

ESE 531: Digital Signal Processing

Lec 17: March 31, 2020

Discrete Fourier Transform



Today

- ❑ Discrete Fourier Series
- ❑ Discrete Fourier Transform (DFT)
- ❑ DFT Properties
- ❑ Circular Convolution

Discrete Fourier Series

Reminder: Eigenvalue (DTFT)

□ $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
 - Re and Im
 - Mag and Phase



Discrete Fourier Series

□ Definition:

- Consider N -periodic signal:

$$\tilde{x}[n + N] = \tilde{x}[n] \quad \forall n$$

- Frequency-domain also periodic in N :

$$\tilde{X}[k + N] = \tilde{X}[k] \quad \forall k$$

- “ $\sim$ ” indicates periodic signal/spectrum

8.1 in text



Discrete Fourier Series

□ Define:

$$W_N \triangleq e^{-j2\pi/N}$$

□ DFS:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$



Discrete Fourier Series

$$W_N \triangleq e^{-j2\pi/N}$$

□ Properties of W_N :

- $W_N^0 = W_N^N = W_N^{2N} = \dots = 1$
- $W_N^{k+r} = W_N^k W_N^r$ and, $W_N^{k+N} = W_N^k$

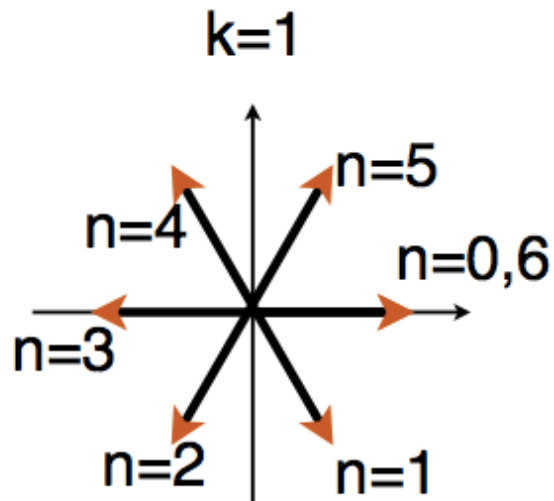
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□ Example: W_N^{kn} ($N=6$)



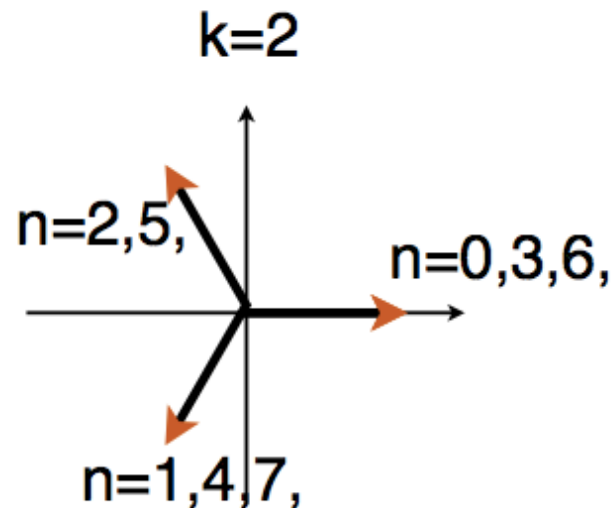
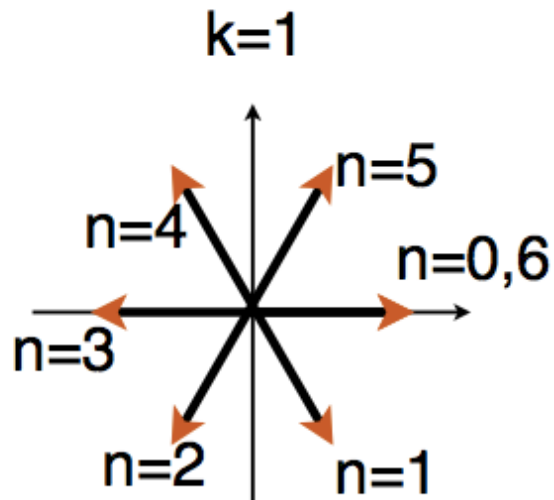
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Discrete Fourier Transform

- By convention, work with one period:

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$
$$X[k] \triangleq \begin{cases} \tilde{X}[k] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Same, but different!



Discrete Fourier Transform

□ The DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

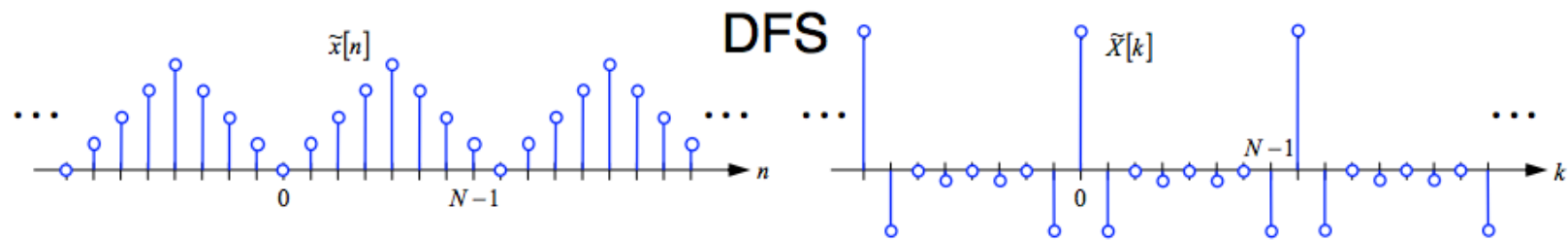
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{DFT, analysis}$$

□ It is understood that,

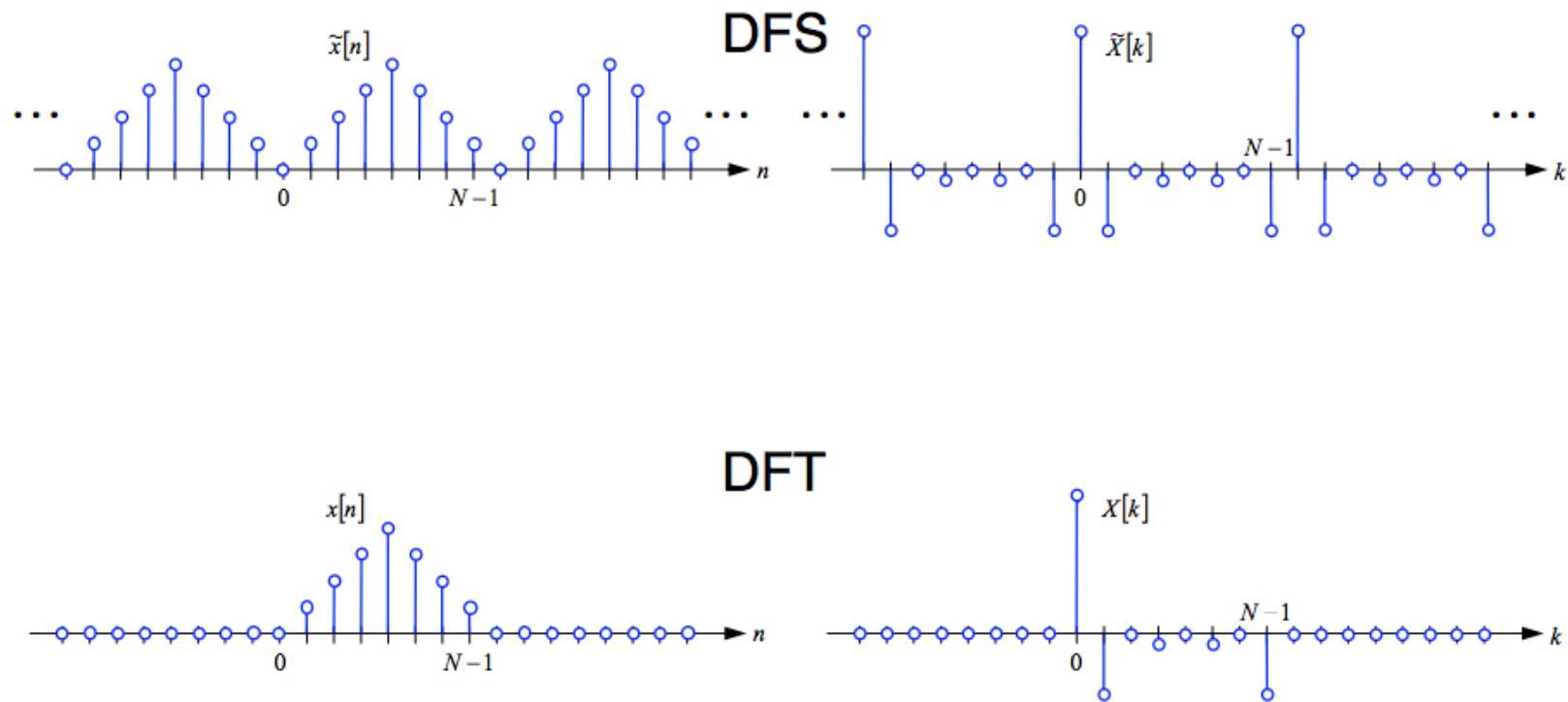
$$x[n] = 0 \quad \text{outside } 0 \leq n \leq N - 1$$

$$X[k] = 0 \quad \text{outside } 0 \leq k \leq N - 1$$

DFS vs. DFT

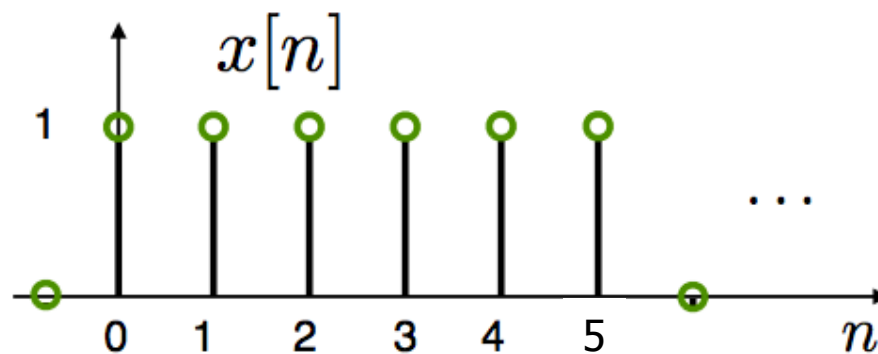


DFS vs. DFT



Example

$$W_N \triangleq e^{-j2\pi/N}$$

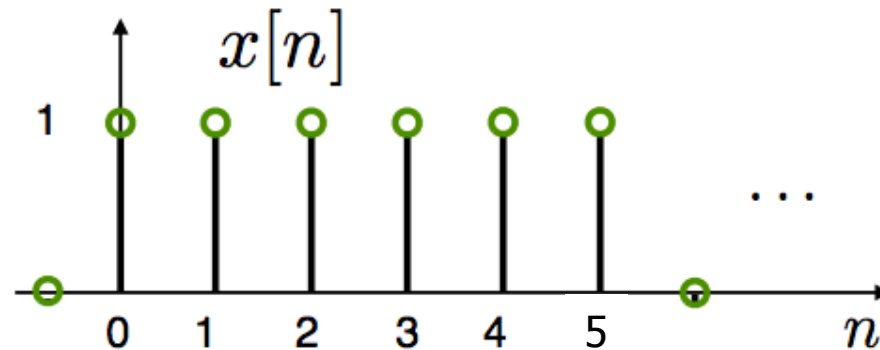


$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

N=6

Example

$$W_N \triangleq e^{-j2\pi/N}$$



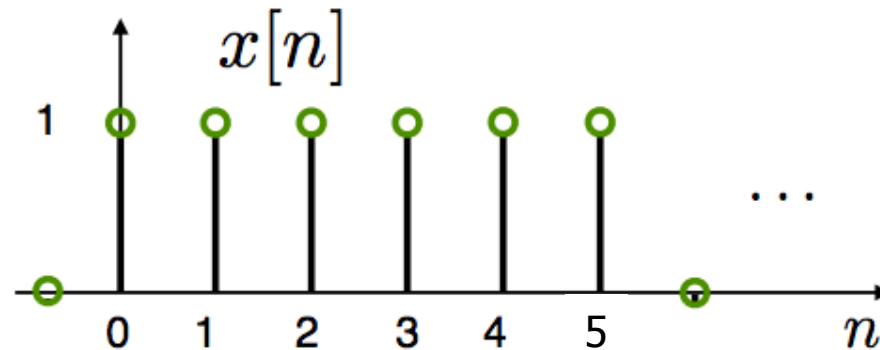
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$W_6 = e^{-j\frac{2\pi}{6}}$$

$$X[k] = \sum_{n=0}^5 (W_6^k)^n$$

Example

$$W_N \triangleq e^{-j2\pi/N}$$

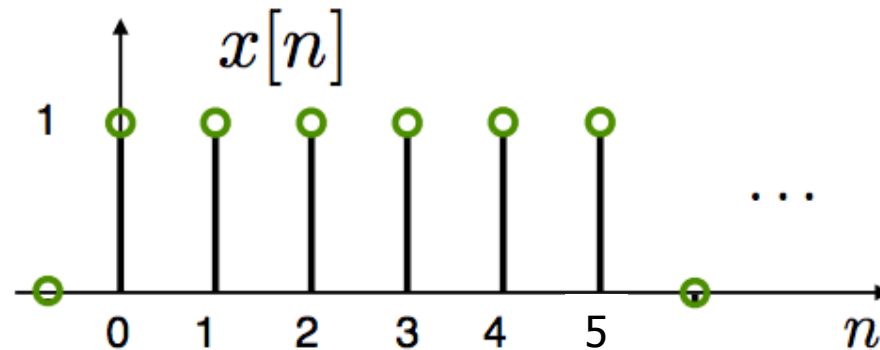


$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$W_6 = e^{-j\frac{2\pi}{6}} \quad \longrightarrow \quad X[0] = \sum_{n=0}^5 (W_6^0)^n = 6$$
$$X[k] = \sum_{n=0}^5 (W_6^k)^n$$

Example

$$W_N \triangleq e^{-j2\pi/N}$$



$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

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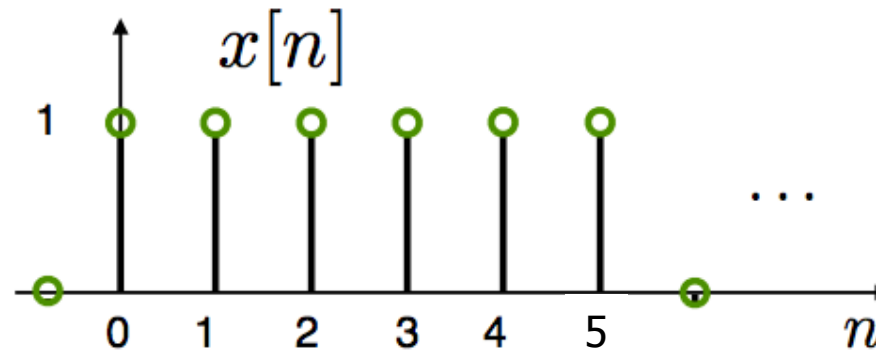
$$X[0] = \sum_{n=0}^5 (W_6^0)^n = 6$$

$$X[k] = \sum_{n=0}^5 (W_6^k)^n$$

$$X[1] = ?$$

Example

$$W_N \triangleq e^{-j2\pi/N}$$

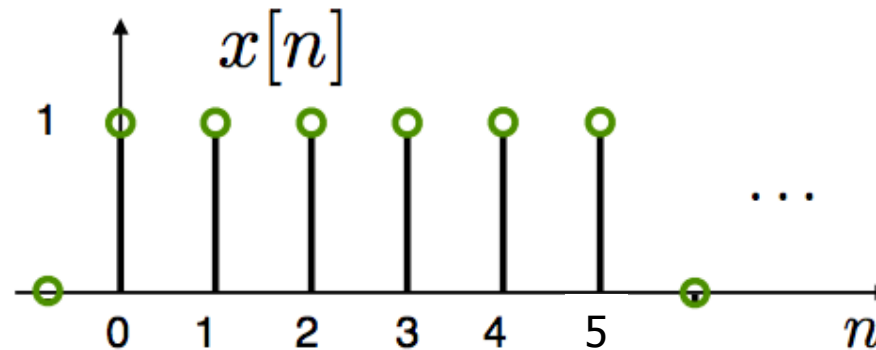


$X[1]=?$

$$X[1] = \sum_{n=0}^5 (W_6)^n = \sum_{n=0}^5 \left(e^{-j\frac{2\pi}{6}} \right)^n$$

Example

$$W_N \triangleq e^{-j2\pi/N}$$

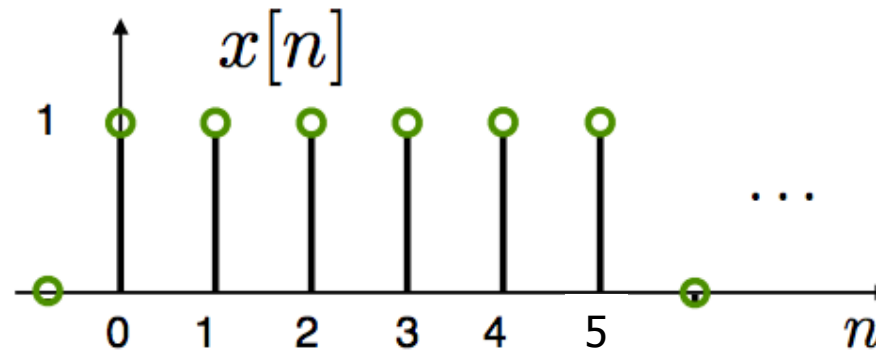


$X[1]=?$

$$\begin{aligned} X[1] &= \sum_{n=0}^5 (W_6)^n = \sum_{n=0}^5 \left(e^{-j\frac{2\pi}{6}} \right)^n \\ &= 1 + e^{-j\frac{2\pi}{6}} + e^{-j\frac{4\pi}{6}} + e^{-j\frac{6\pi}{6}} + e^{-j\frac{8\pi}{6}} + e^{-j\frac{10\pi}{6}} \end{aligned}$$

Example

$$W_N \triangleq e^{-j2\pi/N}$$

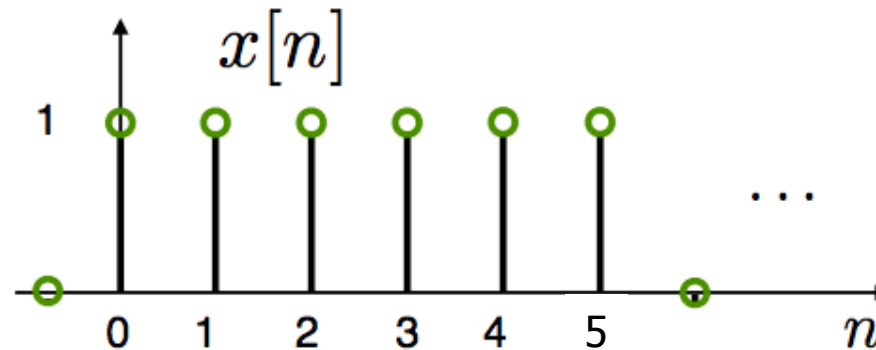


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Example

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$X[1]=?$

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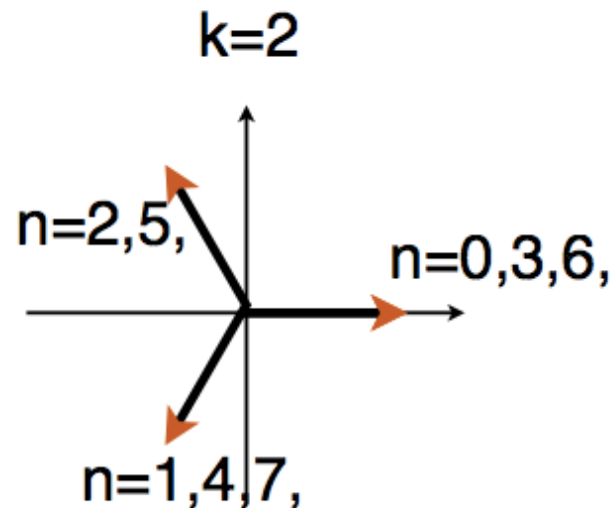
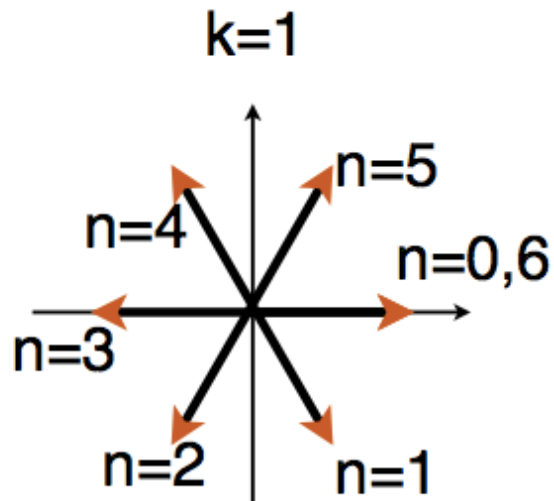
Discrete Fourier Series

$$W_N \triangleq e^{-j2\pi/N}$$

□ Properties of W_N :

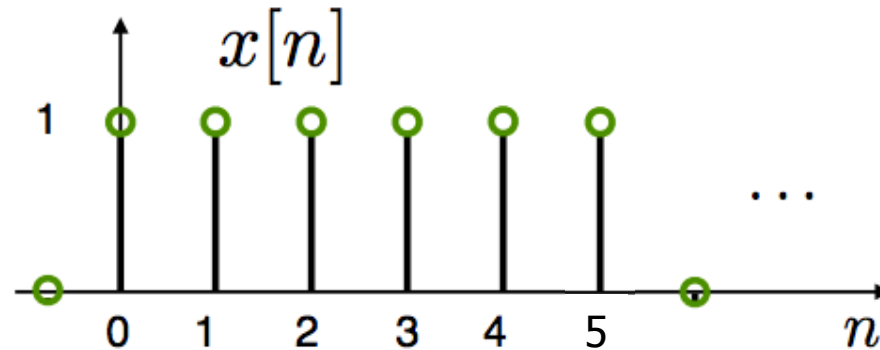
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- $W_N^{k+r} = W_N^k W_N^r$ and, $W_N^{k+N} = W_N^k$

□ Example: W_N^{kn} ($N=6$)



Example

$$W_N \triangleq e^{-j2\pi/N}$$

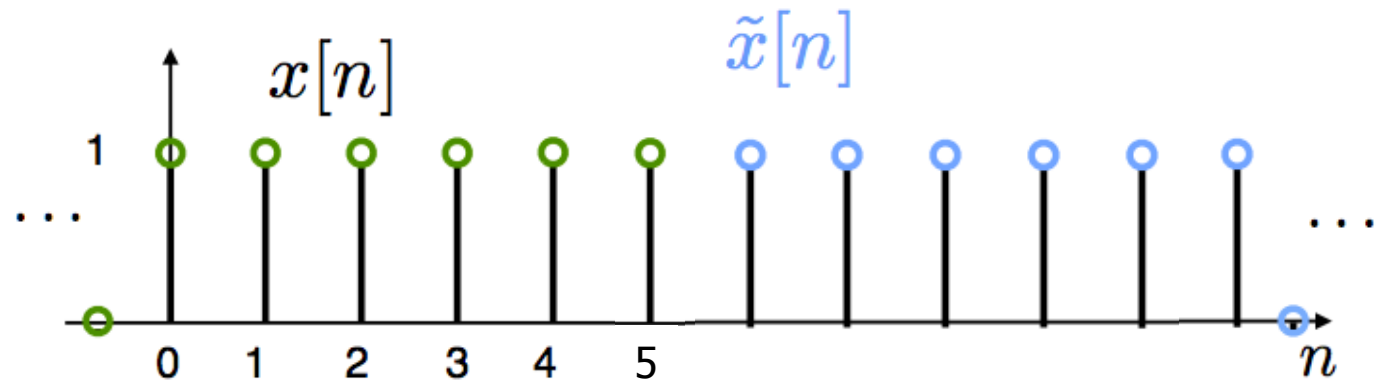


$$X[k] = \begin{cases} \sum_{n=0}^5 W_5^{nk} & k = 0, 1, 2, 3, 4, 5 \\ 0 & \text{else} \end{cases}$$
$$= 6\delta[k]$$

"6-point" DFT

Example

$$W_N \triangleq e^{-j2\pi/N}$$



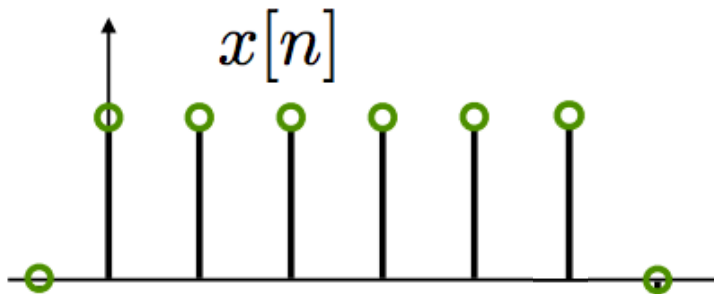
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“6-point” DFT

Example

$$W_N \triangleq e^{-j2\pi/N}$$

- Q: What if we take $N=10$?
- A: $X[k] = \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.



Example

$$W_N \triangleq e^{-j2\pi/N}$$

- Q: What if we take $N=10$?
- A: $X[k] = \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.



$$X[k] = \begin{cases} \sum_{n=0}^9 x[n] W_{10}^{nk} & k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \\ 0 & \text{else} \end{cases}$$

“10-point” DFT

Example

$$W_N \triangleq e^{-j2\pi/N}$$

- Q: What if we take $N=10$?
- A: $X[k] = \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.



$$X[k] = \begin{cases} \sum_{n=0}^5 W_{10}^{nk} & k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \\ 0 & \text{else} \end{cases}$$

“10-point” DFT



Example

□ Now, sum from $n=0$ to 9

$$X[k] = \sum_{n=0}^9 x[n] W_{10}^{nk} = \sum_{n=0}^5 W_{10}^{nk}$$



Example

□ Now, sum from $n=0$ to 9

$$\begin{aligned} X[k] &= \sum_{n=0}^9 x[n] W_{10}^{nk} = \sum_{n=0}^5 W_{10}^{nk} \\ &= \sum_{n=0}^5 \left(e^{-j\frac{2\pi}{10}k} \right)^n = \frac{1 - \left(e^{-j\frac{2\pi}{10}k} \right)^6}{1 - e^{-j\frac{2\pi}{10}k}} = \frac{1 - e^{-j\frac{6\pi}{5}k}}{1 - e^{-j\frac{2\pi}{10}k}} \end{aligned}$$

Example

□ Now, sum from $n=0$ to 9

$$\begin{aligned} X[k] &= \sum_{n=0}^9 x[n] W_{10}^{nk} = \sum_{n=0}^5 W_{10}^{nk} \\ &= \sum_{n=0}^5 \left(e^{-j\frac{2\pi}{10}k} \right)^n = \frac{1 - \left(e^{-j\frac{2\pi}{10}k} \right)^6}{1 - e^{-j\frac{2\pi}{10}k}} = \frac{1 - e^{-j\frac{6\pi}{5}k}}{1 - e^{-j\frac{2\pi}{10}k}} \\ &= \frac{e^{-j\frac{3\pi}{5}k} \left(e^{j\frac{3\pi}{5}k} - e^{-j\frac{3\pi}{5}k} \right)}{e^{-j\frac{\pi}{10}k} \left(e^{j\frac{\pi}{10}k} - e^{-j\frac{\pi}{10}k} \right)} = e^{-j\frac{\pi}{2}k} \frac{\sin\left(\frac{3\pi}{5}k\right)}{\sin\left(\frac{\pi}{10}k\right)} \end{aligned}$$



DFT vs. DTFT

□ For finite sequences of length N :

■ The N -point DFT of $x[n]$ is:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} \quad 0 \leq k \leq N-1$$

■ The DTFT of $x[n]$ is:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \quad -\infty < \omega < \infty$$



DFT vs. DTFT

- The DFT are samples of the DTFT at N equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega=k\frac{2\pi}{N}} \quad 0 \leq k \leq N - 1$$



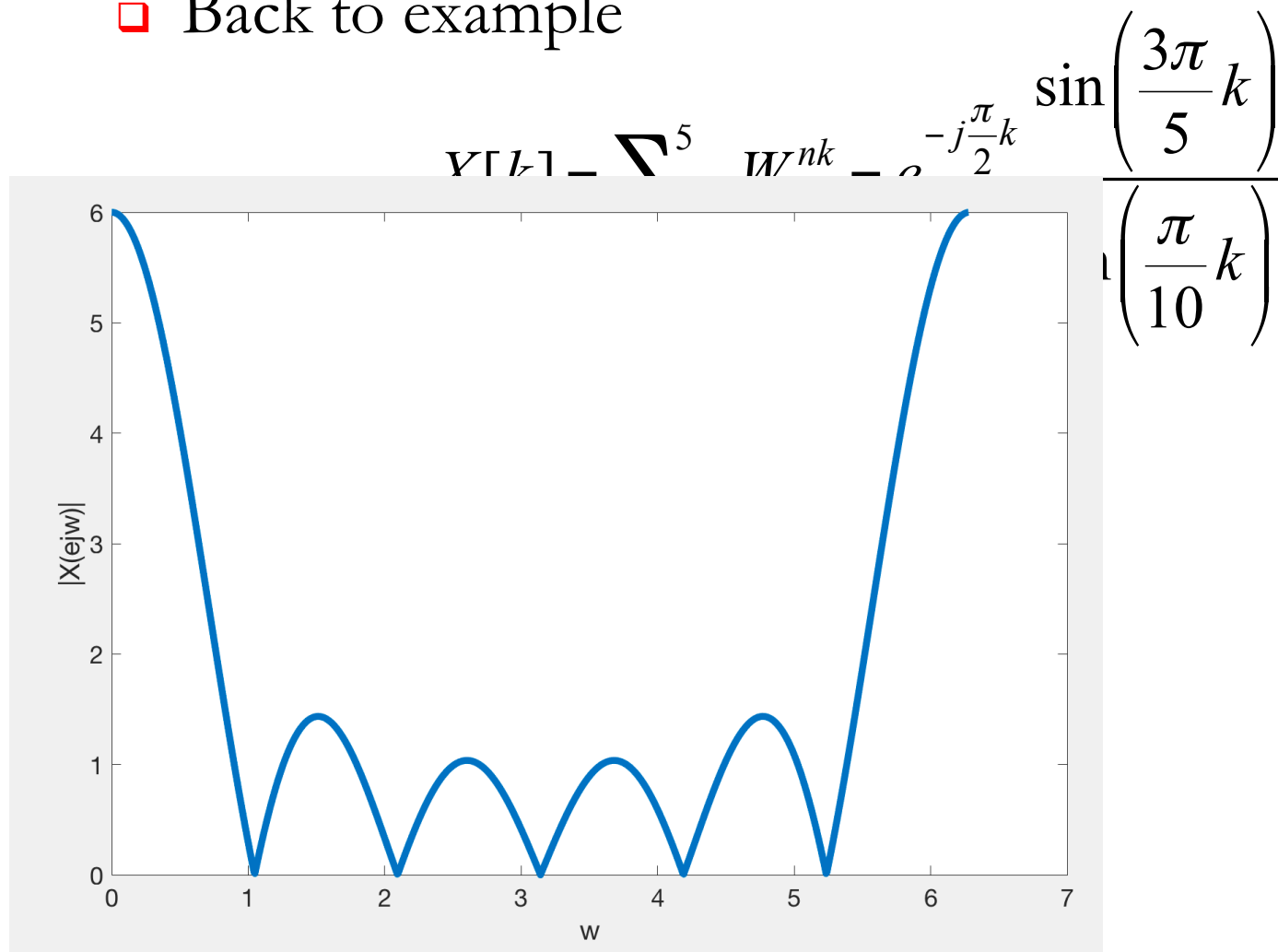
DFT vs DTFT

□ Back to example

$$X[k] = \sum_{n=0}^5 W_{10}^{nk} = e^{-j\frac{\pi}{2}k} \frac{\sin\left(\frac{3\pi}{5}k\right)}{\sin\left(\frac{\pi}{10}k\right)}$$

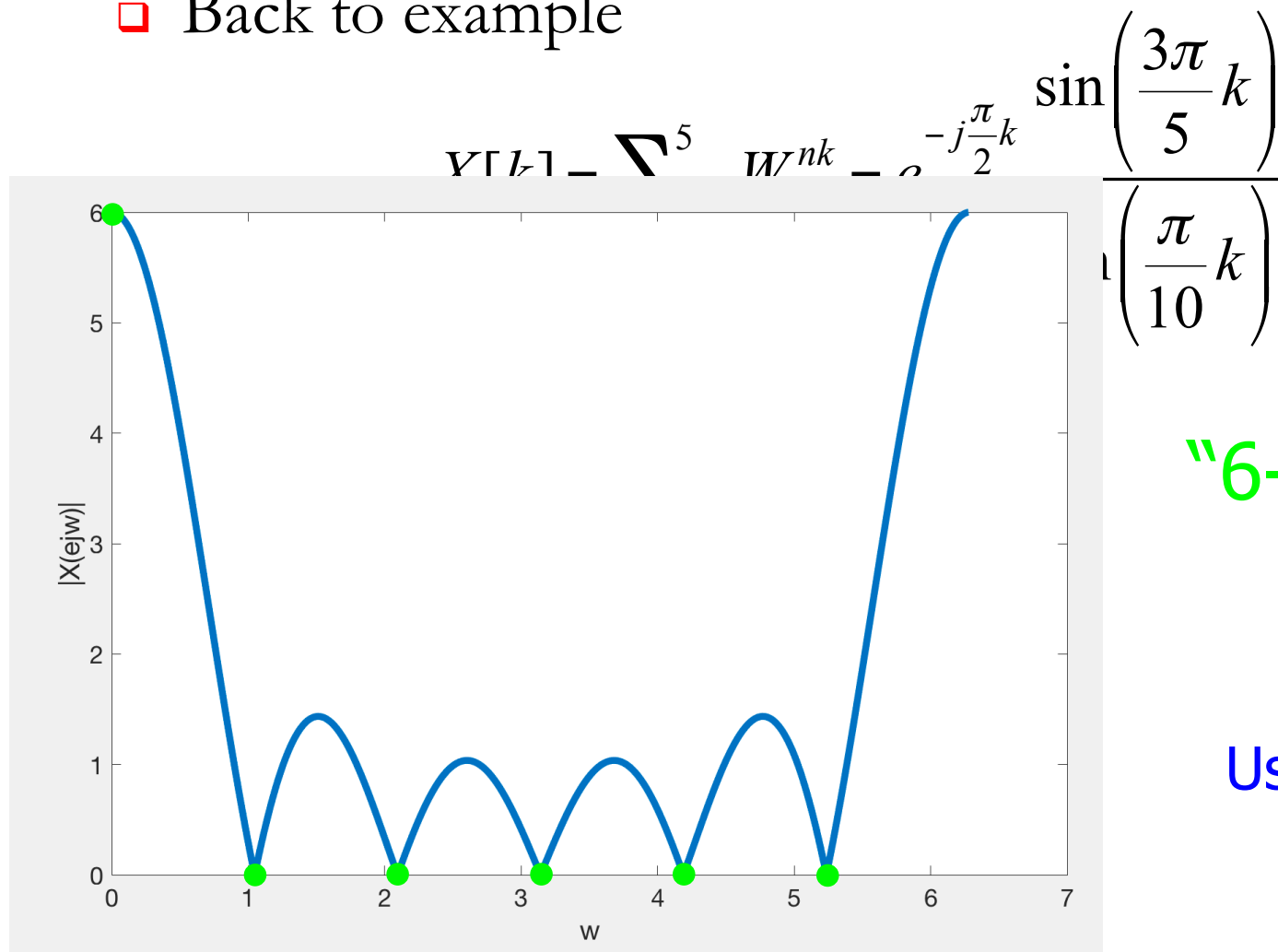
DFT vs DTFT

□ Back to example



DFT vs DTFT

□ Back to example

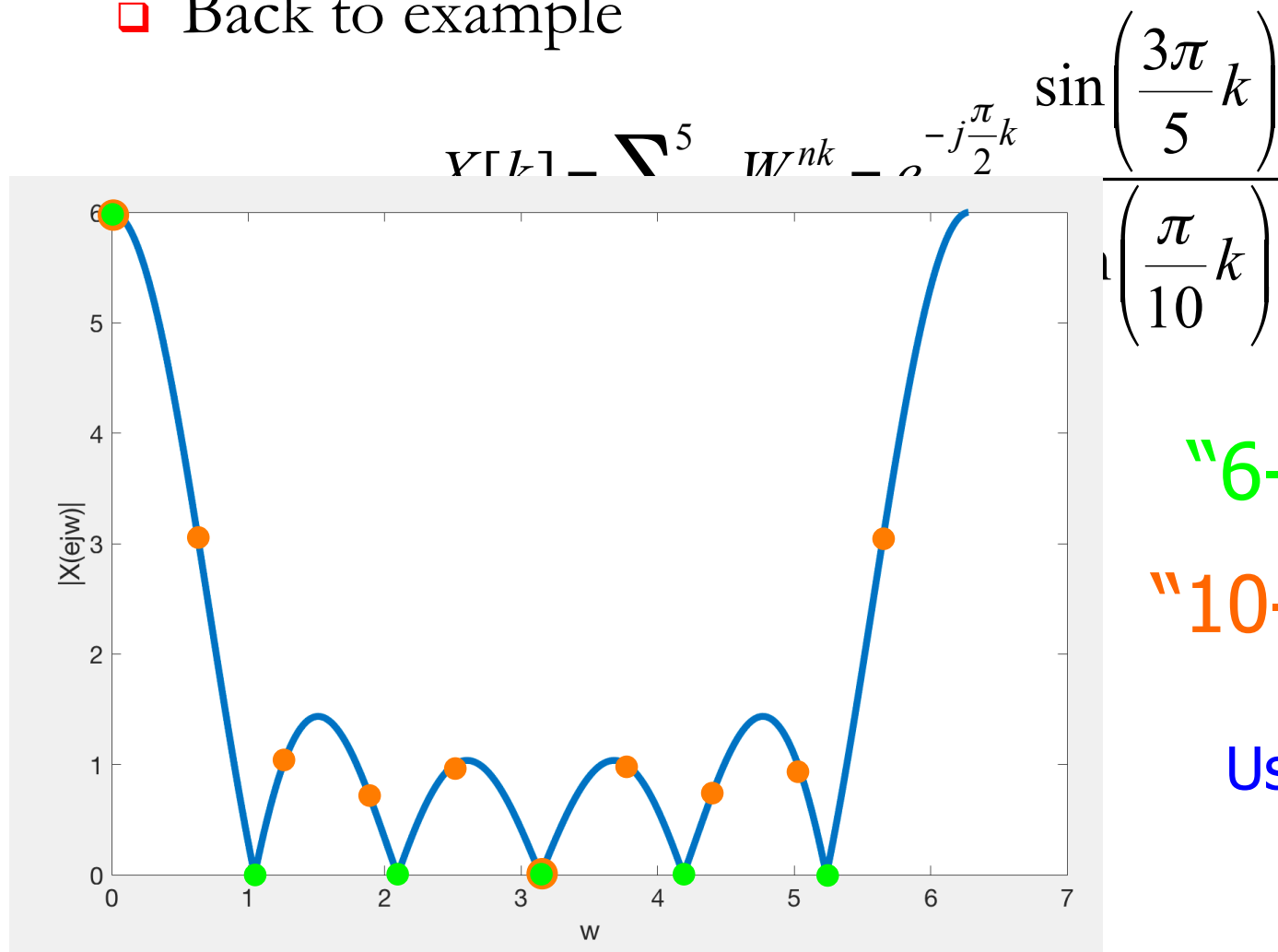


“6-point” DFT

Use `fftshift`
to center
around dc

DFT vs DTFT

□ Back to example

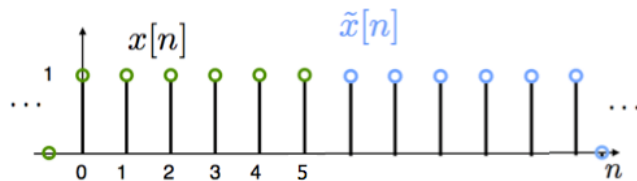


"6-point" DFT

"10-point" DFT

Use `fftshift`
to center
around dc

DFT Examples



$$X[k] = \begin{cases} \sum_{n=0}^5 W_5^{nk} & k = 0, 1, 2, 3, 4, 5 \\ 0 & \text{else} \end{cases}$$

$$= 5\delta[k] \quad \text{"6-point" DFT}$$



$$X[k] = \begin{cases} e^{-j\frac{\pi}{2}k} \frac{\sin\left(\frac{3\pi}{5}k\right)}{\sin\left(\frac{\pi}{10}k\right)} & k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \\ 0 & \text{else} \end{cases}$$



DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{DFT, analysis}$$



DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$N \cdot x^*[n] = N \left(\mathcal{DFT}^{-1} \{X[k]\} \right)^*$$

DFT and Inverse DFT

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DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$N \cdot x^*[n] = N \left(\mathcal{DFT}^{-1} \{X[k]\} \right)^*$$

$$\begin{aligned} & \left((a + jb) + (c + jd) \right)^* \\ &= \left((a + c) + j(b + d) \right)^* \\ &= (a + c) - j(b + d) \\ &= (a - jb) + (c - jd) \\ &= (a + jb)^* + (c + jd)^* \end{aligned} = N \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right)^* = \sum_{k=0}^{N-1} X^*[k] W_N^{kn}$$

DFT and Inverse DFT

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DFT and Inverse DFT

□ So

$$\mathcal{DFT} \{X^*[k]\} = N (\mathcal{DFT}^{-1} \{X[k]\})^*$$



DFT and Inverse DFT

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$$\mathcal{DFT} \{X^*[k]\} = N (\mathcal{DFT}^{-1} \{X[k]\})^*$$

$$\mathcal{DFT}^{-1} \{X[k]\} = \frac{1}{N} (\mathcal{DFT} \{X^*[k]\})^*$$

DFT and Inverse DFT

□ So

$$\mathcal{DFT} \{X^*[k]\} = N (\mathcal{DFT}^{-1} \{X[k]\})^*$$

$$\mathcal{DFT}^{-1} \{X[k]\} = \left(\frac{1}{N}\right) (\mathcal{DFT} \{X^*[k]\})^*$$

- Implement IDFT by:
- Take complex conjugate
 - Take DFT
 - Multiply by $1/N$
 - Take complex conjugate

DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \dots & W_N^{0n} & \dots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \dots & W_N^{kn} & \dots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \dots & W_N^{(N-1)n} & \dots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

freq, $X[k]$ DFT matrix time, $x[n]$

DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \dots & W_N^{0n} & \dots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \dots & W_N^{kn} & \dots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \dots & W_N^{(N-1)n} & \dots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

IDFT:

$$\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \begin{pmatrix} W_N^{-00} & \dots & W_N^{-0k} & \dots & W_N^{-0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-n0} & \dots & W_N^{-nk} & \dots & W_N^{-n(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)0} & \dots & W_N^{-(N-1)k} & \dots & W_N^{-(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix}$$

DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \dots & W_N^{0n} & \dots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \dots & W_N^{kn} & \dots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \dots & W_N^{(N-1)n} & \dots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

IDFT:

$$\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \begin{pmatrix} W_N^{-00} & \dots & W_N^{-0k} & \dots & W_N^{-0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-n0} & \dots & W_N^{-nk} & \dots & W_N^{-n(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)0} & \dots & W_N^{-(N-1)k} & \dots & W_N^{-(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix}$$



DFT as Matrix Operator

- Can write compactly as

$$\begin{aligned}\mathbf{X} &= \mathbf{W}_N \mathbf{x} \\ \mathbf{x} &= \frac{1}{N} \mathbf{W}_N^* \mathbf{X}\end{aligned}$$



Properties of the DFT

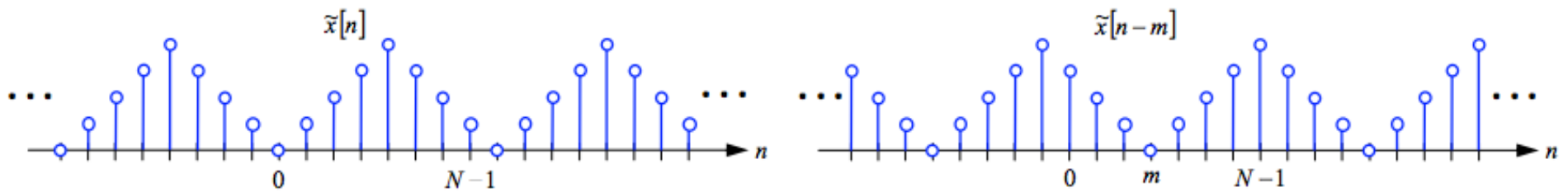
- ❑ Properties of DFT inherited from DFS
- ❑ Linearity

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

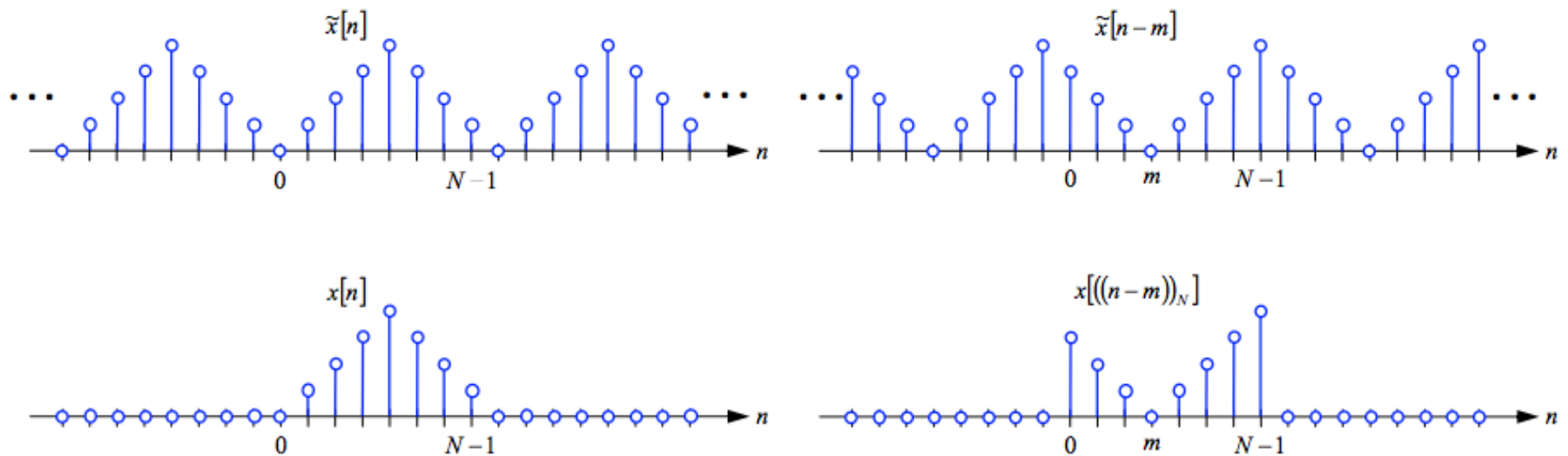
- ❑ Circular Time Shift

$$x[((n - m))_N] \leftrightarrow X[k]e^{-j(2\pi/N)km} = X[k]W_N^{km}$$

Circular Shift



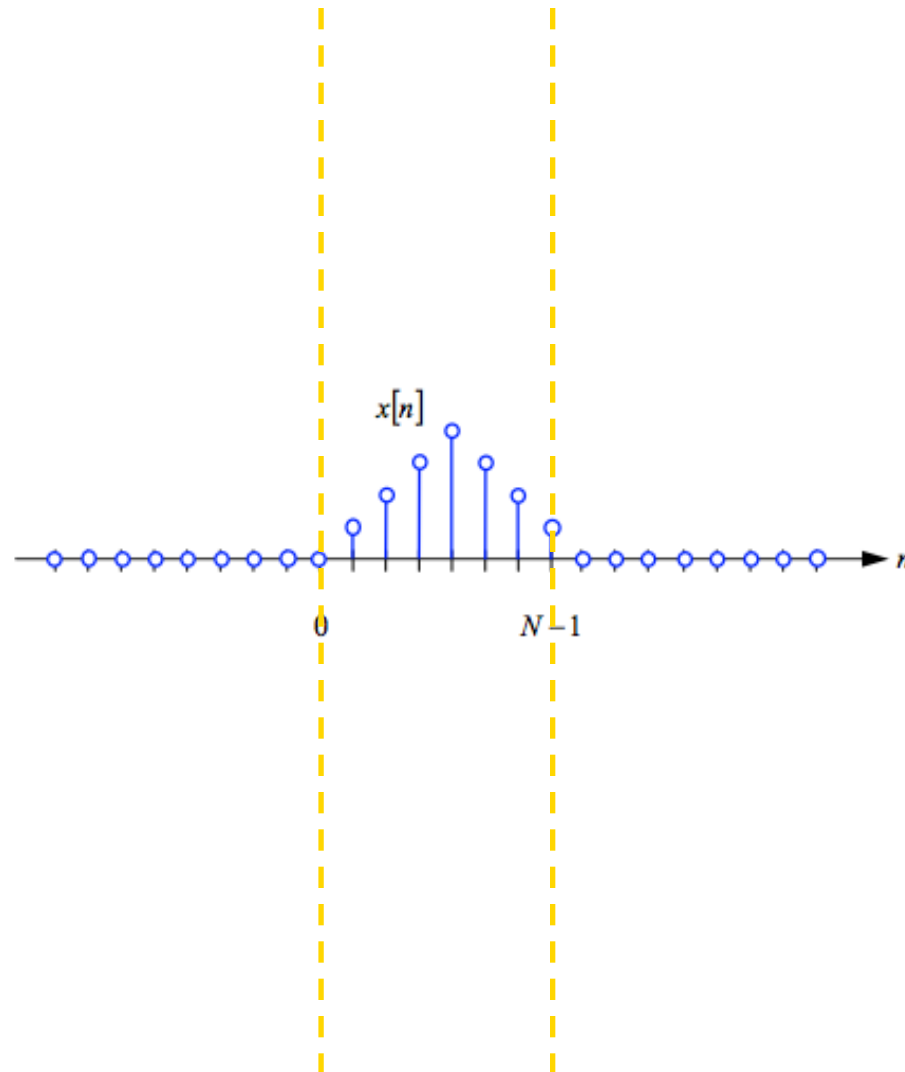
Circular Shift



$m=?$

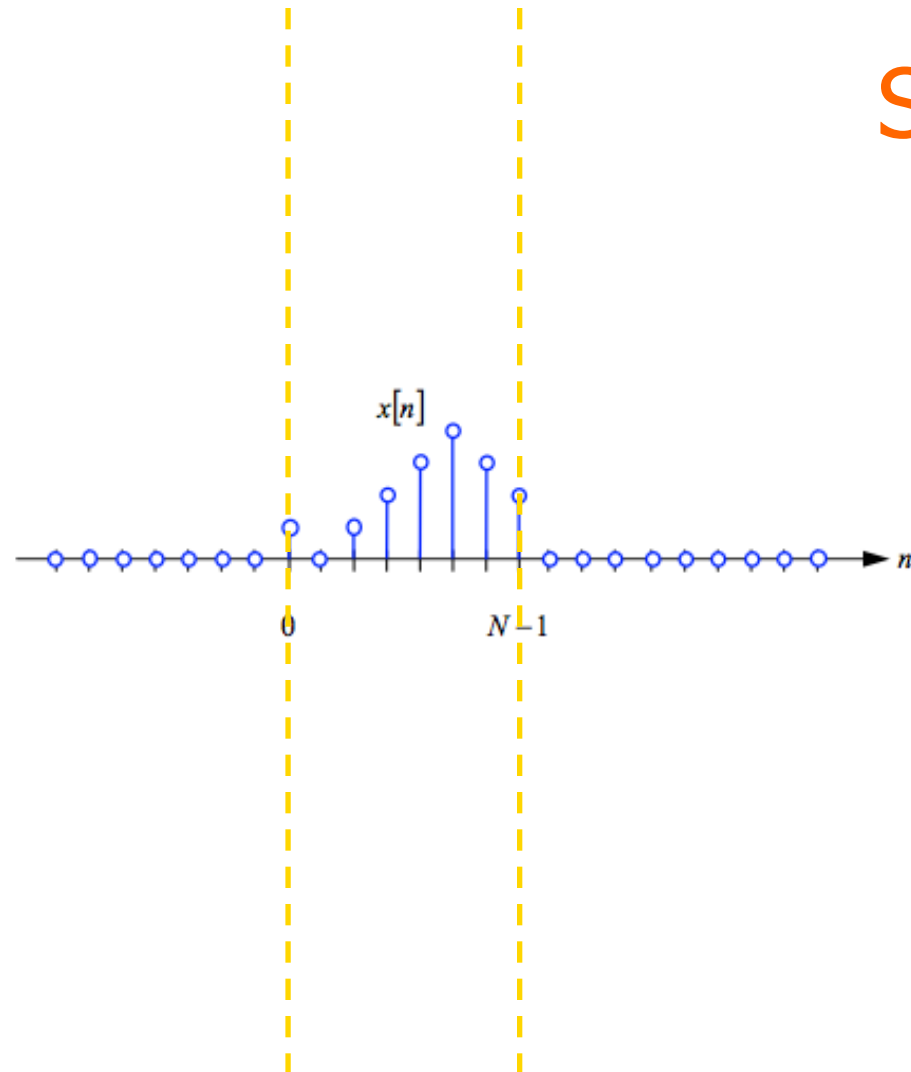
Circular Shift

Shift by 3



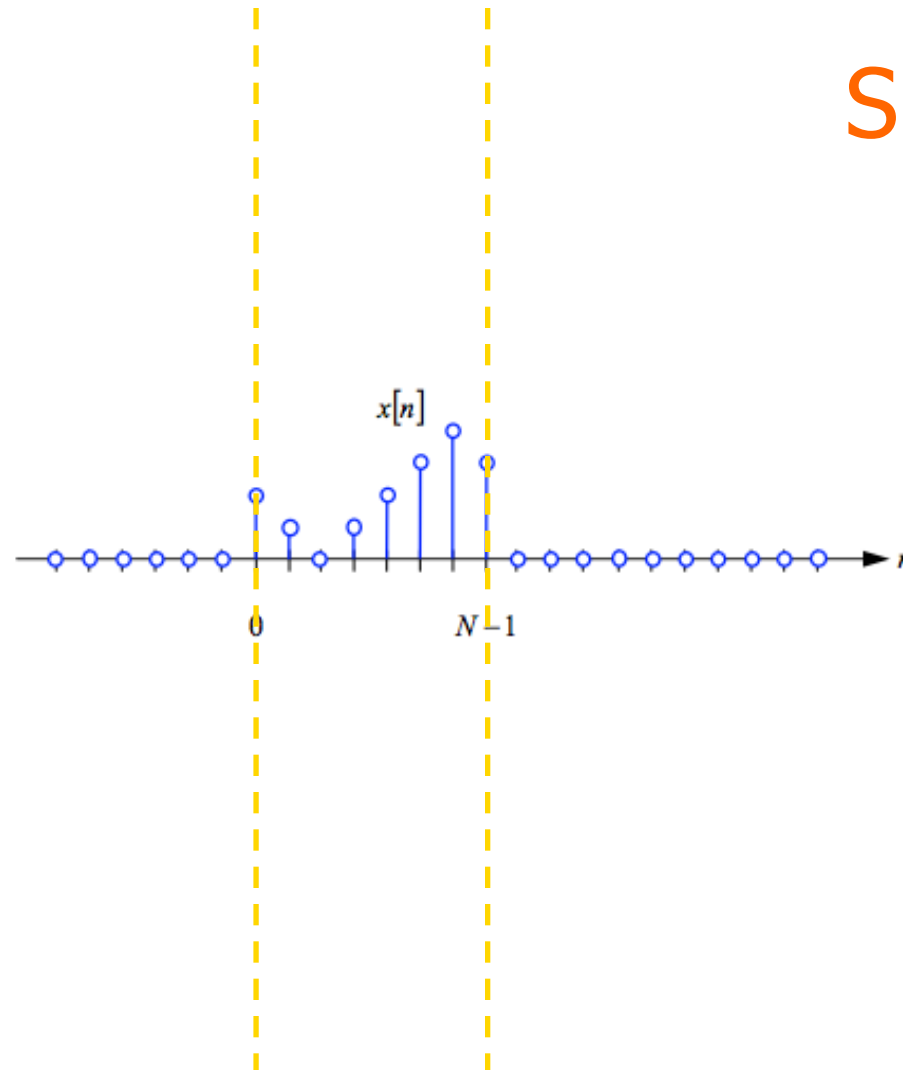
Circular Shift

Shift by 3



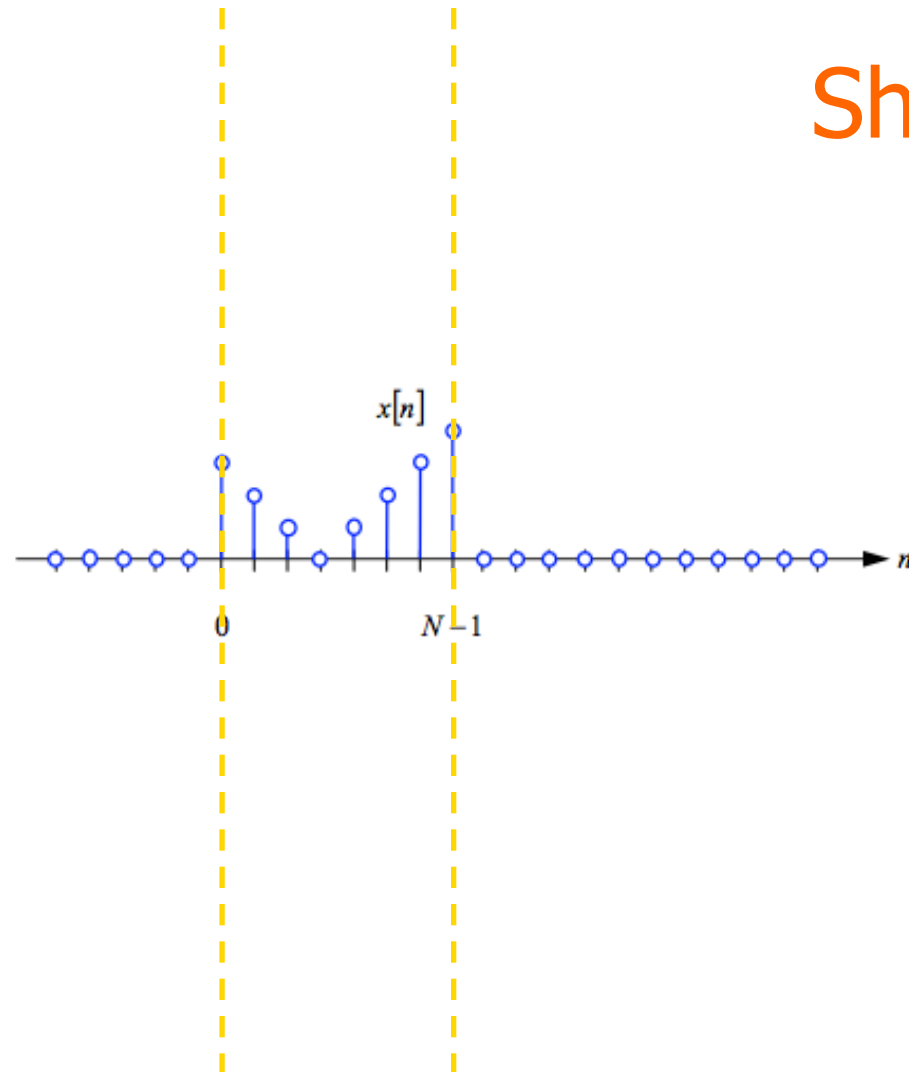
Circular Shift

Shift by 3



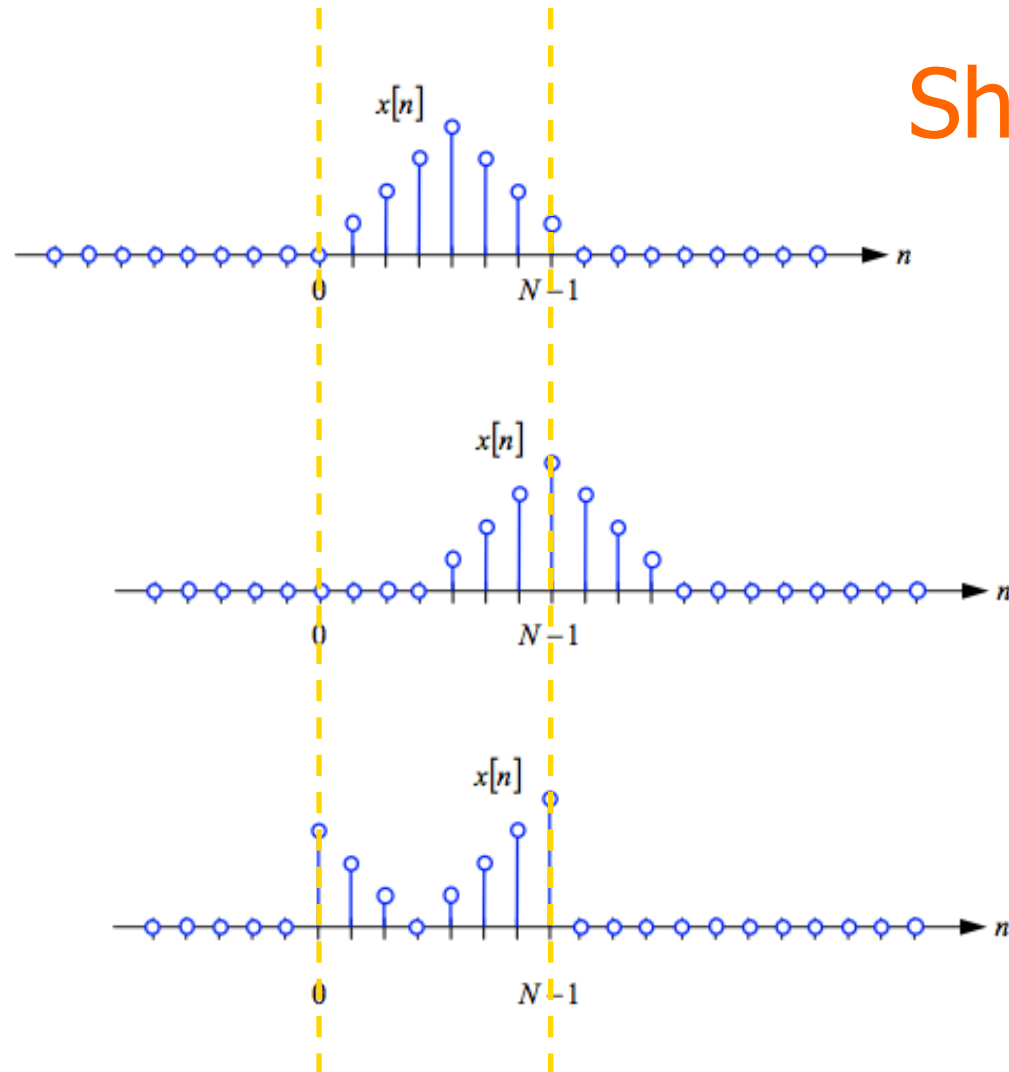
Circular Shift

Shift by 3



Circular Shift

Shift by 3



Properties of DFT

- ❑ Circular frequency shift

$$x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- ❑ Complex Conjugation

$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

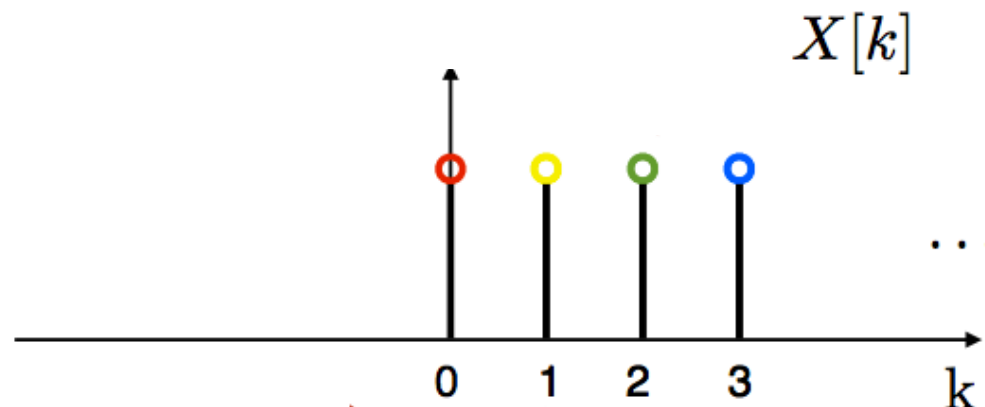
- ❑ Conjugate Symmetry for Real Signals

If $x[n]$ real

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

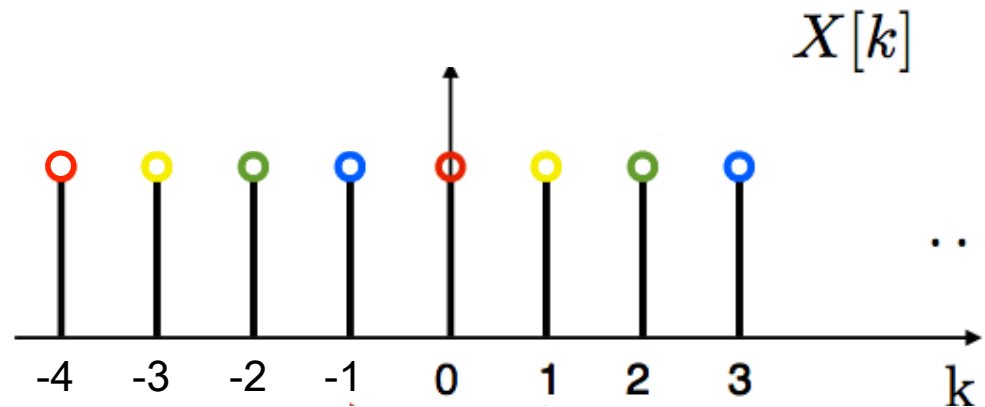
4-point DFT
–Symmetry



$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

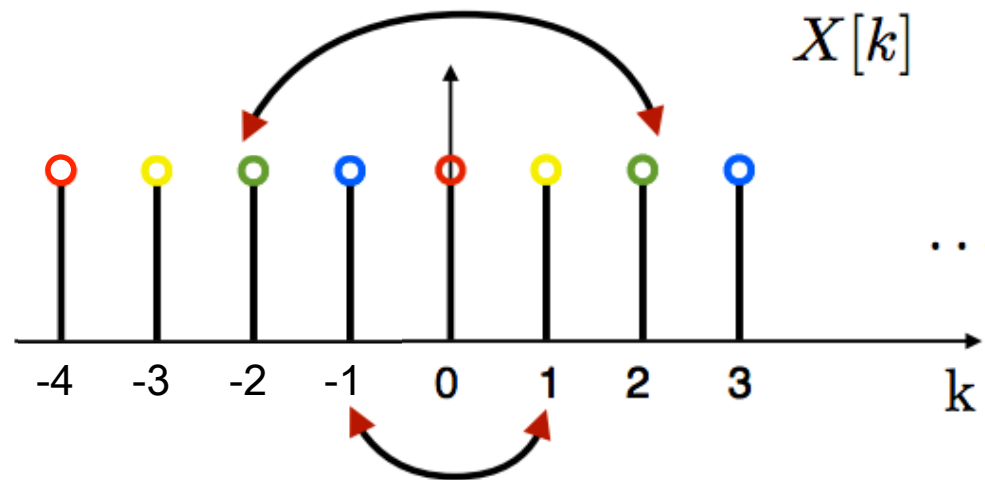
4-point DFT
–Symmetry



$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

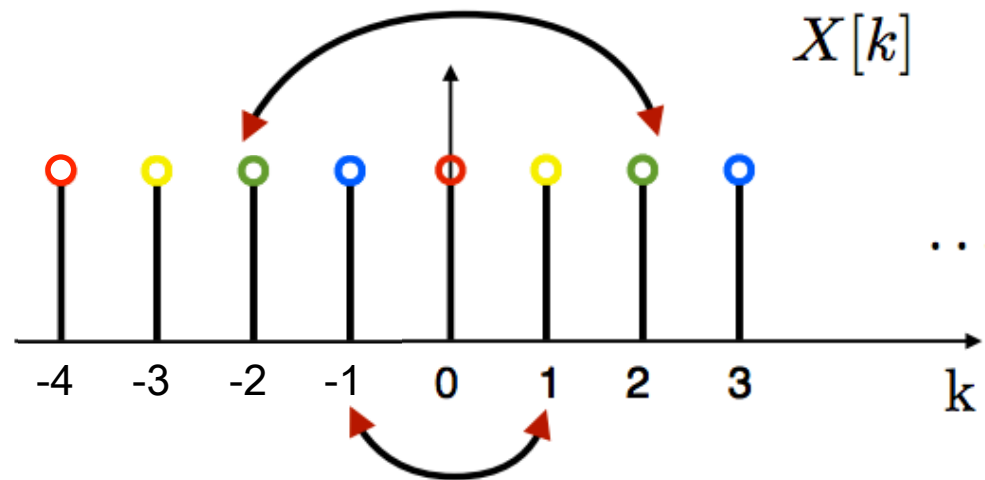
4-point DFT
–Symmetry



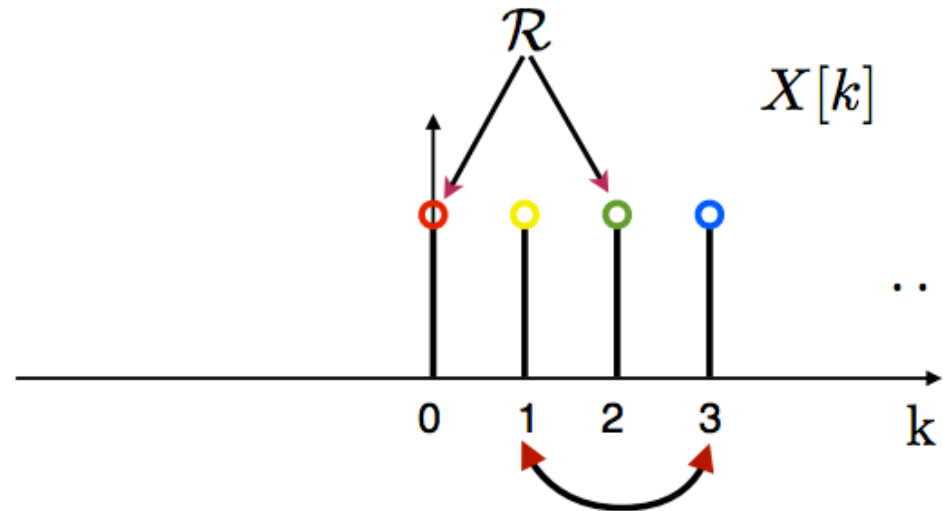
$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

4-point DFT
–Symmetry



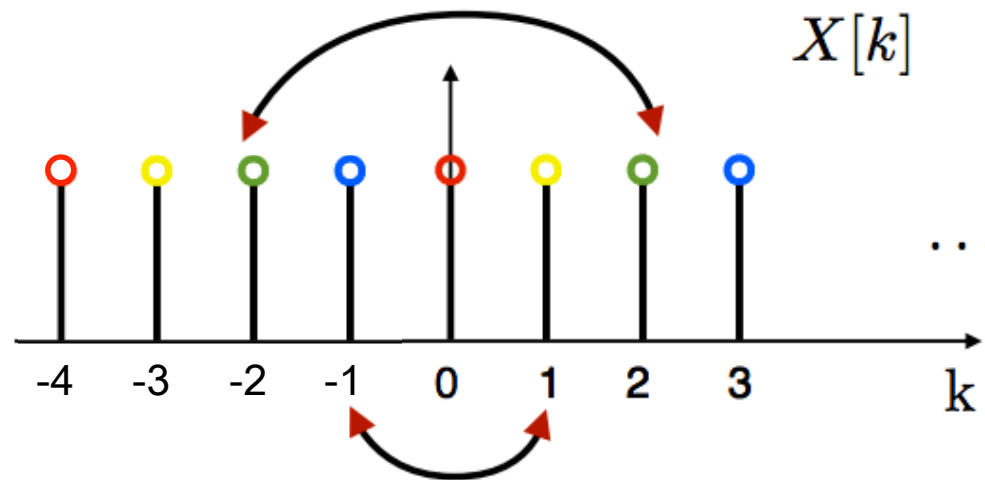
4-point DFT
–Symmetry



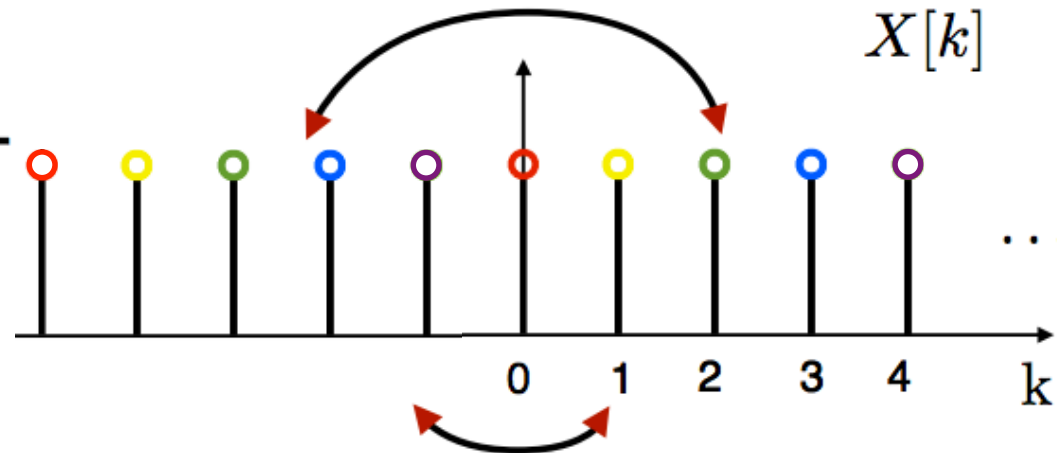
$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

4-point DFT
–Symmetry

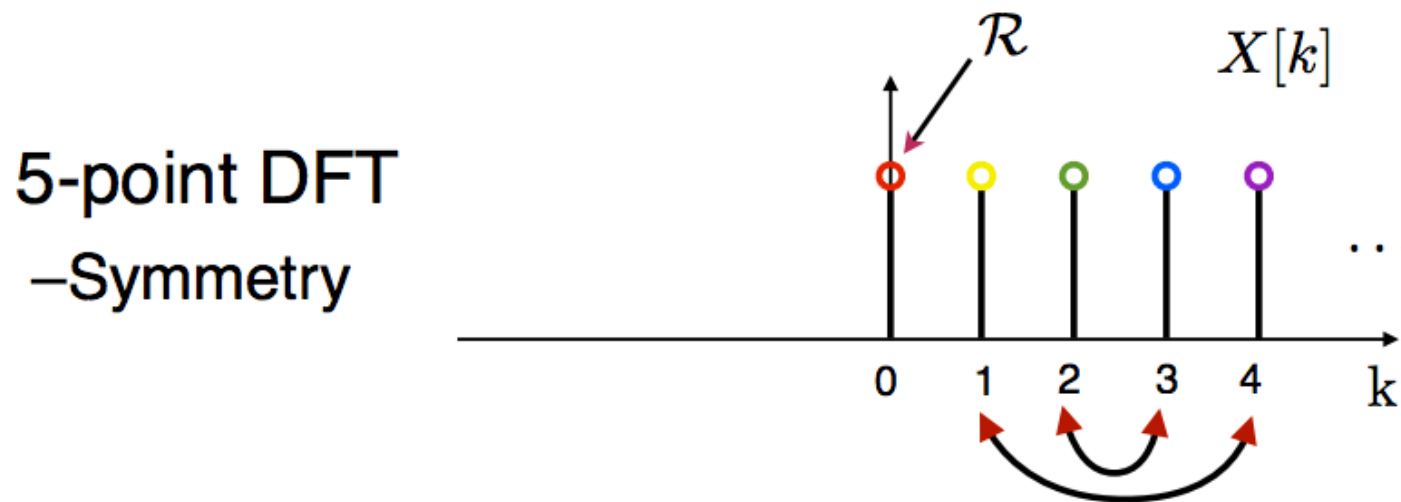
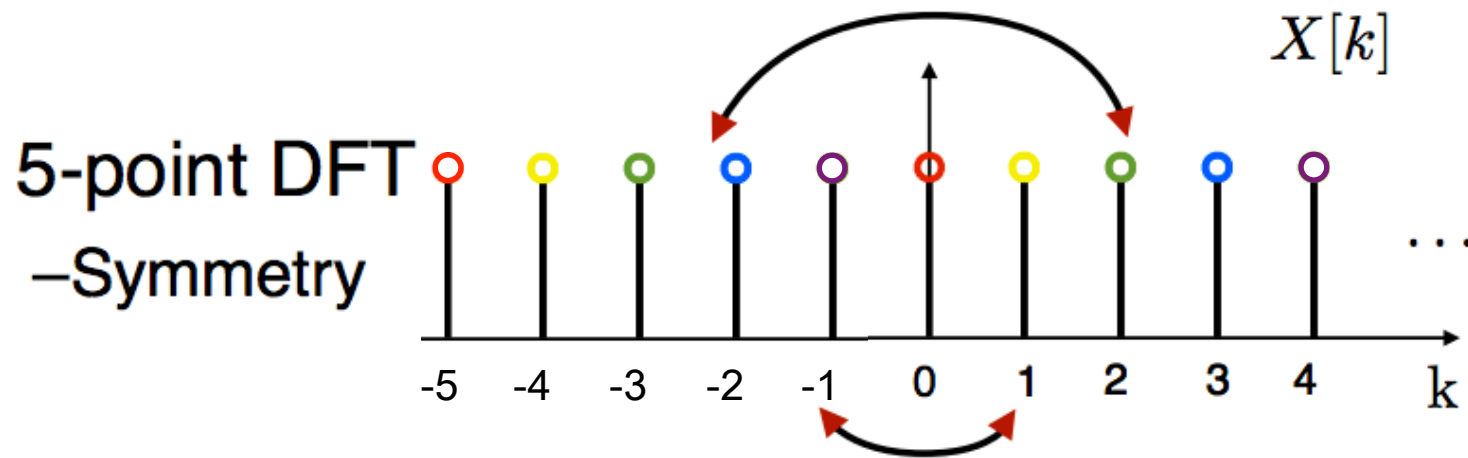


5-point DFT
–Symmetry



$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example



Properties of the DFS/DFT

Discrete Fourier Series			Discrete Fourier Transform		
Property	N -periodic sequence	N -periodic DFS	Property	N -point sequence	N -point DFT
	$\tilde{x}[n]$ $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}[k]$ $\tilde{X}_1[k], \tilde{X}_2[k]$		$x[n]$ $x_1[n], x_2[n]$	$X[k]$ $X_1[k], X_2[k]$
Linearity	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$	Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Duality	$\tilde{X}[n]$	$N\tilde{x}[-k]$	Duality	$X[n]$	$Nx[((-k))_N]$
Time Shift	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$	Circular Time Shift	$x[((n-m))_N]$	$W_N^{km}X[k]$
Frequency Shift	$W_N^{-ln}\tilde{x}[n]$	$\tilde{X}[k-l]$	Circular Frequency Shift	$W_N^{-ln}x[n]$	$X[((k-l))_N]$
Periodic Convolution	$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$	$\tilde{X}_1[k]\tilde{X}_2[k]$	Circular Convolution	$\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$	$X_1[k]X_2[k]$
Multiplication	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}_1[l]\tilde{X}_2[k-l]$	Multiplication	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{l=0}^{N-1} X_1[l]X_2[((k-l))_N]$
Complex Conjugation	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$	Complex Conjugation	$x^*[n]$	$X^*[((-k))_N]$

Properties (Continued)

Time-Reversal and Complex Conjugation	$\tilde{x}^*[-n]$	$\tilde{X}^*[k]$	Time-Reversal and Complex Conjugation	$x^*[((-n))_N]$	$X^*[k]$
Real Part	$\text{Re}\{\tilde{x}[n]\}$	$\tilde{X}_{ep}[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$	Real Part	$\text{Re}\{x[n]\}$	$X_{ep}[k] = \frac{1}{2}(X[k] + X^*[((-k))_N])$
Imaginary Part	$j \text{Im}\{\tilde{x}[n]\}$	$\tilde{X}_{op}[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$	Imaginary Part	$j \text{Im}\{x[n]\}$	$X_{op}[k] = \frac{1}{2}(X[k] - X^*[((-k))_N])$
Even Part	$\tilde{x}_{ep}[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\text{Re}\{\tilde{X}[k]\}$	Even Part	$x_{ep}[n] = \frac{1}{2}(x[n] + x^*[((-n))_N])$	$\text{Re}\{X[k]\}$
Odd Part	$\tilde{x}_{op}[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j \text{Im}\{\tilde{X}[k]\}$	Odd Part	$x_{op}[n] = \frac{1}{2}(x[n] - x^*[((-n))_N])$	$j \text{Im}\{X[k]\}$
Symmetry for Real Sequence	$\tilde{x}[n] = \tilde{x}^*[n]$	$\tilde{X}[k] = \tilde{X}^*[-k]$ $\begin{cases} \text{Re}\{\tilde{X}[k]\} = \text{Re}\{\tilde{X}^*[-k]\} \\ \text{Im}\{\tilde{X}[k]\} = -\text{Im}\{\tilde{X}^*[-k]\} \end{cases}$ $\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \angle \tilde{X}[k] = -\angle \tilde{X}^*[-k] \end{cases}$	Symmetry for Real Sequence	$x[n] = x^*[n]$	$X[k] = X^*[((-k))_N]$ $\begin{cases} \text{Re}\{X[k]\} = \text{Re}\{X^*[((-k))_N]\} \\ \text{Im}\{X[k]\} = -\text{Im}\{X^*[((-k))_N]\} \end{cases}$ $\begin{cases} X[k] = X^*[((-k))_N] \\ \angle X[k] = -\angle X^*[((-k))_N] \end{cases}$
Parseval's Identity	$\sum_{n=0}^{N-1} \tilde{x}_1[n] \tilde{x}_2^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_1[k] \tilde{X}_2^*[k]$ $\sum_{n=0}^{N-1} \tilde{x}[n] ^2 = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] ^2$		Parseval's Identity	$\sum_{n=0}^{N-1} x_1[n] x_2^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_1[k] X_2^*[k]$ $\sum_{n=0}^{N-1} x[n] ^2 = \frac{1}{N} \sum_{k=0}^{N-1} X[k] ^2$	



Duality

If $x \xrightarrow{DFT} X$, then $\{X[n]\}_{n=0}^{N-1} \xrightarrow{DFT} N \{x[((-k))_N]\}_{k=0}^{N-1}$

Duality

If $x \xrightarrow{DFT} X$, then $\{X[n]\}_{n=0}^{N-1} \xrightarrow{DFT} N \{x[((-k))_N]\}_{k=0}^{N-1}$

$$\tilde{x}[n] \xleftrightarrow{\mathcal{DFS}} \tilde{X}[k],$$

$$\tilde{X}[n] \xleftrightarrow{\mathcal{DFS}} N\tilde{x}[-k].$$

Proof of Duality

$$\text{DFT of } \{x[n]\}_{n=0}^{N-1} \text{ is } X[k] = \sum_{p=0}^{N-1} x[p] e^{-j\frac{2\pi}{N}kp}; \quad k \leq 0 \leq N-1$$

$$\text{DFT of } \{X[n]\}_{n=0}^{N-1} \text{ is } \underbrace{\sum_{n=0}^{N-1} \sum_{p=0}^{N-1} x[p] e^{-j\frac{2\pi}{N}pn} e^{-j\frac{2\pi}{N}kn}}_{X[n]}, \quad k \leq 0 \leq N-1$$

$$= \sum_{p=0}^{N-1} x[p] \underbrace{\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(p+k)n}}_{\substack{N \text{ for } ((p+k))_N=0, \\ 0 \text{ otherwise}}}$$

$$((p+k))_N = 0 \text{ for } 0 \leq p \text{ \& } k \leq N-1 \Rightarrow p = ((-k))_N$$

$$p = -k + mN = ((-k))_N + rN + mN = ((-k))_N \text{ because } 0 \leq p \leq N-1$$

$$\therefore \text{DFT of } \{X[n]\}_{n=0}^{N-1} \text{ is } N \{x[((-k))_N]\}_{k=0}^{N-1}$$



Circular Convolution

□ Circular Convolution:

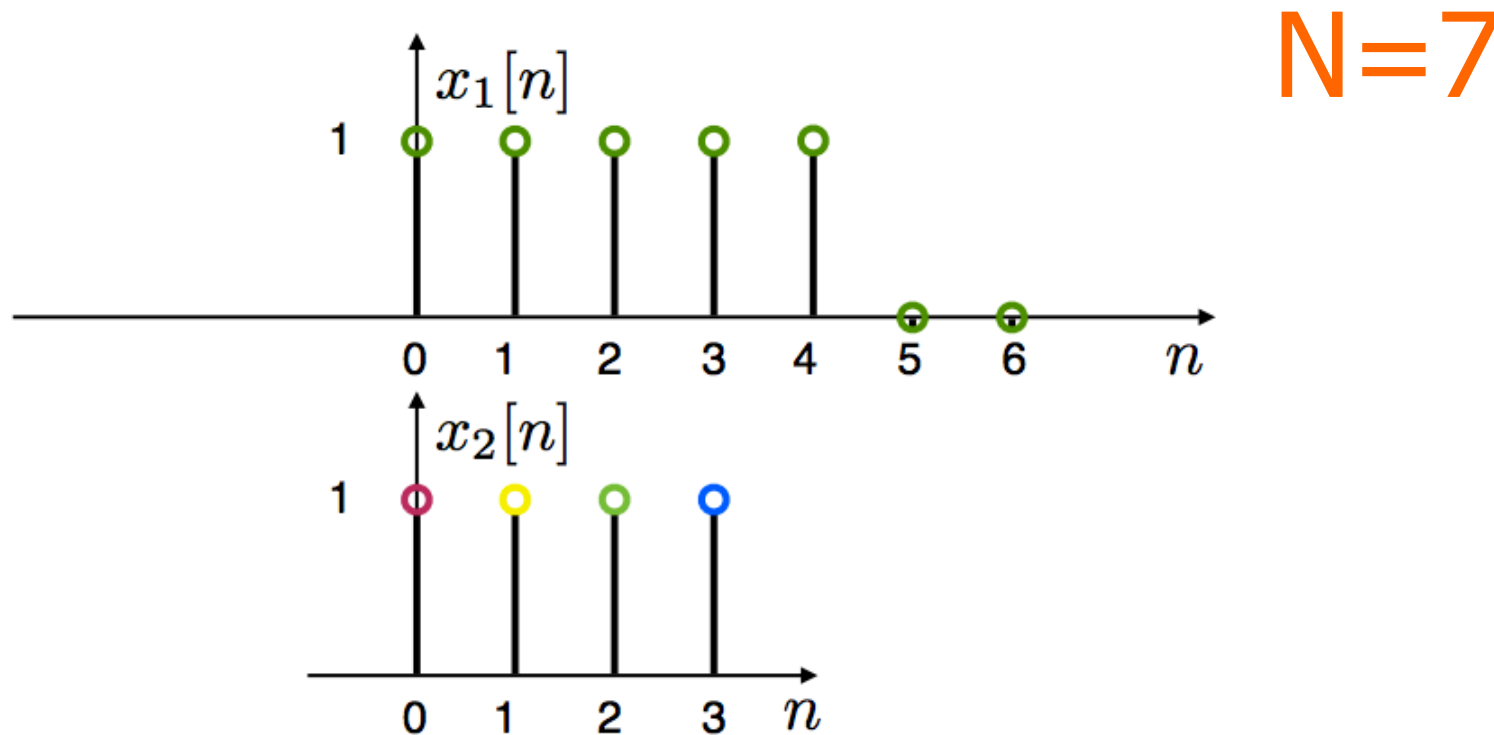
$$x_1[n] \textcircled{N} x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

For two signals of length N

Note: Circular convolution is commutative

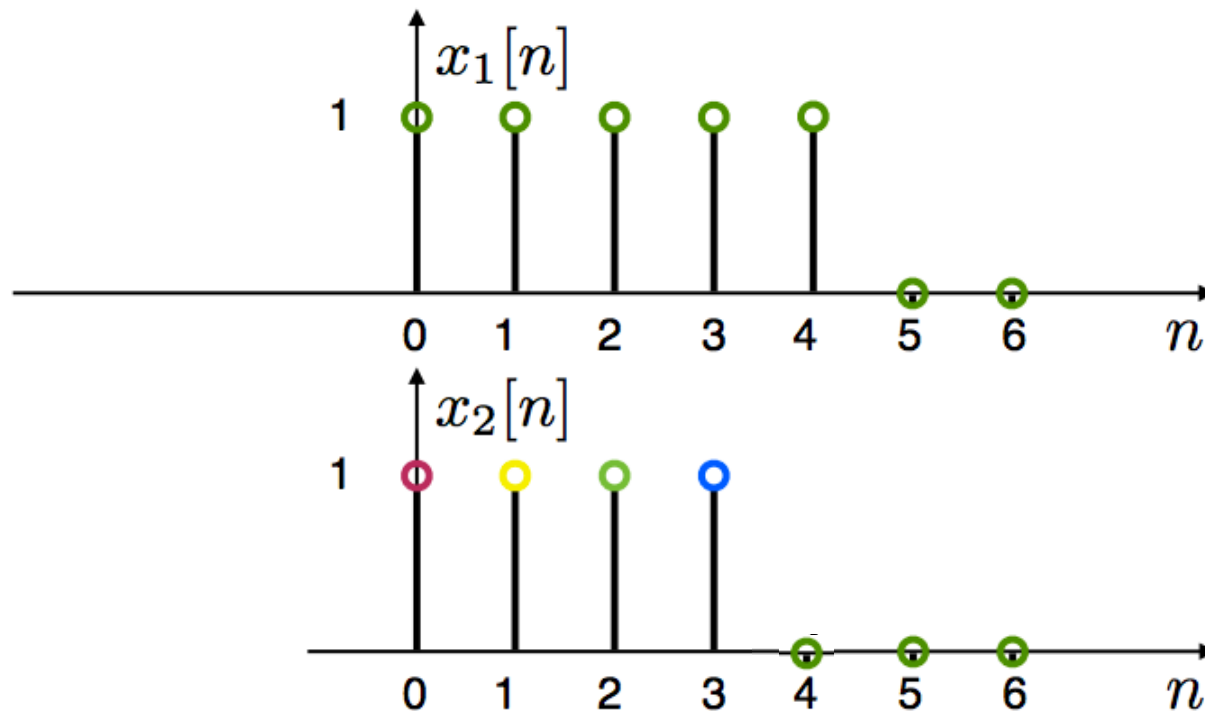
$$x_2[n] \textcircled{N} x_1[n] = x_1[n] \textcircled{N} x_2[n]$$

Compute Circular Convolution Sum



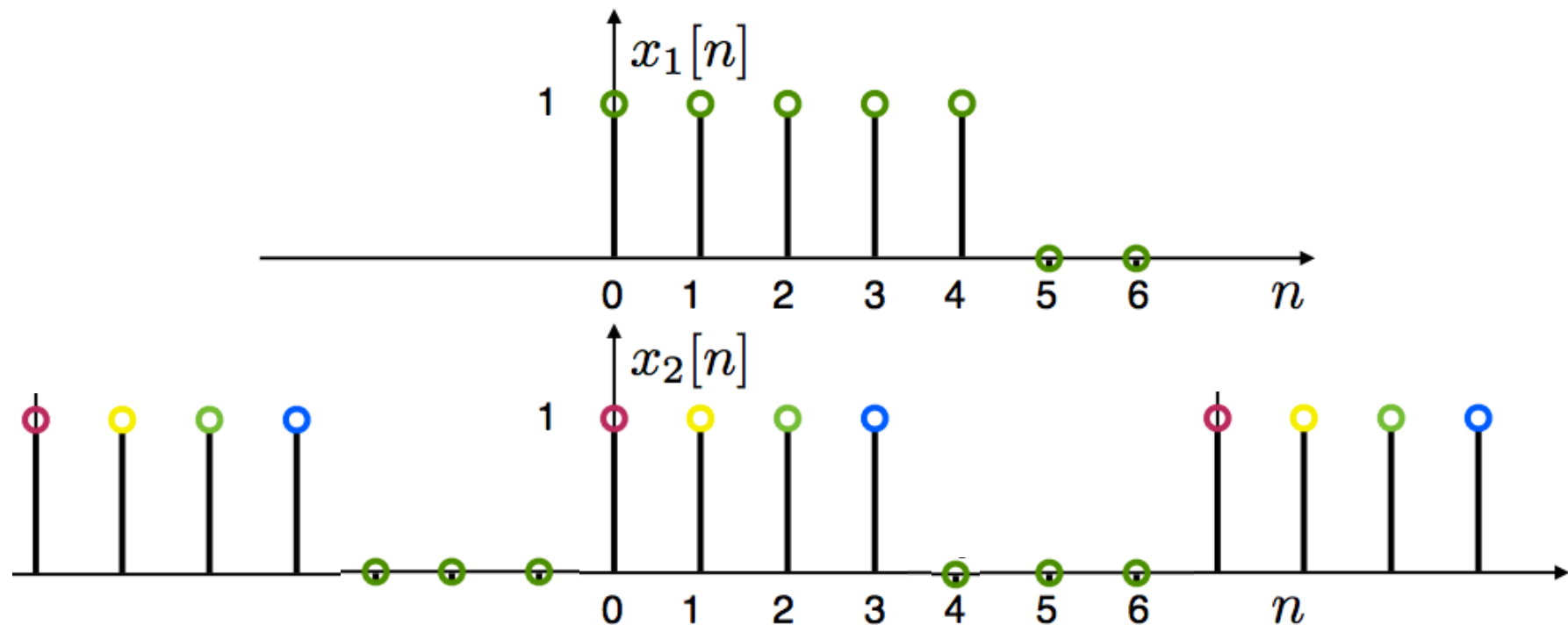
$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum



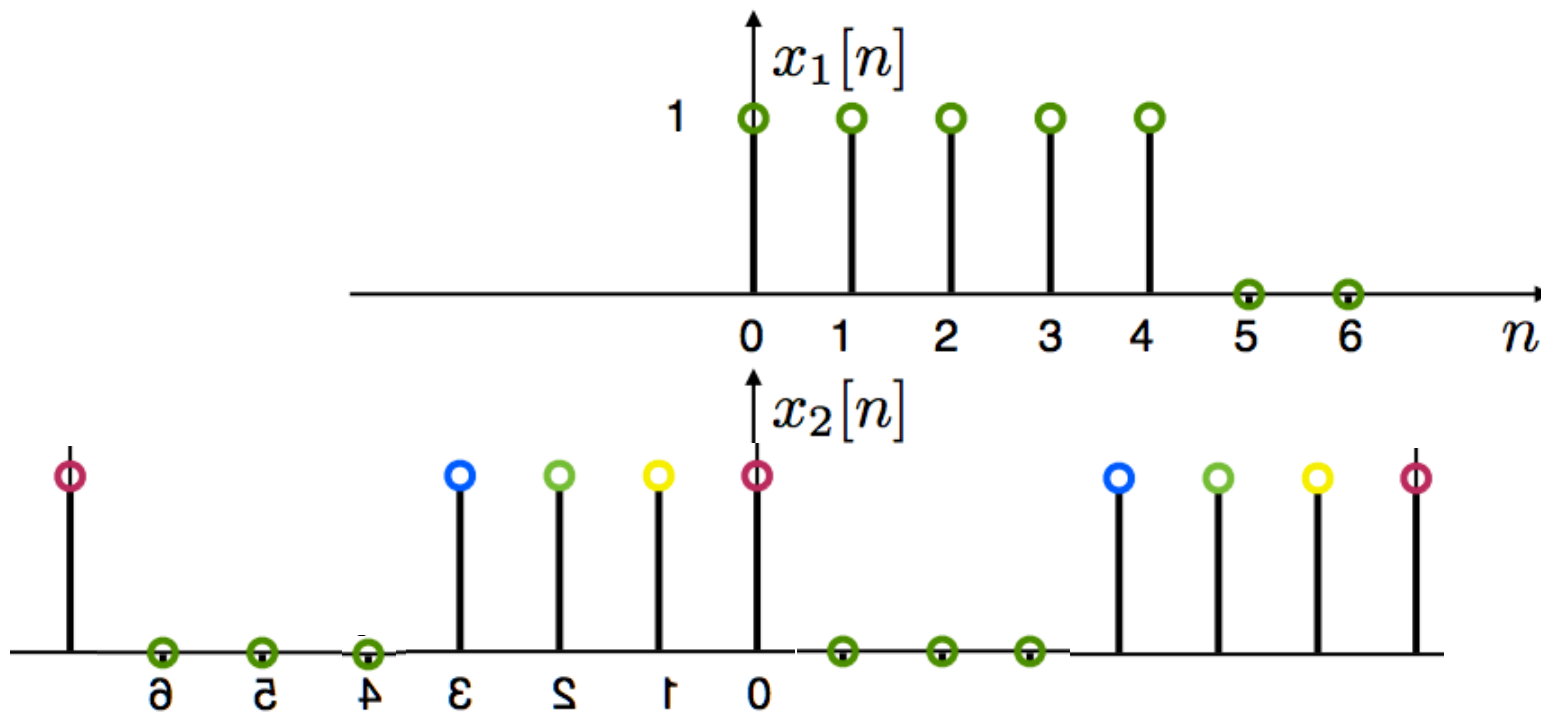
$$x_1[n] \circledcirc x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum



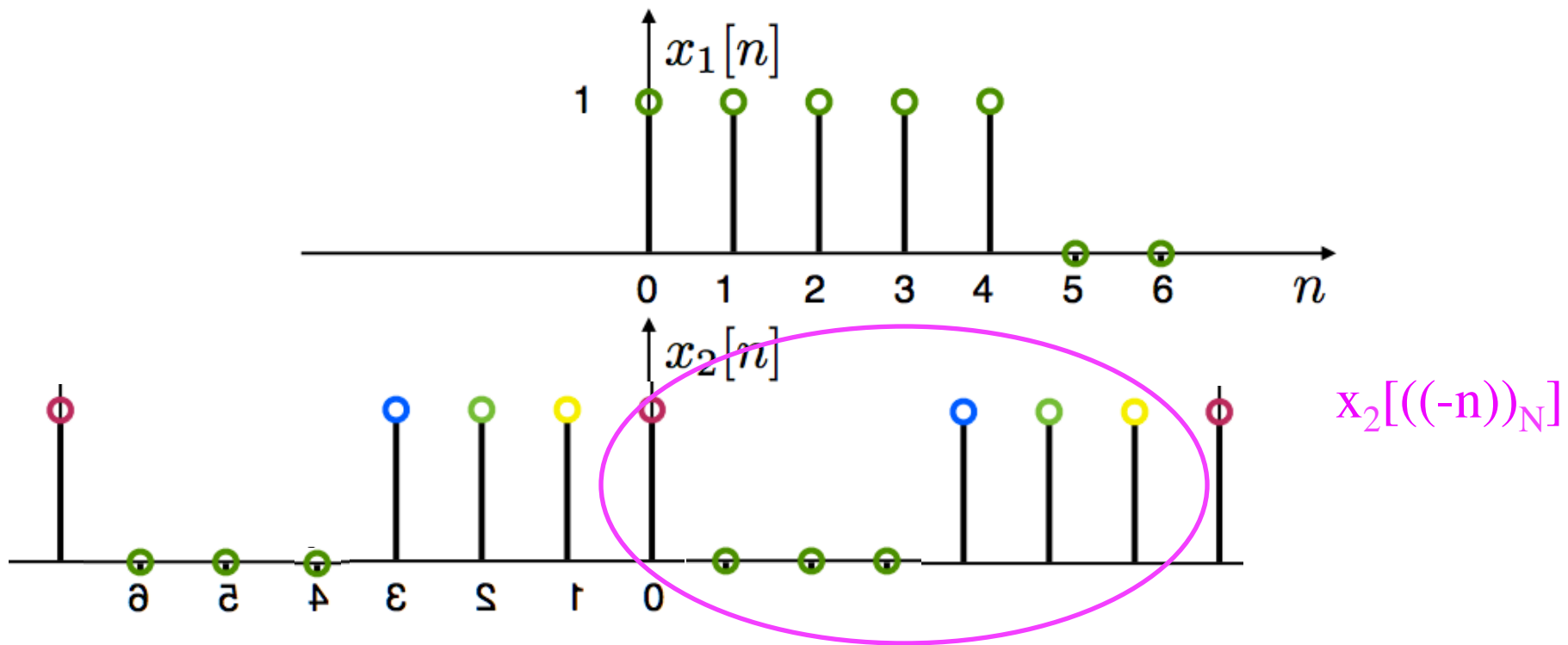
$$x_1[n] \circledcirc x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum



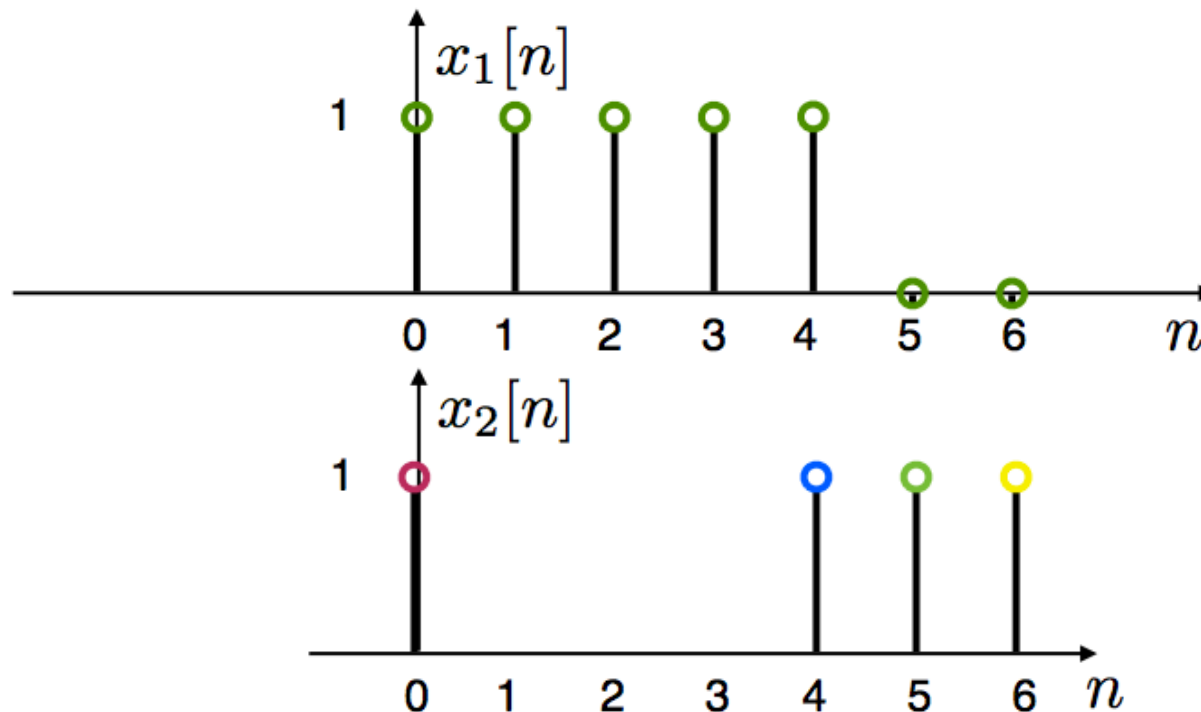
$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

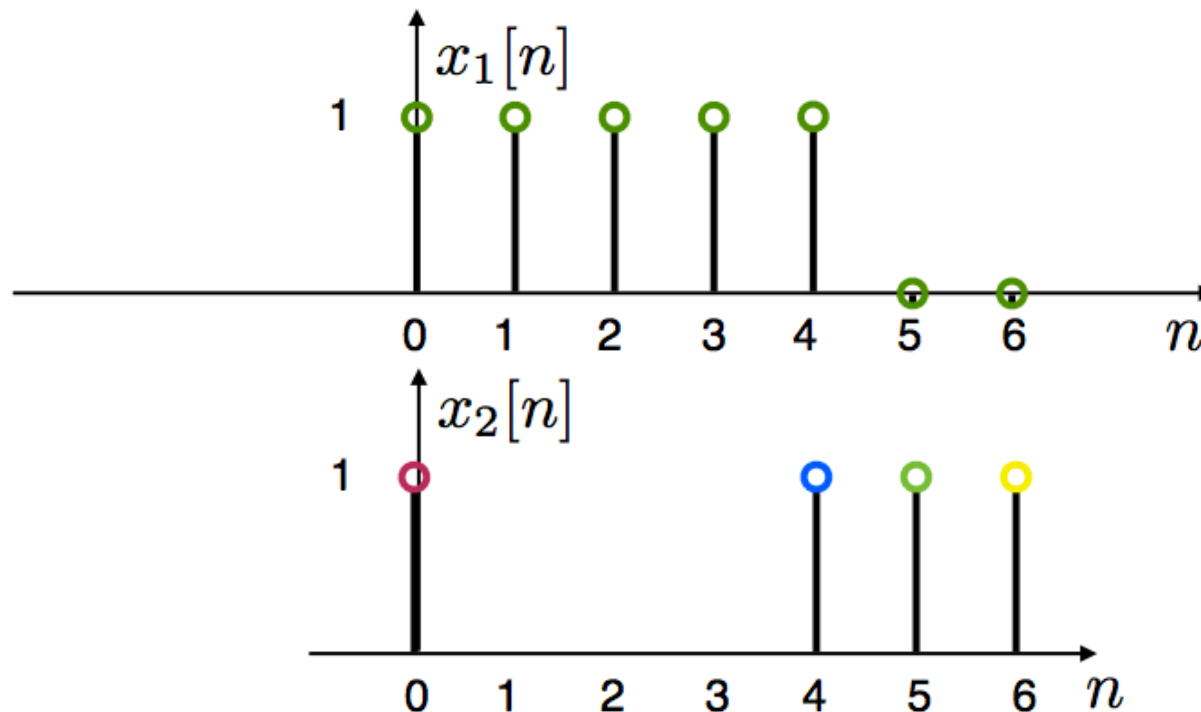
Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum

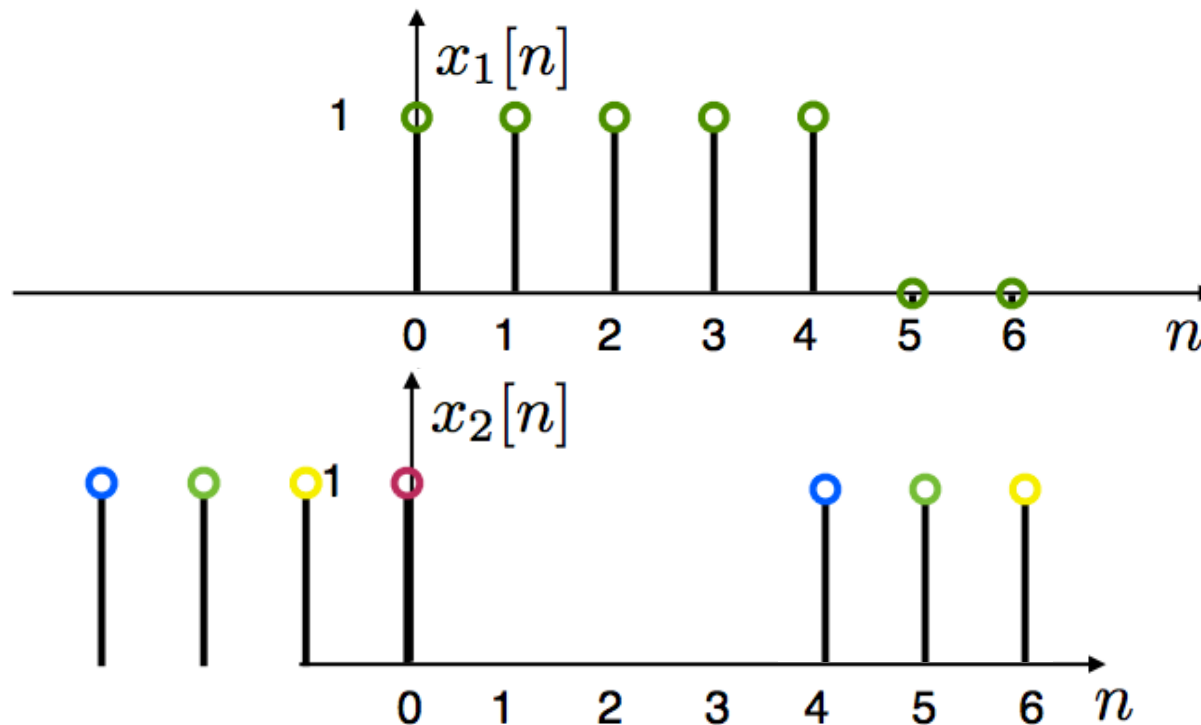
$$y[0]=2$$



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

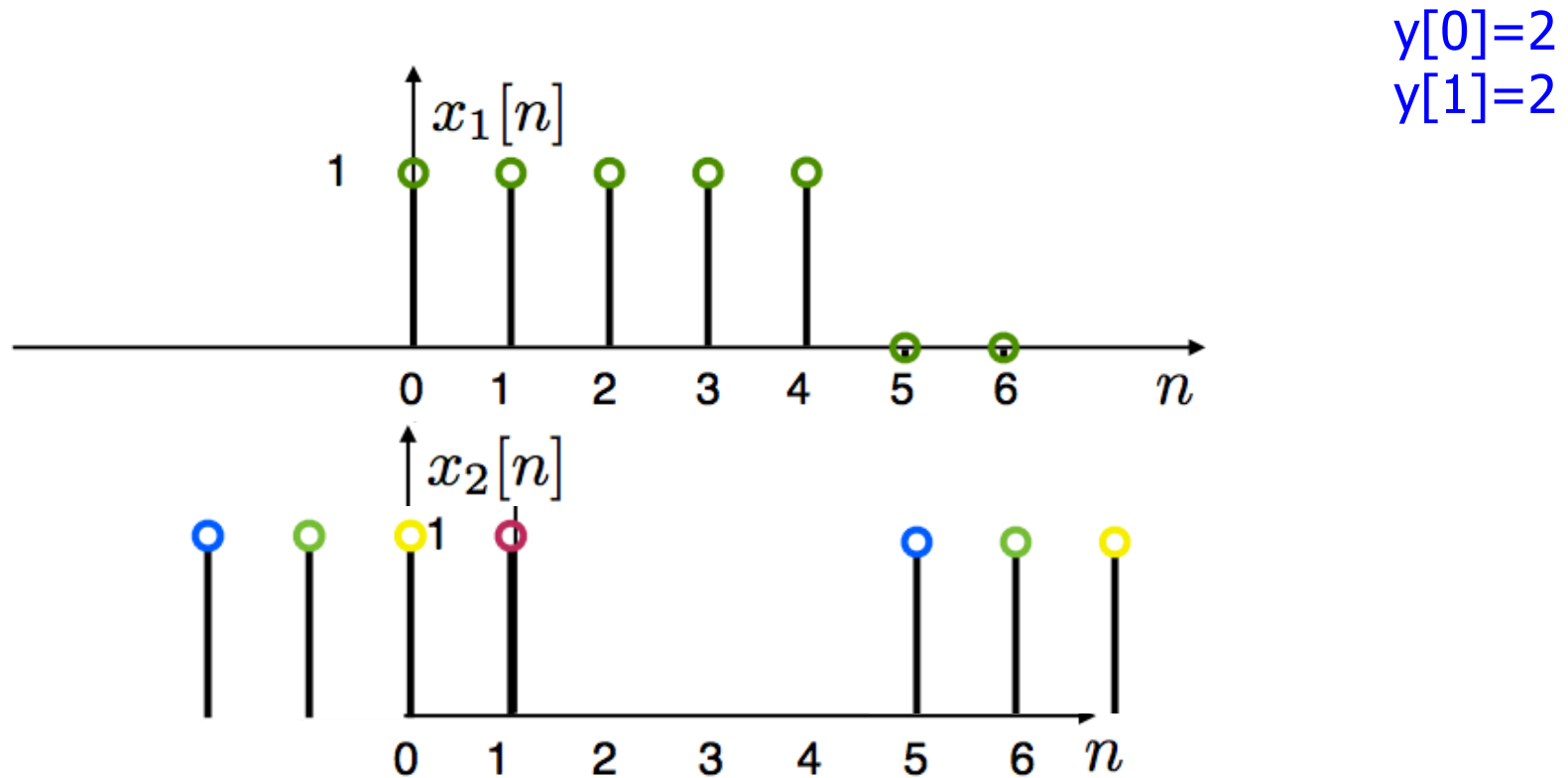
Compute Circular Convolution Sum

$$y[0]=2$$



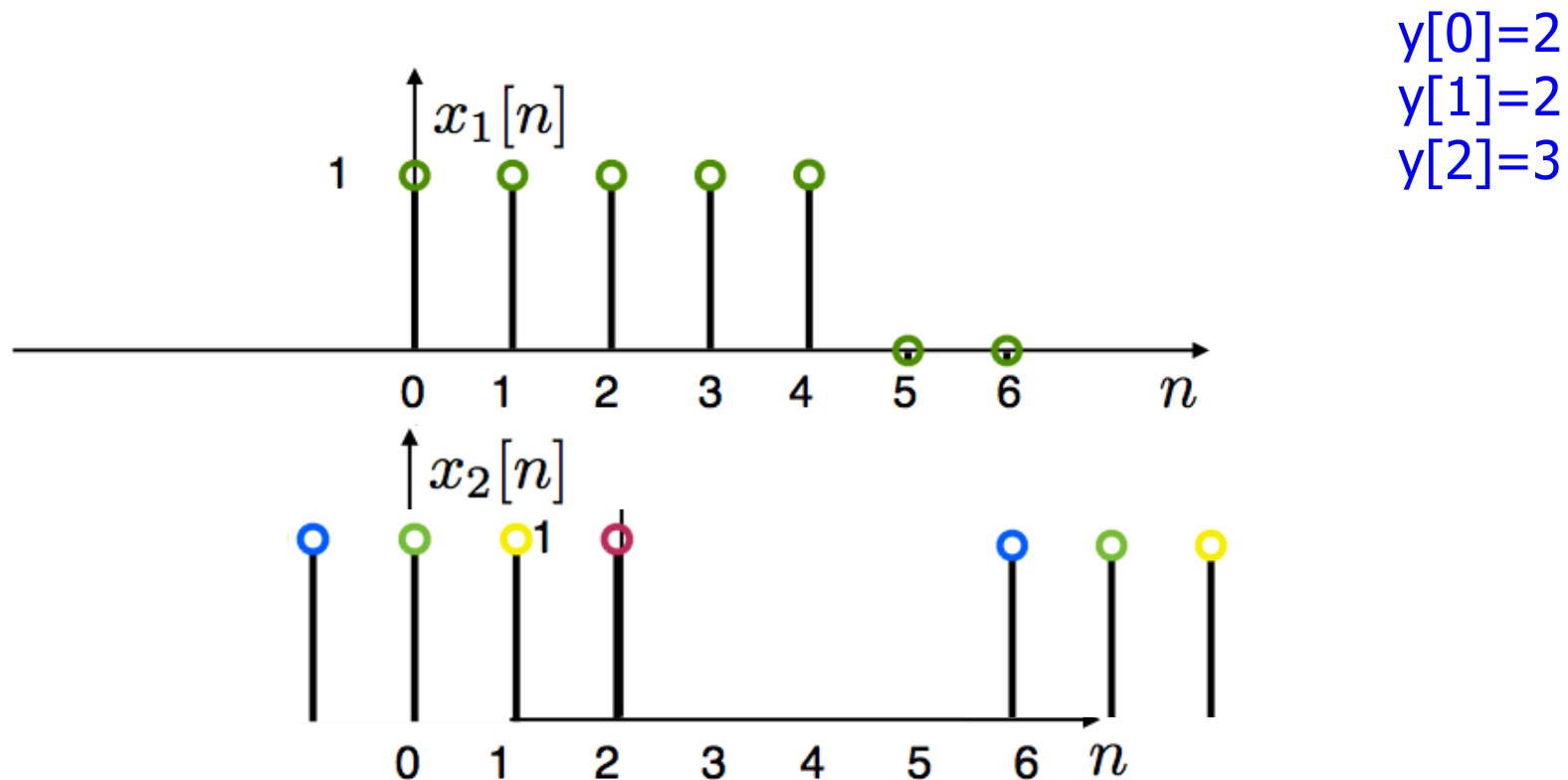
$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum



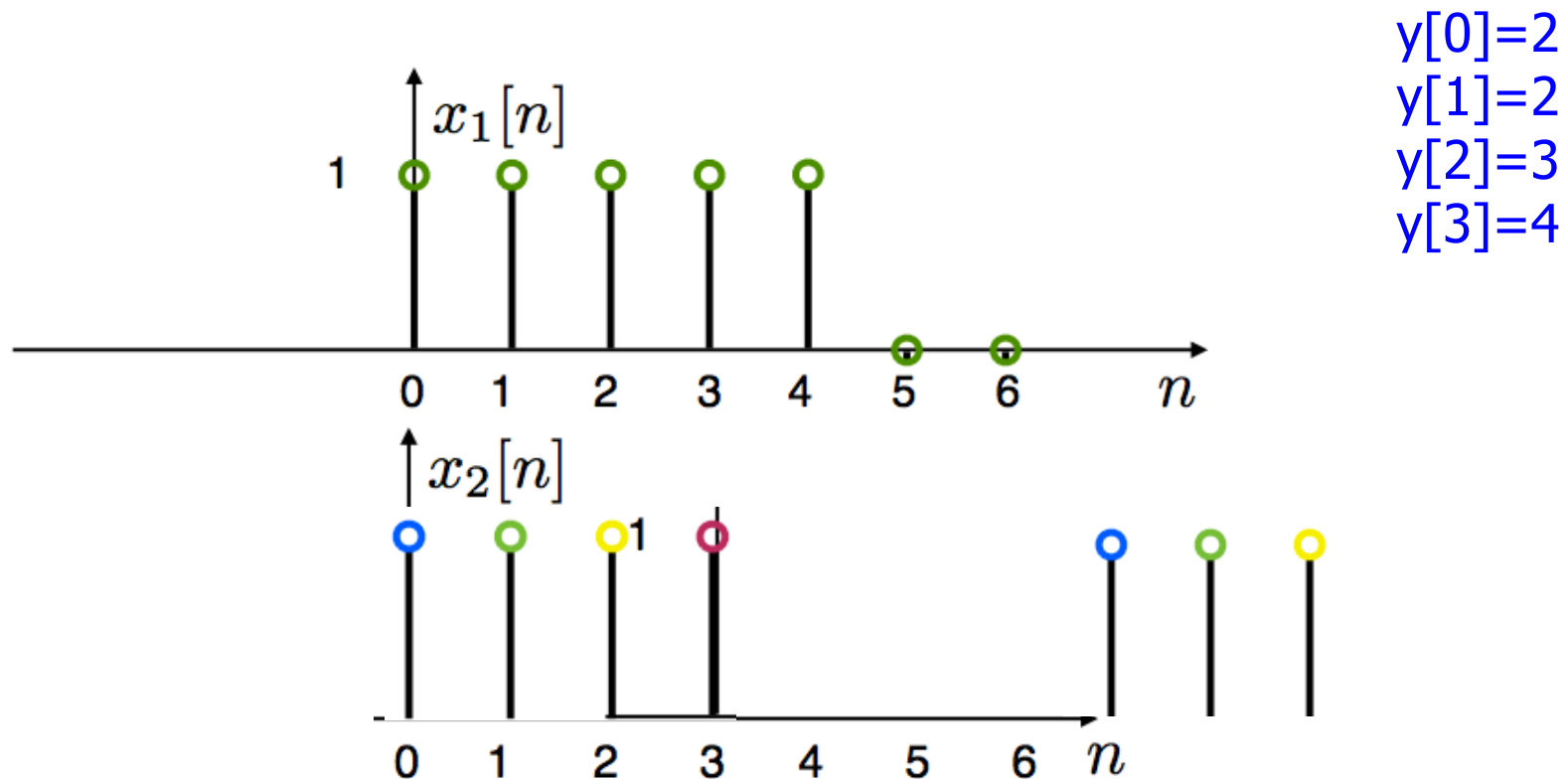
$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

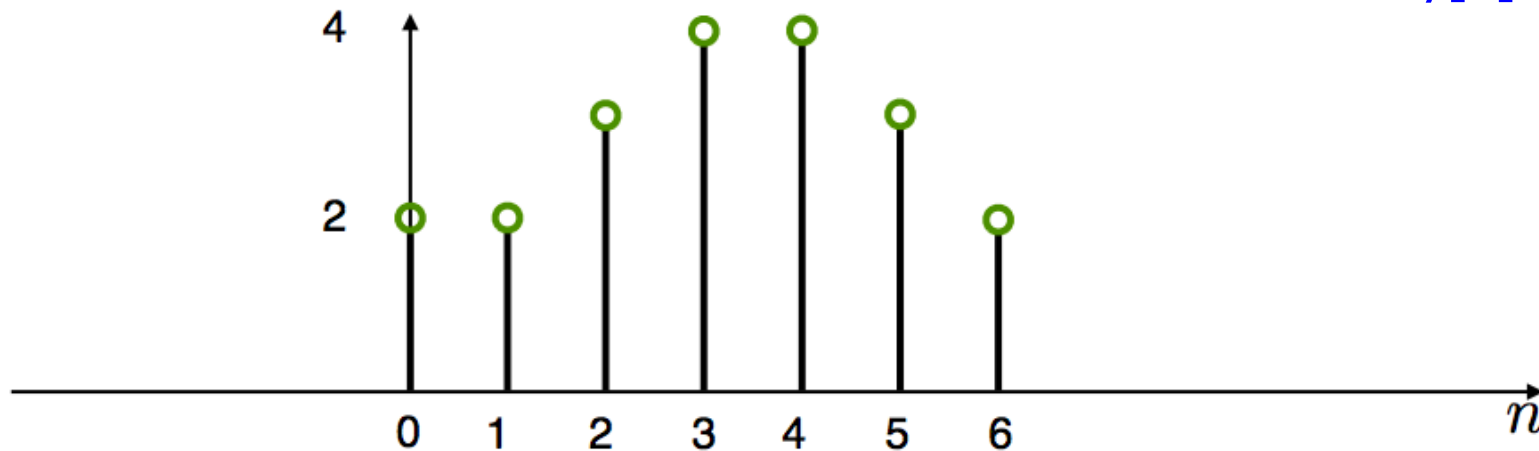
Compute Circular Convolution Sum



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Result

$$\begin{aligned}y[0] &= 2 \\y[1] &= 2 \\y[2] &= 3 \\y[3] &= 4\end{aligned}$$



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$



Circular Convolution

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \textcircled{N} x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

- Very useful!! (for linear convolutions with DFT)



Multiplication

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledast X_2[k]$$



Linear Convolution

□ Next....

- Using DFT, circular convolution is easy...
- ...But want linear convolution not circular
- So, show how to perform linear convolution with circular convolution
- Use DFT to do linear convolution
 - Because FFT is fast implementation of DFT, do linear convolution faster via the DFT



Big Ideas

- ❑ Discrete Fourier Transform (DFT)
 - For finite signals assumed to be zero outside of defined length
 - N-point DFT is sampled DTFT at N points
 - Useful properties allow easier linear convolution
- ❑ DFT Properties
 - Inherited from DFS, but circular operations!



Admin

- HW 7 out now
 - Due Sunday