ESE 531: Digital Signal Processing

Lec 15: March 24, 2020

Generalized Linear Phase Systems, Design of IIR Filters

Your First Priority

- Your first priority is your health
  - You should abide by all health guidelines
    - Wash your hands
    - Cough/Sneeze into your elbow
    - Don't touch your face
    - Maintain ~~social~~ physical distancing
      - Virtual social interaction is encouraged!
    - Stay home except for necessary activities
  - Part of your health is your mental and emotional health
    - See https://caps.wellness.upenn.edu/selfhelp/ for help

Advice for Remote Learning Success

- Attend lecture in real time and stay on schedule
  - Rewatch recorded lectures, attend virtual office hours
  - We are here to help, but you need to let us know you need help
- Start homework early!
- Form a virtual study group with classmates
  - Use piazza to find groups
- Useful study habits reference:
  - https://lsa.umich.edu/content/dam/rll-assets/rll-docs/Study%20Habits.pdf

Course Grading Changes

- Homework
  - 20% of grade (same)
  - Only 8 HW instead of 9
    - Still drop lowest homework (same)
- Final Project
  - 30% of grade (increased)
- Midterm Exam
  - 25% of grade (same)
- Final Exam
  - 25% of grade (decreased)

Course Structure Changes

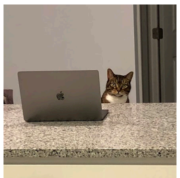
- Homework Canvas submission, course webpage and Piazza all the same
  - Recommend you turn on Piazza email notifications
- Virtual lectures via Zoom
  - See Piazza threads for link
- Virtual Office hours via Google Hangouts/Zoom
  - See Piazza threads for schedule and links
- Big challenge will be staying on pace with material
  - Requires self-motivation

I want to hear from you…

- Will you be in a different time zone?
- Will you have trouble seeing or hearing video lectures?
- Are there any other accessibility issues I should know about?
- Email me any concerns -- I will do everything I can to ensure you have the same learning objectives from Lecture 1

New Course Staff

- New TAs/Graders



Questions?

- Use chat panel to type question
  - Scroll to bottom of Zoom window and click "chat" button

Lecture Outline

- Review: Where did we leave off?
- Generalized Linear Phase Systems
- IIR Filters (if time)

Frequency Response of Systems (Filters)

Frequency Response of LTI System

$$Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- We can define a magnitude response

$$\left|Y\left(e^{j\omega}\right)\right|=\left|H\left(e^{j\omega}\right)\right|\left|X\left(e^{j\omega}\right)\right|$$

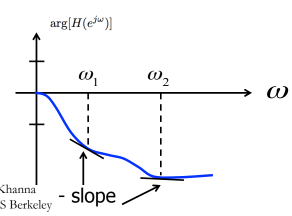
- And a phase response

$$\angle Y\left(e^{j\omega}\right)=\angle H\left(e^{j\omega}\right)+\angle X\left(e^{j\omega}\right)$$

Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

$$y[n]=\frac{x[n-M]+...+x[n]}{M+1}$$

Impulse response

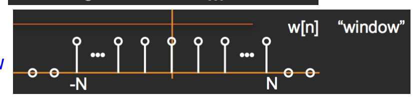
Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

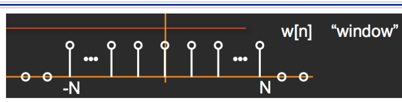
$$y[n]=\frac{x[n-M]+...+x[n]}{M+1}$$

Impulse response

Scaled &Time Shifted window

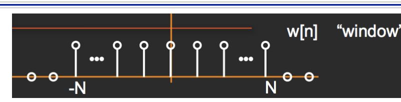


Example: Moving Average



$$w[n]\leftrightarrow W(e^{j\omega})=\frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

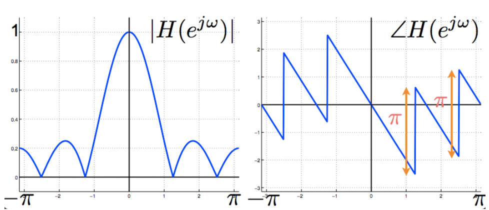
Example: Moving Average



$$w[n]\leftrightarrow W(e^{j\omega})=\frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$\frac{1}{M+1}w[n-M/2]\leftrightarrow W(e^{j\omega})=\frac{e^{-j\omega M/2}}{M+1}\frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average



$$W(e^{j\omega})=\frac{e^{-j\omega M/2}}{M+1}\frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

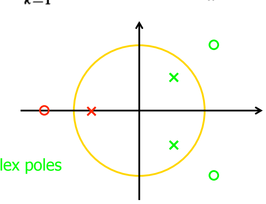
General All-Pass Systems

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\text{ap}}(z)=A\prod_{k=1}^{M_r}\frac{z^{-1}-d_k}{1-d_kz^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1}-e_k^*)(z^{-1}-e_k)}{(1-e_kz^{-1})(1-e_k^*z^{-1})}$$

- Example:

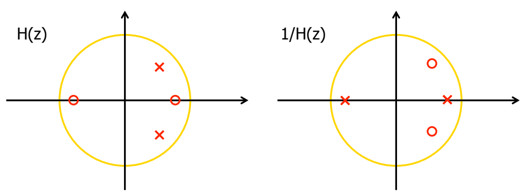
$$d_k=-\frac{3}{4}$$
$$e_k=0.8e^{j\pi/4}$$



Complex poles

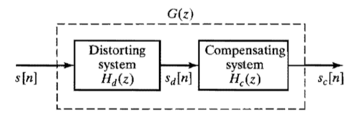
Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle



Min-Phase Decomposition Purpose

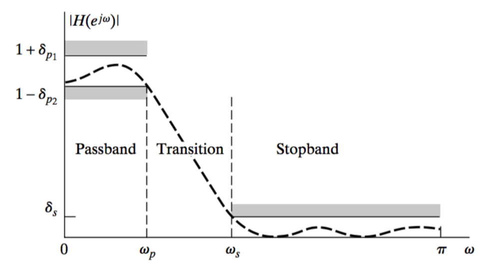
- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z)$ → $H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

FIR Filter Design

Filter Specifications



What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

- What does it mean to design a filter?
  - Determine the parameters of a transfer function or difference equation that approximates a desired impulse response (h[n]) or frequency response ($H(e^{j\omega})$).

What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
  - Both non-linear and linear phase

Generalized Linear Phase Systems

Generalized Linear Phase

- An LTI system has generalized linear phase if the frequency response $H(e^{j\omega})$ can be expressed as:

$$\boxed{H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha + j\beta}, \quad |\omega| < \pi}$$

- Where $A(\omega)$ is a real function.

Generalized Linear Phase

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- Where $A(\omega)$ is a real function.
- What is the group delay?

Causal FIR Systems

$$y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M], \quad \text{all } n$$

$$H(z) = b_0 + b_1 z^{-1} + \ldots + b_M z^{-M} = b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})$$

$$h[n] = \begin{cases} b_n, & n = 0, 1, \ldots, M \\ 0, & \textit{otherwise} \end{cases}$$

Causal FIR Systems

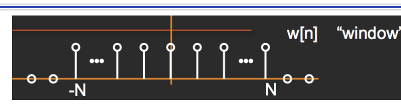
- Causal FIR systems have generalized linear phase if they have impulse response length (M+1) and some symmetry
- It can be shown if

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$

- then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

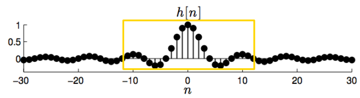
$$\frac{1}{M+1} w[n - M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \le |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate and shift $\quad h_{LP}[n] = w_N[n-N] \cdot h[n-N]$

Causal FIR Systems

- Causal FIR systems have generalized linear phase if they have impulse response length (M+1)
- It can be shown if

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$

- Then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

- Sufficient conditions to guarantee GLP, not necessary
  - Other systems can also have GLP

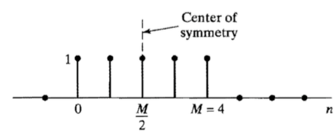
FIR GLP Systems

- Four types of FIR GLP
  - Type I
    - Even Symmetry, M even
  - Type II
    - Even Symmetry, M odd
  - Type III
    - Odd Symmetry, M even
  - Type IV
    - Odd Symmetry, M odd

FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



FIR GLP: Type I

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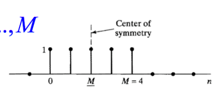

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$$

integer delay

FIR GLP: Type I – Example, M=4

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$

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integer delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$

FIR GLP: Type I – Example, M=4

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$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$

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$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[3]e^{-j\omega} + h[4]e^{-j2\omega}\right]$$

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$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega}\right]$$

FIR GLP: Type I – Example, M=4

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$

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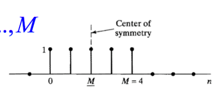
integer delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$
$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega}\right]$$
$$= \underbrace{\left[2h[0]\cos(2\omega) + 2h[1]\cos(\omega) + h[2]\right]}_{A(\omega)\ (even)} \; e^{-j2\omega}$$

FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$$

integer delay

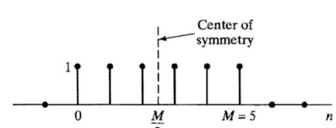
$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right)$$

$$a[0] = h[M/2],$$
$$a[k] = 2h[(M/2) - k], \qquad k = 1, 2, \ldots, M/2.$$

FIR GLP: Type II

Type II Even Symmetry, M odd

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



FIR GLP: Type II

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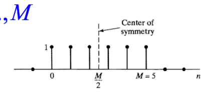
$$h[n]=h[M-n], \quad n=0,1,\ldots,M$$



Then $H(e^{j\omega})=\sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

FIR GLP: Type II – Example, M=3

Type II Even Symmetry, M odd

Then $H(e^{j\omega})=\sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

$$H(e^{j\omega})=h[0]+h[1]e^{-j\omega}+h[2]e^{-j2\omega}+h[3]e^{-j3\omega}$$

FIR GLP: Type II – Example, M=3

Type II Even Symmetry, M odd

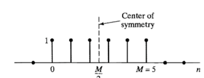
Then $H(e^{j\omega})=\sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

$$H(e^{j\omega})=h[0]+h[1]e^{-j\omega}+h[2]e^{-j2\omega}+h[3]e^{-j3\omega}$$

$$H(e^{j\omega})=\underbrace{\left[2h[0]\cos(3\omega/2)+2h[1]\cos(\omega/2)\right]}_{A(\omega)} e^{-j3\omega/2}$$

FIR GLP: Type II

Type II Even Symmetry, M odd

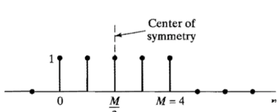


Then $H(e^{j\omega})=\sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

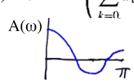
$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right\}$$

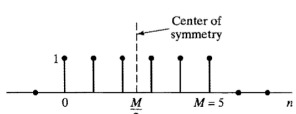
$$b[k]=2h[(M+1)/2-k], \qquad k=1,2,\ldots,(M+1)/2.$$

FIR GLP: Type I and II



$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right)$$





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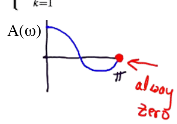
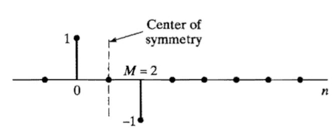


FIR GLP: Type III

Type III Odd Symmetry, M even

$$h[n]=-h[M-n], \quad n=0,1,\ldots,M \qquad (note\ h[M/2]=0)$$



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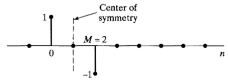

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

FIR GLP: Type III – Example, M=4

Type III Odd Symmetry, M even

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note\ \ h[M/2] = 0)$

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FIR GLP: Type III – Example, M=4

Type III Odd Symmetry, M even

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note\ \ h[M/2] = 0)$

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$
$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} - h[1]e^{-j\omega} - h[0]e^{-j2\omega}\right]$$

FIR GLP: Type III – Example, M=4

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$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} - h[1]e^{-j\omega} - h[0]e^{-j2\omega}\right]$$
$$= \underbrace{\left[2h[0]\sin(2\omega) + 2h[1]\sin(\omega)\right]}_{A(\omega)\ (odd)}\ j\,e^{-j2\omega}$$

FIR GLP: Type III

Type III Odd Symmetry, M even

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$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

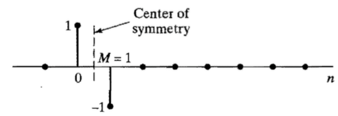
$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin\omega k\right]$$

$$c[k] = 2h[(M/2)-k], \qquad k = 1,2,\ldots,M/2.$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$



FIR GLP: Type IV – Example, M=3

Type IV Odd Symmetry, M Odd

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$ fractional delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$$

FIR GLP: Type IV – Example, M=3

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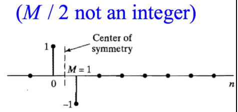
$$H(e^{j\omega}) = \underbrace{\left[2h[0]\sin(3\omega/2) + 2h[1]\sin(\omega/2)\right]}_{A(\omega)}\ je^{-j3\omega/2}$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$



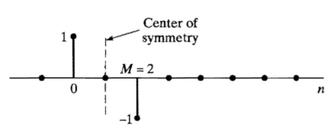
$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$ fractional delay

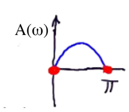
$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right]$$

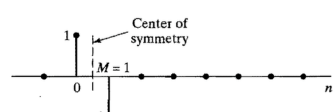
$$d[k] = 2h[(M+1)/2-k], \qquad k = 1,2,\ldots,(M+1)/2.$$

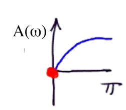
FIR GLP: Type III and IV

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin\omega k\right]$$









$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right]$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}$$

$$H(z)=\sum_{n=0}^{M}h[M-n]z^{-n}$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}$$

$$H(z)=\sum_{n=0}^{M}h[M-n]z^{-n}=\sum_{k=M}^{0}h[k]z^{k}z^{-M}$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

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$$H(z)=\sum_{n=0}^{M}h[M-n]z^{-n}=\sum_{k=M}^{0}h[k]z^{k}z^{-M}$$

$$=z^{-M}H(z^{-1}).$$

- If z_0 is a zero of H(z), then z_0^{-1} is also a zero of H(z)

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}$$

If z_0 is a zero then z_0^{-1} is also a zero.

- If h[n] is real,

If z_0 is a zero then z_0^{*} is also a zero.

(Use conjugate symmetry property of real signals)

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1-\underbrace{re^{j\theta}}_{z_0}z^{-1})(1-\underbrace{re^{-j\theta}}_{(z_0)^*}z^{-1})(1-\underbrace{r^{-1}e^{j\theta}}_{(z^{-1}_0)^*}z^{-1})(1-\underbrace{r^{-1}e^{-j\theta}}_{z^{-1}_0}z^{-1})$$

Zeros of GLP System – Type I and II

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$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

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- If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

- If zero is on unit circle (r=1)

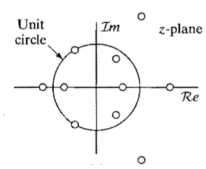
$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

- If zero is real and not on unit circle (θ =0)

$$(1 + rz^{-1})(1 + r^{-1}z^{-1}).$$

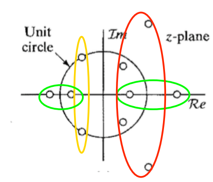
Zeros of GLP System – Type I

Type I



Zeros of GLP System – Type I

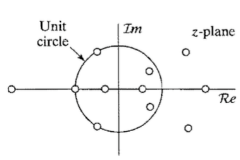
Type I



Group of 4 reciprocal and conjugate zeros
Group of 2 real and not on unit circle zeros
Group of 2 on unit circle zeros

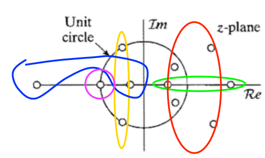
Zeros of GLP System – Type II

Type II



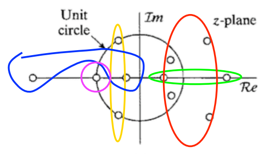
Zeros of GLP System – Type II

Type II



Group of 4 reciprocal and conjugate zeros
Group of 2 real and not on unit circle zeros
Group of 2 real and not on unit circle zeros
Group of 2 on unit circle zeros

Zeros of GLP System – Type II



Single real and on unit circle zero

Zeros of GLP System – Type II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M}$$
$$= z^{-M}H(z^{-1}).$$

Zeros of GLP System – Type II

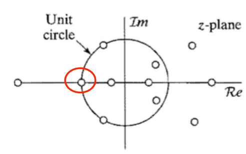
- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M}$$
$$= z^{-M}H(z^{-1}).$$

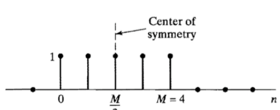
Consider $z=-1$: $H(-1)=(-1)^{-M}H(-1)$

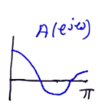
$\boxed{\Rightarrow \text{ for } M \text{ odd, } z=-1 \text{ must be a zero (Type II)}}$

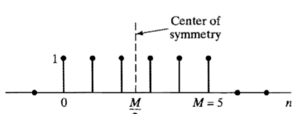
Zeros of GLP System – Type II

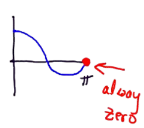


FIR GLP: Type I and II









Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

If z_0 is a zero then z_0^{-1} is also a zero.

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Real system $\rightarrow$ zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

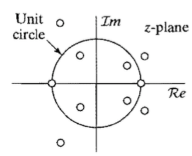
- If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

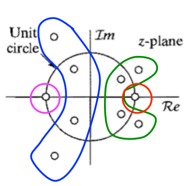
- If zero is real and not on unit circle (θ =0)

$$(1 + rz^{-1})(1 + r^{-1}z^{-1}).$$

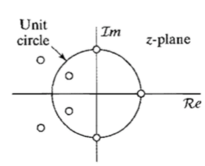
Zeros of GLP System – Type III



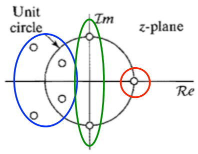
Zeros of GLP System – Type III



Zeros of GLP System – Type IV



Zeros of GLP System – Type IV



Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = -z^{-M} H(z^{-1}).$$

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = -z^{-M} H(z^{-1}).$$

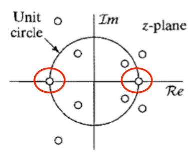
$H(1) = -H(1) \;\Rightarrow\; z = 1$ must be a zero (Type III and IV)

$H(-1) = -(-1)^{-M} H(-1)$

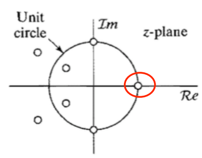
$\boxed{\Rightarrow \text{ for } M \text{ even, } z = -1 \text{ must be a zero (Type III)}}$

Zeros of GLP System – Type III and IV

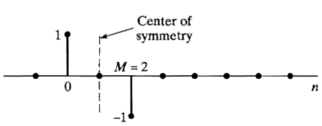
Type III

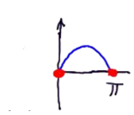


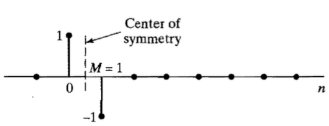
Type IV

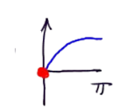


FIR GLP: Type III and IV









GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

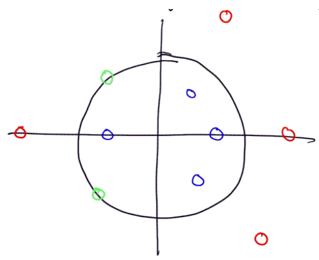
$$H(z) = H_{\min}(z) H_{uc}(z) H_{\max}(z)$$

- A min phase system, system containing only zeros on unit circle, and max phase system

GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

$H(z) =$



- A min phase s
  on unit circle, :

What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
  - Both non-linear and linear phase

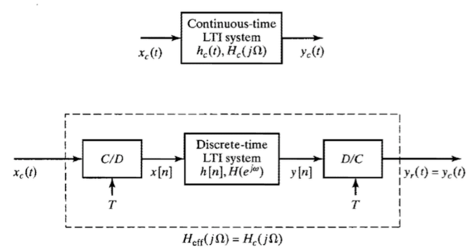
IIR Filter Design

- Transform continuous-time filter into a discrete-time filter meeting specs
  - Pick suitable transformation from s (Laplace variable) to z (or t to n)
  - Pick suitable analog $H_c(s)$ allowing specs to be met, transform to $H(z)$

- We've seen this before… impulse invariance

Impulse Invariance

- Want to implement continuous-time system in discrete-time



Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

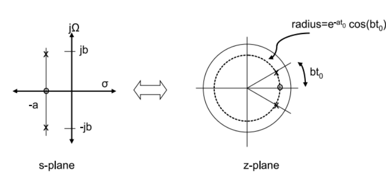
$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

$$\boxed{h[n] = Th_c(nT)}$$

S-Plane Mapping to Z-Plane



Impulse Invariance

- Sampling the impulse response is equivalent to mapping the s-plane to the z-plane using:
  - $z = e^{sTd} = r\,e^{j\omega}$
- The entire Ω axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times

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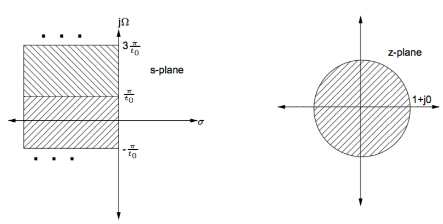
- The left half s-plane maps to the interior of the unit circle and the right half plane to the exterior

Impulse Invariance Mapping

Mapping

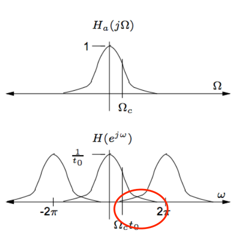


Impulse Invariance

- Sampling the impulse response is equivalent to mapping the s-plane to the z-plane using:
  - $z = e^{sTd} = r\,e^{j\omega}$
- The entire Ω axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times
- The left half s-plane maps to the interior of the unit circle and the right half plane to the exterior
- This means stable filters (poles in LHP) will transform to stable digital filters (poles inside unit circle)
- This is a many-to-one mapping of strips of the s-plane to regions of the z-plane
  - Not a conformal mapping
  - The poles map according to $z = e^{sTd}$, but the zeros do not always

Impulse Invariance

- Limitation of Impulse Invariance: overlap of images of the frequency response. This prevents it from being used for high-pass filter design



Bilinear Transformation

- The technique uses an algebraic transformation between the variables *s* and *z* that maps the entire $j\Omega$-axis in the s-plane to one revolution of the unit circle in the z-plane.

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right).$$

Bilinear Transformation

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right);$$

- Substituting s = σ + j Ω and z = $e^{j\omega}$

Bilinear Transformation

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right);$$

- Substituting s = σ + j Ω and z = $e^{j\omega}$

$$s = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right),$$

$$s = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j\sin\omega/2)}{2e^{-j\omega/2}(\cos\omega/2)}\right] = \frac{2j}{T_d}\tan(\omega/2).$$

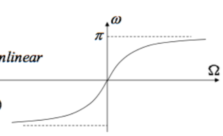
Bilinear Transformation

$$\Omega = \frac{2}{T_d}\tan(\omega/2),$$

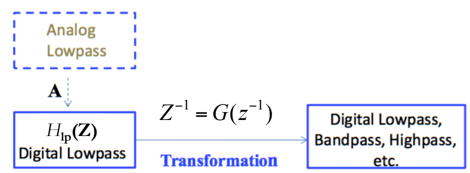
$$\omega = 2\arctan(\Omega T_d/2).$$

No aliasing, but mapping nonlinear

(Impulse invariance:
linear mapping, but with aliasing)

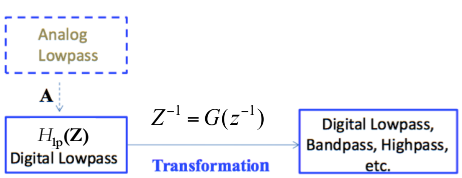


Transformation of DT Filters



- Z – complex variable for the LP filter
- z – complex variable for the transformed filter
- Map Z-plane→z-plane with transformation G

Transformation of DT Filters



- Map Z-plane→z-plane with transformation G

$$H(z) = H_{lp}(Z)\big|_{Z^{-1}=G(z^{-1})}$$

General Transformations

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1}-\alpha}{1-az^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p-\omega_p}{2}\right)}{\sin\left(\frac{\theta_p+\omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p+\omega_p}{2}\right)}{\cos\left(\frac{\theta_p-\omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2}-\frac{2\alpha k}{k+1}z^{-1}+\frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2}-\frac{2\alpha k}{k+1}z^{-1}+1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2}+\omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2}-\frac{2\alpha}{1+k}z^{-1}+\frac{1-k}{1+k}}{\frac{1-k}{1+k}z^{-2}-\frac{2\alpha}{1+k}z^{-1}+1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2}+\omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2}-\omega_{p1}}{2}\right)\tan\left(\frac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

7.2-7.4 in textbook

Big Ideas

- Generalized Linear Phase Systems
  - Useful for design of causal FIR filters
  - Have linear phase → constant group delay
  - Type I, II, III, IV suited for different types
    - Low pass, band pass, high pass
- IIR
  - Design from continuous time filters with mapping of s-plane onto z-plane
- DT filter transformations
  - Transform z-plane with rational function $G(z^{-1})$

Admin

- HW 6
  - Out now
  - Due Sunday 3/29
- I want to hear from you
  - Concerns
  - Suggestions
  - Feedback