ Will you be in a different time zone?
- ☐ Will you have trouble seeing or hearing video lectures?
- ☐ Are there any other accessibility issues I should know about?
- ☐ Email me any concerns -- I will do everything I can to ensure you have the same learning objectives from Lecture 1

New Course Staff

❑ New TAs/Graders





Questions?

- ❑ Use chat panel to type question
 - Scroll to bottom of Zoom window and click “chat” button



Lecture Outline

- ❑ Review: Where did we leave off?
- ❑ Generalized Linear Phase Systems
- ❑ IIR Filters (if time)

Frequency Response of Systems (Filters)



Frequency Response of LTI System

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- We can define a magnitude response

$$|Y(e^{j\omega})| = |H(e^{j\omega})| |X(e^{j\omega})|$$

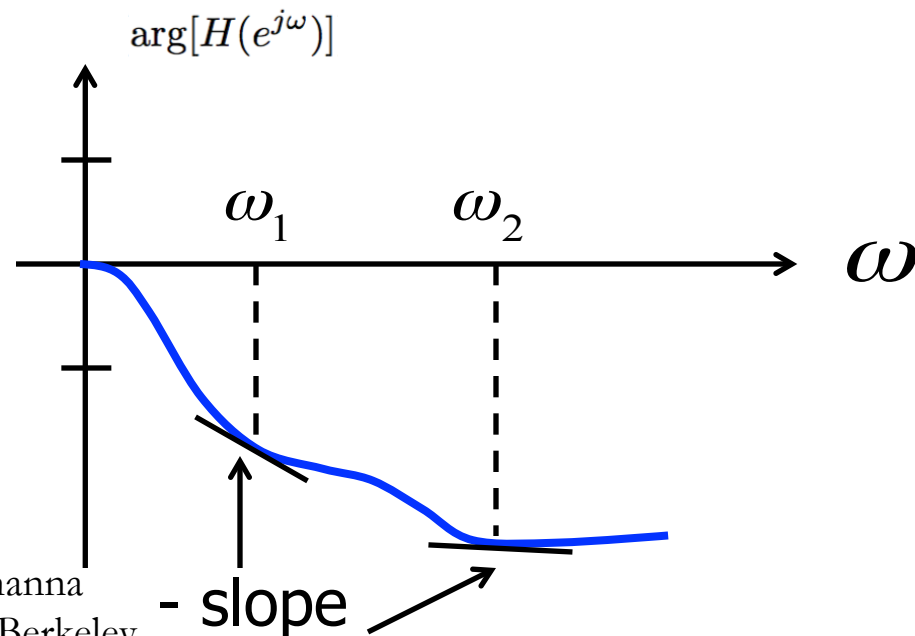
- And a phase response

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$



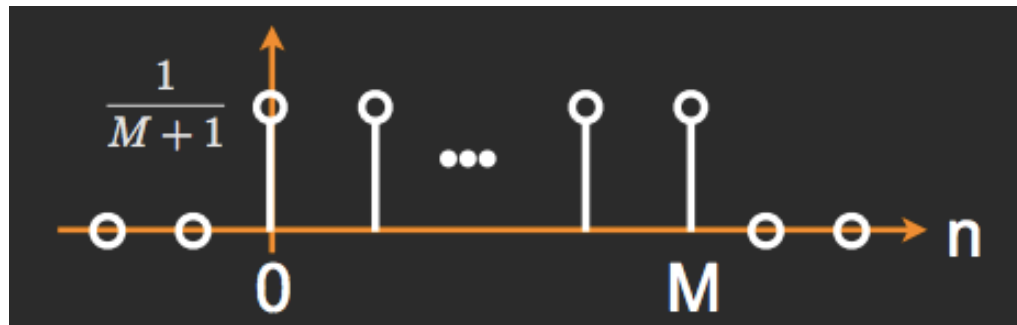
Example: Moving Average

□ Moving Average Filter

- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

Impulse
response



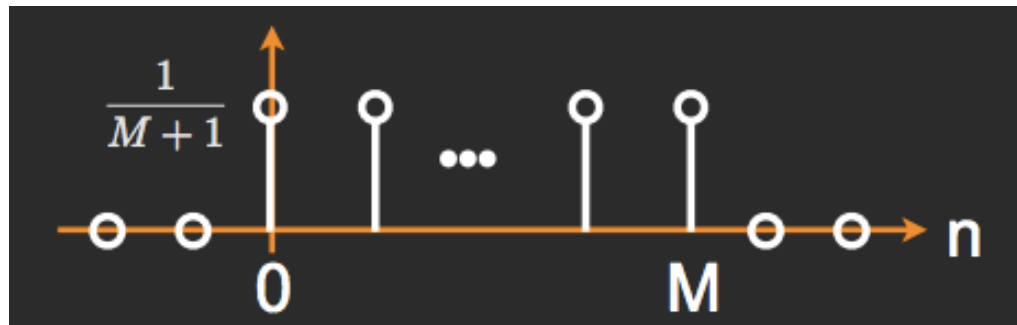
Example: Moving Average

□ Moving Average Filter

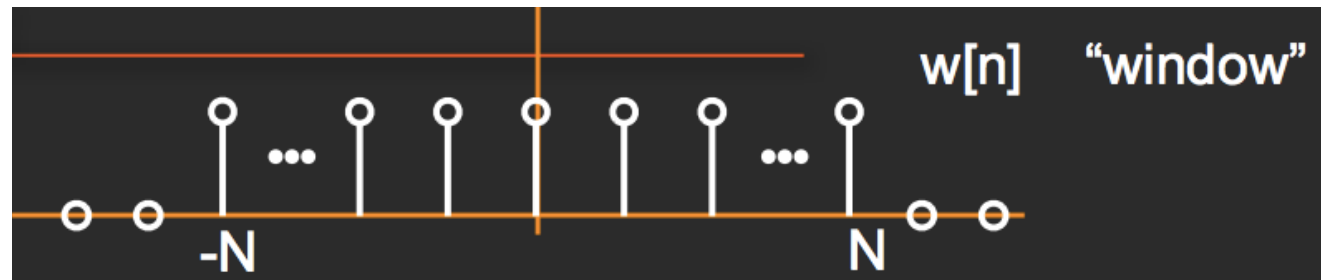
- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

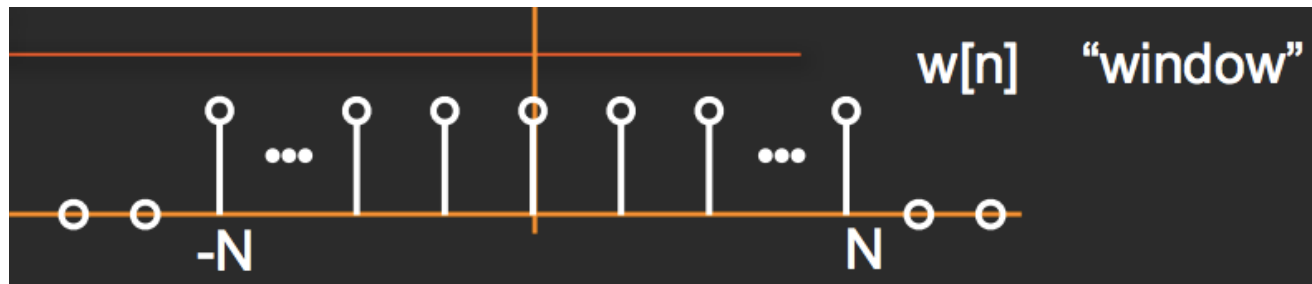
Impulse
response



Scaled & Time
Shifted window

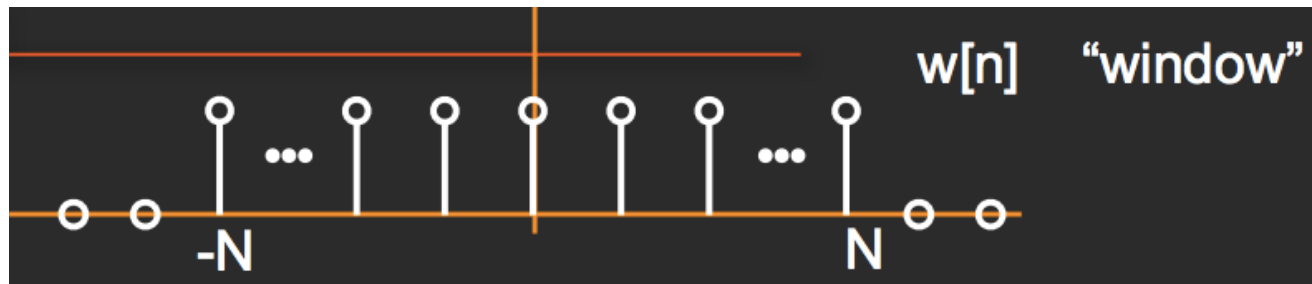


Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N + 1/2)\omega\right)}{\sin(\omega/2)}$$

Example: Moving Average

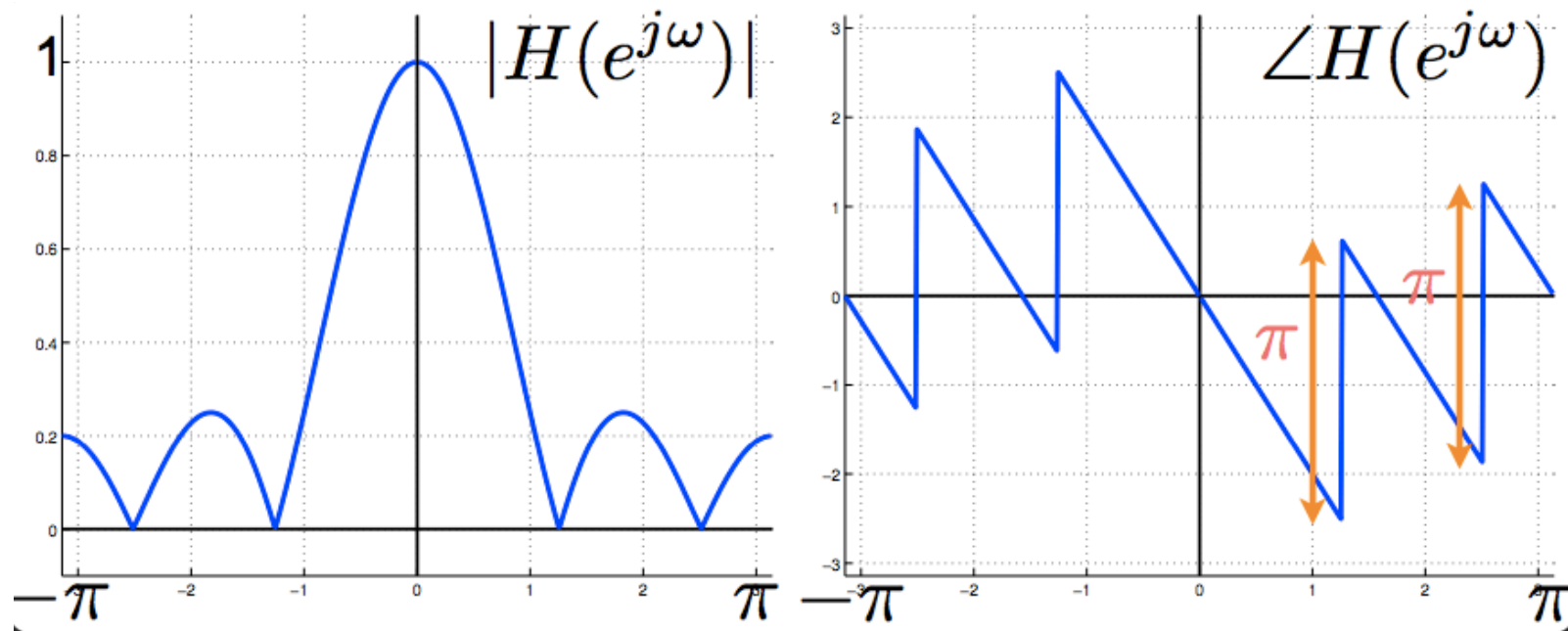


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$\frac{1}{M+1} w[n - M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Moving Average



$$W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

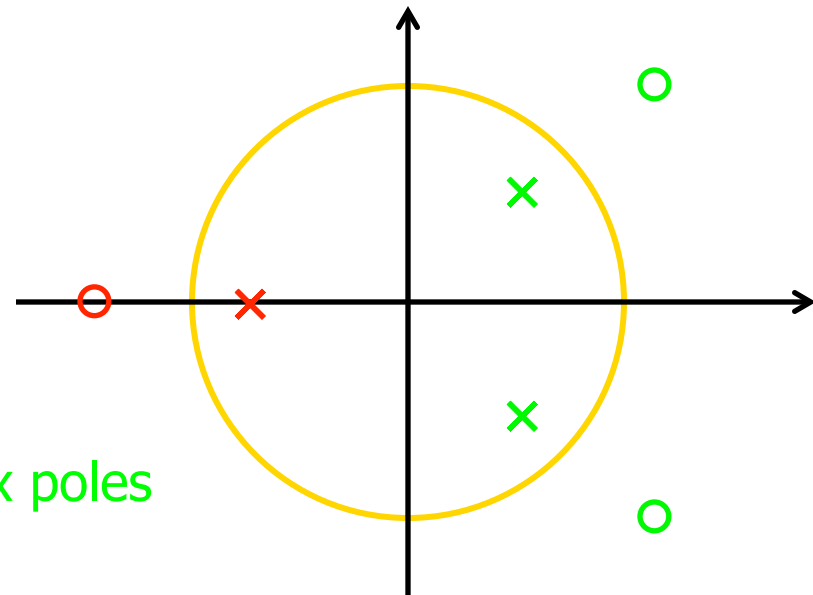
General All-Pass Systems

- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

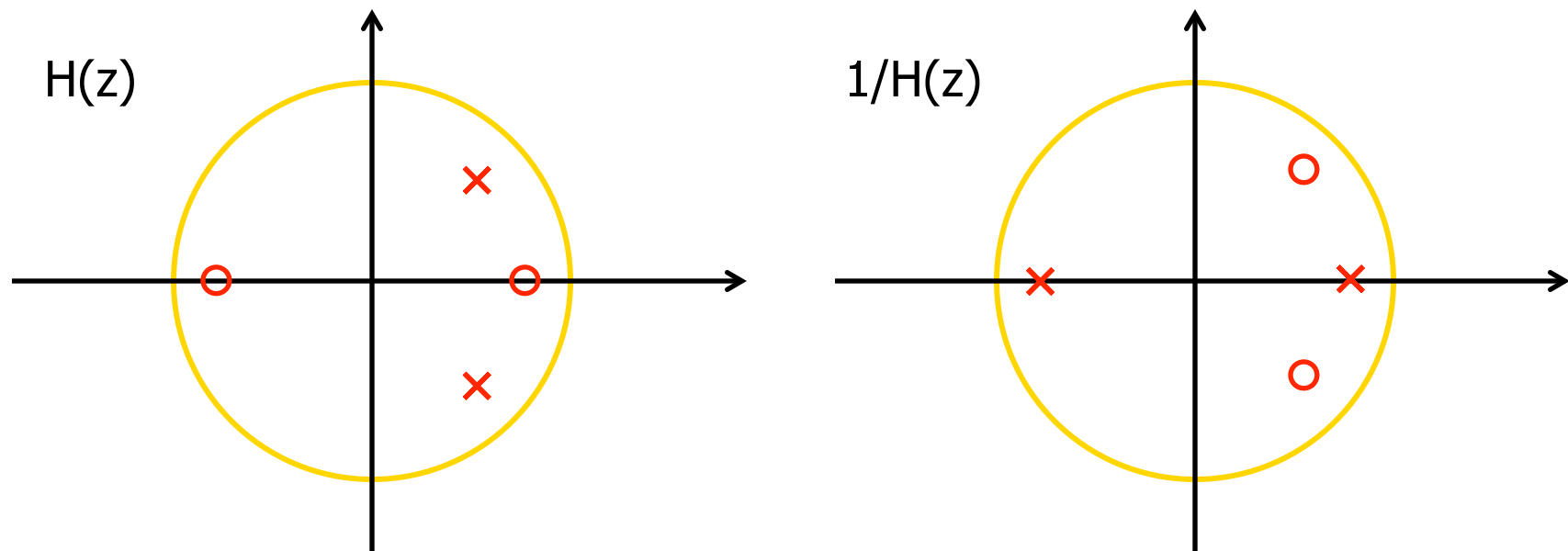
$$d_k = -\frac{3}{4}$$
$$e_k = 0.8e^{j\pi/4}$$



Complex poles

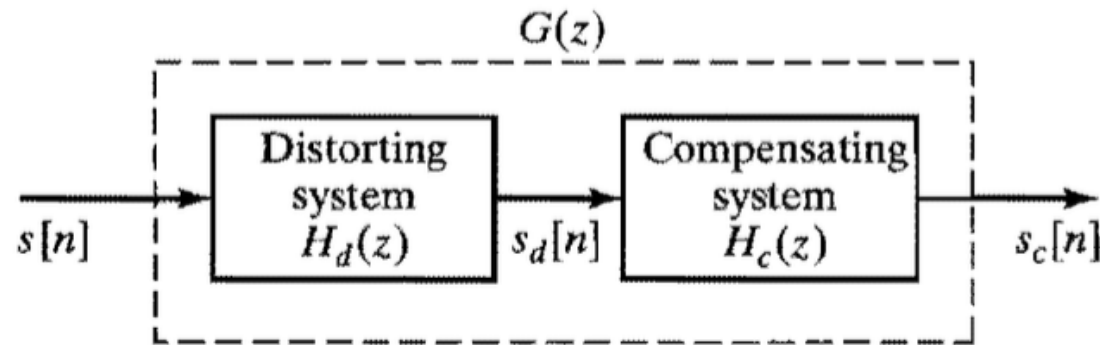
Minimum-Phase Systems

- Definition: A stable and causal system $H(z)$ (i.e. poles inside unit circle) whose inverse $1/H(z)$ is also stable and causal (i.e. zeros inside unit circle)
 - All poles and zeros inside unit circle



Min-Phase Decomposition Purpose

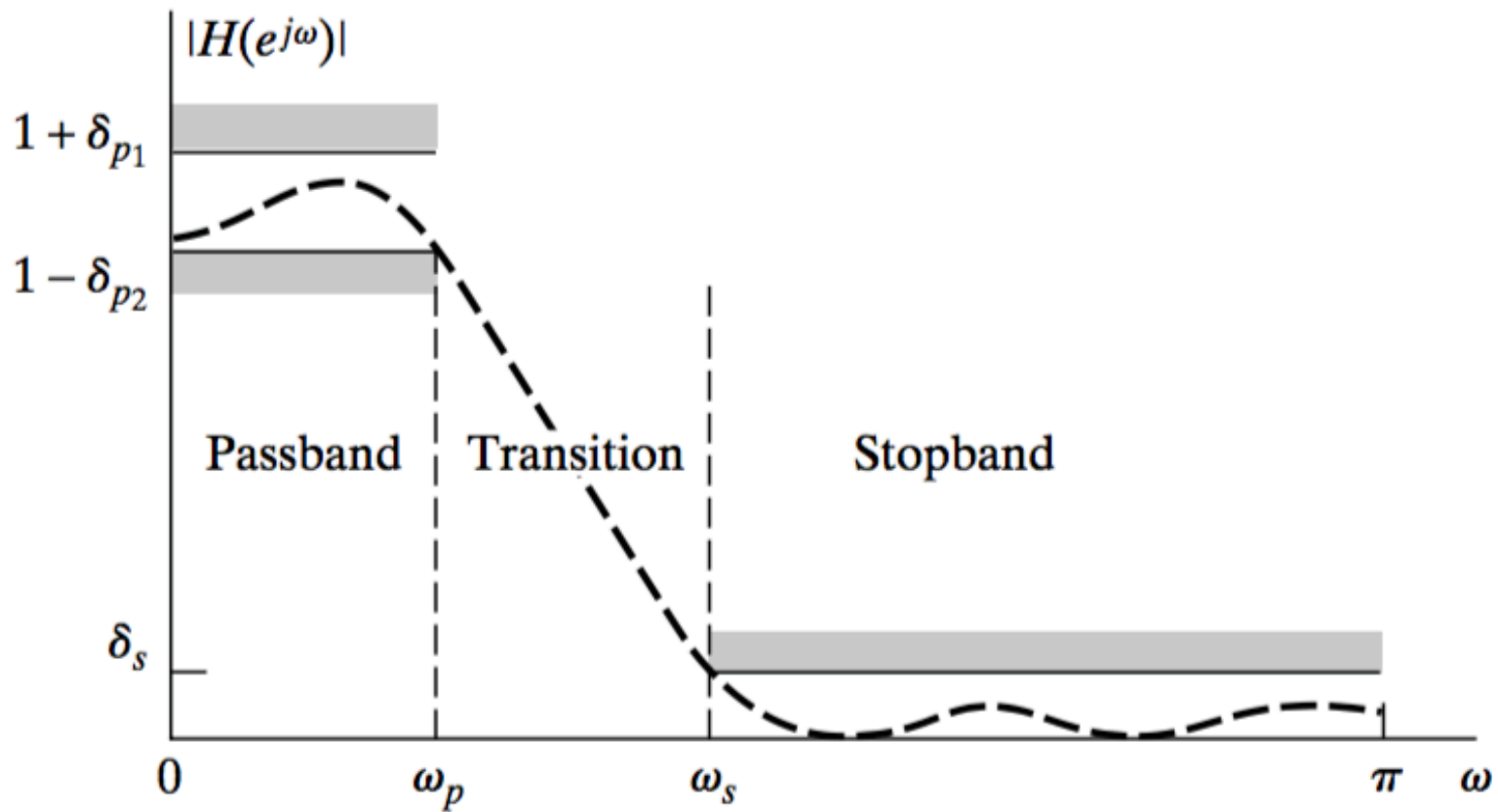
- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
 - $H_c(z) = 1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z) = H_{d,\min}(z) H_{d,\text{ap}}(z)$
 - $H_c(z) = 1/H_{d,\min}(z) \rightarrow H_d(z)H_c(z) = H_{d,\text{ap}}(z)$
 - Compensate for magnitude distortion

FIR Filter Design

Filter Specifications





What is a Linear Filter?

- ❑ Attenuates certain frequencies
- ❑ Passes certain frequencies
- ❑ Affects both phase and magnitude

- ❑ What does it mean to **design** a filter?
 - Determine the parameters of a transfer function or difference equation that approximates a desired impulse response ($h[n]$) or frequency response ($H(e^{j\omega})$).



What is a Linear Filter?

- ❑ Attenuates certain frequencies
- ❑ Passes certain frequencies
- ❑ Affects both phase and magnitude

- ❑ IIR
 - Mostly non-linear phase response
 - Could be linear over a range of frequencies
- ❑ FIR
 - Much easier to control the phase
 - Both non-linear and linear phase

Generalized Linear Phase Systems



Generalized Linear Phase

- An LTI system has generalized linear phase if the frequency response $H(e^{j\omega})$ can be expressed as:

$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha+j\beta}, \quad |\omega| < \pi$$

- Where $A(\omega)$ is a real function.

Generalized Linear Phase

- An LTI system has generalized linear phase if frequency response $H(e^{j\omega})$ can be expressed as:

$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha+j\beta}, \quad |\omega| < \pi$$

- Where $A(\omega)$ is a real function.
- What is the group delay?



Causal FIR Systems

$$y[n] = b_0x[n] + b_1x[n-1] + \dots + b_Mx[n-M], \quad \text{all } n$$

$$H(z) = b_0 + b_1z^{-1} + \dots + b_Mz^{-M} = b_0 \prod_{k=1}^M (1 - c_k z^{-1})$$

$$h[n] = \begin{cases} b_n, & n = 0, 1, \dots, M \\ 0, & \text{otherwise} \end{cases}$$



Causal FIR Systems

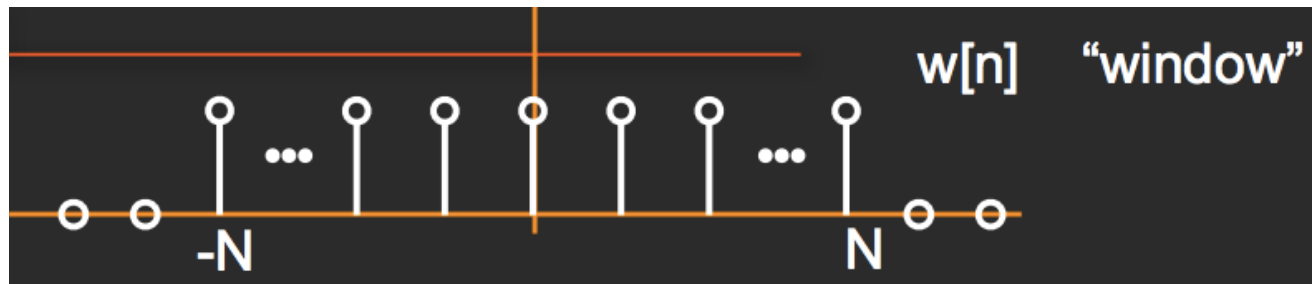
- Causal FIR systems have generalized linear phase if they have impulse response length $(M+1)$ and some symmetry
- It can be shown if

$$h[n] = \begin{cases} h[M - n], & 0 \leq n \leq M, \\ 0, & \text{otherwise,} \end{cases}$$

- then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



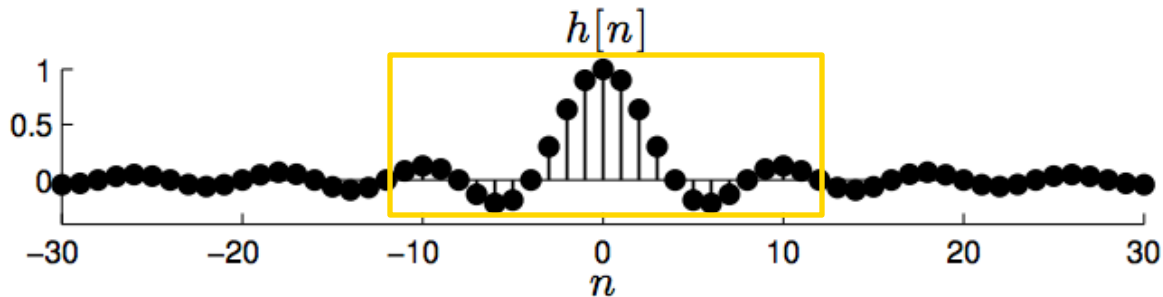
$$\frac{1}{M+1} w[n - M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate
and shift

$$h_{LP}[n] = w_N[n - N] \cdot h[n - N]$$



Causal FIR Systems

- ❑ Causal FIR systems have generalized linear phase if they have impulse response length $(M+1)$
- ❑ It can be shown if

$$h[n] = \begin{cases} h[M - n], & 0 \leq n \leq M, \\ 0, & \text{otherwise,} \end{cases}$$

- ❑ Then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

- ❑ Sufficient conditions to guarantee GLP, not necessary
 - Other systems can also have GLP



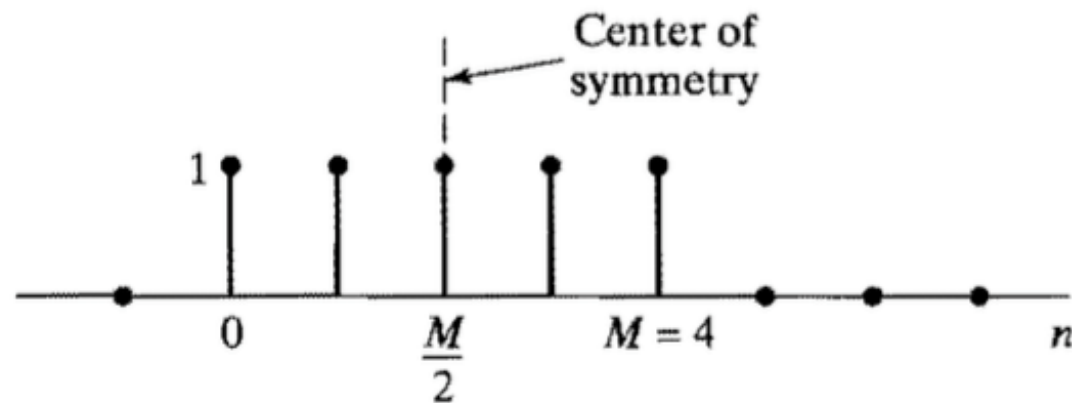
FIR GLP Systems

- ❑ Four types of FIR GLP
 - Type I
 - Even Symmetry, M even
 - Type II
 - Even Symmetry, M odd
 - Type III
 - Odd Symmetry, M even
 - Type IV
 - Odd Symmetry, M odd

FIR GLP: Type I

Type I Even Symmetry, M even

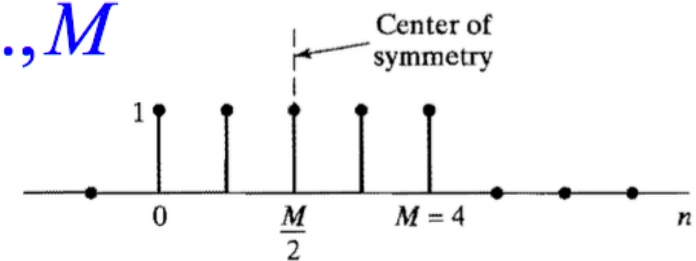
$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$



FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$



$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

← integer delay



FIR GLP: Type I – Example, $M=4$

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

← integer delay



FIR GLP: Type I – Example, $M=4$

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2} \quad \leftarrow \text{integer delay}$$

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$



FIR GLP: Type I – Example, $M=4$

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2} \quad \leftarrow \text{integer delay}$$

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} \left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[3]e^{-j\omega} + h[4]e^{-j2\omega} \right] \end{aligned}$$



FIR GLP: Type I – Example, $M=4$

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2} \quad \leftarrow \text{integer delay}$$

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} \left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega} \right] \end{aligned}$$

FIR GLP: Type I – Example, $M=4$

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$

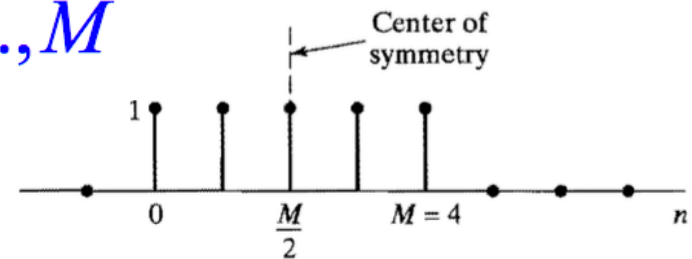
$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2} \quad \leftarrow \text{integer delay}$$

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} \left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega} \right] \\ &= \underbrace{\left[2h[0]\cos(2\omega) + 2h[1]\cos(\omega) + h[2] \right]}_{A(\omega) \text{ (even)}} e^{-j2\omega} \end{aligned}$$

FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$



$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

← integer delay

$$H(e^{j\omega}) = e^{-j\omega M/2} \left(\sum_{k=0}^{M/2} a[k] \cos \omega k \right)$$

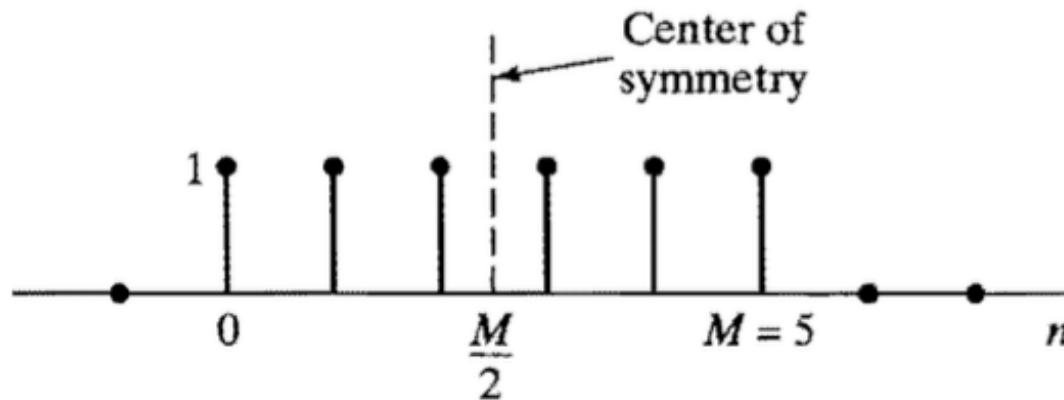
$$a[0] = h[M/2],$$

$$a[k] = 2h[(M/2) - k], \quad k = 1, 2, \dots, M/2.$$

FIR GLP: Type II

Type II Even Symmetry, M odd

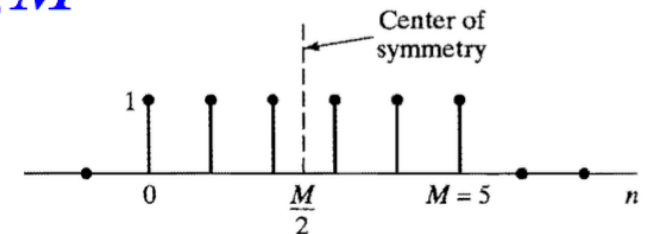
$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$



FIR GLP: Type II

Type II Even Symmetry, M odd

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$



$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

← integer delay



FIR GLP: Type II – Example, $M=3$

Type II Even Symmetry, M odd

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n]e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

← integer delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$$



FIR GLP: Type II – Example, $M=3$

Type II Even Symmetry, M odd

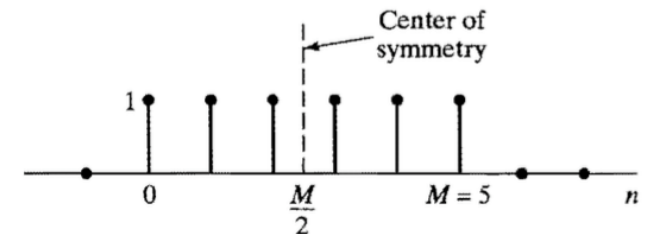
$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n]e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2} \quad \leftarrow \text{integer delay}$$

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$$

$$H(e^{j\omega}) = \underbrace{[2h[0]\cos(3\omega/2) + 2h[1]\cos(\omega/2)]}_{A(\omega)} e^{-j3\omega/2}$$

FIR GLP: Type II

Type II Even Symmetry, M odd



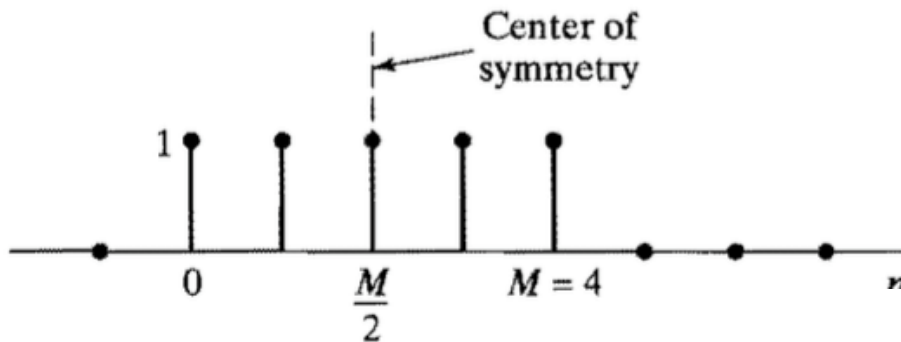
$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

integer delay

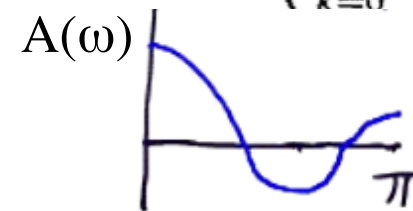
$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=1}^{(M+1)/2} b[k] \cos \left[\omega \left(k - \frac{1}{2} \right) \right] \right\}$$

$$b[k] = 2h[(M+1)/2 - k], \quad k = 1, 2, \dots, (M+1)/2.$$

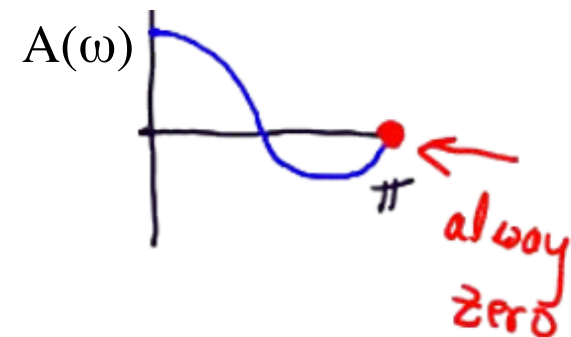
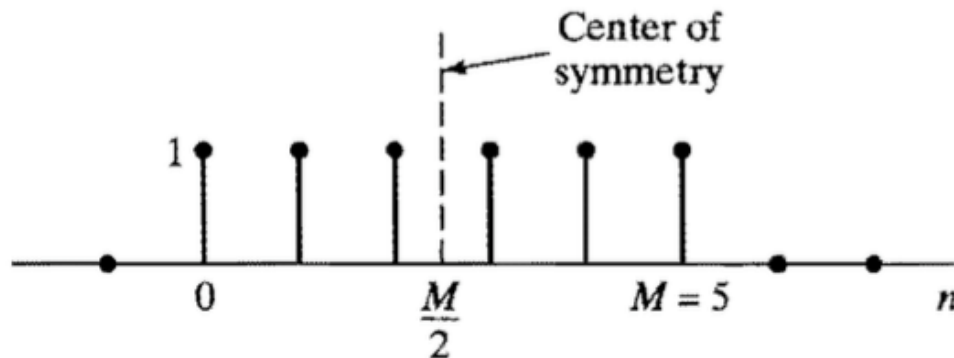
FIR GLP: Type I and II



$$H(e^{j\omega}) = e^{-j\omega M/2} \left(\sum_{k=0}^{M/2} a[k] \cos \omega k \right)$$



$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=1}^{(M+1)/2} b[k] \cos \left[\omega \left(k - \frac{1}{2} \right) \right] \right\}$$

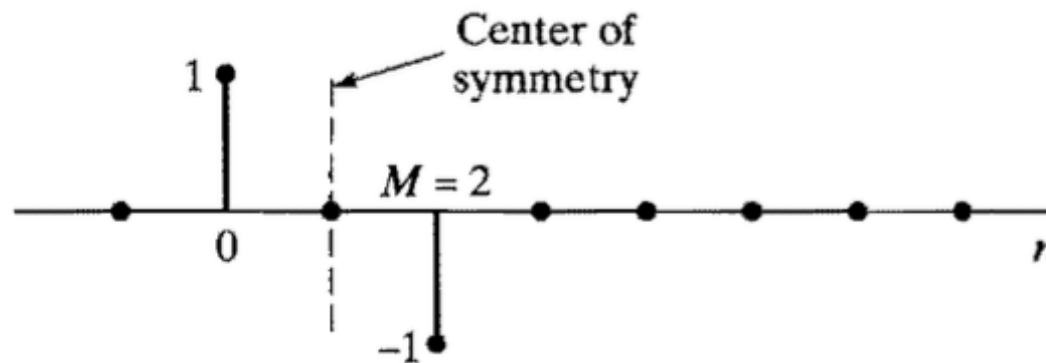




FIR GLP: Type III

Type III Odd Symmetry, M even

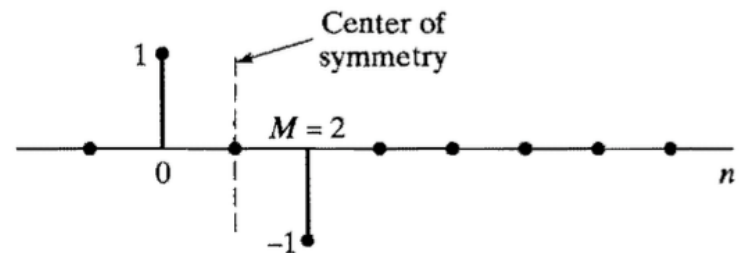
$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$



FIR GLP: Type III

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$



$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

FIR GLP: Type III – Example, $M=4$

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

FIR GLP: Type III – Example, $M=4$

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$



FIR GLP: Type III – Example, $M=4$

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} \left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[3]e^{-j\omega} + h[4]e^{-j2\omega} \right] \end{aligned}$$



FIR GLP: Type III – Example, $M=4$

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n]e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} \left[h[0]e^{j2\omega} + h[1]e^{j\omega} - h[1]e^{-j\omega} - h[0]e^{-j2\omega} \right] \end{aligned}$$



FIR GLP: Type III – Example, $M=4$

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n]e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} \left[h[0]e^{j2\omega} + h[1]e^{j\omega} - h[1]e^{-j\omega} - h[0]e^{-j2\omega} \right] \\ &= \underbrace{\left[2h[0]\sin(2\omega) + 2h[1]\sin(\omega) \right]}_{A(\omega) \text{ (odd)}} j e^{-j2\omega} \end{aligned}$$

FIR GLP: Type III

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

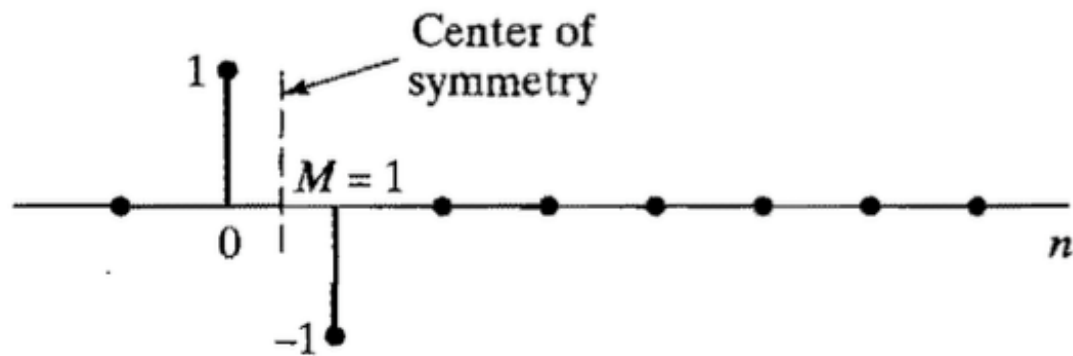
$$H(e^{j\omega}) = j e^{-j\omega M/2} \left[\sum_{k=1}^{M/2} c[k] \sin \omega k \right]$$

$$c[k] = 2h[(M/2) - k], \quad k = 1, 2, \dots, M/2.$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (M / 2 \text{ not an integer})$$





FIR GLP: Type IV – Example, $M=3$

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (M / 2 \text{ not an integer})$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

← fractional delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$$



FIR GLP: Type IV – Example, $M=3$

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (M / 2 \text{ not an integer})$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

fractional delay

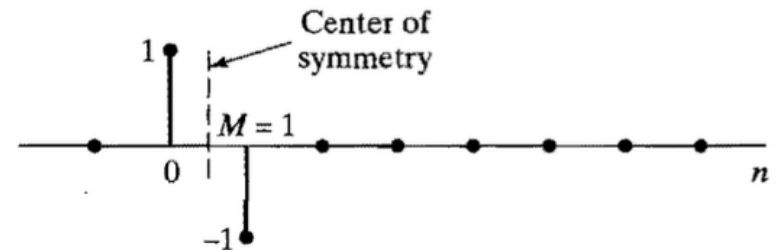
$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$$

$$H(e^{j\omega}) = \underbrace{[2h[0]\sin(3\omega/2) + 2h[1]\sin(\omega/2)]}_{A(\omega)} j e^{-j3\omega/2}$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (M/2 \text{ not an integer})$$



$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

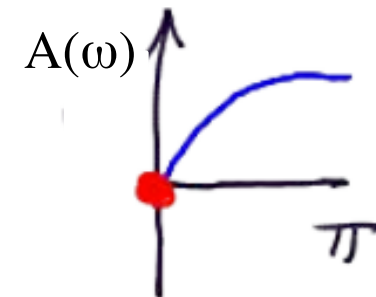
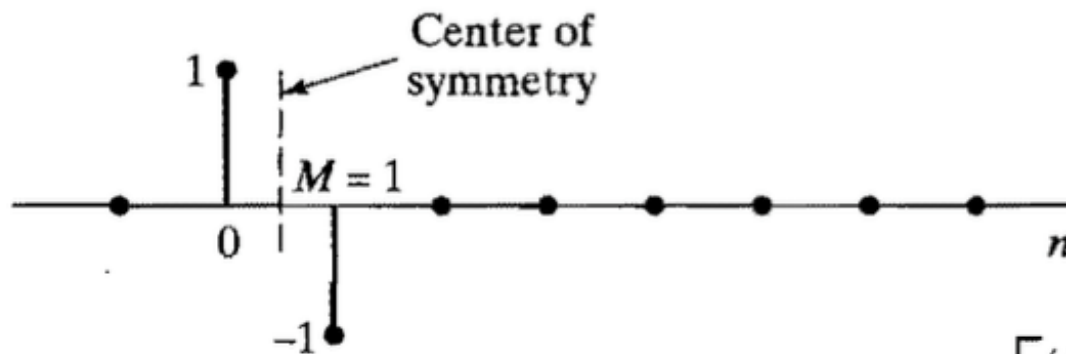
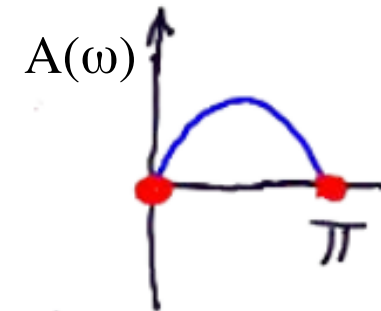
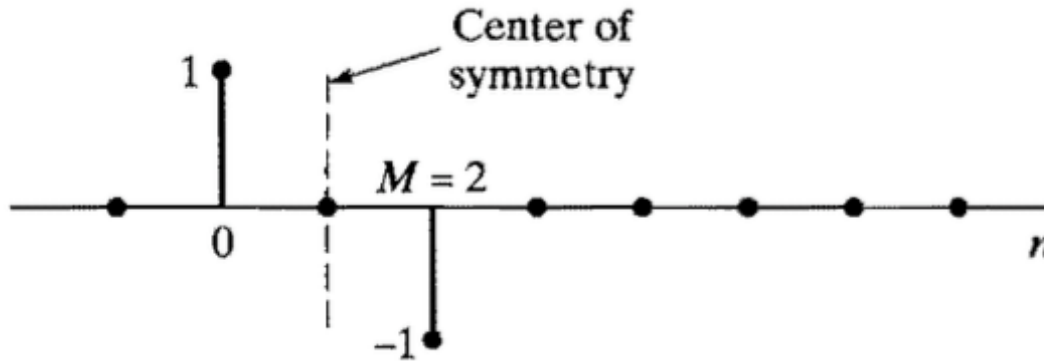
fractional delay

$$H(e^{j\omega}) = j e^{-j\omega M/2} \left[\sum_{k=1}^{(M+1)/2} d[k] \sin \left[\omega \left(k - \frac{1}{2} \right) \right] \right]$$

$$d[k] = 2h[(M+1)/2 - k], \quad k = 1, 2, \dots, (M+1)/2.$$

FIR GLP: Type III and IV

$$H(e^{j\omega}) = je^{-j\omega M/2} \left[\sum_{k=1}^{M/2} c[k] \sin \omega k \right]$$



$$H(e^{j\omega}) = je^{-j\omega M/2} \left[\sum_{k=1}^{(M+1)/2} d[k] \sin \left[\omega \left(k - \frac{1}{2} \right) \right] \right]$$



Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$



Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

$$H(z) = \sum_{n=0}^M h[M - n]z^{-n}$$



Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

$$H(z) = \sum_{n=0}^M h[M-n]z^{-n} = \sum_{k=M}^0 h[k]z^k z^{-M}$$



Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

$$\begin{aligned} H(z) &= \sum_{n=0}^M h[M-n]z^{-n} = \sum_{k=M}^0 h[k]z^k z^{-M} \\ &= z^{-M} H(z^{-1}). \end{aligned}$$

□ If z_0 is a zero of $H(z)$, then z_0^{-1} is also a zero of $H(z)$



Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

If z_0 is a zero then z_0^{-1} is also a zero.

- If $h[n]$ is real,

If z_0 is a zero then z_0^* is also a zero.

(Use conjugate symmetry property of real signals)

Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - \underbrace{re^{j\theta} z^{-1}}_{z_0})(1 - \underbrace{re^{-j\theta} z^{-1}}_{(z_0)^*})(1 - \underbrace{r^{-1}e^{j\theta} z^{-1}}_{(z_0^{-1})^*})(1 - \underbrace{r^{-1}e^{-j\theta} z^{-1}}_{z_0^{-1}})$$

Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

□ If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

□ If zero is on unit circle ($r=1$)

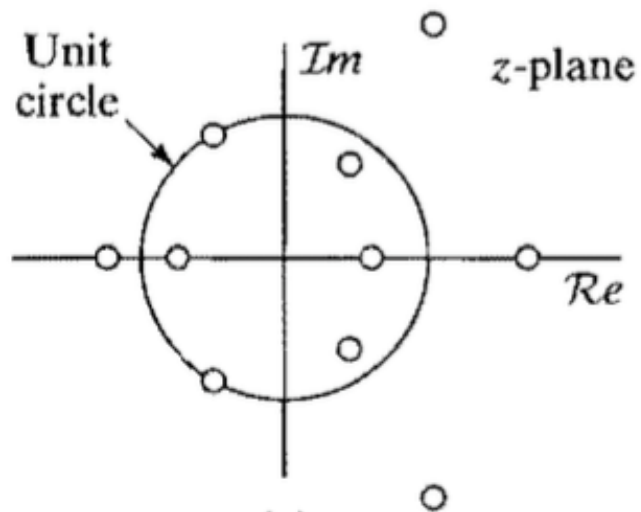
$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

□ If zero is real and not on unit circle ($\theta=0$)

$$(1 + rz^{-1})(1 + r^{-1}z^{-1}).$$

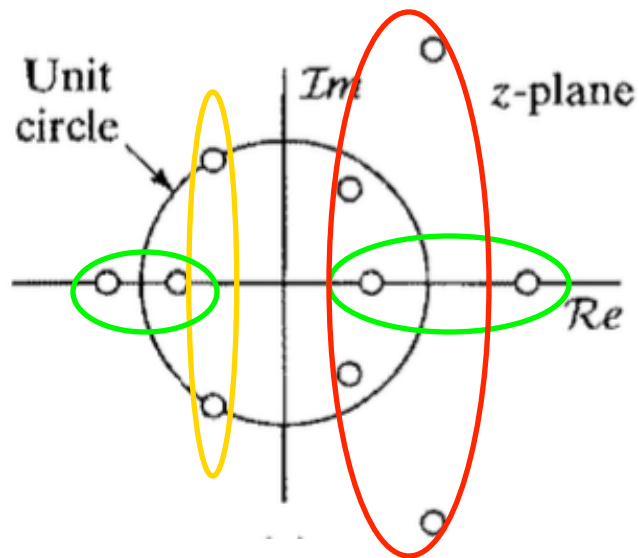
Zeros of GLP System – Type I

Type I



Zeros of GLP System – Type I

Type I

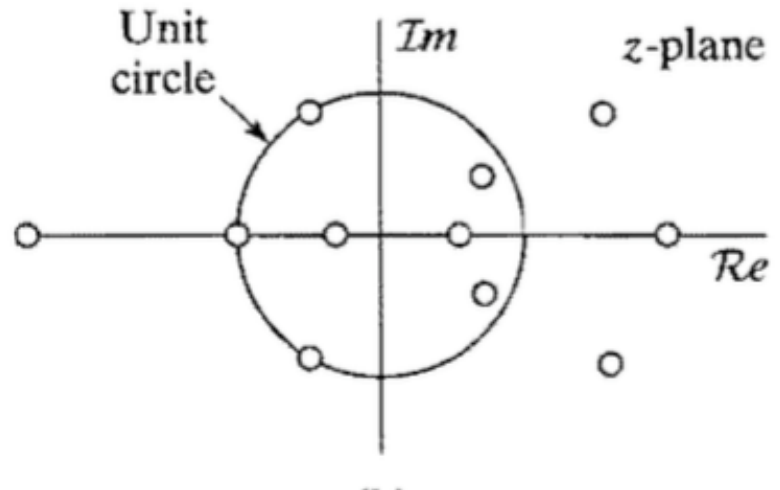


Group of 4 reciprocal and conjugate zeros

Group of 2 real and not on unit circle zeros

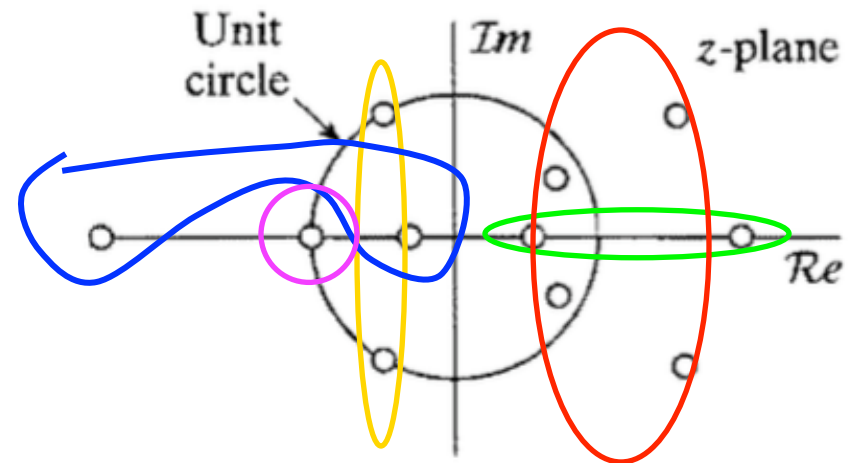
Group of 2 on unit circle zeros

Type II



Zeros of GLP System – Type II

Type II



Group of 4 reciprocal and conjugate zeros

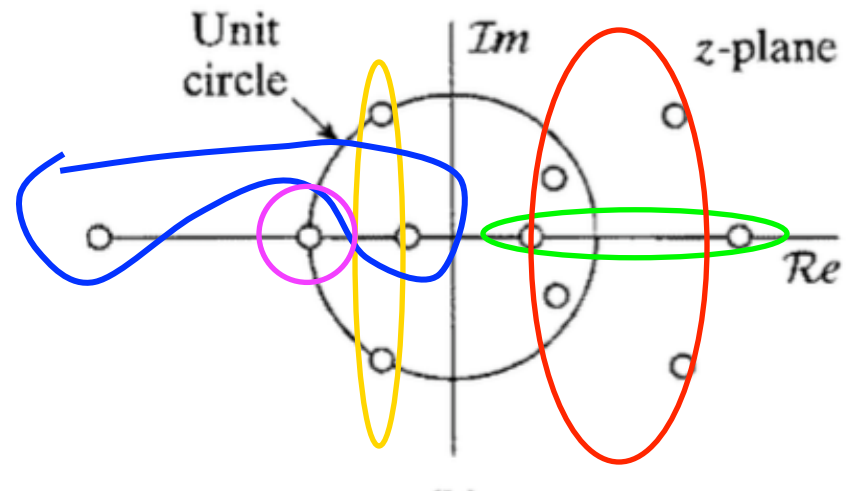
Group of 2 real and not on unit circle zeros

Group of 2 real and not on unit circle zeros

Group of 2 on unit circle zeros

Zeros of GLP System – Type II

Type II



Single real and on unit circle zero



Zeros of GLP System – Type II

□ FIR GLP System Function

$$\begin{aligned} H(z) &= \sum_{n=0}^M h[M-n]z^{-n} = \sum_{k=M}^0 h[k]z^k z^{-M} \\ &= z^{-M} H(z^{-1}). \end{aligned}$$

Zeros of GLP System – Type II

□ FIR GLP System Function

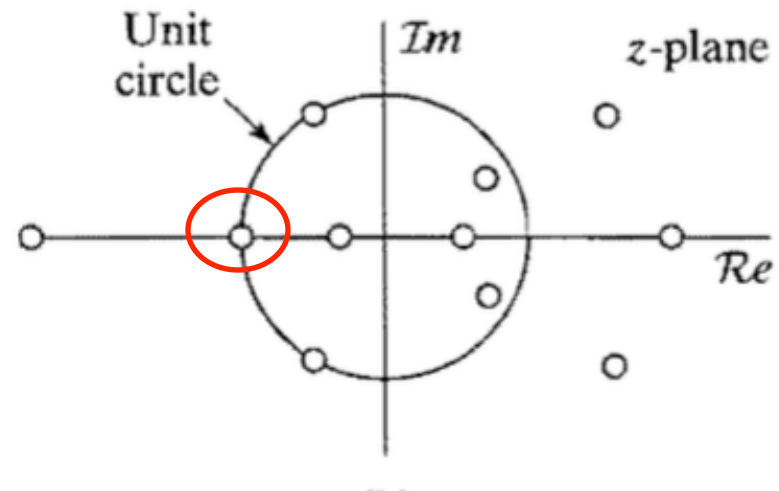
$$\begin{aligned} H(z) &= \sum_{n=0}^M h[M-n]z^{-n} = \sum_{k=M}^0 h[k]z^k z^{-M} \\ &= z^{-M} H(z^{-1}). \end{aligned}$$

Consider $z = -1$: $H(-1) = (-1)^{-M} H(-1)$

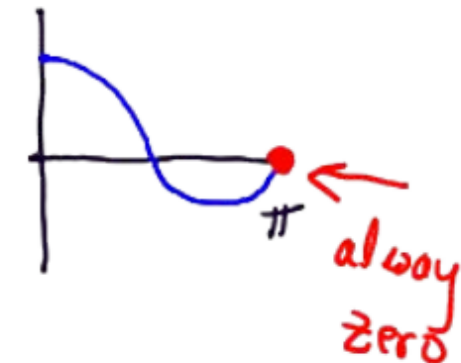
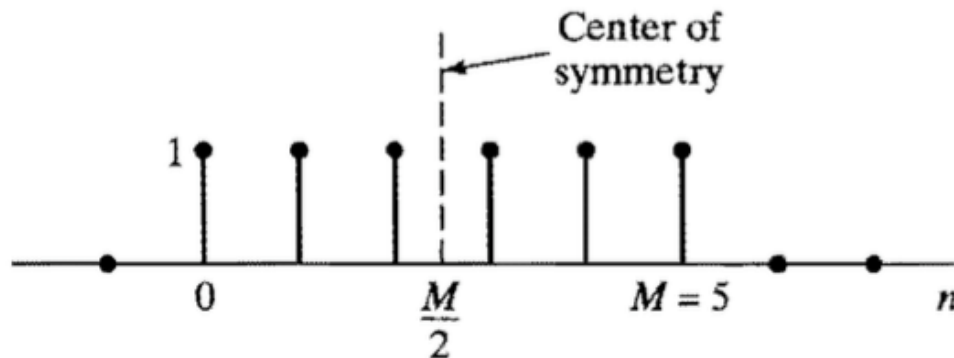
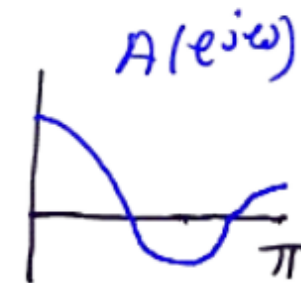
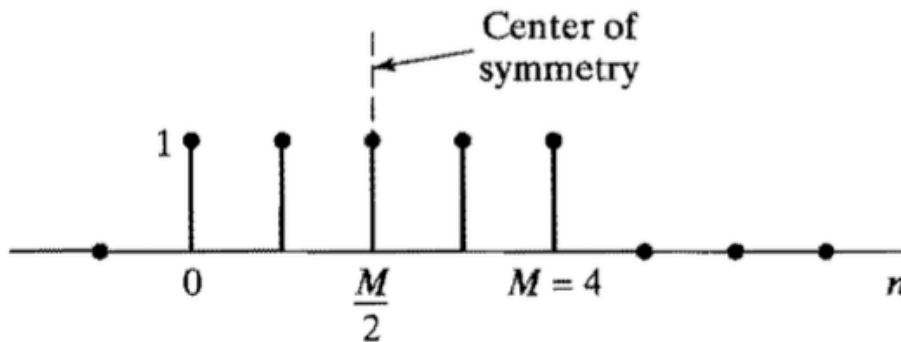
$\Rightarrow$ for M odd, $z = -1$ must be a zero (Type II)

Zeros of GLP System – Type II

Type II



FIR GLP: Type I and II





Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$



Zeros of GLP System – Type III and IV

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

Zeros of GLP System – Type III and IV

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

If z_0 is a zero then z_0^{-1} is also a zero.

Zeros of GLP System – Type III and IV

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

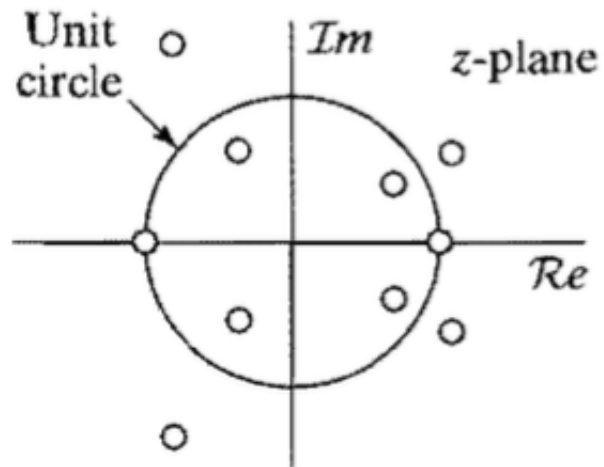
□ If zero is on unit circle ($r=1$)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

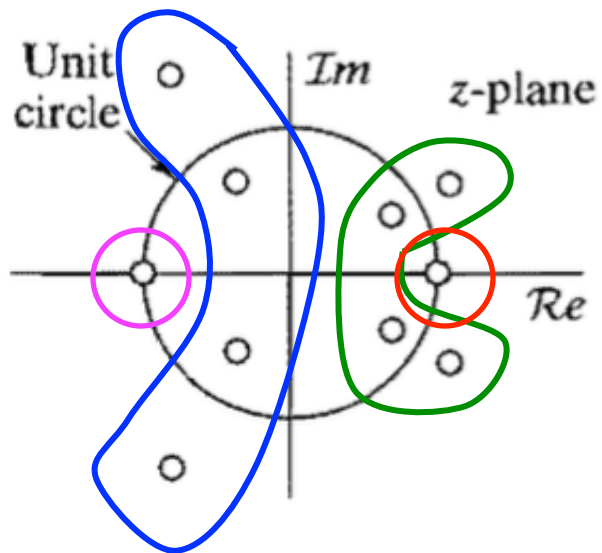
□ If zero is real and not on unit circle ($\theta=0$)

$$(1 + rz^{-1})(1 + r^{-1}z^{-1}).$$

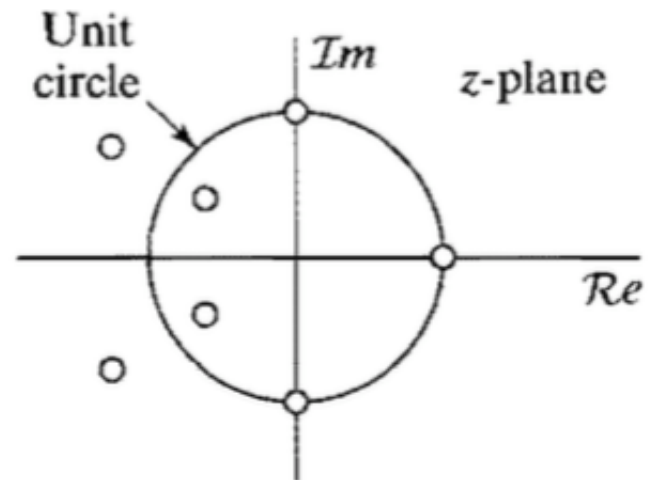
Zeros of GLP System – Type III



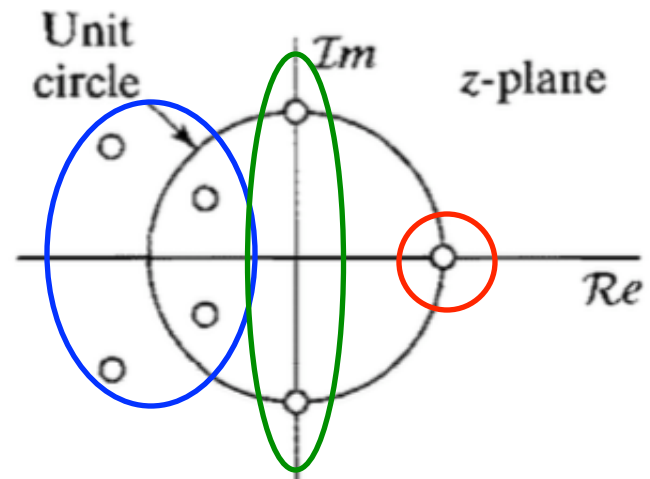
Zeros of GLP System – Type III



Zeros of GLP System – Type IV



Zeros of GLP System – Type IV





Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = -z^{-M} H(z^{-1}).$$

Zeros of GLP System – Type III and IV

□ FIR GLP System Function

$$H(z) = -z^{-M} H(z^{-1}).$$

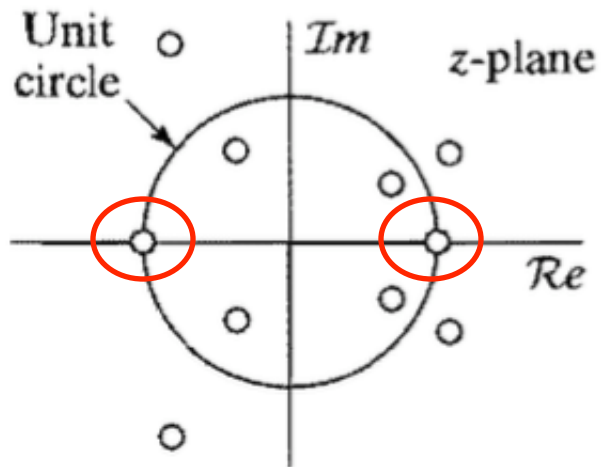
$$H(1) = -H(1) \Rightarrow z = 1 \text{ must be a zero (Type III and IV)}$$

$$H(-1) = -(-1)^{-M} H(-1)$$

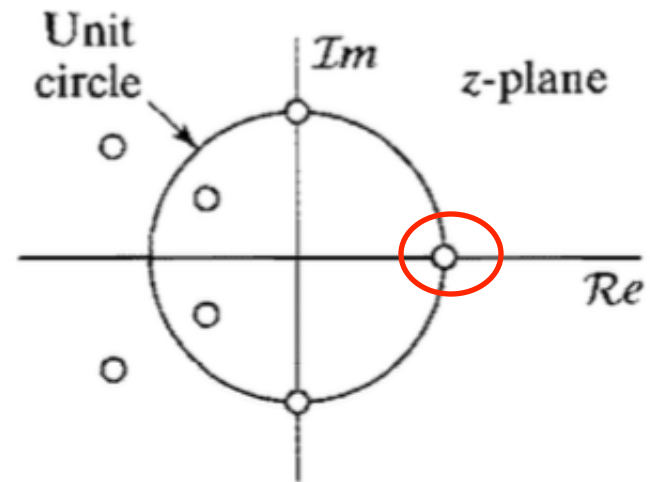
$$\Rightarrow \text{for } M \text{ even, } z = -1 \text{ must be a zero (Type III)}$$

Zeros of GLP System – Type III and IV

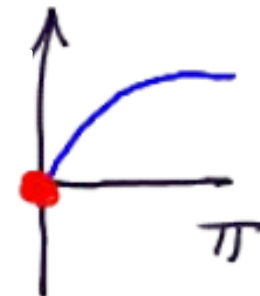
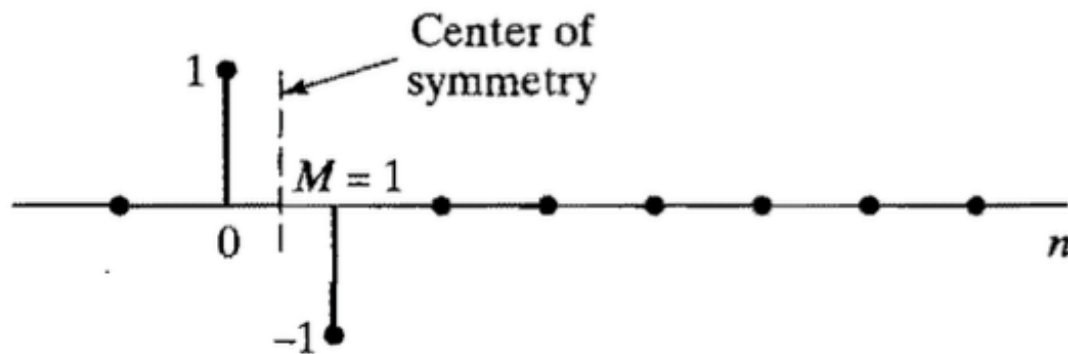
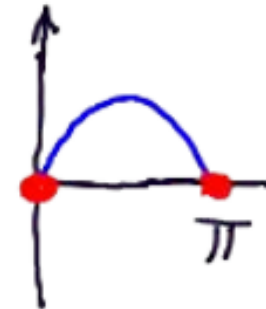
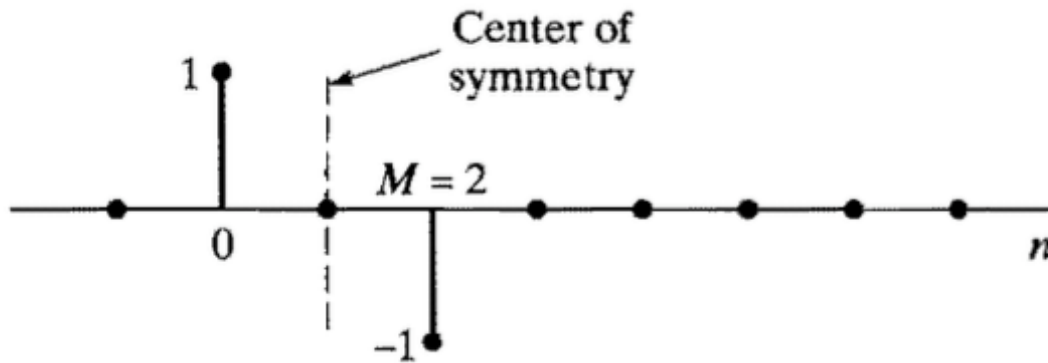
Type III



Type IV



FIR GLP: Type III and IV





GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

$$H(z) = H_{\min}(z)H_{uc}(z)H_{\max}(z)$$

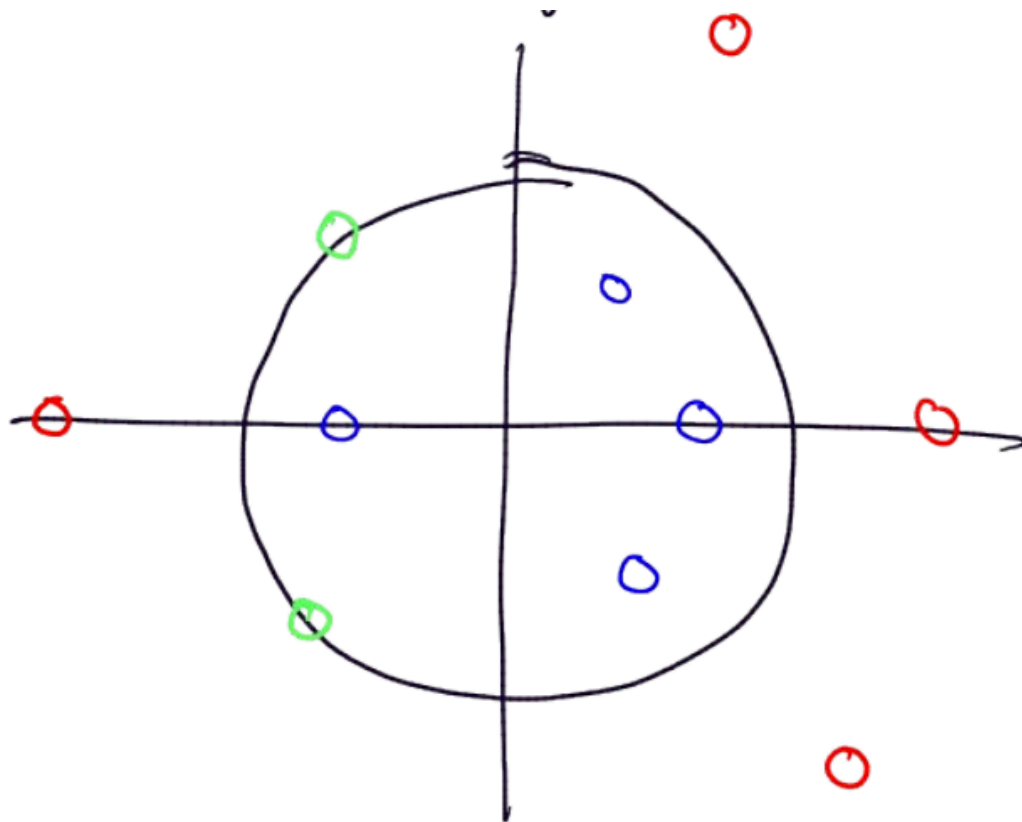
- A min phase system, system containing only zeros on unit circle, and max phase system

GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

$$H(z) =$$

- A min phase system with all poles and zeros inside the unit circle, and a unit delay.





What is a Linear Filter?

- ❑ Attenuates certain frequencies
- ❑ Passes certain frequencies
- ❑ Affects both phase and magnitude

- ❑ IIR
 - Mostly non-linear phase response
 - Could be linear over a range of frequencies
- ❑ FIR
 - Much easier to control the phase
 - Both non-linear and linear phase



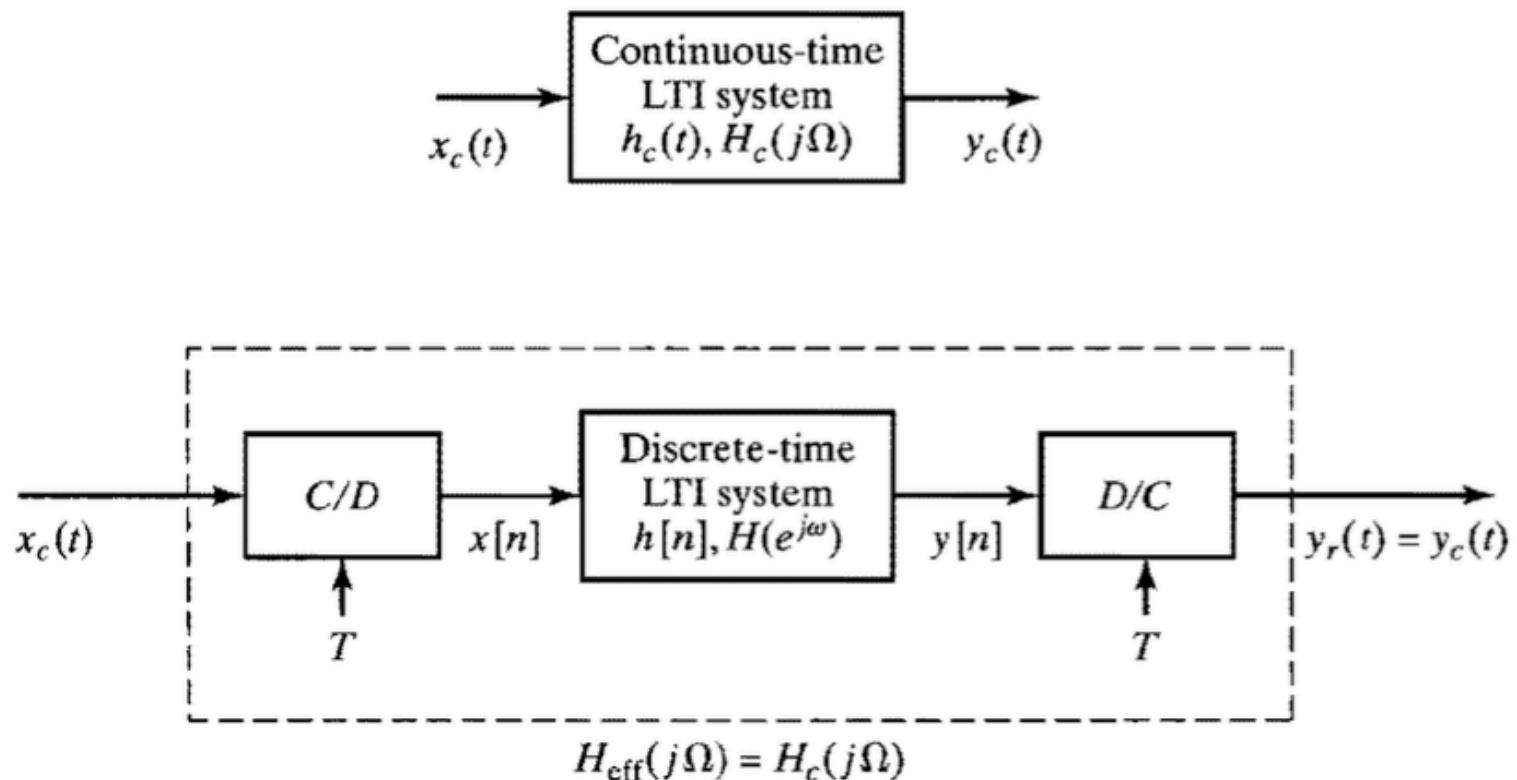
IIR Filter Design

- ❑ Transform continuous-time filter into a discrete-time filter meeting specs
 - Pick suitable transformation from s (Laplace variable) to z (or t to n)
 - Pick suitable analog $H_c(s)$ allowing specs to be met, transform to $H(z)$

- ❑ We've seen this before... impulse invariance

Impulse Invariance

- Want to implement continuous-time system in discrete-time



Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

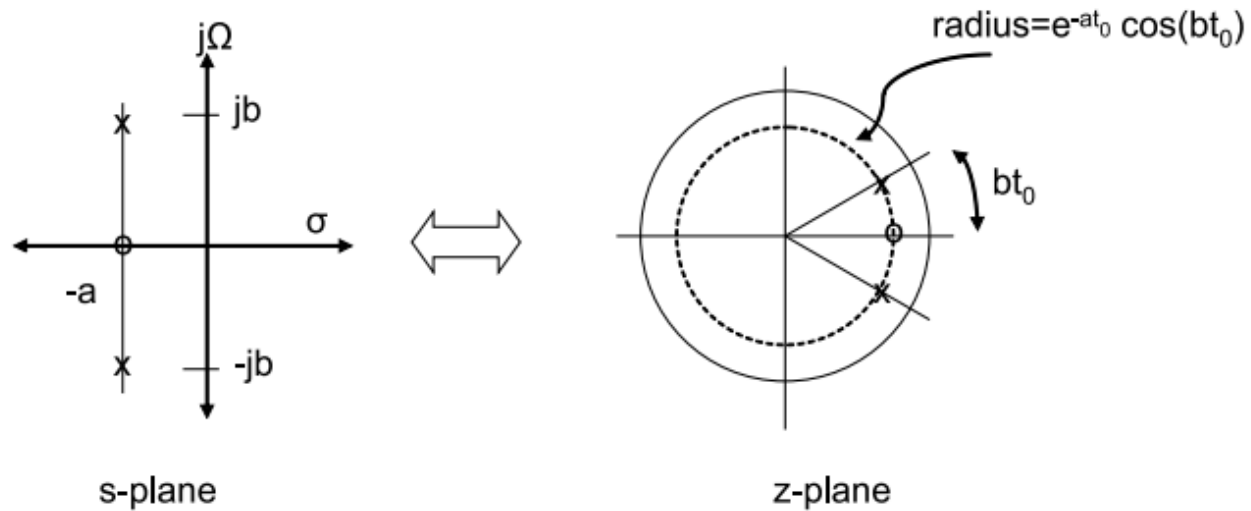
$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

$$h[n] = Th_c(nT)$$

S-Plane Mapping to Z-Plane





Impulse Invariance

- ❑ Sampling the impulse response is equivalent to mapping the s-plane to the z-plane using:
 - $z = e^{sT_d} = r e^{j\omega}$
- ❑ The entire Ω axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times

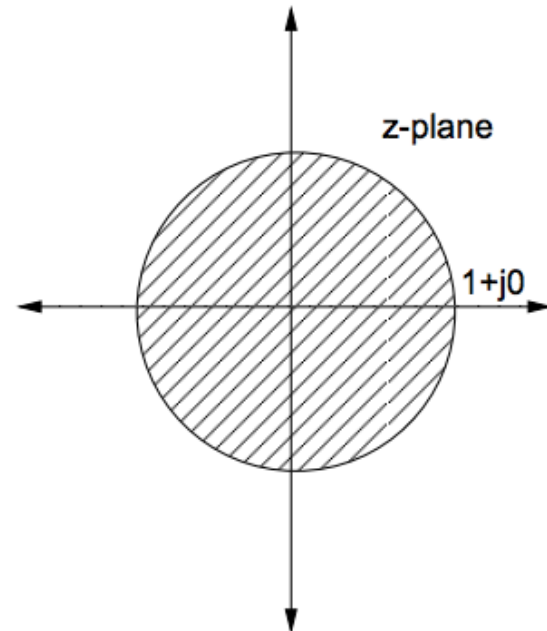
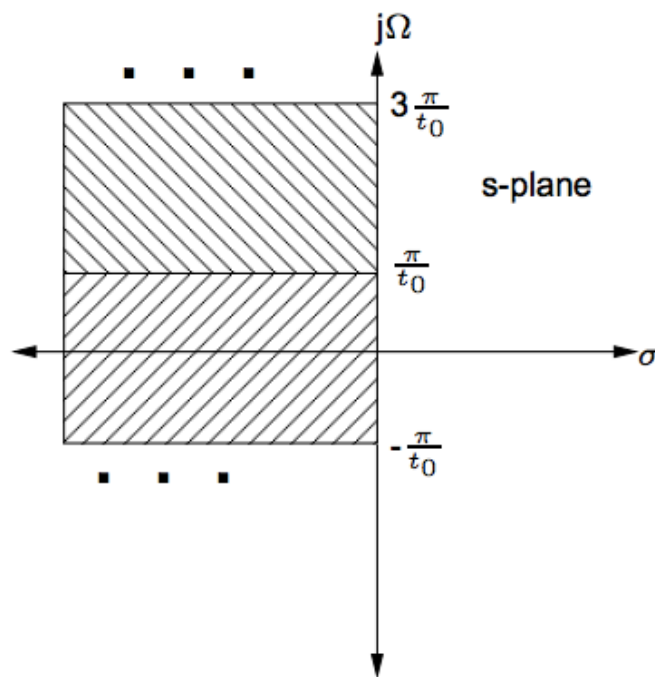


Impulse Invariance

- ❑ Sampling the impulse response is equivalent to mapping the s-plane to the z-plane using:
 - $z = e^{sT_d} = r e^{j\omega}$
- ❑ The entire Ω axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times
- ❑ The left half s-plane maps to the interior of the unit circle and the right half plane to the exterior

Impulse Invariance Mapping

Mapping



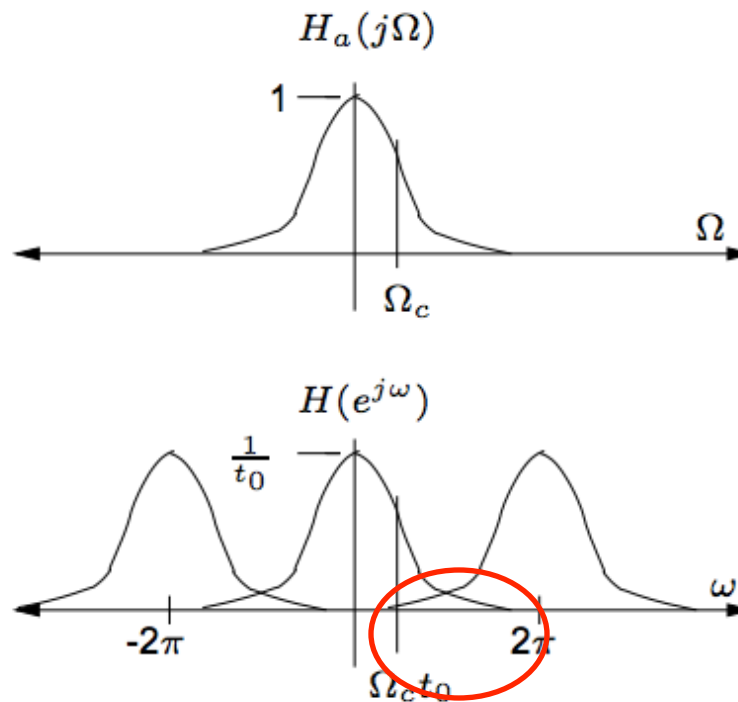


Impulse Invariance

- ❑ Sampling the impulse response is equivalent to mapping the s-plane to the z-plane using:
 - $z = e^{sT_d} = r e^{j\omega}$
- ❑ The entire Ω axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times
- ❑ The left half s-plane maps to the interior of the unit circle and the right half plane to the exterior
- ❑ This means stable analog filters (poles in LHP) will transform to stable digital filters (poles inside unit circle)
- ❑ This is a many-to-one mapping of strips of the s-plane to regions of the z-plane
 - Not a conformal mapping
 - The poles map according to $z = e^{sT_d}$, but the zeros do not always

Impulse Invariance

- ❑ Limitation of Impulse Invariance: overlap of images of the frequency response. This prevents it from being used for high-pass filter design





Bilinear Transformation

- The technique uses an algebraic transformation between the variables s and z that maps the entire $j\Omega$ -axis in the s -plane to one revolution of the unit circle in the z -plane.

$$s = \frac{2}{T_d} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right);$$

$$H(z) = H_c \left(\frac{2}{T_d} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) \right).$$



Bilinear Transformation

$$s = \frac{2}{T_d} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right);$$

- Substituting $s = \sigma + j\Omega$ and $z = e^{j\omega}$

Bilinear Transformation

$$s = \frac{2}{T_d} \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right);$$

□ Substituting $s = \sigma + j\Omega$ and $z = e^{j\omega}$

$$s = \frac{2}{T_d} \left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \right),$$

$$s = \sigma + j\Omega = \frac{2}{T_d} \left[\frac{2e^{-j\omega/2}(j \sin \omega/2)}{2e^{-j\omega/2}(\cos \omega/2)} \right] = \frac{2j}{T_d} \tan(\omega/2).$$

Bilinear Transformation

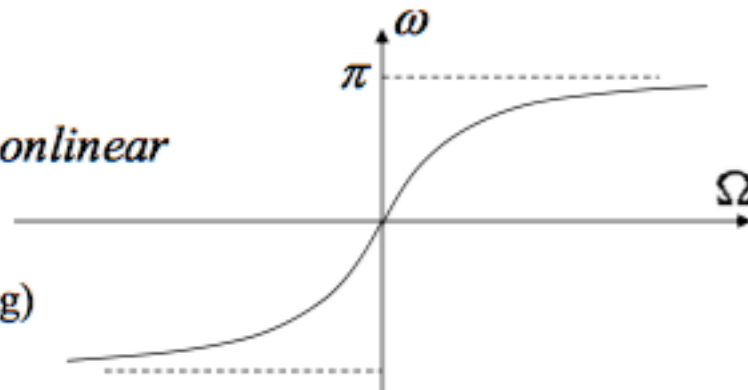
$$\Omega = \frac{2}{T_d} \tan(\omega/2),$$

$$\omega = 2 \arctan(\Omega T_d/2).$$

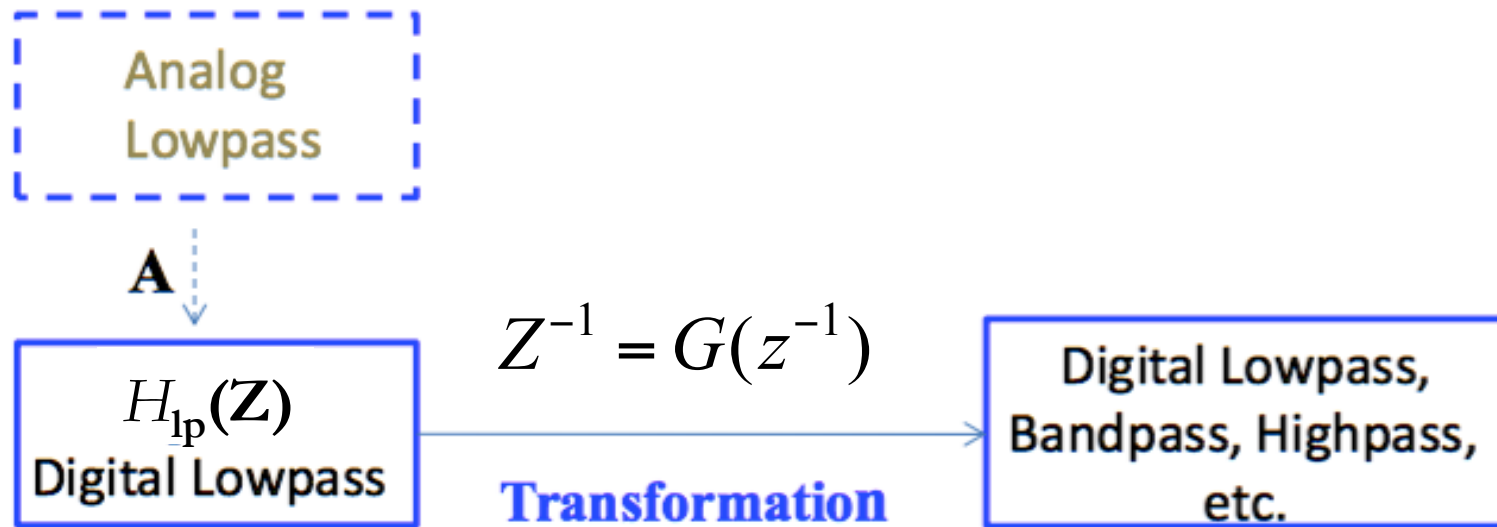
No aliasing, but mapping nonlinear

(Impulse invariance:

linear mapping, but with aliasing)

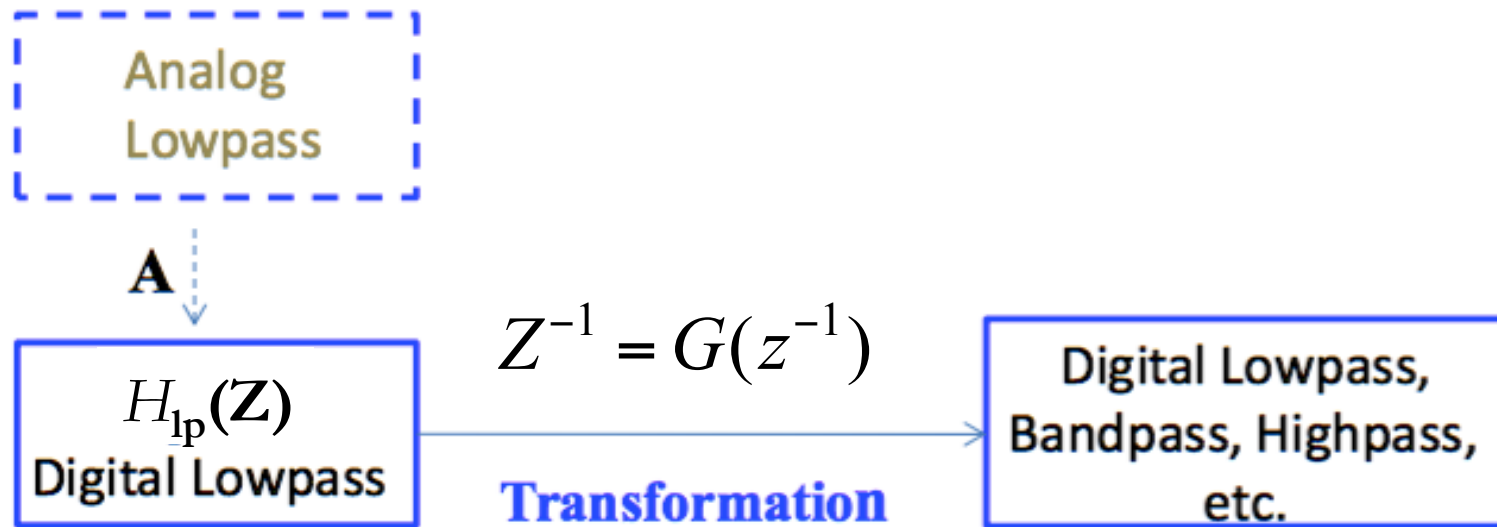


Transformation of DT Filters



- ❑ Z – complex variable for the LP filter
- ❑ z – complex variable for the transformed filter
- ❑ Map Z -plane $\rightarrow$ z -plane with transformation G

Transformation of DT Filters



- ❑ Map Z-plane $\rightarrow$ z-plane with transformation G

$$H(z) = H_{lp}(Z) \big|_{Z^{-1}=G(z^{-1})}$$



General Transformations

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ $\omega_p = \text{desired cutoff frequency}$
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ $\omega_p = \text{desired cutoff frequency}$
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} = \text{desired lower cutoff frequency}$ $\omega_{p2} = \text{desired upper cutoff frequency}$
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k}z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2} - \frac{2\alpha}{1+k}z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} = \text{desired lower cutoff frequency}$ $\omega_{p2} = \text{desired upper cutoff frequency}$

7.2-7.4 in textbook



Big Ideas

- ❑ Generalized Linear Phase Systems
 - Useful for design of causal FIR filters
 - Have linear phase $\rightarrow$ constant group delay
 - Type I, II, III, IV suited for different types
 - Low pass, band pass, high pass
- ❑ IIR
 - Design from continuous time filters with mapping of s-plane onto z-plane
- ❑ DT filter transformations
 - Transform z-plane with rational function $G(z^{-1})$



Admin

- HW 6

- Out now
- Due Sunday 3/29

- I want to hear from you

- Concerns
- Suggestions
- Feedback