ESE 531: Digital Signal Processing

Lec 17: March 31, 2020

Discrete Fourier Transform

Today

- Discrete Fourier Series
- Discrete Fourier Transform (DFT)
- DFT Properties
- Circular Convolution

Discrete Fourier Series

Reminder: Eigenvalue (DTFT)

- $x[n]=e^{j\omega n}$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$
$$= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k]$$
$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$
$$= H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase

Discrete Fourier Series

- Definition:
  - Consider N-periodic signal:

$$\tilde{x}[n+N] = \tilde{x}[n] \quad \forall n$$

  - Frequency-domain also periodic in N:

$$\tilde{X}[k+N] = \tilde{X}[k] \quad \forall k$$

  - "~" indicates periodic signal/spectrum

8.1 in text

Discrete Fourier Series

- Define:

$$W_N \triangleq e^{-j2\pi/N}$$

- DFS:

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}$$
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]W_N^{kn}$$

Discrete Fourier Series

$W_N \triangleq e^{-j2\pi/N}$

- Properties of W_N:
  - $W_N^0 = W_N^N = W_N^{2N} = ... = 1$
  - $W_N^{k+r} = W_N^k W_N^r$ and, $W_N^{k+N} = W_N^k$

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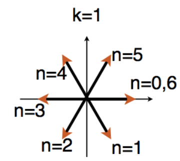
- Example: W_N^{kn} (N=6)



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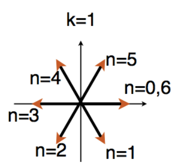
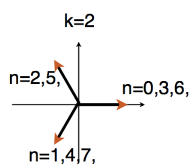




Discrete Fourier Transform

- By convention, work with one period:

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X[k] \triangleq \begin{cases} \tilde{X}[k] & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Same, but different!

Discrete Fourier Transform

- The DFT

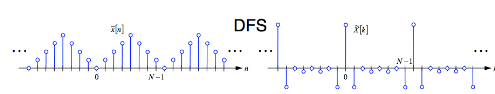
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$$ Inverse DFT, synthesis

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$ DFT, analysis

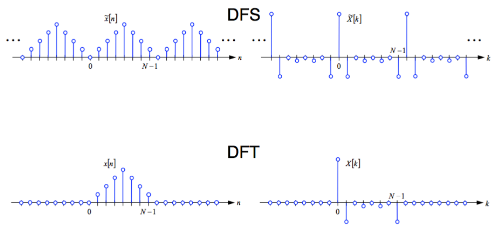
- It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \le n \le N-1$$
$$X[k] = 0 \quad \text{outside } 0 \le k \le N-1$$

DFS vs. DFT

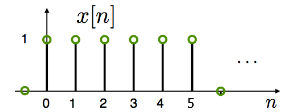


DFS vs. DFT



Example

$W_N \triangleq e^{-j2\pi/N}$



$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

N=6

Example

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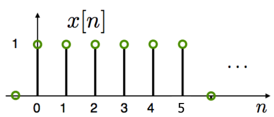


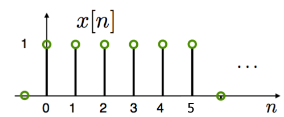
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$W_6 = e^{-j\frac{2\pi}{6}}$$

$$X[k] = \sum_{n=0}^{5} (W_6^k)^n$$

Example

$W_N \triangleq e^{-j2\pi/N}$



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$$\rightarrow \quad X[0] = \sum_{n=0}^{5} (W_6^0)^n = 6$$

Example

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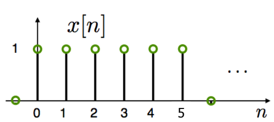

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Example

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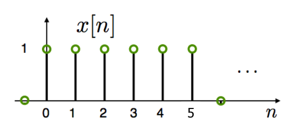

X[1]=?

$$X[1] = \sum_{n=0}^{5} (W_6)^n = \sum_{n=0}^{5} \left(e^{-j\frac{2\pi}{6}} \right)^n$$

Example

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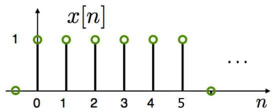


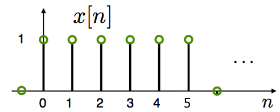
X[1]=?

$$X[1] = \sum_{n=0}^{5} (W_6)^n = \sum_{n=0}^{5} \left(e^{-j\frac{2\pi}{6}}\right)^n$$

$$= 1 + e^{-j\frac{2\pi}{6}} + e^{-j\frac{4\pi}{6}} + e^{-j\frac{6\pi}{6}} + e^{-j\frac{8\pi}{6}} + e^{-j\frac{10\pi}{6}}$$

Example

$W_N \triangleq e^{-j2\pi/N}$



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$$= e^{-j\frac{5\pi}{6}}\left(e^{j\frac{5\pi}{6}} + e^{j\frac{3\pi}{6}} + e^{j\frac{\pi}{6}} + e^{-j\frac{\pi}{6}} + e^{-j\frac{3\pi}{6}} + e^{-j\frac{5\pi}{6}}\right)$$

Example

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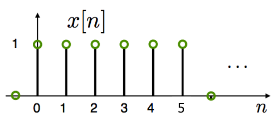

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$$= e^{-j\frac{5\pi}{6}}\left(2\cos\left(\frac{\pi}{6}\right) + 2\cos\left(\frac{3\pi}{6}\right) + 2\cos\left(\frac{5\pi}{6}\right)\right)$$ =0!

Discrete Fourier Series

$W_N \triangleq e^{-j2\pi/N}$

- Properties of WN:
  - $W_N^0 = W_N^N = W_N^{2N} = \ldots = 1$
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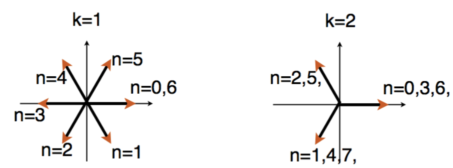
- Example: W_N^{kn} (N=6)



Example

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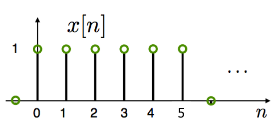


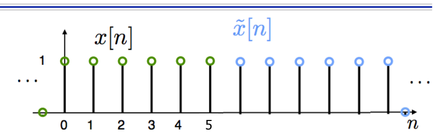
$$X[k] = \begin{cases} \sum_{n=0}^{5} W_5^{nk} & k = 0,1,2,3,4,5 \\ 0 & else \end{cases}$$

$$= 6\delta[k]$$

"6-point" DFT

Example

$W_N \triangleq e^{-j2\pi/N}$



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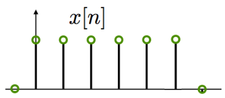
Example

$W_N \triangleq e^{-j2\pi/N}$

- Q: What if we take N=10?
- A: $X[k] \;=\; \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.



Example

$W_N \triangleq e^{-j2\pi/N}$

- Q: What if we take N=10?
- A: $X[k] \;=\; \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.

$$X[k] = \begin{cases} \sum_{n=0}^{9} x[n] W_{10}^{nk} & k = 0,1,2,3,4,5,6,7,8,9 \\ 0 & else \end{cases}$$

"10-point" DFT

Example

$W_N \triangleq e^{-j2\pi/N}$

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- A: $X[k] \;=\; \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.

$$X[k] = \begin{cases} \sum_{n=0}^{5} W_{10}^{nk} & k = 0,1,2,3,4,5,6,7,8,9 \\ 0 & else \end{cases}$$

"10-point" DFT

Example

- Now, sum from n=0 to 9

$$X[k] = \sum_{n=0}^{9} x[n] W_{10}^{nk} = \sum_{n=0}^{5} W_{10}^{nk}$$

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$$X[k] = \sum_{n=0}^{9} x[n] W_{10}^{nk} = \sum_{n=0}^{5} W_{10}^{nk}$$

$$= \sum_{n=0}^{5} \left(e^{-j\frac{2\pi}{10}k}\right)^{n} = \frac{1-\left(e^{-j\frac{2\pi}{10}k}\right)^{6}}{1-e^{-j\frac{2\pi}{10}k}} = \frac{1-e^{-j\frac{6\pi}{5}k}}{1-e^{-j\frac{2\pi}{10}k}}$$

Example

- Now, sum from n=0 to 9

$$X[k] = \sum_{n=0}^{9} x[n] W_{10}^{nk} = \sum_{n=0}^{5} W_{10}^{nk}$$

$$= \sum_{n=0}^{5} \left(e^{-j\frac{2\pi}{10}k}\right)^{n} = \frac{1-\left(e^{-j\frac{2\pi}{10}k}\right)^{6}}{1-e^{-j\frac{2\pi}{10}k}} = \frac{1-e^{-j\frac{6\pi}{5}k}}{1-e^{-j\frac{2\pi}{10}k}}$$

$$= \frac{e^{-j\frac{3\pi}{5}k}\left(e^{j\frac{3\pi}{5}k}-e^{-j\frac{3\pi}{5}k}\right)}{e^{-j\frac{\pi}{10}k}\left(e^{j\frac{\pi}{10}k}-e^{-j\frac{\pi}{10}k}\right)} = e^{-j\frac{\pi}{2}k}\frac{\sin\left(\frac{3\pi}{5}k\right)}{\sin\left(\frac{\pi}{10}k\right)}$$

"10-point" DFT

DFT vs. DTFT

- For finite sequences of length N:
  - The N-point DFT of x[n] is:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} \qquad 0 \le k \le N-1$$

  - The DTFT of x[n] is:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \qquad -\infty < \omega < \infty$$

DFT vs. DTFT

- The DFT are samples of the DTFT at N equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega = k\frac{2\pi}{N}} \qquad 0 \le k \le N-1$$

DFT vs DTFT

- Back to example

$$X[k] = \sum_{n=0}^{5} W_{10}^{nk} = e^{-j\frac{\pi}{2}k} \frac{\sin\left(\frac{3\pi}{5}k\right)}{\sin\left(\frac{\pi}{10}k\right)}$$

DFT vs DTFT

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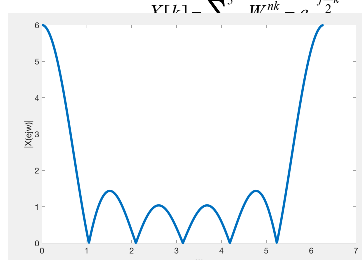

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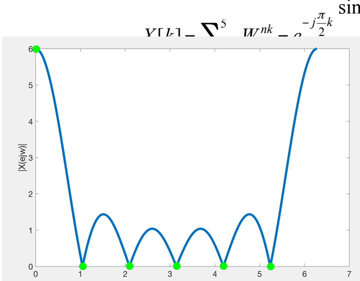

"6-point" DFT

Use `fftshift` to center around dc

DFT vs DTFT

- Back to example

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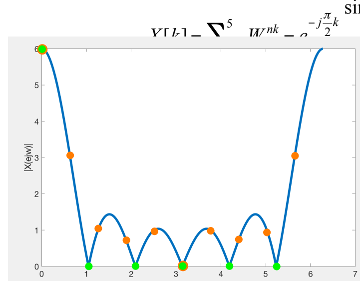
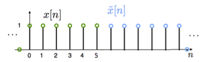


"6-point" DFT

"10-point" DFT

Use `fftshift` to center around dc

DFT Examples



$$X[k] = \begin{cases} \sum_{n=0}^{5} W_5^{nk} & k = 0,1,2,3,4,5 \\ 0 & else \end{cases}$$

$$= 5\delta[k]$$

"6-point" DFT

$$X[k] = \begin{cases} e^{-j\frac{\pi}{2}k} \dfrac{\sin\left(\frac{3\pi}{5}k\right)}{\sin\left(\frac{\pi}{10}k\right)} & k = 0,1,2,3,4,5,6,7,8,9 \\ 0 & else \end{cases}$$

DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{DFT, analysis}$$

DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$N \cdot x^*[n] = N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^*$$

DFT and Inverse DFT

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$$\begin{aligned} N \cdot x^*[n] &= N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^* \\ &= N\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}\right)^* \end{aligned}$$

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DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$\begin{aligned} &\left((a+jb)+(c+jd)\right)^* \\ &= \left((a+c)+j(b+d)\right)^* \\ &= (a+c) - j(b+d) \\ &= (a-jb)+(c-jd) \\ &= (a+jb)^* + (c+jd)^* \end{aligned}$$

$$\begin{aligned} N \cdot x^*[n] &= N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^* \\ &= N\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}\right)^* \\ &= \sum_{k=0}^{N-1} X^*[k] W_N^{kn} \end{aligned}$$

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DFT and Inverse DFT

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DFT and Inverse DFT

- So

$$\mathcal{DFT}\{X^*[k]\} = N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^*$$

DFT and Inverse DFT

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$$\mathcal{DFT}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{DFT}\{X^*[k]\}\right)^*$$

DFT and Inverse DFT

- So

$$\mathcal{DFT}\{X^*[k]\} = N\left(\mathcal{DFT}^{-1}\{X[k]\}\right)^*$$

$$\mathcal{DFT}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{DFT}\{X^*[k]\}\right)^*$$

- Implement IDFT by:
  - Take complex conjugate
  - Take DFT
  - Multiply by 1/N
  - Take complex conjugate

DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

freq, X[k] DFT matrix time, x[n]

DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

IDFT:

$$\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \begin{pmatrix} W_N^{-00} & \cdots & W_N^{-0k} & \cdots & W_N^{-0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-n0} & \cdots & W_N^{-nk} & \cdots & W_N^{-n(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)0} & \cdots & W_N^{-(N-1)k} & \cdots & W_N^{-(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix}$$

DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

IDFT:

$$\begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N} \begin{pmatrix} W_N^{-00} & \cdots & W_N^{-0k} & \cdots & W_N^{-0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-n0} & \cdots & W_N^{-nk} & \cdots & W_N^{-n(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{-(N-1)0} & \cdots & W_N^{-(N-1)k} & \cdots & W_N^{-(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix}$$

N^2 complex multiples

DFT as Matrix Operator

- Can write compactly as

$$\mathbf{X} = \mathbf{W}_N \, \mathbf{x}$$
$$\mathbf{x} = \frac{1}{N} \mathbf{W}_N^* \, \mathbf{X}$$

Properties of the DFT

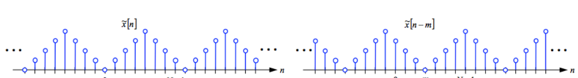
- Properties of DFT inherited from DFS
- Linearity

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

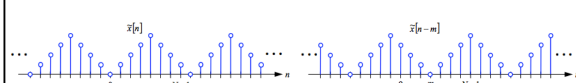
- Circular Time Shift

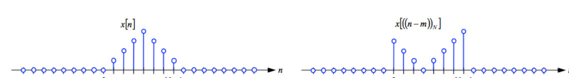
$$x[((n-m))_N] \leftrightarrow X[k] e^{-j(2\pi/N)km} = X[k] W_N^{km}$$

Circular Shift



Circular Shift

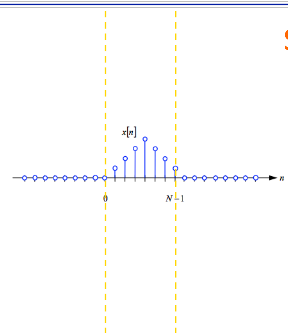




m=?

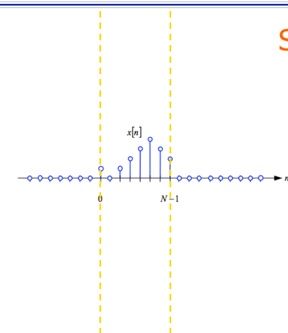
Circular Shift

Shift by 3



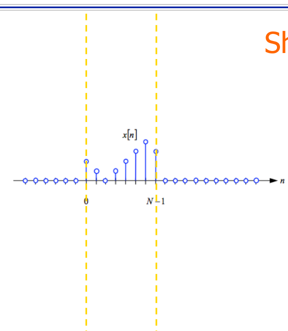
Circular Shift

Shift by 3



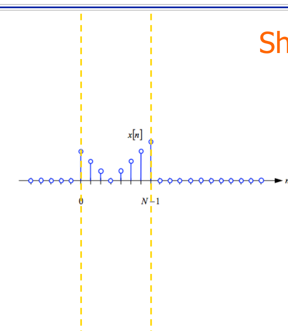
Circular Shift

Shift by 3



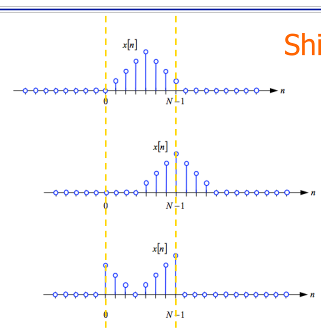
Circular Shift

Shift by 3



Circular Shift

Shift by 3



Properties of DFT

- Circular frequency shift

$$x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- Complex Conjugation

$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

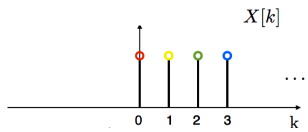
- Conjugate Symmetry for Real Signals

If x[n] real

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

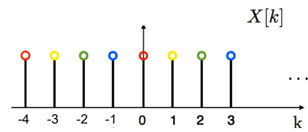
4-point DFT
–Symmetry



$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$

Example: Conjugate Symmetry

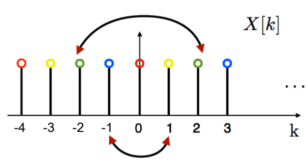
4-point DFT
–Symmetry



$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$

Example: Conjugate Symmetry

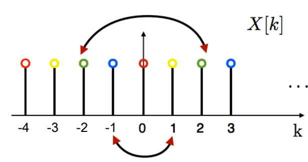
4-point DFT
–Symmetry



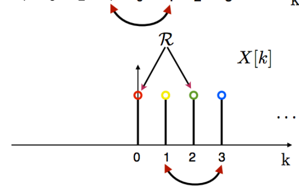
$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$

Example: Conjugate Symmetry

4-point DFT
–Symmetry



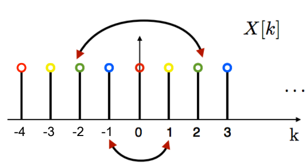
4-point DFT
–Symmetry



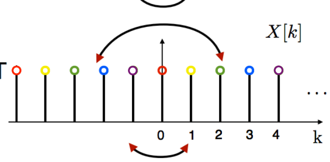
$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$

Example: Conjugate Symmetry

4-point DFT
–Symmetry



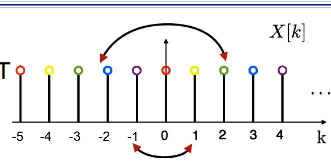
5-point DFT
–Symmetry



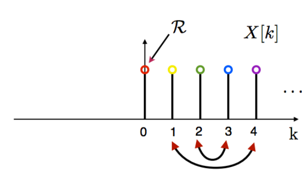
$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$

Example

5-point DFT
–Symmetry



5-point DFT
–Symmetry



Properties of the DFS/DFT

Discrete Fourier Series			Discrete Fourier Transform		
Property	***N*-periodic sequence**	***N*-periodic DFS**	**Property**	***N*-point sequence**	***N*-point DFT**
	$\tilde{x}[n]$ $\tilde{x}_1[n],\ \tilde{x}_2[n]$	$\tilde{X}[k]$ $\tilde{X}_1[k],\ \tilde{X}_2[k]$		$x[n]$ $x_1[n],\ x_2[n]$	$X[k]$ $X_1[k],\ X_2[k]$
Linearity	$a\tilde{x}_1[n]+b\tilde{x}_2[n]$	$a\tilde{X}_1[k]+b\tilde{X}_2[k]$	Linearity	$ax_1[n]+bx_2[n]$	$aX_1[k]+bX_2[k]$
Duality	$\tilde{X}[n]$	$N\tilde{x}[-k]$	Duality	$X[n]$	$N\,x[((-k))_N]$
Time Shift	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$	Circular Time Shift	$x[((n-m))_N]$	$W_N^{km}X[k]$
Frequency Shift	$W_N^{-ln}\tilde{x}[n]$	$\tilde{X}[k-l]$	Circular Frequency Shift	$W_N^{-ln}x[n]$	$X[((k-l))_N]$
Periodic Convolution	$\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$	$\tilde{X}_1[k]\tilde{X}_2[k]$	Circular Convolution	$\sum_{m=0}^{N-1}x_1[m]x_2[((n-m))_N]$	$X_1[k]X_2[k]$
Multiplication	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{l=0}^{N-1}\tilde{X}_1[l]\tilde{X}_2[k-l]$	Multiplication	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{l=0}^{N-1}X_1[l]X_2[((k-l))_N]$
Complex Conjugation	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$	Complex Conjugation	$x^*[n]$	$X^*[((-k))_N]$

Properties (Continued)

Time-Reversal and Complex Conjugation	$\tilde{x}^*[-n]$	$\tilde{X}^*[k]$	Time-Reversal and Complex Conjugation	$x^*[((-n))_N]$	$X^*[k]$
Real Part	$\mathrm{Re}\{\tilde{x}[n]\}$	$\tilde{X}_{ep}[k]=\frac{1}{2}(\tilde{X}[k]+\tilde{X}^*[-k])$	Real Part	$\mathrm{Re}\{x[n]\}$	$X_{ep}[k]=\frac{1}{2}(X[k]+X^*[((-k))_N])$
Imaginary Part	$j\,\mathrm{Im}\{\tilde{x}[n]\}$	$\tilde{X}_{op}[k]=\frac{1}{2}(\tilde{X}[k]-\tilde{X}^*[-k])$	Imaginary Part	$j\,\mathrm{Im}\{x[n]\}$	$X_{op}[k]=\frac{1}{2}(X[k]-X^*[((-k))_N])$
Even Part	$\tilde{x}_{ep}[n]=\frac{1}{2}(\tilde{x}[n]+\tilde{x}^*[-n])$	$\mathrm{Re}\{\tilde{X}[k]\}$	Even Part	$x_{ep}[n]=\frac{1}{2}(x[n]+x^*[((-n))_N])$	$\mathrm{Re}\{X[k]\}$
Odd Part	$\tilde{x}_{op}[n]=\frac{1}{2}(\tilde{x}[n]-\tilde{x}^*[-n])$	$j\,\mathrm{Im}\{\tilde{X}[k]\}$	Odd Part	$x_{op}[n]=\frac{1}{2}(x[n]-x^*[((-n))_N])$	$j\,\mathrm{Im}\{X[k]\}$
Symmetry for Real Sequence	$\tilde{x}[n]=\tilde{x}^*[n]$	$\tilde{X}[k]=\tilde{X}^*[-k]$ $\begin{cases}\mathrm{Re}\{\tilde{X}[k]\}=\mathrm{Re}\{\tilde{X}[-k]\}\\ \mathrm{Im}\{\tilde{X}[k]\}=-\mathrm{Im}\{\tilde{X}[-k]\}\end{cases}$ $\begin{cases}\lvert\tilde{X}[k]\rvert=\lvert\tilde{X}[-k]\rvert\\ \angle\tilde{X}[k]=-\angle\tilde{X}[-k]\end{cases}$	Symmetry for Real Sequence	$x[n]=x^*[n]$	$X[k]=X^*[((-k))_N]$ $\begin{cases}\mathrm{Re}\{X[k]\}=\mathrm{Re}\{X[((-k))_N]\}\\ \mathrm{Im}\{X[k]\}=-\mathrm{Im}\{X[((-k))_N]\}\end{cases}$ $\begin{cases}\lvert X[k]\rvert=\lvert X[((-k))_N]\rvert\\ \angle X[k]=-\angle X[((-k))_N]\end{cases}$
Parseval's Identity	$\sum_{n=0}^{N-1}\tilde{x}_1[n]\tilde{x}_2^*[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}_1[k]\tilde{X}_2^*[k]$ $\sum_{n=0}^{N-1}\lvert\tilde{x}[n]\rvert^2=\frac{1}{N}\sum_{k=0}^{N-1}\lvert\tilde{X}[k]\rvert^2$		Parseval's Identity	$\sum_{n=0}^{N-1}x_1[n]x_2^*[n]=\frac{1}{N}\sum_{k=0}^{N-1}X_1[k]X_2^*[k]$ $\sum_{n=0}^{N-1}\lvert x[n]\rvert^2=\frac{1}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2$	

Duality

If $x \xrightarrow{DFT} X$, then $\{X[n]\}_{n=0}^{N-1} \xrightarrow{DFT} N\{x[((-k))_N]\}_{k=0}^{N-1}$

Duality

If $x \xrightarrow{DFT} X$, then $\{X[n]\}_{n=0}^{N-1} \xrightarrow{DFT} N\{x[((-k))_N]\}_{k=0}^{N-1}$

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k],$$

$$\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k].$$

Proof of Duality

DFT of $\{x[n]\}_{n=0}^{N-1}$ is $X[k]=\sum_{p=0}^{N-1}x[p]e^{-j\frac{2\pi}{N}kp}$; $\quad k\le 0\le N-1$

DFT of $\{X[n]\}_{n=0}^{N-1}$ is $\underbrace{\sum_{n=0}^{N-1}\sum_{p=0}^{N-1}x[p]e^{-j\frac{2\pi}{N}pn}}_{X[n]}e^{-j\frac{2\pi}{N}kn}$, $\quad k\le 0\le N-1$

$$=\sum_{p=0}^{N-1}x[p]\underbrace{\sum_{n=0}^{N-1}e^{-j\frac{2\pi}{N}(p+k)n}}_{N \text{ for } ((p+k))_N=0,\ 0 \text{ otherwise}}$$

$((p+k))_N=0$ for $0\le p$ & $k\le N-1$ $\Rightarrow p=((-k))_N$

$p=-k+mN=((-k))_N+rN+mN=((-k))_N$ *because* $0\le p\le N-1$

$\therefore$ DFT of $\{X[n]\}_{n=0}^{N-1}$ is $N\{x[((-k))_N]\}_{k=0}^{N-1}$

Circular Convolution

- Circular Convolution:

$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

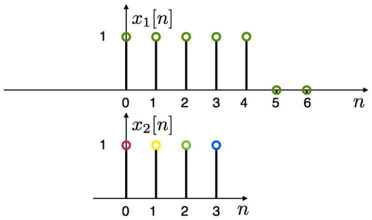
For two signals of length N

Note: Circular convolution is commutative

$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$

Compute Circular Convolution Sum

N=7



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

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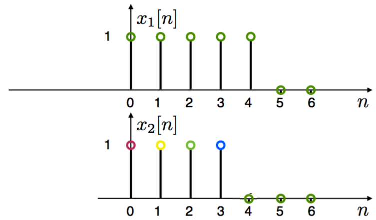
Adapted from M. Lustig, EECS Berkeley

Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

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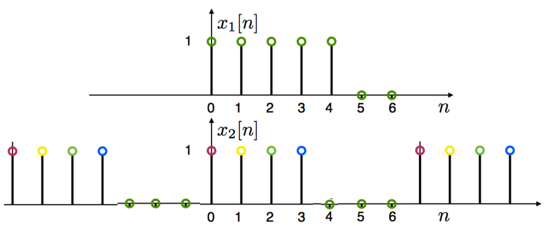
Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

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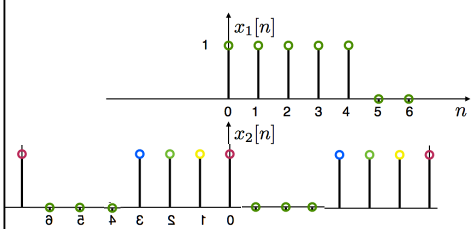
Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

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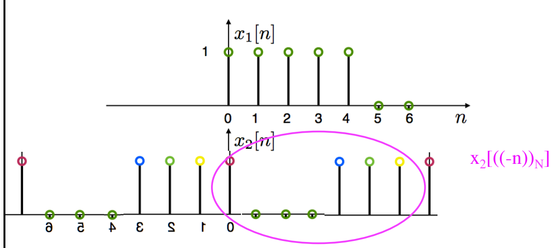
Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

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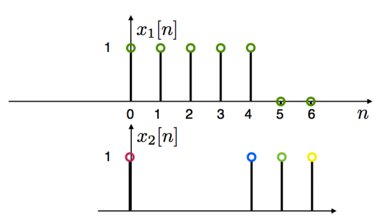
Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

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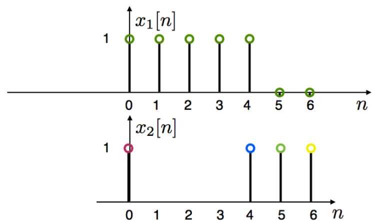
Compute Circular Convolution Sum

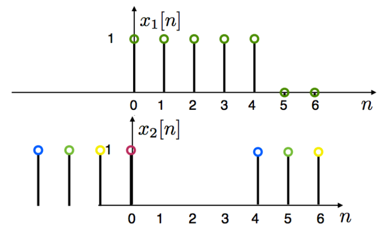
y[0]=2



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

Compute Circular Convolution Sum

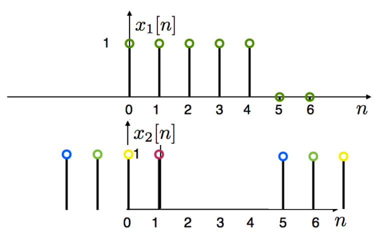
y[0]=2



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

Compute Circular Convolution Sum

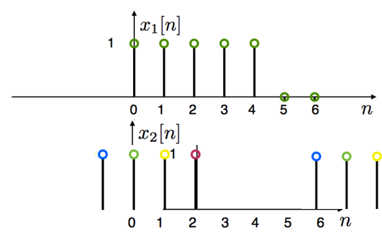
y[0]=2
y[1]=2



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

Compute Circular Convolution Sum

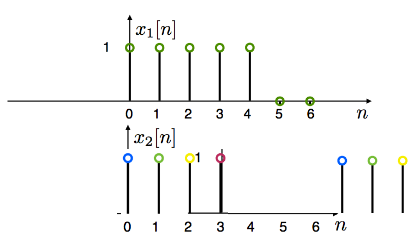
y[0]=2
y[1]=2
y[2]=3



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

Compute Circular Convolution Sum

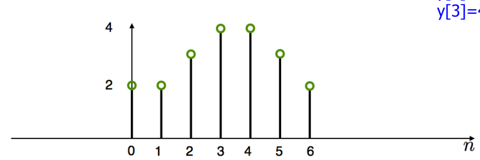
y[0]=2
y[1]=2
y[2]=3
y[3]=4



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

Result

y[0]=2
y[1]=2
y[2]=3
y[3]=4



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

Circular Convolution

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

  - Very useful!! (for linear convolutions with DFT)

Multiplication

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledN X_2[k]$$

Linear Convolution

- Next....
  - Using DFT, circular convolution is easy…
  - …But want linear convolution not circular
  - So, show how to perform linear convolution with circular convolution
  - Use DFT to do linear convolution
    - Because FFT is fast implementation of DFT, do linear convolution faster via the DFT

Big Ideas

- Discrete Fourier Transform (DFT)
  - For finite signals assumed to be zero outside of defined length
  - N-point DFT is sampled DTFT at N points
  - Useful properties allow easier linear convolution
- DFT Properties
  - Inherited from DFS, but circular operations!

Admin

- HW 7 out now
  - Due Sunday